Background - 1

Gromov-Witten Invariants

X = compact (almost) Kähler mfold

$X_{g,n,d}$ = space of stable curves in X

genus (g); number of marked pts $p_1, \dots, p_n$ (n); degree $\in H_2(X)$ (d)

The primary GW invariants are

$$\int_{X_{g,n,d}} \Lambda ev_i^*(\alpha_i)$$

where $\alpha_i \in H^*(X)$, $ev_i : X_{g,n,d} \to X$ evaluation at p_i

Descendants. At p_i have tangent line to curve $\to$ Line bundle L_i on $X_{g,n,d}$ with Euler class ψ_i

$$\int_{X_{gnd}} \Lambda_i \left[ev_i^*(\alpha_i) \cdot \psi_i^{k_i} \right] = \text{descendants}$$

Background - 2

Descendent Potential.

Complete invariant is encoded in a generating function

$$\mathcal{D}_X = \exp\left\{ \sum_{g \geq 0} \hbar^{g-1} \mathcal{F}_g(X) \right\}$$

$$\mathcal{F}_g = \sum_{n,d} \frac{Q^d}{n!} \int_{X_{g,n,d}} \bigwedge_{i=1}^{n} \left(\Sigma\, ev_i^*(a_k) \psi_i^k \right)$$

a function of the formal power series

$$\sum_{k \geq 0} a_k \psi^k \in H^*(X)[[\psi]]$$

This is conjectured (and sometimes known) to satisfy miraculous identities, such as a family of differential equations forming $\frac{1}{2}$ Virasoro algebra.

Backgrnd 3

Ancestors. A standard topologist's approach is to factor integration on $X_{g,n,d}$ via the push-forward to Deligne-Mumford space $\overline{M_g^n}$. **Problems:**

- No DM spaces for low g, n
- ψ on $X_{g,n,d} \neq \psi$ on $\overline{M_g^n}$.

One then defines the ancestor potential similarly to the descendent one (slight changes..) but using the DM ψ's and existing DM spaces only.

Kontsevich-Manin solved the problem of relating descendents to ancestors in principle, in terms of genus 0 data.

Givental gave an explicit formula using his Fock space formalism.

Backgrnd 4

Givental's Fock space constructions

On $H^*(X)((\psi)) = H((\psi))$ have skew form

$$\Omega(f,g) = \oint \langle f(-\psi), g(\psi) \rangle \, d\psi$$

$H[[\psi]]$ is Lagrangian for that $\Rightarrow$ can interpret $\mathcal{D}_X$ as vector in Fock space.

That carries an action of the loop group

$$L_\psi GL(H) \cap Sp(H((\psi)))$$

Givental observed that many known relations about $\mathcal{D}_X$ are expressible in terms of quantised quadratic Hamiltonians:

- Virasoro equations
- Genus 0 "Reconstruction Theorem" (Dubrovin; Dijkgraaf-Witten)
- Ancestor/Descendent relation

Backgrd 5

Flavour of formulae. Involve the "2-point gravitational descendent in genus 0"

$S_a(1/z)$ an $a \in H$ - dependent formal series in $\mathrm{End}(H)[[1/z]]$ satisfying

$$S_a^*(-1/z)\, S_a(1/z) = Id$$

and defined by

$$(\alpha, S_a\beta) = (\alpha,\beta) + \sum_{n,d} \frac{Q^d}{n!} \int_{X_{0,n+2,d}} ev_1^*(\alpha) \cdot \bigwedge_{i=2}^{n+1} ev_i^*(a) \cdot \frac{ev_{n+2}^*\beta}{z-\psi_{n+2}}$$

Then, roughly

$$\mathcal{D}_X = \hat{S}_a^{-1} \cdot \mathcal{A}_{X,a}$$

where $\mathcal{A}$ is the ancestor potential and $\hat{S}_a$ = action of S_a on Fock space.

NB: There is a correction term from genus 1.

Backgrd 6

Higher genus Reconstruction.

Givental also conjectured (and proved in some cases) a formula for $\mathcal{D}_X$ in terms of S_a in the case when the quantum cohomology of X is generically semi-simple; essentially

$$\mathcal{D}_X = \hat{S}_a^{-1} \circ \hat{\Psi} \circ \hat{R}_a \left(\mathcal{D}_{N\ \text{points}} \right)$$

where the constant matrix $\Psi \in \mathrm{End}(H)$ and the "upper-triangular" loop group element $R_a \in \mathrm{End}(H)[[\psi]] \cap Sp$ are defined from S_a, using Dubrovin's theory of Frobenius structures.

Note: Givental Conjecture implies the Virasoro relations.

Tautological classes on $\overline{M_g^n}$

$\overline{C_g^n}$ universal curve

$\downarrow$

$\overline{M_g^n}$

· has tangent line L_i at i^{th} marked point

$\to$ Euler classes $\psi_i \in H^2(\overline{M_g^n})$

· has fibrewise canonical bundle K, $e(K) = -\psi$,

$$\kappa_i = \int_{fibre} \psi^{i+1} \in H^{2i}(\overline{M_g^n}).$$

The rational Mumford conjecture (Madsen-Weiss) + Harer stability $\Rightarrow$

$$H^*(M_g) = \mathbb{Q}[\kappa_i] \quad \text{if } * < g/2$$

Older result of Looijenga (Bödigheimer-Tillman: over $\mathbb{Z}$)

$$H^*(M_g^n) = H^*(M_g)[\psi_1, \ldots, \psi_n]$$

in stable range

1

Structure of semi-simple FTFT's

- Smooth surfaces, parametrised boundaries:

$\widetilde{M}_g^{p,q}$ = moduli orbifold of such

Need $\widetilde{Z}_g^{p,q} : A^{\otimes p} \to H^*(\widetilde{M}_g^{p,q}) \otimes A^{\otimes q}$

satisfying all gluing laws

Datum: $\widetilde{Z}_\infty \in H^*(M_\infty; A)$

~~primitive~~ group-like for the monoid structure of M_g

$$\widetilde{Z}_\infty = \exp\left\{\sum_{i \geq 0} a_i \kappa_i\right\} \qquad a_i \in A$$

and with $a_0 = \frac{1}{2} \sum_j \log \theta_j \cdot P_j$.

Note: a fixed $\Sigma_g^{1,1}$ gives $\sum_j \theta_j^{-g} P_j$ α^{-g}

Formula for $\widetilde{Z}_g^{p,q}$:

$$A^{\otimes p} \xrightarrow{\text{mult.}} A \xrightarrow{\times Z_\infty} A \xrightarrow{\text{comult}} A^{\otimes q}$$

2

Structure of FTFT's

Smooth surfaces, unparametrised ∂:

Need $Z_g^{p,q} : A^{\otimes p} \longrightarrow H^*(M_g^{p,q}) \otimes A^{\otimes q}$

Data: $\tilde{Z}_\infty$ as before

$E(\psi) : A \longrightarrow A[[\psi]]$ $(= Id \bmod \psi)$

(Think of E as implementing the "bordism" from a point to a circle )

Apply $E(\psi)$ to incoming states and $E(\psi)^{-1}$ to outgoing ones

$$Z_g^{p,q} : A^{\otimes p} \xrightarrow{\boxtimes_{i=1}^{p} E(\psi_i)} A^{\otimes p} \xrightarrow{\tilde{Z}_g^{p,q}} A^{\otimes q} \xrightarrow{\boxtimes_{p+1}^{q} E^{-1}(\psi_i)} A^{\otimes q}$$

Example: $Z_0^{2,0}(a_1, a_2) = \theta\left(E(\psi)a_1 \cdot E(-\psi)a_2\right)$

a_1

a_2

$A^{\otimes 2} \longrightarrow k[[\psi]]$

3

Structure of FTFT's

- Deligne-Mumford theories:

Need $\overline{Z}_g^{p,q} : A^{\otimes p} \longrightarrow H^*(\overline{M}_g^{p,q}) \otimes A^{\otimes q}$

Data: $\tilde{Z}_\infty$, E as before

- $F : A \longrightarrow A[[\psi]]$ $(= Id \bmod \psi)$

F describes the extension of Z-classes to the Deligne-Mumford boundary strata.

$Z = Id$ $\qquad (FE)^{-1}(\psi_2) \circ FE(\psi_1)$

A boundary stratum in $\overline{M}_g$ has the form $M^1_{g_1} \times M^1_{g_2}$. Inserting the above formula between the propagators $Z_{g_{1,2}}$ gives the restriction of $\overline{Z}$ to that stratum.

4

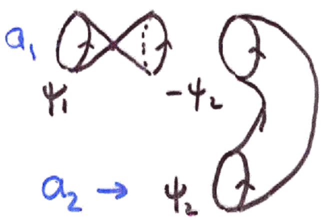
The Symplectic Condition

Theorem. On the data for DM theories, we have the single constraint

$$F(-\psi)\,F(\psi)^* = Id.$$

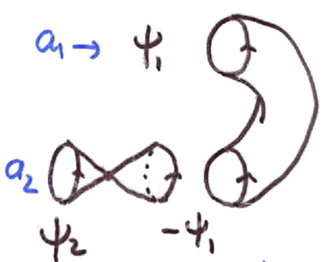
(So $F \in LGL(A) \cap Sp$)

Proof: By pictures



$$\Theta\left(F^{-1}(-\psi_2)F(\psi_1)E(\psi_1)a_1 \cdot E(\psi_2)a_2\right)$$

$$\|$$

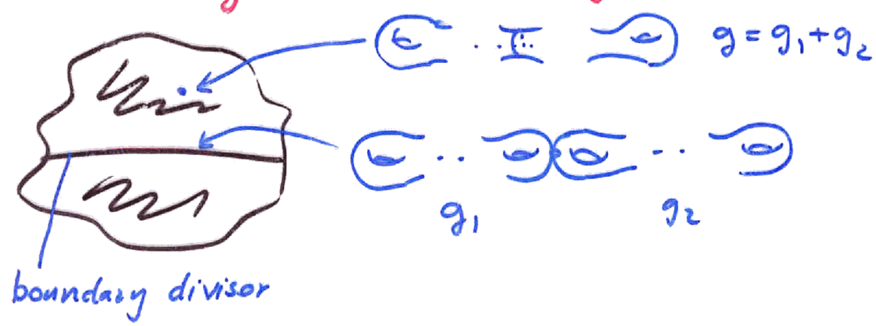


$$\Theta\left(E(\psi_1)a_1 \cdot F^{-1}(-\psi_1)F(\psi_2)E(\psi_2)a_2\right)$$

Equating for all a_1, a_2

$\Rightarrow$ $F(-\psi)F(\psi)^* =$ indep of ψ.

Extending Z over boundary DM strata



From $\tilde{Z}$, E and F we know the restriction of $\overline{Z}$ to the "bulk" and to the boundary divisor.

The Euler class of the normal bundle is $\psi_1 + \psi_2$.

If either g_1 or g_2 is large, this is not a zero-divisor through a range of degrees

$\Rightarrow$ Patched class is unique (in the range)

Stabilisation trick allows us to increase genus w/o loss of information

Large genus stabilisation

- When the surface has a parametrised boundary, this is obvious:

  Attach a large genus surface to embed M_g to M_{g+G}.

  Propagator α^G is invertible in A

  $\Rightarrow$ no loss of information.

  (*Note:* α invertible $\Leftrightarrow$ A semi-simple)

- When the surface has only a marked point (= unparametrized boundary): attach a "moving" surface of large genus G.

  Resulting moduli space: circle bundle over $M_g^1 \times M_G^1$, with Euler class $\psi_1 + \psi_2$. But ψ_2 is a free generator!

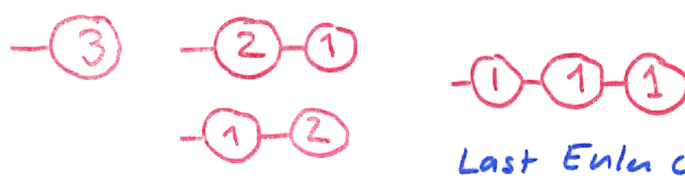
  $\Rightarrow$ No loss of information.

Inductive extension of Z to $\overline{M}$

The Euler class observation is the basis of an inductive procedure showing uniqueness of $\overline{Z}$ from our data.

We single out a marked point for stabilisation and take care that each Euler class involves a ψ on that irreducible component.

Bad example: (4 strata in $\overline{M}_3^1$)

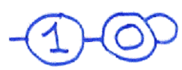
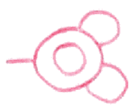
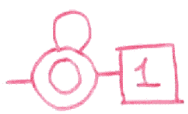
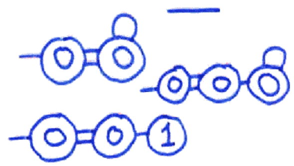


Last Euler class **is** 0-divisor even after stabilisation!!

Correct: First glue –(3) + –(2)–(1)

Then glue –(1)–(2) + –(1)–(1)–(1). Then glue together

Stratification of $\overline{M_2^1}$

attaching order	d	graph	substrata
0	4		—
1	3		—
2	2		
2	2		—
4	1		
4	1		—
6	0		

Stratification of $\overline{M_2^2}$