# Counting black hole microstates as open string flux vacua

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## Outline

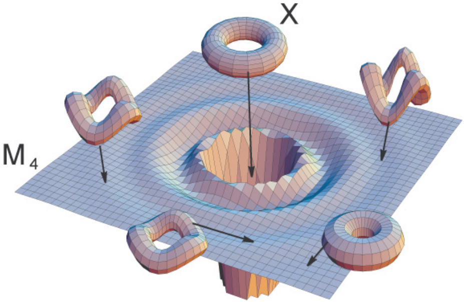
Setting and formulation of the problem

Black hole microstates and open string flux vacua

Counting BPS states (or open string flux vacua)

# Setting and formulation of the problem

# Setting




| | | |
|---|---|---|
| • IIA on Calabi-Yau $X$ | $\rightsquigarrow$ | 4d $\mathcal{N}=2$ sugra + vect. mult. |
| • D6-D4-D2-D0 BPS bound st. (D-branes + gauge flux) | $\rightsquigarrow$ | BPS black holes with magn. and el. charges $(p^0, p^A, q_A, q_0)$ |

# General problem

Count number of BPS states with given charge $(p^i, q_i)$, $\equiv \Omega(p,q)$, and compare with gravity prediction (beyond Bekenstein-Hawking).
$\rightsquigarrow$ Conjecture [Ooguri-Strominger-Vafa]:

$$\Omega(p,q) = \int d\phi \, e^{\mathcal{F}(p,\phi)+\pi q_i \phi^i} \quad \text{(+ exp. small)}$$

where $\mathcal{F}(p,\phi) \equiv F_{top}(g, t^A) + c.c.$, and $F_{top}$ = topological string free energy:

$$F_{\rm top} = \frac{1}{g^2} D_{ABC} t^A t^B t^C + c_{2A} t^A + \sum_{h,\beta} N_\beta^h e^{2\pi i \beta_A t^A} g^{2h-2}.$$

with following substitutions:

$$g \to \frac{4\pi i}{p^0 + i\phi^0}, \quad t^A \to \frac{p^A + i\phi^A}{p^0 + i\phi^0}$$

Motivation: leading order saddle point approximation reproduces Bekenstein-Hawking-Wald entropy:

$$\int d\phi \, e^{\mathcal{F}(p,\phi)+\pi q_i \phi^i} \approx e^{S_{BHW}(p,q)}$$

## Specific problem

To test conjecture, we need to find $\Omega(p, q)$ *beyond* leading order.

Problem: difficult in general.

Known cases:

- D4-D2-D0 dual to pert. heterotic states $\rightsquigarrow$ "small" black holes [Dabholkar,Dabholkar-Denef-Moore-Pioline]
- some D4-D2-D0 in noncompact CY (BH interpret.?) [Vafa,Aganagic-Ooguri-Saulina-Vafa]
- $T^6$, $K3 \times T^2$ [Dijkgraaf-Moore-Verlinde-Verlinde,Strominger-Shih-Yin]

This talk: *arbitrary* D4-D2-D0 system on *arbitrary, compact* Calabi-Yau.

$\rightsquigarrow$ $\Omega(p, q)$ in large $q_0$ (D0-charge) expansion, computable using... "landscape techniques" (flux vacua counting methods of [Ashok-Douglas, Denef-Douglas])

## Specific problem for D4-D2-D0

For general D4-D2-D0:

$$\mathcal{F}(p,\phi) = \frac{\pi}{\phi^0}\left(-\frac{1}{6}(D_{ABC}p^A p^B p^C + c_{2A}p^A) + \frac{1}{2}D_{ABC}p^A\phi^B\phi^C\right) + O\left(\frac{1}{(\phi^0)^{2h}}e^{\beta\cdot p/\phi^0}\right)$$

Saddle point of OSV integral:

$$\phi^0 \sim -\sqrt{\frac{p^3}{q_0}}, \qquad \phi^A \sim D^{AB}q_B\phi^0$$

So $q_0 \to \infty \Leftrightarrow \phi^0 \to 0 \rightsquigarrow$ instanton corrections exp. small.

Hence we need to compute

$$Z \equiv \sum_q \Omega(p,q)e^{\pi\phi\cdot q}$$

and show that this reduces to $e^{\mathcal{F}_0(p,\phi)}$ at small $\phi^0$, where $\mathcal{F}_0$ corresponds to $\mathcal{F}$ above without the instanton corrections.

# Black hole microstates and open string flux vacua

## From flux to charge

Consider D4-brane wrapped on divisor $P = p^A J_A$, with $N$ D0-branes bound to it and $U(1)$ flux $F$ turned on.

- Total D0-brane charge:

$$-q_0 = N - \frac{1}{2}F^2 - \frac{\chi}{24}$$

  where

$$\chi = P^3 + c_2 \cdot P = \text{Euler characteristic of } P$$

- Conserved D2-brane charges:

$$q_A = -J_A \cdot F.$$

  Here scalar product = intersection product on $H^2(P)$.

Note: typically dim $H^2(P) \gg$ dim $H^2(X)$, so many different fluxes $F$ can give rise to equal charges! $\Rightarrow$ need to count different flux realizations of given charge.

# Supersymmetric configurations

Supersymmetry requires [Marino-Minasian-Moore-Strominger]:

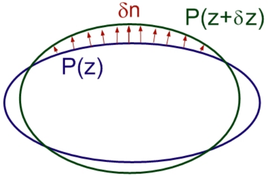
$$F^{(0,2)} = F^{(2,0)} = 0$$

- For generic fluxes $F$ at generic points in the D4-brane deformation moduli space, this will not be satisfied.
- Exceptions: fluxes $F$ which are pulled back from $H^2(X) = H^{1,1}(X)$: for these, $F^{(0,2)} = 0$ identically.
- But many $F \in H^2(P)$ not pulled back from $H^2(X)$. Then condition $F^{0,2} = 0$ imposes $h^{2,0}$ equations on the $h^{2,0}$ geometric moduli of $P$.

  $\rightsquigarrow$ generically restricts moduli to set of isolated points: "open string flux vacua".

# Divisor moduli

Divisor $P$ has deformation moduli space $\mathcal{M}$, parametrized locally by coordinates $z^i$, $i = 1, \ldots, n$.



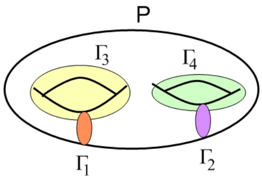

1-1 correspondence infinitesimal holomorphic deformations of $P$ (given by holomorphic normal vector fields $\delta_i n$ on $P$) and $H^{2,0}(P)$:

$$\omega_i^{2,0} = \Omega^{3,0} \cdot \delta_i n$$

$\Rightarrow$ $n = h^{2,0}(P)$.

# Special geometry structure

Moduli space $\mathcal{M}$ has "$\mathcal{N} = 1$ special geometry" [Lerche-Mayr-Warner]:




- Choose basis $C_\alpha$ of $H_2(P)$, and corresponding 3-chains $\Gamma_\alpha$ with $\partial\Gamma_\alpha|_P = C_\alpha$ (and possibly other, fixed, $z$-independent boundary components).
- Define chain periods

$$\Pi_\alpha(z) \equiv \int_{\Gamma_\alpha(z)} \Omega.$$

- Then

$$\partial_i \Pi_\alpha = \int_{\delta_i \Gamma_\alpha} \Omega = \int_{C_\alpha} \delta_i n \cdot \Omega = \int_{C_\alpha} \omega_i^{2,0}$$

# Special geometry structure

- Natural Kähler metric on $\mathcal{M}$ determined by periods:

$$g_{i\bar{j}} \equiv \int_P \omega_i \wedge \bar{\omega}_{\bar{j}} = \partial_i \Pi_\alpha \, Q^{\alpha\beta} \, \bar{\partial}_{\bar{j}} \bar{\Pi}_\beta = \partial_i \bar{\partial}_{\bar{j}} (\Pi_\alpha \, Q^{\alpha\beta} \, \bar{\Pi}_\beta)$$

  where $Q^{\alpha\beta} \equiv (Q_{\alpha\beta})^{-1}$ and

$$Q_{\alpha\beta} \equiv C_\alpha \cdot C_\beta,$$

  i.e. the intersection form on $H_2(P)$.
- $\partial_i \Pi$ is period vector of $(2,0)$-form, and by Griffiths transversality:

$$\nabla_i \partial_j \Pi \sim (1,1).$$

  By orthogonality of $(2,0)$ and $(1,1)$ forms, this implies e.g.

$$\nabla_i \partial_j \Pi_\alpha Q^{\alpha\beta} \partial_k \Pi_\beta = 0.$$

## Susy conditions from superpotential

Given flux $F \rightsquigarrow$ Poincaré dual 2-cycle $\Sigma_F$ on $P$.

Expand $\Sigma_F = m^\alpha C_\alpha$.

Define superpotential

$$W_F(z) \equiv m^\alpha \Pi_\alpha(z).$$

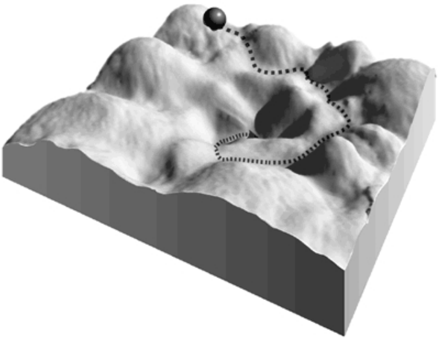
Then

$$\partial_i W = m^\alpha \int_{C_\alpha} \omega_i = \int_{\Sigma_F} \omega_i = \int_P F \wedge \omega_i$$

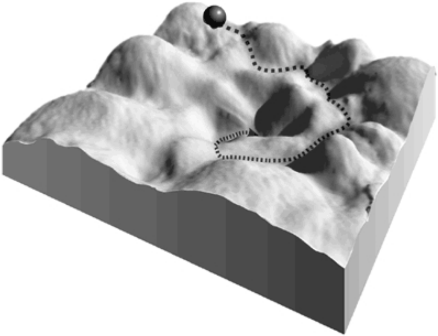
so

$$\partial_i W(z) = 0 \Leftrightarrow F^{0,2} = 0 \Leftrightarrow \text{susy}.$$

# Closed string landscape



# Open string landscape



Same form $\Rightarrow$ same techniques applicable.

# Counting BPS states (or open string flux vacua)

## Counting critical points

At fixed $F$, number of isolated critical points of $W_F$ given by

$$\int_{\mathcal{M}} d^{2n}z\, \delta^{2n}(\partial W_F)\, |\det \nabla_i \partial_j W_F|^2.$$

Determinant ensures each isolated zero of the delta function contributes $+1$ to the integral.

At any such critical point, the divisor is frozen, so the only remaining moduli are the positions of the $N$ D0-branes bound to $P$.

Upon quantization $\rightsquigarrow$ number (index) of susy ground states corresponding to this critical point $=$ Euler characteristic of Hilbert scheme of $N$ points on $P$. This is $p_\chi(N)$, where

$$\sum_N p_\chi(N) q^{N-\chi/24} = \frac{1}{\eta(q)^\chi}$$

# Black hole partition sum

Using this, we get for the OSV partition sum

$$
\begin{aligned}
Z &= \sum_q \Omega(p,q)\, e^{-\pi\phi^0 q_0 - \pi\phi^A q_A} \\
&= \sum_{N,F} p_\chi(N)\, e^{\pi\phi^0 (N - \frac{1}{2}F^2 - \frac{\chi}{24}) - \pi\Phi\cdot F} \\
&\quad \times \int_{\mathcal{M}} d^{2n}z\, \delta^{2n}(\partial W_F)\, |\det \nabla_i \partial_j W_F|^2 \\
&= \frac{1}{\eta^\chi(e^{\pi\phi^0})} \int_{\mathcal{M}} d^{2n}z \sum_m e^{-\pi\frac{\phi^0}{2} Q_{\alpha\beta} m^\alpha m^\beta - \pi\Phi_\alpha m^\alpha} \\
&\quad \times \delta^{2n}(m^\alpha \partial_i \Pi_\alpha)\, |\det m^\alpha \nabla_i \partial_j \Pi_\alpha|^2
\end{aligned}
$$

## Gaussian form of $Z$

Both the delta-function and the determinant can be rewritten as integrals of exponentials linear in $m^\alpha$:

$$\begin{aligned}
\delta^{2n}(m^\alpha \partial_i \Pi_\alpha) &= \int d^{2n}\lambda \, e^{i\pi m^\alpha (\lambda^i \partial_i \Pi_\alpha + \bar\lambda^{\bar i} \bar\partial_{\bar i} \bar\Pi_\alpha)} \\
|\det m^\alpha \nabla_i \partial_j \Pi_\alpha|^2 &= \frac{1}{\pi^{2n}} \int d^n\theta \, d^n\psi \, d^n\bar\theta \, d^n\bar\psi \\
&\quad \times e^{\pi m^\alpha (\nabla_i \partial_j \Pi_\alpha \, \theta^i \psi^j + \bar\nabla_{\bar i} \bar\partial_{\bar j} \bar\Pi_\alpha \, \bar\theta^{\bar i} \bar\psi^{\bar j})} .
\end{aligned}$$

Second integral is over fermionic variables.

$\rightsquigarrow$ Gaussian ensemble with boson-fermion-fermion interactions.

# Large $q_0$ (small $\phi^0$) approximation

$\rightsquigarrow$ large fluxes $m^\alpha$

$\rightsquigarrow$ continuum approximation: replace $\sum_m \to \int d^{b_2} m$.

$\Rightarrow$ Integral over $m^\alpha$ is straightforward Gaussian:

$$Z = \frac{1}{\eta^\chi(e^{\pi\phi^0})} \int d^{2n}z\, d^{2n}\lambda\, d^n\theta\, d^n\psi\, d^n\bar{\theta}\, d^n\bar{\psi}\, \frac{1}{\pi^{2n}} \left(\frac{2}{\phi^0}\right)^{b_2/2}$$
$$\times e^{\frac{\pi}{2\phi^0}\left(\Phi_\alpha - i\lambda^i\partial_i\Pi_\alpha - \nabla_i\partial_j\Pi_\alpha\psi^i\theta^j + \mathrm{c.c.}\right) Q^{\alpha\beta} \left(\Phi_\beta - i\lambda^i\partial_i\Pi_\beta - \nabla_i\partial_j\Pi_\beta\psi^i\theta^j + \mathrm{c.c.}\right)}$$

At first sight: 25 complicated cross terms in exponential $\to$ ???

But recall $\partial_i\Pi \sim (2,0)$, $\nabla_i\partial_j\Pi \sim (1,1)$ and $\Phi \sim (1,1)$, and only products of (1,1) with (1,1) or (2,0) with (0,2) can be nonzero.

$\Rightarrow$ Most cross terms are zero.

# Geometrization

The only nontrivial intersection products contributing are:

$$
\begin{aligned}
\partial_i \Pi_\alpha \, Q^{\alpha\beta} \, \bar{\partial}_{\bar{j}} \bar{\Pi}_\beta &= g_{i\bar{j}} \\
\nabla_i \partial_j \Pi_\alpha \, Q^{\alpha\beta} \, \bar{\nabla}_{\bar{k}} \bar{\partial}_{\bar{l}} \bar{\Pi}_\beta &= R_{i\bar{k}j\bar{l}}
\end{aligned}
$$

with $R$ the curvature of $g$.

Hence exponential becomes simply

$$
e^{\frac{\pi}{\phi^0}\left(\frac{1}{2}\Phi^2 - g_{i\bar{j}}\lambda^i\bar{\lambda}^{\bar{j}} + R_{i\bar{k}j\bar{l}}\psi^i\bar{\psi}^k\theta^j\bar{\theta}^l\right)}.
$$

Doing gaussian integrals over $\lambda$ and $\psi$ turns this in

$$
\pi^n \, e^{\frac{\pi}{2\phi^0}\Phi^2} (\det g_{i\bar{j}})^{-1} \det(R_{i\bar{k}j\bar{l}}\theta^j\bar{\theta}^l)
$$

Integrated over $\theta$ and combined with $d^{2n}z$, this produces measure

$$
\pi^n \, e^{\frac{\pi}{2\phi^0}\Phi^2} \det R
$$

with $R_i^k \equiv \frac{i}{2} R^k_{ij\bar{l}} dz^j \wedge d\bar{z}^{\bar{l}}$ the curvature 2-form on $\mathcal{M}$.

# Final result

After doing a modular transformation on $\eta$:

$$\frac{1}{\eta^{\chi}(e^{\pi\phi^0})} = \left(\frac{\phi^0}{2}\right)^{\chi/2} \frac{1}{\eta^{\chi}(e^{\frac{4\pi}{\phi^0}})},$$

where $\chi = b_2(P) - 2b_1 + 2 = P^3 + c_2 \cdot P$, we get for small $\phi^0$,

$$\begin{aligned} Z &\approx \left(\frac{\phi^0}{2}\right)^{1-b_1} \frac{e^{\frac{\pi}{2\phi^0}\Phi^2}}{\eta^{\chi}(e^{\frac{4\pi}{\phi^0}})} \int_{\mathcal{M}} \frac{1}{\pi^n} \det R \\ &\approx \hat{\chi}(\mathcal{M}) \left(\frac{\phi^0}{2}\right)^{1-b_1} \exp\left(\frac{\pi}{\phi^0}(-\frac{1}{6}(P^3 + c_2 \cdot P) + \frac{1}{2}\Phi^2)\right). \end{aligned}$$

where we defined the "Euler characteristic" $\hat{\chi}(\mathcal{M})$ of divisor moduli space as

$$\hat{\chi}(\mathcal{M}) \equiv \int_{\mathcal{M}} \frac{1}{\pi^n} \det R.$$

# Comparison to OSV

• OSV prediction partition function derived from topological string, dropping all instanton corrections:

$$Z_{OSV} = \exp\left(\frac{\pi}{\phi^0}\left(-\frac{1}{6}(P^3 + c_2 \cdot P) + \frac{1}{2}\Phi^2\right)\right).$$

• our microscopic partition function at small $\phi^0$:

$$Z = \hat{\chi}(\mathcal{M}) \left(\frac{\phi^0}{2}\right)^{1-b_1} Z_{OSV}$$

So essentially confirms conjecture in this regime, up to prefactor refinement.

Recall that degeneracies are given as Laplace transform of $Z$, so this encodes an infinite series of $1/N$ corrections to the Bekenstein-Hawking entropy formula!

## The $\hat{\chi}$ factor

Results agree with [Shih-Yin] for $X = T^6$ and $X = T^2 \times K3$ in limit $\phi^0 \to 0$, provided

- For $T^6$: $\hat{\chi}(p) = 1/(p)^3$ with $(p)^3 \equiv P^3/6$
  so $\hat{\chi} = 1/p_1 p_2 p_3$ if $P = \sum_i p_i [z_i = 0]$ on $(T^2)^3$.
- For $T^2 \times K3$: $\hat{\chi}(p_{T^2}, p_{K3}) = ((p_{K3})^2 + 4)/2p_{T^2}$

Shih-Yin result is totally independent derivation, so this gives *prediction* for $\hat{\chi}$.

Nonintegral? = crazy? No!

$\hat{\chi}(\mathcal{M})$ need not be integral, because $\mathcal{M}$ has singularities. Similar for closed strings: $\hat{\chi}(\mathcal{M}) = 1/5$ for mirror quintic.

Problem: because of singularities, not known how to compute $\hat{\chi}$ directly.

Are these values plausible?

## The $\hat{\chi}$ factor

For $T^6$, relevant moduli space = deformations of $P$ modulo $T^6$ translations (since translations never get frozen by flux and trivial $T^6$ factor otherwise gives $\hat{\chi} = 0$).

Divisors in class $P = (p_1, p_2, p_3)$ as above can be described as zero locus of linear combination of theta functions:

$$P : \sum_{\vec{\mu}=1}^{\vec{p}} a_{\vec{\mu}} \, \Theta_{\vec{\mu},\vec{p}}(\vec{\tau}, \vec{z}) = 0$$

$$\Theta_{\vec{\mu},\vec{p}}(\vec{\tau}, \vec{z}) \equiv \prod_{i=1}^{3} \Theta_{\mu_i,p_i}(\tau_i, z_i); \quad \Theta_{\mu,p}(\tau, z) \equiv \sum_{k \in \mu + p\mathbb{Z}} e^{\pi i \tau k^2 + 2\pi i k z}.$$

Naive moduli space $= \{a_{\vec{\mu}}\}/\mathbb{C}^* = \mathbb{CP}^{p_1 p_2 p_3 - 1}$, but there is residual $\mathbb{Z}_{p_1}^2 \times \mathbb{Z}_{p_2}^2 \times \mathbb{Z}_{p_3}^2$ translation symmetry group to mod out: $z_i \to z_i + \frac{n_i + m_i \tau_i}{p_i}$ acts as permutations and phase shifts on the $a_{\vec{\mu}}$.

$\Rightarrow \mathcal{M} = \mathbb{CP}^{p_1 p_2 p_3 - 1} / \mathbb{Z}_{p_1}^2 \times \mathbb{Z}_{p_2}^2 \times \mathbb{Z}_{p_3}^2$.

# The $\hat{\chi}$ factor

Thus *if* $\hat{\chi}(\mathbb{CP}^{p_1 p_2 p_3 - 1}) = \chi(\mathbb{CP}^{p_1 p_2 p_3 - 1})$, we get

$$\hat{\chi}(\mathcal{M}) = \frac{p_1 p_2 p_3}{(p_1 p_2 p_3)^2} = \frac{1}{p_1 p_2 p_3}$$

$=$ exactly as required for compatibility with [Shih-Yin].

Similar for $T^2 \times K3$, but now only residual translation symmetry for $T^2$ factor:

$$\hat{\chi}(\mathcal{M}) = \frac{\frac{P^3}{6} + \frac{c_2 \cdot P}{12}}{p_{T^2}^2} = \frac{(p_{K3})^2 + 4}{2 p_{T^2}}.$$

again exactly as required.

$\Rightarrow$ Conjecture: For $\mathcal{M} = \mathbb{CP}^d$, $\hat{\chi}(\mathcal{M}) = d + 1$.

# Application to counting D7-D3 flux vacua

By letting D-branes fill noncompact part of spacetime, (i.e. $D0 \to D3$, $D4 \to D7$), these computations can be adapted to counting open string flux vacua of IIB orientifold compactifications in weak string coupling limit.

Large $q_0$ = large D3-tadpole $L = \chi(X_4)/24$.

Related subtleties: divisor $P$ constrained to be compatible with involution, and flux $F$ must be odd.

Joint treatment open + closed flux vacua at arbitrary coupling: F-theory on fourfold. OSV counts subsector (counting components weighted by their Euler characteristics).

Main conclusion: working on black holes and the topological string = working on landscape statistics!