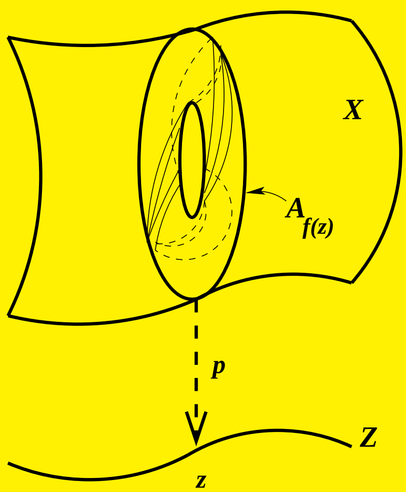
T-duality in string theory via noncommutative geometry

type IIA $\Longleftrightarrow$ type IIB



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***Disclaimer**: Only references of importance to this talk are listed and is **not** meant to be an account of the history of the subject.

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The idea of T-duality

The simplest example is a free theory on a torus $\mathbb{T}^n = \mathbb{R}^n/\Gamma$, where Γ is a lattice in $\mathbb{R}^n$. The partition function is a theta function,

$$Z_\Gamma(\beta) = \sum_{z \in \widehat{\Gamma}} e^{-2\pi^2 r|z|^2}$$

where $\widehat{\Gamma}$ is the dual lattice in the dual vector space $\widehat{\mathbb{R}}^n$.

By the Poisson summation formula, this is equivalent to the partition function $Z_{\widehat{\Gamma}}$ on the dual torus $\widehat{\mathbb{T}}^n = \widehat{\mathbb{R}}^n/\widehat{\Gamma}$, and $\quad r \Leftrightarrow 1/r$.

The situation however gets much more complicated when a background flux H is turned on, where $[H] \in H^3(\mathbb{T}^n, \mathbb{Z})$, and $[H] \neq 0$, and will be discussed later.

T-duality in the literature

Spacetime is $M \times \mathbb{T}$, with trivial background flux - then the T-dual is topologically the same space $M \times \widehat{\mathbb{T}}$, and T-duality is realized by using the correspondence

$$\begin{array}{ccccc} & & M \times \mathbb{T} \times \widehat{\mathbb{T}} & & \\ & \swarrow^{p} & & \searrow^{\widehat{p}} & \\ M \times \mathbb{T} & & & & M \times \widehat{\mathbb{T}} \end{array} \qquad (1)$$

Poincaré line bundle $\mathcal{P}$: There is a canonical line bundle $\mathcal{P}$ over the torus $\mathbb{T} \times \widehat{\mathbb{T}}$, defined as follows: Consider the free action of $\mathbb{Z}$ on $\mathbb{R} \times \widehat{\mathbb{T}} \times \mathbb{C}$ given by,

$$\begin{aligned} \mathbb{Z} \times (\mathbb{R} \times \widehat{\mathbb{T}} \times \mathbb{C}) &\rightarrow \mathbb{R} \times \widehat{\mathbb{T}} \times \mathbb{C} \\ (n, (r, \rho, z)) &\rightarrow (r + n, \rho, \rho(n) z) \end{aligned}$$

The Poincaré line bundle is defined as $\mathcal{P} = (\mathbb{R} \times \widehat{\mathbb{T}} \times \mathbb{C})/\mathbb{Z}$, its curvature is $\mathcal{F} = d\theta \wedge d\widehat{\theta}$.

In this case, T-dualizing on $\mathbb{T}$, the Buscher rules for the RR fields can be conveniently encoded in the formula on $M \times \mathbb{T} \times \widehat{\mathbb{T}}$,

$$T_* G = \int_{\mathbb{T}} e^{\mathcal{F}} \, G \,, \qquad (2)$$

$G \in \Omega^\bullet(M \times \mathbb{T})$ is the total RR fieldstrength,

$G \in \Omega^{even}(M \times \mathbb{T})$ for **Type IIA**;
$G \in \Omega^{odd}(M \times \mathbb{T})$ for **Type IIB**.

Here $\mathcal{F} = d\theta \wedge d\widehat{\theta}$ is the curvature of the Poincaré line bundle $\mathcal{P}$ on $\mathbb{T} \times \widehat{\mathbb{T}}$, so that $e^{\mathcal{F}} = ch(\mathcal{P})$ is the Chern character of $\mathcal{P}$.

Note that G is a closed form if and only if its T-dual $T_* G$ is a closed form. So the Buscher rules (2) can be interpreted as an isomorphism

$$T_* : H^\bullet(M \times \mathbb{T}) \stackrel{\cong}{\longrightarrow} H^{\bullet+1}(M \times \widehat{\mathbb{T}}). \qquad (3)$$

Recently, it was argued by Minasian-Moore, Horava and Moore-Witten that,

Type IIA string theory

RR fields are classified by $K^0(X)$;

Charges are classified by $K^1(X)$;

whereas,

Type IIB string theory

RR fields are classified by $K^1(X)$;

Charges are classified by $K^0(X)$.

Note the **parity shift**!

The T-duality discussion given earlier can also be realized also in K-theory, and thus to the classification of D-branes on $M \times \mathbb{T}$ and $M \times \widehat{\mathbb{T}}$, by using the correspondence

$$\begin{array}{ccc} & M \times \mathbb{T} \times \widehat{\mathbb{T}} & \\ {\scriptstyle p} \swarrow & & \searrow {\scriptstyle \widehat{p}} \\ M \times \mathbb{T} & & M \times \widehat{\mathbb{T}} \end{array} \tag{4}$$

Induces a T-duality isomorphism of K-theories

$$T_! : K^\bullet(M \times \mathbb{T}) \xrightarrow{\cong} K^{\bullet+1}(M \times \widehat{\mathbb{T}}) \tag{5}$$

given by $T_! = \widehat{p}_! \left(p^!(\, \cdot \,) \otimes \mathcal{P} \right)$.

That is, T-duality in the absence of a background field, gives an equivalence

Type IIA theory $\Longleftrightarrow$ Type IIB theory

N.B. **No** change in topology!

T-duality in the absence of a background flux can be summarized as the commutativity of the following diagram,

$$\begin{array}{ccc} K^\bullet(M\times\mathbb{T}) & \xrightarrow[\cong]{T_!} & K^{\bullet+1}(M\times\widehat{\mathbb{T}}) \\ ch\Big\downarrow & & \Big\downarrow ch \\ H^\bullet(M\times\mathbb{T}) & \xrightarrow[\cong]{T_*} & H^{\bullet+1}(M\times\widehat{\mathbb{T}}) \end{array}$$

where the horizontal arrows are isomorphisms, and ch is the Chern character. That is,

$$ch(T_!(Q)) = T_* ch(Q)$$

for all $Q \in K^\bullet(M\times\mathbb{T})$.

The aim: to generalize this to the case when there is a **non-trivial background flux**.

The case of circle bundles

In [BEM], we investigated the general case where E is an oriented $\mathbb{T}$-bundle over M

$$\begin{array}{ccc} \mathbb{T} & \longrightarrow & E \\ & & \downarrow{\scriptstyle \pi} \\ & & M \end{array} \tag{6}$$

classified by its first Chern class $c_1(E) \in H^2(M, \mathbb{Z})$, with H-flux $H \in H^3(E, \mathbb{Z})$.

The **T-dual** of E is another oriented $\mathbb{T}$-bundle over M, denoted by $\widehat{E}$,

$$\begin{array}{ccc} \widehat{\mathbb{T}} & \longrightarrow & \widehat{E} \\ & & \downarrow{\scriptstyle \widehat{\pi}} \\ & & M \end{array} \tag{7}$$

which has first Chern class $c_1(\widehat{E}) = \pi_* H$.

The Gysin sequence for E enables us to define a T-dual H-flux $\widehat{H} \in H^3(\widehat{E}, \mathbb{Z})$, satisfying

$$c_1(E) = \widehat{\pi}_* \widehat{H}\,, \tag{8}$$

where $\pi_* : H^k(E, \mathbb{Z}) \to H^{k-1}(M, \mathbb{Z})$, and similarly $\widehat{\pi}_*$, denote the pushforward maps.

N.B. $\widehat{H}$ is not fixed by this data, since any integer degree 3 cohomology class on M that is pulled back to $\widehat{E}$ also satisfies (8). However, $\widehat{H}$ is determined uniquely upon choosing connections A on $E \to M$ and $\widehat{A}$ on $\widehat{E} \to M$. Explicit formulae will be given later.

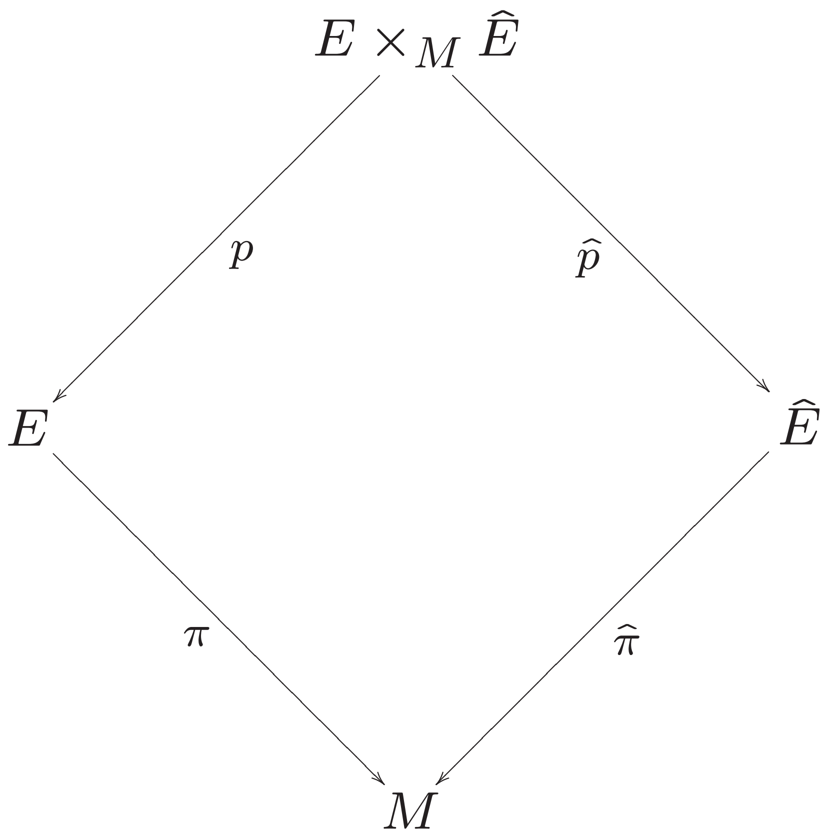
The surprising **new** phenomenon is that there is a **change in topology** when the H-flux is non-trivial. N.B. this can also happen when spacetime is a product $E = AdS^7 \times \mathbb{T}^3$ (a trivial circle bundle over $M = AdS^7 \times \mathbb{T}^2$) with H-flux $H = a \cup b$, where $a = k.\text{vol} \in H^2(\mathbb{T}^2, \mathbb{Z})$, b the generator of $H^1(\mathbb{T}, \mathbb{Z})$. Then the T-dual circle bundle is $\hat{E} = AdS^7 \times Heis(1,k)$ with trivial H-flux, where $Heis(1,k)$ denotes the Heisenberg nilmanifold, with first Chern class equal to a.

Another example is $AdS^5 \times S^5$ with trivial H-flux, is T-dual to $AdS^5 \times \mathbb{C}P^2 \times \mathbb{T}$ with H-flux $H = a \cup b$ where $a = \text{vol} \in H^2(\mathbb{C}P^2, \mathbb{Z})$, b the generator of $H^1(\mathbb{T}, \mathbb{Z})$.

T-duality for **circle bundles** is the exchange,

$$\boxed{\textbf{background H-flux} \Longleftrightarrow \textbf{Chern class}}$$

T-duality & correspondence spaces



T-duality in a background flux

Choosing connection 1-forms A and $\widehat{A}$, on the $\mathbb{T}$-bundles E and $\widehat{E}$, respectively, the rules for transforming the RR fields can be encoded in the formula

$$T_*G = \int_{\mathbb{T}} e^{A\wedge\widehat{A}}\; G\,, \tag{9}$$

$G \in \Omega^\bullet(E)$ is the total RR fieldstrength,

$$G \in \Omega^{even}(E) \quad \text{for } \textbf{Type IIA};$$
$$G \in \Omega^{odd}(E) \quad \text{for } \textbf{Type IIB},$$

where the right hand side is a form on $E \times_M \widehat{E}$, and the integration is along the $\mathbb{T}$-fiber of E.

Recall that the twisted cohomology $H^\bullet(E, H)$ is defined as the cohomology of the complex

$$(\Omega^\bullet(E), d_H = d - H\wedge).$$

Let $F = dA$ and $\widehat{F} = d\widehat{A}$,

$$H = A \wedge \widehat{F} - \Omega\,, \tag{10}$$

for some $\Omega \in \Omega^3(M)$, while the T-dual $\widehat{H}$ is given by

$$\widehat{H} = F \wedge \widehat{A} - \Omega\,. \tag{11}$$

We note that

$$d(A \wedge \widehat{A}) = -H + \widehat{H}\,, \tag{12}$$

so that T_* indeed maps d_H-closed forms G to $d_{\widehat{H}}$-closed forms T_*G. Therefore T-duality T_* induces a map

$$T_* : H^\bullet(E, H) \to H^{\bullet+1}(\widehat{E}, \widehat{H}).$$

The inverse is similarly defined, and using the fact that locally, we have $A = d\theta + \widehat{\pi}_*\widehat{B}$, $\widehat{A} = d\widehat{\theta} + \pi_*B$, one shows after a small computation that T-duality T_* is indeed an **isomorphism**.

Proposal by Witten, Kapustin (torsion twist), Bouwknegt-Mathai, Atiyah-Segal (general twist) - that in the presence of a background H-flux,

Type IIA string theory

RR Fields are classified by $K^0(X, H)$

Charges are classified by $K^1(X, H)$

whereas,

Type IIB string theory

RR Fields are classified by $K^1(X, H)$

Charges are classified by $K^0(X, H)$

Note the **parity shift**!

Dixmier-Douady theory asserts that isomorphism classes of locally trivial algebra bundles $\mathcal{K}_P$ with fiber the algebra of compact operators $\mathcal{K}$ and structure group $PU = U/\mathbb{T}$ over a manifold X are in bijective correspondence with $H^3(X, \mathbb{Z})$. Moreover since $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$, such algebra bundles form a group called the **infinite Brauer group**, $\mathrm{Br}(X)$, which is isomorphic to $H^3(X, \mathbb{Z})$.

This is proved by noticing that U is contactible in the weak operator topology so PU is a $B\mathbb{T} = K(\mathbb{Z}, 2)$ since $\mathbb{T} = K(\mathbb{Z}, 1)$. Therefore $BPU = K(\mathbb{Z}, 3)$. Therefore principal PU bundles P are classified up to isomorphism by

$$[X, BPU] = [X, K(\mathbb{Z}, 3)] = H^3(X, \mathbb{Z}).$$

Then $\mathcal{K}_P = (P \times \mathcal{K})/PU$ and $DD(\mathcal{K}_P) \in H^3(X, \mathbb{Z})$ is its Dixmier-Douady invariant.

Also relevant here is the theorem of Serre, which says that a cohomology class $H^3(X, \mathbb{Z})$ can be represented by a locally trivial algebra bundle with fiber the algebra of finite dimensional matrices if and only if the cohomology class is torsion.

Decomposable nontorsion example. Let $\alpha \in H^1(X, \mathbb{Z})$ and $\beta \in H^2(X, \mathbb{Z})$. Then α can be thought of as a character $\chi_\alpha : \pi_1(X) \to \mathbb{Z}$ with associated $\mathbb{Z}$-covering space $\hat{X}$. Similarly, β can be thought of as a principal bundle $U(1) \to P \to X$. $\pi : P \times_X \hat{X} \to X$ is a principal $U(1) \times \mathbb{Z}$-bundle over X with DD invariant $\alpha \cup \beta$. Now $(\gamma', n) \in U(1) \times \mathbb{Z}$ acts on $L^2(U(1))$,

$$(\sigma(n)f)(\gamma) = \gamma^n f(\gamma), \quad (\sigma(\gamma')f)(\gamma) = f(\gamma'\gamma).$$

Since $[\sigma(\gamma), \sigma(n)] = \gamma^n I$, this is a projective action, i.e. $\sigma : U(1) \times \mathbb{Z} \to PGL(L^2(U(1)))$ is a homomophism. Equivalently, σ is a representation of the Heisenberg group H in this context, i.e. the central extension,

$$1 \to U(1) \to H \to U(1) \times \mathbb{Z} \to 1.$$

So $P \times_X \hat{X} \times_\sigma PGL(L^2(U(1))$ is a principal $PGL(L^2(U(1))$-bundle over X with DD invariant $\alpha \cup \beta$.

Twisted K-theory. Twisted K-theory was defined by J. Rosenberg as the K-theory of the C^*-algebra of continuous sections, $C(X, \mathcal{K}_P)$ - we will also denote this algebra by $CT(X, H)$, where $H = DD(\mathcal{K}_P)$. Twisted K-theory will be denoted by $K^\bullet(X, H)$. It is a module over $K^0(X)$ and possesses many nice functorial properties.

Back to T-duality in an H-flux. The earlier discussion in twisted cohomology can also be realized in twisted K-theory and, in this more general setting, T-duality gives an isomorphism of the twisted K-theories of E and $\hat{E}$,

$$T_! : K^\bullet(E, H) \to K^{\bullet+1}(\hat{E}, \hat{H})$$

defined by

$$T_! = \hat{p}_! \ (p^!(\ \cdot\ ) \otimes \mathcal{L})$$

where $\mathcal{L}$ is the substitute for the Poincaré line bundle. It is no longer necessarily a line bundle on the correspondence space $E \times_M \hat{E}$, but rather a line bundle on the total space of a principal PU bundle Q over the correspondence space $E \times_M \hat{E}$ with Dixmier-Douady invariant $\widehat{H} - H$. The first Chern class of $\mathcal{L}$ is $c_1(\mathcal{L}) = A \wedge \hat{A} - \hat{B} + B$ determines $\mathcal{L}$, , where we recall that $d(A \wedge \hat{A}) = \widehat{H} - H = d(\hat{B} - B)$.

Several of the constructions used in the definition of T-duality on twisted K-theory are adapted from different joint work of V.M. with **Melrose and Singer.**

T-duality in the presence of a background flux H can be summarized as the commutativity of the following diagram,

$$\begin{array}{ccc} K^\bullet(E,H) & \xrightarrow{T_!} & K^{\bullet+1}(\hat{E},\hat{H}) \\ ch_H \downarrow & & \downarrow ch_{\hat{H}} \\ H^\bullet(E,H) & \xrightarrow{T_*} & H^{\bullet+1}(\hat{E},\hat{H}) \end{array} \qquad (13)$$

That is,

$$ch_H(T_!(Q)) = T_* ch_{\hat{H}}(Q)$$

Sample calculations

Lens spaces $L(1,p) = S^3/\mathbb{Z}_p$, which is the total space of the circle bundle over the 2-sphere with Chern class equal to p times the generator of $H^2(S^2,\mathbb{Z}) \cong \mathbb{Z}$.

$$K^i(L(1,j), H = k) \cong K^{i+1}(L(1,k), H = j)\,.$$

In particular, since $L(1,1) = S^3 = SU(2)$ we obtain an isomorphism

$$K^i(SU(2), H = k) \cong K^{i+1}(L(1,k), H = 1).$$

In particular, since $L(1,0) = S^2 \times \mathbb{T}$, we obtain an isomorphism

$$K^i(S^2 \times \mathbb{T}, H = k) \cong K^{i+1}(L(1,k)) \qquad (14)$$

which shows that it is a **ring**.

The case of principal $\mathbb{T}^2$-bundles

In [MR], we investigated the general case where E is an oriented $\mathbb{T}^2$-bundle over M,

$$\begin{array}{ccc} \mathbb{T}^2 & \longrightarrow & E \\ & & \Big\downarrow \pi \\ & & M \end{array} \tag{15}$$

which is classified by its first Chern class $c_1(E) \in H^2(M, \mathbb{Z}^2)$. We again assume that E comes with an H-flux $H \in H^3(E, \mathbb{Z})$.

However, in this case, the T-dual of E is **not** in general another oriented $\mathbb{T}^2$-bundle over M!

A famous example of a principal torus bundle with **non T-dualizable** H-flux is provided by

$$\begin{array}{ccc} \mathbb{T}^2 & \longrightarrow & \mathbb{T}^3 \\ & & \downarrow p \\ & & \mathbb{T} \end{array} \tag{16}$$

with H-flux $H = k\text{vol} \in H^3(\mathbb{T}^3, \mathbb{Z})$, $k \neq 0$. Since $p_*[H] = [\int_{\mathbb{T}} H] \neq 0$, and there are no non-trivial principal $\mathbb{T}^2$-bundles over $\mathbb{T}$ since $H^2(\mathbb{T}, \mathbb{Z}^2) = 0$, it follows that there is no way to get a T-dual that is another principal torus bundle with H-flux over $\mathbb{T}$. Upon doing T-duality one circle at a time, the 2nd circle disappears, as noticed by S. Kachru, M. Schulz, P. Tripathy and S. Trivedi, [hep-th/0211182].

In [MR], we proposed that the T-dual in this case is instead a continuous field of stabilized **noncommutative tori** fibered over $\mathbb{T}$. This will be justified shortly.

Noncommutative torus

For each $\theta \in [0, 1]$, the **noncommutative torus** A_θ is defined abstractly as the C^*-algebra generated by two unitaries U and V in an infinite dimensional separable Hilbert space satisfying the **Weyl commutation relation**

$$UV = \exp(2\pi i\theta)VU.$$

Elements f in A_θ can be represented by infinite power series, where $a_{(n,m)} \in \mathbb{C}$,

$$f = \sum_{(m,n)\in\mathbb{Z}^2} a_{(n,m)} U^m V^n. \qquad (17)$$

The stabilized noncommutative torus $A_\theta \otimes \mathcal{K}$ is isomorphic to the **foliation algebra** of the space of leaves of the **Kronecker foliation** on $\mathbb{T}^2$ defined by the equation $dx = \theta\, dy$ on $\mathbb{T}^2$.

An important realization of A_θ in physics is as the norm completion of the algebra of Schwartz functions $\mathcal{S}(\mathbb{R}^2)$ on $\mathbb{R}^2$ with the **Moyal product**: for all $f, g \in \mathcal{S}(\mathbb{R}^2)$,

$$(f \star g)(z) = \left(e^{\pi i \theta \frac{\partial}{\partial x} \frac{\partial}{\partial y}} f(x) g(y) \right)_{x=y=z}$$

or equivalently,

$$(f \star g)(z) = c \int dx\, dy\, e^{2\pi i x y} f(z + \theta x) g(z + y).$$

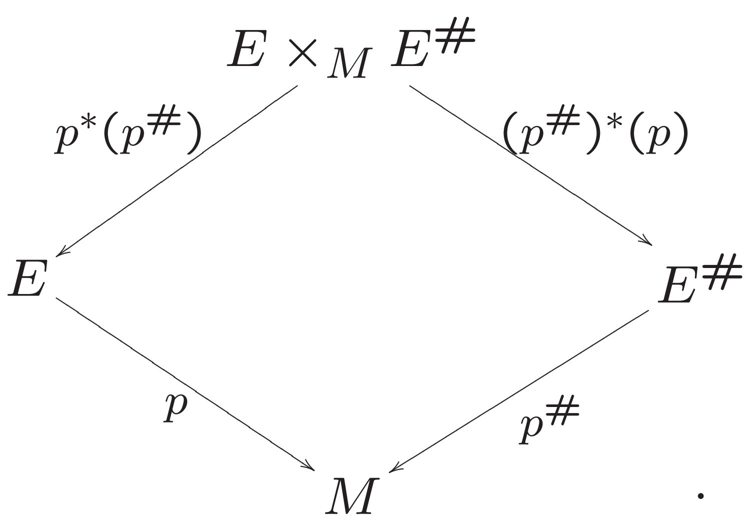
Upon taking the partial Fourier transform, there is an isomorphism $A_\theta \cong C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$, where the generator of $\mathbb{Z}$ acts on $\mathbb{T}$ by rotation by the angle $2\pi\theta$. Because of this A_θ is also known as the **rotation algebra**.

A couple of key properties:

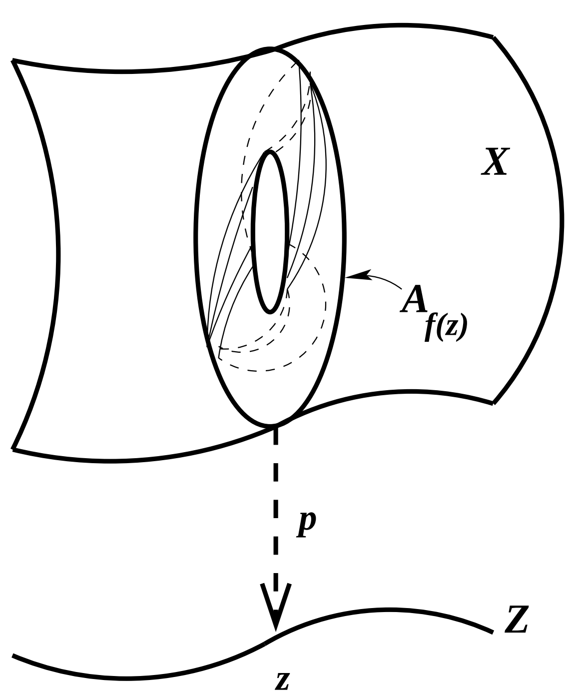
When $\theta \in \mathbb{Q}$, A_θ is Morita equivalent to $C(\mathbb{T}^2)$. However, when $\theta \notin \mathbb{Q}$, A_θ is a simple algebra (i.e. spectrum is a single point).

Theorem (Rank 2 bundles) Let $\pi\colon E \to M$ be a principal $\mathbb{T}^2$-bundle and $H \in H^3(E, \mathbb{Z})$ an H-flux on E. Let $\pi_* H = \int_{\mathbb{T}^2} H \in H^1(M, \mathbb{Z})$.

1. [BHM] If $\pi_* H = 0 \in H^1(M, \mathbb{Z})$, then there is a uniquely determined classical T-dual to (π, H), consisting of $\pi^\# : E^\# \to M$, which is a another principal $\mathbb{T}^2$-bundle over Z, and $H^\# \in H^3(E^\#, \mathbb{Z})$, the "T-dual H-flux" on $E^\#$. T-dualizing a circle at a time works - the picture is exactly as in the case of circle bundles.



2. [MR] If $\pi_* H \neq 0 \in H^1(M, \mathbb{Z}) = [M, \mathbb{T}]$, then a classical T-dual as above does **not** exist, i.e., T-dualizing a circle at a time does **not** work. In this case however, there is a "nonclassical" T-dual which is a continuous field of **noncommutative tori** over M whose fibre over $m \in M$ is $A_{f(m)}$, where $f : M \to \mathbb{T}$ is a continuous function, $[f] = \pi_* H$. Geometrically, the T-dual can be viewed as a rank 2 bundle of (Kronecker) foliated tori over M.



The mathematical theorem proved in [MR] is that for $\mathbb{T}^2$-torus bundles, the $\mathbb{R}^2$ action on E lifts to a $\mathbb{R}^2$ action on $CT(E,H)$, where $CT(E,H)$ is the algebra of sections of a bundle of compact operators with Dixmier-Douady invariant equal to H.

Then the **T-dual** of (E,H) is defined to be the crossed product $CT(E,H) \rtimes \mathbb{R}^2$. It has an action of the dual group $\widehat{\mathbb{R}}^2$ and **Takai duality** asserts the Morita equivalence

$$\boxed{CT(E,H) \Longleftrightarrow CT(E,H) \rtimes \mathbb{R}^2 \rtimes \widehat{\mathbb{R}}^2}$$

The implication is that T-duality, when applied twice, returns us to a physically equivalent algebra. This is the **1st justification** of the crossed product algebra as the T-dual.

Brief digression on crossed products Let A be a C^*-algebra, and α an action of a locally compact abelian group G on A. Then the crossed product $A \rtimes_\alpha G$ is the norm completion of $C_c(G, A)$ with product given by convolution multiplication on G and the formal relation $g.a.g^{-1} = \alpha_g(a),\ g \in G, a \in A$.

Now on the crossed product $A \rtimes_\alpha G$, there is an action $\widehat{\alpha}$ of $\widehat{G}$ given by multiplication by $\widehat{G}$ on functions on G, with formal relations $\gamma.a = a.\gamma$, $\gamma.g.\gamma^{-1} = \langle \gamma, g \rangle g$ for all $\gamma \in \widehat{G}, g \in G, a \in A$.

Then **Takai duality** says that there is a canonical isomorphism,

$$A \rtimes_\alpha G \rtimes_{\widehat{\alpha}} \widehat{G} \cong A \otimes \mathcal{K}.$$

We have the isomorphism of K-theories

$$T_! : K^\bullet(E,H) \xrightarrow{\cong} K_\bullet(CT(E,H) \rtimes \mathbb{R}^2)$$

which is **Connes Thom isomorphism** theorem, giving the **2nd justification** of the crossed product algebra as the T-dual.

We also have the commutative diagram,

$$\begin{array}{ccc} K^\bullet(E,H) & \underset{\cong}{\overset{T_!}{\longrightarrow}} & K_\bullet(CT(E,H) \rtimes \mathbb{R}^2) \\ {\scriptstyle Ch_H}\Big\downarrow & & \Big\downarrow{\scriptstyle Ch} \\ H^\bullet(E,H) & \underset{\cong}{\overset{T_*}{\longrightarrow}} & HP_\bullet(CT(E,H)^\infty \rtimes \mathbb{R}^2) \end{array}$$

where the horizontal arrows are isomorphisms, Ch_H is the twisted Chern character, Ch is the Connes-Chern character and the lower horizontal arrow is the Elliott-Natsume-Nest Thom isomorphism in periodic cyclic homology.

In the example of the trivial torus bundle,

$$\begin{array}{ccc} \mathbb{T}^2 & \longrightarrow & \mathbb{T}^3 \\ & & \downarrow p \\ & & \mathbb{T} \end{array}$$

with H-flux $H = k\text{vol} \in H^3(\mathbb{T}^3, \mathbb{Z})$, $k \neq 0$, then it turns out that the T-dual is

$$CT(\mathbb{T}^3, H) \rtimes \mathbb{R}^2 \cong C^*(H_{\mathbb{Z}}) \otimes \mathcal{K},$$

where $H_{\mathbb{Z}}$ is the integer Heisenberg group,

$$H_{\mathbb{Z}} = \left\{ \begin{pmatrix} 1 & x & \frac{1}{k}z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\},$$

which is a $\mathbb{Z}$-central extension of $\mathbb{Z}^2$,

$$0 \to \mathbb{Z} \to H_{\mathbb{Z}} \to \mathbb{Z}^2 \to 0.$$

Mackey induction via the central $\mathbb{Z}$ subgroup of $H_{\mathbb{Z}}$ gives the direct integral decomposition,

$$C^*(H_{\mathbb{Z}}) \otimes \mathcal{K} = \int_{\theta \in \mathbb{T}} A_\theta \, d\theta \otimes \mathcal{K}.$$

3rd justification of T-dual:

Can reformulate T-duality for circle bundles discussed earlier, via C^*-algebras as follows. Let

$$\begin{array}{ccc} \mathbb{T} & \longrightarrow & E \\ & & \Big\downarrow p \\ & & M \end{array}$$

be a principal circle bundle and H a closed, integral 3-form on E. Then there is a continuous trace C^*-algebra $CT(E, H)$ with spectrum equal to E and Dixmier-Douady invariant equal to $[H] \in H^3(E, \mathbb{Z})$. Using a connection on the associated principal PU bundle, the $\mathbb{R}$ action on E lifts to an $\mathbb{R}$ action on $CT(E, H)$ (uniquely!, cf. **Raeburn-Rosenberg**), and one has a commutative diagram,

$$
\begin{array}{ccc}
 & \text{spec}(CT(E,H) \rtimes \mathbb{Z}) & \\
 \swarrow^{p} & & \searrow^{\widehat{p}} \\
\text{spec}(CT(E,H)) & & \text{spec}(CT(E,H) \rtimes \mathbb{R}) \\
 \searrow_{\pi} & & \swarrow_{\widehat{\pi}} \\
 & \text{spec}(CT(E,H))/\mathbb{R} &
\end{array}
\tag{18}
$$

That is, **Raeburn-Rosenberg** show that the C^*-algebras $CT(E,H) \rtimes \mathbb{Z}$ and $CT(E,H) \rtimes \mathbb{R}$ are also continuous trace C^*-algebras with spec$(CT(E,H) \rtimes \mathbb{R}) = \widehat{E}$ a circle bundle over $M = \text{spec}(CT(E,H))/\mathbb{R}$, such that $c_1(\widehat{E}) = \pi_*[H]$ and the Dixmier-Douady invariant of $CT(E,H) \rtimes \mathbb{R}$ is $[\hat{H}] \in H^3(\widehat{E}, \mathbb{Z})$, such that $c_1(E) = \widehat{\pi}_*[\widehat{H}]$, and spec$(CT(E,H) \rtimes \mathbb{Z}) = E \times_M \widehat{E}$ is the correspondence space, i.e.

we recover the T-duality for circle bundles.

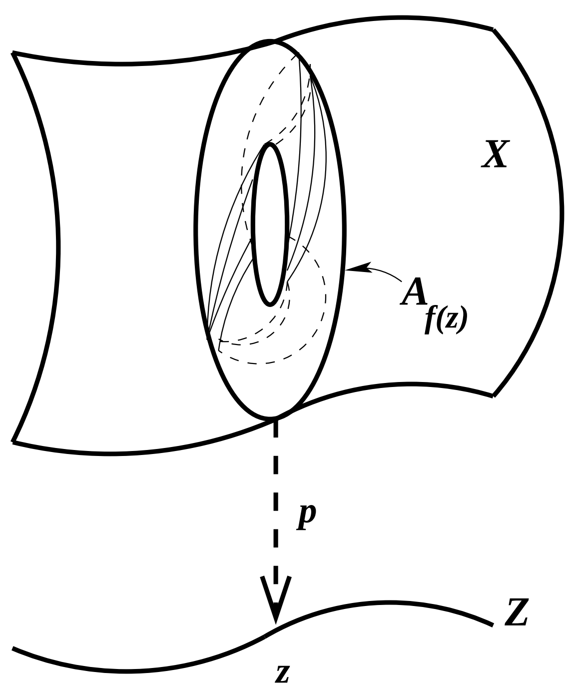
Theorem (rank n case, part 1) [MR2] Let

$$\begin{array}{ccc} \mathbb{T}^n & \xrightarrow{i} & E \\ & & \downarrow p \\ & & M \end{array}$$

be a principal torus bundle over M, $[H] \in H^3(M, \mathbb{Z})$. Then the $\mathbb{R}^n$ action on E lifts to a $\mathbb{R}^n$ action on $CT(E, H)$ **if and only if** the restriction, $i^*[H] = 0 \in H^3(\mathbb{T}^n, \mathbb{Z})$ is trivial. This is a **nontrivial obstruction** $\Leftrightarrow n \geq 3$.

(1) If $0 = \pi_*(H) = \left(\int_{\mathbb{T}^2_1} H, ..., \int_{\mathbb{T}^2_k} H\right) \in H^1(M, H^2(\mathbb{T}^n, \mathbb{Z}))$, where $k = \binom{n}{2}$ and $\mathbb{T}^2_j$ are the subtori of rank 2 in the torus fibers, then there is a uniquely determined classical T-dual consisting of $\pi^{\#}$: $E^{\#} \to M$, which is a another principal $\mathbb{T}^n$-bundle over Z, and $H^{\#} \in H^3(E^{\#}, \mathbb{Z})$, the "T-dual H-flux" on $E^{\#}$. T-dualizing a circle at a time works and the picture is exactly as in the case of circle bundles.

(2) If $0 \neq \pi_*(H) \in H^1(M, H^2(\mathbb{T}^n, \mathbb{Z})) = [M, \mathbb{T}^k]$, then the T-dual is again defined as the C^*-algebra $CT(E, H) \rtimes \mathbb{R}^n$, which is again a continuous field of noncommutative tori, whose fiber at the point $x \in M$ is the noncommutative torus $A_{f(x)}$ of rank n where $f : M \to \mathbb{T}^k$ is a continuous map corresponding to $\pi_*(H)$. Geometrically, the T-dual can be viewed as a rank n bundle of (Kronecker) foliated tori over M.



We also have a commutative diagram

$$\begin{array}{ccc} K^\bullet(E,H) & \overset{T_!}{\underset{\cong}{\longrightarrow}} & K_{\bullet+n}(CT(E,H)\rtimes\mathbb{R}^n) \\ {\scriptstyle Ch_H}\Big\downarrow & & \Big\downarrow{\scriptstyle Ch} \\ H^\bullet(E,H) & \overset{T_*}{\underset{\cong}{\longrightarrow}} & HP_{\bullet+n}(CT(E,H)^\infty\rtimes\mathbb{R}^n) \end{array}$$

where the horizontal arrows are isomorphisms, Ch_H is the twisted Chern character and Ch is the Connes-Chern character.

Observations so far: A **striking fact** is that starting off with a type II string theory on a (compactified) **classical spacetime** which is a non-trivial torus bundle with topologically nontrivial background H-flux, then the T-dual is a type II string theory on a **noncommutative spacetime**, which is a continuous field of non-commutative tori. The action etc has been studied in special situations cf. D. Lowe, H. Nastase, S. Ramgoolam, [hep-th/030317]

Nonassociative tori

Definition in 3D. Let U_1, U_2, U_3 be generators satisfying the relations

$$e^{2\pi i\phi} U_1(U_2 U_3) = (U_1 U_2) U_3,$$

where $\phi \in \mathbb{R}$ is a tricharacter and $e^{2\pi i\phi} \in H^3(\mathbb{Z}^3, U(1))$ is the associator. The algebra generated by U_1, U_2, U_3 is what we call the 3D **nonassociative torus**. Jackiw's nonassociative anomaly in QFT/gauge theory is related.

The right context for studying it and its hybrids is the theory of **generalized C^*-algebras** in the monoidal/tensor category of $\widehat{G}$-modules (where $G = \mathbb{R}^n$) with non-trivial associator given by a tricharacter ϕ of G, which is what is developed in [BHM2] and in some work in progress.

Theorem (rank n case, part 2) [BHM2] Let

$$\begin{array}{ccc} \mathbb{T}^n & \xrightarrow{i} & E \\ & & \downarrow p \\ & & M \end{array}$$

be a principal torus bundle over M, $[H] \in H^3(E, \mathbb{Z})$. Now suppose that the restriction, $i^*([H]) \neq 0 \in H^3(\mathbb{T}^n, \mathbb{Z})$.

Then the $\mathbb{R}^n$ action on E lifts to a **twisted** $\mathbb{R}^n$ action on $CT(E, H)$ and the T-dual of (E, H) is defined to be the twisted crossed product $CT(E, H) \rtimes_{twist} \mathbb{R}^n$ which is a nonassociative algebra, or what we call a generalized C^*-algebra. It is in general is a continuous field of hybrids of noncommutative tori & **nonassociative tori**.

Justification of the T-dual

First of all, when $i^*([H]) = 0 \in H^3(\mathbb{T}^n, \mathbb{Z})$, then the twisted crossed product is the standard crossed product and we are reduced to the earlier definition of the T-dual. Then we prove in [BHM2] the following **untwisting trick**:

$(CT(E,H) \rtimes_{twist} \mathbb{R}^n) \otimes \mathcal{K} \cong \left(CT(E,H) \otimes \mathcal{K}_\phi\right) \rtimes \mathbb{R}^n.$

$\mathcal{K}_\phi$ is a nonassociative deformation of the algebra of compact operators $\mathcal{K}$ on $L^2(\mathbb{R}^n)$, such that on the smoothing operators it looks like

$$T_1 \star T_2(x,z) = \int_{y \in \mathbb{R}^n} \phi(x,y,z) T_1(x,y) T_2(y,z) dy.$$

We also prove a **new Takai duality theorem** in this context, which says in particular that,

$$\left(CT(E,H) \otimes \mathcal{K}_\phi\right) \rtimes \mathbb{R}^n \rtimes \widehat{\mathbb{R}}^n \cong CT(E,H) \otimes \mathcal{K}_\phi.$$

This enables us to conclude that T-duality applied twice is the identity.

Finally, in some work in progress, we first develop the K-theory of our nonassociative generalized C^*-algebras, and use the Takai duality theorem proved in [BHM2] to prove a **new Connes-Thom isomorphism theorem** in this nonassociative context of generalized C^*-algebras, thereby fully justifying the twisted crossed product algebra as the T-dual of a general principal torus bundle with H-flux.

Summary of 4 important special cases:

1) The T-dual of the torus $\mathbb{T}^3$ with no background flux is the **dual torus** $\widehat{\mathbb{T}}^3$. Similar if the background flux is topologically trivial.

2) $(\mathbb{T}^3, k\text{vol})$ considered as a trivial circle bundle over $\mathbb{T}^2$. The T-dual of $(\mathbb{T}^3, k\text{vol})$ is the **Heisenberg nilmanifold** $(H_{\mathbb{R}}/H_{\mathbb{Z}}, 0)$, where $H_{\mathbb{R}}$ is the 3D Heisenberg group, $H_{\mathbb{Z}}$ a lattice in it.

3) $(\mathbb{T}^3, k\text{vol})$ considered as a trivial $\mathbb{T}^2$-bundle over $\mathbb{T}$. The T-dual of $(\mathbb{T}^3, k\text{vol})$ is a continuous field over $\mathbb{T}$ of stabilized **noncommutative tori**, $C^*(H_{\mathbb{Z}}) \otimes \mathcal{K}$, since $\int_{\mathbb{T}^2} kvol \neq 0$.

4) $(\mathbb{T}^3, k\text{vol})$ considered as a trivial $\mathbb{T}^3$-bundle over a point. The T-dual of $(\mathbb{T}^3, k\text{vol})$ is a **nonassociative torus**, A_ϕ, where ϕ is the tricharacter associated to $H = kvol$, where $\int_{\mathbb{T}^3} kvol \neq 0$.

Other results covered in our papers

Because of time constraints, the following relevant topics could not be covered.

1) Just as principal torus bundles are classified by their 1st Chern class, the fields of noncommutative tori (and nonassociative tori) conjecturally ([BHM3] are classified by 'twisted' cohomology classes (cf. [BHM3]). In the latest paper with J. Rosenberg [MR2], we have proved this for rank 2 torus bundles, but the general case is an open problem.

2) There is a classifying space for T-dual pairs, whose automorphism group is the T-duality group $O(n, n, \mathbb{Z})$. This acts to give a whole orbit of T-dual pairs and isomorphisms in K-theory (with J. Rosenberg [MR2]). This work generalizes some work of Bunke-Schick.