Integrality of instanton numbers and p-adic topological sigma-model

Maxim Kontsevich, Albert Schwarz, Vadim Vologodsky

We prove that instanton numbers for quintic become integers after multiplication by 12. Our methods can be used to prove integrality in general case.

We give an expression of instanton numbers in terms of Frobenius map.

In basic example of mirror symmetry one starts with holomorphic curves on the quintic $\mathcal{A}$ (in other words, one considers A-model on this quintic). Mirror simmetry relates this A-model to the B-model on $\mathcal{B}$ (on the quintic factorized with respect to the finite symmetry group $(\mathbb{Z}_5)^3$). Instanton numbers are defined mathematically in terms of Gromov-Witten invariants, i.e. by means of integration over the moduli space of curves. The moduli space is an orbifold, therefore it is not clear that this construction gives integer numbers. The mirror conjective proved by Givental permits us to express the instanton numbers in terms of solutions of Picard-Fuchs equations on mirror quinfic $\mathcal{B}$; however, integrality is not clear from this expression.

Starting point :formula relating the Yukawa coupling Y in canonical coordinates (normalized Yukawa coupling) to instanton numbers n_k:

$$Y(q) = const + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d}$$

(This formula is valid in the case when the moduli space of complex structures is one-dimensional; for the quintic const=5.)

Lemma .

Let us assume that

$$\sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d} = \sum_{k=1}^{\infty} m_k q^k \qquad (1)$$

If the numbers n_k are integers then for every prime number p the difference $m_{kp}-m_k$ is divisible by $p^{3(\alpha+1)}$ where α is defined as the number of factors equal to p in the prime decomposition of k. Conversely, if

$$p^{3(\alpha+1)} | m_{kp} - m_k \qquad (2)$$

for every prime p and every k, the numbers m_k are integers.

Proof. The following expression for m_k in terms of n_k can be derived from (1):

$$m_k = \sum_{d|k} n_d d^3 \tag{3}$$

Let us suppose that $k = p^\alpha r$ where r is not divisible by p. Then

$$m_{kp} - m_k = \sum n_{p^{\alpha+1}s} (p^{\alpha+1} s)^3 \tag{4}$$

(We are summing over all divisors of kp that are not divisors of k, i.e. over all $p^{\alpha+1}s$, where $s|r$.)

We see immediately that $m_{kp} - m_k$ is divisible by $p^{3(\alpha+1)}$.

To derive integrality of n_k from this property one can use the Moebius inversion formula

$$n_k k^3 = \sum_{d|k} \mu(d) m_{\frac{k}{d}}$$

p- adic numbers -completion of $\mathbb{Q}$ with respect to p-adic norm

$$||x||_p = p^{-\text{ord}_p x}$$

$\text{ord}_p x = \alpha$ if $x = p^\alpha s$

Integer p-adic number $= ||x||_p \leq 1 =$

$$a_0 + a_1 p + ... + a_n p^n + ...$$

Rational number is an integer iff all corresponding p-adic numbers are integers

Formulation of the lemma in terms of p-adic numbers.

The numbers n_k defined in terms of $Y(q)$ are integers if and only if for every prime p there exists such a series $\psi(q) = \sum s_k q^k$ having p-adic integer coefficients that

$$Y(q) - Y(q^p) = \delta^3 \psi(q) \qquad (5)$$

Here δ stands for the logarithmic derivative $q\frac{d}{dq}$

Proof. The coefficients of the decomposition of $Y(q) - Y(q^p)$ into q-series are equal to

$$m_k - m_{\frac{k}{p}} = m_{p^\alpha s} - m_{p^{\alpha-1}s}$$

if $k = p^\alpha s$ and $\alpha \geqslant 1$, and to m_k if k is not divisible by p. From the other side, the coefficients of $\delta^3\psi(q)$ are equal to $k^3 s_k$.

Frobenius map transforms a q into q^p; this map is an automorphism of a field of characteristic p:

$$(x+y)^p = x^p + y^p \mod p$$

One can consider Frobenius map on p-adic numbers; corresponding map φ^* on functions of p-adic variable q transforms $f(q)$ into $f(q^p)$.

The lemma can be rewritten in the form

$$Y - \varphi^* Y = \delta^3 \psi.$$

B-model $\to$ the theory of variations of complex structure and corresponding variations of Hodge filtration on cohomology

$\mathcal{M}$ - moduli space of complex structures on a Calabi -Yau threefold

Cohomology does not depend on the choice of complex structure, but the Hodge filtration on cohomology does depend.

Gauss-Manin connection $\to$ flat connection on the bundle of three-dimensional cohomology

If we work with mirror quintic $\mathcal{B}$ (and more generally if the moduli space $\mathcal{M}$ of complex structures on a Calabi -Yau threefold is one-dimensional) then one can find a coordinate q on $\mathcal{M}$ (canonical coordinate) and a symplectic basis $e^0(q), e^1(q), e_1(q), e_0(q)$ in three-dimensional cohomology that satisfy the following conditions:

1) Gauss-Manin connection acts in the following way:

$$\nabla_\delta e^0 = 0,$$

$$\nabla_\delta e^1 = e^0,$$

$$\nabla_\delta e_1 = Y(q) e^1,$$

$$\nabla_\delta e_0 = e_1$$

Here δ stands for logarithmic derivative $q\frac{d}{dq}$ and ∇_δ for corresponding Gauss-Manin covariant derivative

2)

$$e^0 \in \mathcal{F}^0 \cap \mathcal{W}_3$$

$$e^1 \in \mathcal{F}^1 \cap \mathcal{W}_1$$

$$e_1 \in \mathcal{F}^2 \cap \mathcal{W}_{-1}$$

$$e_0 \in \mathcal{F}^3 \cap \mathcal{W}_{-3}$$

Here $\mathcal{F}$ stands for the Hodge filtration and $\mathcal{W}_k$ for the weight filtration

3) $Y(q) = \text{const} + \sum m_k q^k$ is a q-series with integer coefficients m_k.

We assume that $q = 0$ corresponds to maximally unipotent boundary point of the space $\mathcal{M}$ and that we are working in a neighborhood of this point.

We considered quintic as a Calabi-Yau complex threefold. However, it is possible to consider it over $\mathbb{Z}$ or over the ring $\mathbb{Z}_p$ of integer p-adic numbers and to study its cohomology over $\mathbb{Z}_p$. It is natural to assume that in p-adic setting all of the statements 1)-4) remain valid. This can be proven (although the proof is not simple).

Notice, that the Yukawa coupling in p-adic situation remains the same, but the coefficients m_k are considered as p-adic integers.

In the p-adic theory there exists an additional symmetry: the Frobenius map. Namely, the map $q \to q^p$ of the moduli space of quintics into itself can be lifted to a homomorphism Fr of cohomology groups of corresponding quintics. Here q is considered as a p-adic integer; cohomology are taken with coefficients in $\mathbb{Z}_p$. We will express instanton numbers in terms of this map, assuming that $p > 3$.

Frobenius map Fr preserves the weight filtration. It does not preserve the Hodge filtration, but :

$$\mathrm{Fr}\mathcal{F}^s \subset p^s \mathcal{F}^0.$$

The Frobenius map is compatible with symplectic structure on 3-dimensional cohomology:

$$< \mathrm{Fr}a, \mathrm{Fr}b >= p^3 < a, b >,$$

where $< a, b >$ stands for the inner product of cohomology classes.

It is compatible with Gauss- Manin connection ∇:

$$\nabla_\delta \mathrm{Fr}a = p\mathrm{Fr}\nabla_\delta a$$

The matrix of Frobenius map is triangular; this follows from the fact that Fr preserves the weight filtration.

$$\text{Fr} e^0 = e^0,$$

$$\text{Fr} e^1 = p e^1 + p m_{12} e^0,$$

$$\text{Fr} e_1 = p^2 e_1 + p^2 m_{23} e^1 + p^2 m_{13} e^0,$$

$$\text{Fr} e_0 = p^3 e_0 + p^3 m_{34} e_1 + p^3 m_{24} e^1 + p^3 m_{14} e^0$$

where $m_{ij} \in \mathbb{Z}_p[q]$ are q-series with integer p-adic coefficients,

$$-m_{34} + m_{12} = 0, -m_{23} m_{34} + m_{24} + m_{13} = 0.$$

Using compatibilty of Fr with Gauss-Manin connection we obtain

$$pe^0 + p\delta m_{12} e^0 = pe^0.$$

This means that m_{12} is a constant; one can prove that this constant vanishes. Hence

$$m_{34} = m_{12} = 0,$$

$$m_{24} + m_{13} = 0.$$

Similarly,

$$Y(q) - Y(q^p) = \delta m_{23},$$

$$m_{23} + \delta m_{13} = 0.$$

$$m_{34} Y + \delta m_{24} = m_{23}, \delta m_{34} = 0, m_{24} + \delta m_{14} = m_{13}$$

We conclude,that

$$m_{23} = \delta m_{24}$$

,

$$2m_{13} = \delta m_{14},$$

Hence

$$Y(q^p) - Y(q) = \frac{1}{2}\delta^3 m_{14}$$

This means that

Instanton numbers are p-adic integers if $p > 3$.

Instanton numbers are genus 0 Gopakumar-Vafa invariants.

$$F' = \Sigma_{g\geq 0, d>0} GW_{g,d} \cdot q^d \lambda^{2g-z} =$$

$$\Sigma_{g\geq 0,d>0} GV_{g,d} \cdot \Sigma_{m\geq 1} \frac{1}{m} (2\sin\frac{m\lambda}{2})^{2g-2} q^{md}$$

GW Gromov-Witten invariants

GV Gopakumar-Vafa invariants

The condition of integrality of GV invariants:

$$F'(\lambda, q) - \frac{1}{p} F'(p\lambda, q^p) \in \nu^{-2}\mathbb{Z}_p[[\nu^2, q]]$$

where $\nu = 2\sin\frac{\lambda}{2}$.

Another form of integrality condition

$$G(z,q) - \frac{1}{p} G(z^p, q^p) \in \frac{1}{(z-1)^2} \cdot \mathbb{Z}[[z-1, q]]$$

where $G(z,q) = F'(e^{i\lambda}, q)$

p-adic B-model for arbitrary genus

Polarization of symplectic vector space= representation of the space as a direct sum of isotropic subspaces

Quantization $\to$ for every polarization we construct a vector space of states; state is defined up to a constant factor. Spaces of states for different polarizations can be identified (under certain conditions)

Middle-dimensional cohomology of a threefold can be considered as symplectic vector space. Partition function of B-model can be interpreted as a state in the corresponding space of states. In the construction of the space of states we use polarization ; to fix polarization we choose a point in moduli space, but the state corresponding to partition function does not depend on this choice.