TOPOLOGICAL MEMBRANES

with
Lilia Anguelova
Paul de Medeiros

hep-th/0507089

TOPOLOGICAL M-THEORY*

- topological theory on 7-manifolds with G_2-structure
- suggested as unification of A/B model topological strings
- unification of lower dim form theories of gravity

* hep-th/0411073
R. Dijkgraaf, A. Neitzke, S. Gukov, C. Vafa
hep-th/0412021
N. Nekrasov

HITCHIN'S THEORY

M with G_2-holonomy $\leftrightarrow$

associative 3-form ϕ

$d\phi = 0 \qquad d * \phi = 0$

classical action which extremizes on these conditions

$$V(\phi) = \int_M \phi \wedge * \phi$$

[$*$ in terms of ϕ]

variation $\phi = \phi_0 + dB$

$d\phi_0 = 0$

Reduction to 6d:

$M_7 = M_6 \times S^1$

Hitchin's ansatz

$\phi = \rho + k \wedge dt$

$*\phi = \hat{\rho} \wedge dt + \frac{1}{2} k \wedge k$

$V(\phi) = \int \phi \wedge * \phi =$

$= \frac{1}{2} \int k \wedge k \wedge k \, dt - \frac{i}{4} \int \Omega \wedge \bar{\Omega} \wedge dt$

$\Omega = \rho + i\hat{\rho}$

A and $B + \bar{B}$ model actions.

- topological $\mathcal{F}$-theory on Spin(7)-manifolds

Spin(7) holonomy metric

$\leftrightarrow$ Cayley 4-form Ψ

$d\Psi = 0$ $\qquad$ $*\Psi = \Psi$

- action is more difficult to define

For $CY \times T^2 = M_8$

$$V(\Psi) = \frac{1}{2}\int_{M_8} *\Psi \wedge \Psi$$

torus reduction of Cayley-form

$$V(\Psi) = \int_{M_6 \times T^2} dx\,dy \Big(\underbrace{\tfrac{1}{2} k \wedge k \wedge k}_{A} \; \underbrace{- \tfrac{i}{4} \Omega \wedge \bar{\Omega}}_{B+\bar{B}} \Big)$$

modular transformations of extra torus

$$\begin{pmatrix} dx \\ dy \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$ad - bc = 1$$

$$dX = \begin{pmatrix} dx \\ dy \end{pmatrix} \qquad \xi = \begin{pmatrix} \hat{s} \\ s \end{pmatrix}$$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad SL(2, \mathbb{Z}) \text{ invariant}$$

Cayley form

$$\Psi = dX^T J \xi + \frac{1}{2} dX^T J\, dX + \frac{1}{2} k \wedge k$$

modular invariant if

$$\begin{pmatrix} \hat{s} \\ s \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \hat{s} \\ s \end{pmatrix} \qquad k \to k$$

topological S-duality?

Construct topological theory of membranes wrapping associative cycles.

Strategy:

Similar to topological A-model.

Consider space of maps from 3-manifold to G_2.

Construct topologically invariant P.I. that localize on associative 3-cycles.

USE MATHAI - GUILLEN FORMALISM

FUNDAMENTAL FORMULATION OF TOPOLOGICAL M-THEORY ?

IS THERE A FORMULATION IN TERMS OF TOPOLOGICAL MEMBRANES ?

- membranes wrapping associative cycles contribute $N=1$ superpotential *
- in 7d membrane dual to string → topological G_2-string *

* hep-th/9808060 Harvey and Moore
/0304115 Beasley and Witten

* hep-th/0506211 de Boer, Naqvi, Shomer

MATHAI - QUILLEN FORMALISM

- integral representation of Euler class of vector bundles
- constructs topological invariants of infinite dim vector bundles

manifold M, vector bundle $E \to M$

Euler number $\chi(M)$

1. topological

choose generic section $\bar{s}$

$$\chi(M) = \sum_{x,\, \bar{s}(x)=0} \pm 1$$

count # of zeros

MG representative of Euler class

$$e_{\nabla, s}(E) =$$

$$= \frac{1}{(2\pi)^n} \int d\chi\, e^{-\frac{1}{2} g_{ab} s^a s^b + i ds^a \chi_a + \frac{1}{2} \Omega_\nabla^{ab} \chi_a \chi_b}$$

- Euler number obtained for any choice of ∇, s
- interpolates between two expression of Euler number

$$s \to \gamma \bar{s}$$

- infinite dimensional vector bundles: $e_\nabla(E)$ has no meaning

MG expression is used to define regularized Euler number

2. differential geometric

if E equipped with connection ∇

Euler class

$$e_\nabla(E) = \frac{1}{(2\pi)^n} \mathrm{Pf}(\Omega_\nabla) =$$

$$= \frac{1}{(2\pi)^n} \int d\chi \, e^{\frac{1}{2} \Omega_\nabla^{ab} \chi_a \chi_b}$$

Ω_∇ : curvature of connection

$$\chi(M) = \int_M e_\nabla(E)$$

MATHAI - QUILLEN

$$e_{\nabla,s}(E)$$

Regularized Euler number

choose appropriate section so that zero locus $M_s \subset M$ is finite dimensional

$$\chi_s(M) = \chi(M_s) = \int_M e_{s,\nabla}(E)$$

- depends on section chosen

But natural choices of sections may carry interesting topological information.

Example SQM

$M = \mathrm{Map}(S^1, X)$ space of loops

$E = TM \quad \{x^i(t)\} \qquad x^i(1) = x^i(0)$

section associated to vector field $\dot{x}^i$ generating constant shifts

$\psi^i = dx^i \quad s^a = e_i^a \dot{x}^i \quad \chi_a = e_a^i \bar{\psi}_i$

MQ representation of Euler number is a path integral with action

$$I_{SQM} = \int_0^1 dt \left(\frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + i \bar{\psi}_i D_t \psi^i - \frac{1}{4} R_{ijkl} \psi^i \psi^j \bar{\psi}^k \bar{\psi}^l \right)$$

$D_t \psi^i = \dot{\psi}^i + \Gamma^i_{jk} \psi^j \dot{x}^k$

invariant under BRST

$\delta x^i = \psi^i \qquad \delta \psi^i = 0$

$\delta \bar{\psi}_i = i g_{ij} \dot{x}^j + \Gamma^k_{ij} \psi^j \bar{\psi}_k$

path integral computes Euler number of manifold

- topological A-model
  - $M = \mathrm{Map}(\Sigma_2, X)$
  - M_S = space of holom maps

TOPOLOGICAL MEMBRANES

$M = \mathrm{Map}(\Sigma_3, X)$

Σ_3 : membrane world volume $\{\sigma^a\}$ $a = 1,2,3$

X : manifold of G_2-holonomy $\{x^I(\sigma)\}$ $I = 1..7$

$TX = T_X(\Sigma_3) \oplus N_X(\Sigma_3)$

$\{x^a(\sigma), y^i(\sigma)\}$ $a = 1,2,3$, $i = 1..4$

static gauge $x^a(\sigma) = \sigma^a$

M_S = maps corresponding to associative cycles embedded in X

localize P.I. on M_S

associative 3-cycles calibrated

$$\phi\big|_{X_3} \leq \mathrm{vol}\, X_3$$

[equality for assoc. 3-cycle $\tilde{X}_3$]

equivalently

$$* \phi_{Ijkl}\, dx^j\, dx^k\, dx^l \big|_{\tilde{X}_3} = 0$$

$$X_3 = x(\Sigma_3)$$

section

$$\Theta_I = \frac{1}{6} * \phi_{Ijkl}\, \partial_a x^j \partial_b x^k \partial_c x^l \varepsilon^{abc}$$

in static gauge

$$\Theta_i = \phi_{ij}{}^a \partial_a y^j + \frac{1}{6} \phi^{abc} * \phi_{ijkl}\, \partial_a y^j \partial_b y^k \partial_c y^l$$

- on circle reduces to A-model holomorphic curves

note

linearized equations can be seen equivalent

$$\partial_a y^i = * \phi_a{}^{bi}{}_j\, \partial_b y^j$$

→ deformation theory of associative 3-cycles

- moduli space of associative 3-cycles is generically obstructed

⇝ cokernel of twisted Dirac operator

Mathai - Quillen action

$$I_M = \int_{\Sigma_3} d^3\sigma \left(\frac{1}{2} g^{IJ} \theta_I \theta_J + \right.$$

$$+ i \chi^I \left(\delta\theta_I - \Gamma_{IJ}^K \psi^J \theta_K \right)$$

$$\left. - \frac{1}{4} R_{IJKL} \psi^I \psi^J \chi^K \chi^L \right)$$

$$\delta\theta_I - \Gamma_{IJ}^K \psi^J \theta_K =$$

$$= \frac{1}{2} * \phi_{IJKL} \nabla_a \psi^J \partial_b x^K \partial_c x^L \varepsilon^{abc}$$

invariant under BRST

$$\delta x^I = \psi^I \qquad \delta\psi^I = 0$$

$$\delta\chi^I = i g^{IJ} \theta_J - \Gamma^I_{JK} \psi^J \chi^K$$

Topological term

$$\int_{\Sigma_3} x^*(\phi) = \int_{\Sigma_3} d^3\sigma \frac{1}{6} \phi_{IJK} \partial_a x^I \partial_b x^J \partial_c x^K \varepsilon^{abc}$$

- reduces on circle to A-model

$$\int_{\Sigma_2} x^*(k) \qquad k : \text{kähler form}$$

measures membrane instantons

- also required for agreement with physical supermembrane action upto quadratic order

$$\int_{\Sigma_3} d^3\sigma \sqrt{\det g_{IJ} \partial_a x^I \partial_b x^J}$$

Observables

- action has global $U(1)$ ghost number symmetry

$(x, \psi, \chi) \sim (0, 1, -1)$

$$\langle \mathcal{O}_{p_1} \ldots \mathcal{O}_{p_s} \rangle = \int_M \mathcal{O}_{p_1} \ldots \mathcal{O}_{p_s} \, e^{-t S_M}$$

$$S_M = I_M + \int_{\Sigma_3} x^*(\phi)$$

restricted to $M_{[x]} \subset M$ for maps of a given homotopy class

membrane instanton weight

$$e^{-t \int_{\Sigma_3} x^*(\phi)}$$

LOCAL OPERATORS

BRST cohomology isomorphic to de Rahm cohomology

$$\mathcal{O}_{\omega_p} = \frac{1}{p!} \omega_{I_1 \ldots I_p}(x) \psi^{I_1} \ldots \psi^{I_p}$$

$$\delta \mathcal{O}_{\omega_p} = \mathcal{O}_{d\omega_p}$$

G_2-holonomy decomposition

$$H^0(X, R) = R$$

$$H^1(X, R) = H^1_7(X, R)$$

$$H^2(X, R) = H^2_7(X, R) + H^2_{14}(X, R)$$

$$H^3(X, R) = H^3_1(X, R) + H^3_7(X, R) + H^3_{27}(X, R)$$

[$H^3_7 = 0$ for smooth compact G_2]

de Rahm cohomology spanned by

$$\mathcal{O}_\phi = \frac{1}{6}\phi_{IJK}\psi^I\psi^J\psi^K$$

$$\mathcal{O}_\alpha = \frac{1}{2}\phi_{IJ}{}^{A}\xi_{KA}\psi^I\psi^J\psi^K$$

$$\mathcal{O}_\beta = \frac{1}{2}\beta_{IJ}\psi^I\psi^J \qquad \beta = P^2_{14}\,\overset{*}{\beta}$$

NON-LOCAL OPERATORS

descent equation

$$\delta W_n = d W_{n-1}$$

$$\mathcal{O}^{(n)} = \int_{\gamma_n} W_n \qquad n = 1, 2, 3$$

3 sets of non-local descendants

$$* \; P^{2}_{14\,IJ}{}^{PG} = \delta^P_{[I}\,\delta^G_{J]} - \frac{1}{6}\phi_{IJA}\,\phi^{PGA}$$

$$\mathcal{O}_\phi^{(1)} = \int_{\gamma_1} \frac{1}{2}\phi_{IJK}\,\partial_a x^I\psi^J\psi^K d\sigma^a$$

$$\mathcal{O}_\phi^{(2)} = \int_{\gamma_2} -\frac{1}{2}\phi_{IJK}\,\partial_a x^I\partial_b x^J\psi^K d\sigma^a d\sigma^b$$

$$\mathcal{O}_\phi^{(3)} = -\int_{\Sigma_3} x^*(\phi)$$

- local observables same as for G_2 - string

But no non-local operators corresponding to ours in G_2-string [no localization on 3 cycles]

CONCLUDING REMARKS

- analogy to topological A-model

string worldsheet	$\sim$	membrane world volume
holomorphic curves	$\sim$	associative cycles

A-model free energy

$$F = \sum_{g=0}^{\infty} g_s^{2g-2} F_g$$

what is the analogue of genus expansion?

$\rightsquigarrow$ classification of closed orientable 3-manifolds

A-model genus-zero free energy

$$F_0 = \int_X k \wedge k \wedge k + \sum_{\Sigma \in H_2(X,\mathbb{Z})} d_\Sigma \sum_{m=1}^{\infty} \frac{1}{m^3} e^{-m k/\Sigma}$$

contribution of constant maps?

anticipate

$$F^{G_2}_{const} = \int_X \phi \wedge * \phi$$

membrane instanton weights

$$e^{-\int_{\Sigma_3} x^*(\phi)}$$

OPEN QUESTIONS

- RELATION TO G2-STRING?
  S-DUALITY?
  B-MODEL FROM MEMBRANES?
- COUPLING TO TOPOLOGICAL GRAVITY?
  ONE LOOP?
- BACKGROUND 3-FORM FIELD?
  → GENERALIZED CALIBRATIONS
- D0-BRANE FORMULATION?
  MATRIX THEORY DESCRIPTION?

CAN WE LEARN ABOUT FULL PHYSICAL M-THEORY?