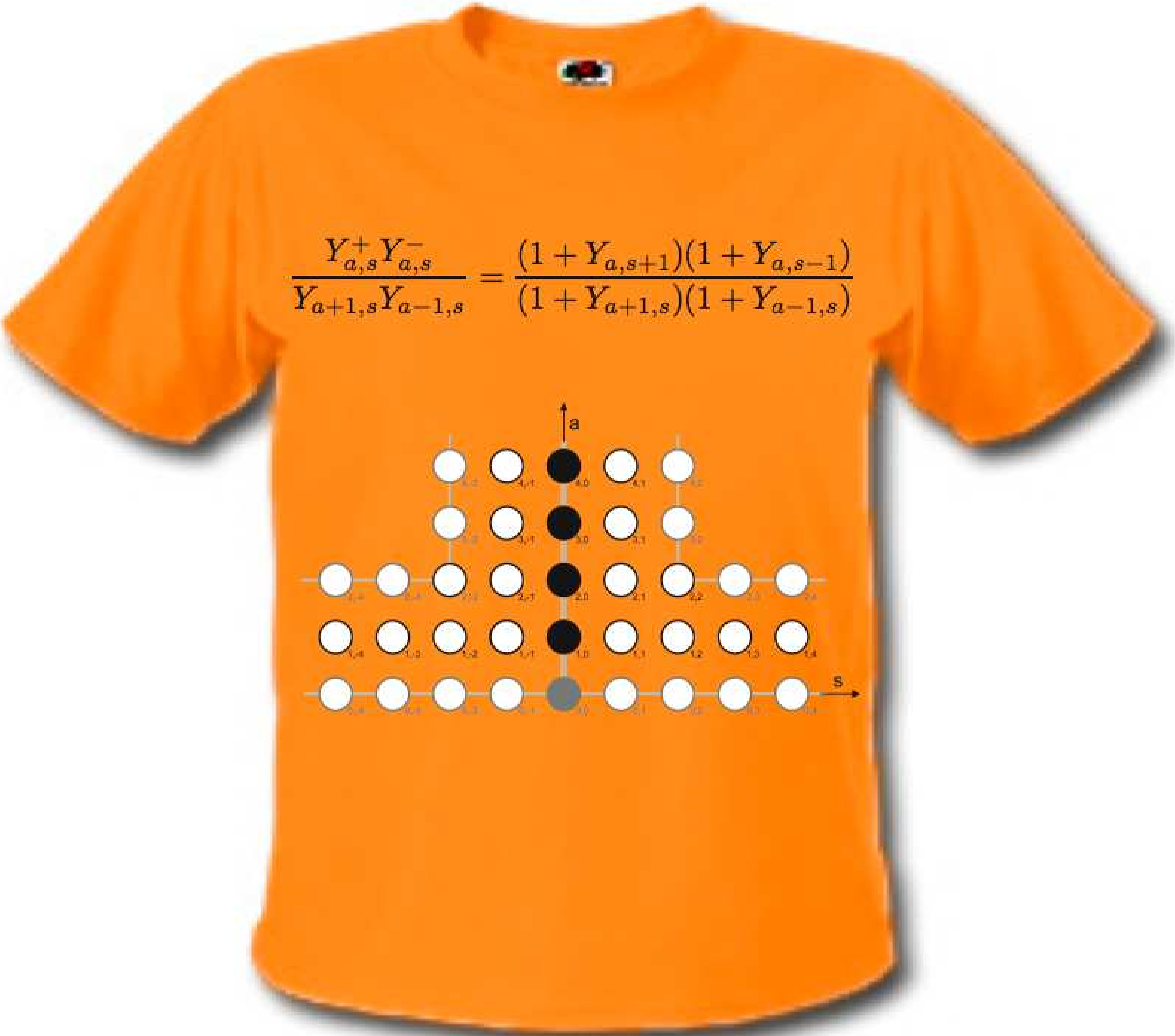
KITP, Santa Barbara, 5 February 2009

Integrability for the Full Spectrum of Planar AdS/CFT

Vladimir Kazakov (ENS,Paris)

with N.Gromov and P.Vieira, arXiv:0812.5091
arXiv:0901.3753

for Joe's store…



Outline

- We want to know the anomalous dimensions of short operators in N=4 SYM (Konishi, tr F^2 , etc) at any coupling. Use the AdS dual superstring sigma-model.

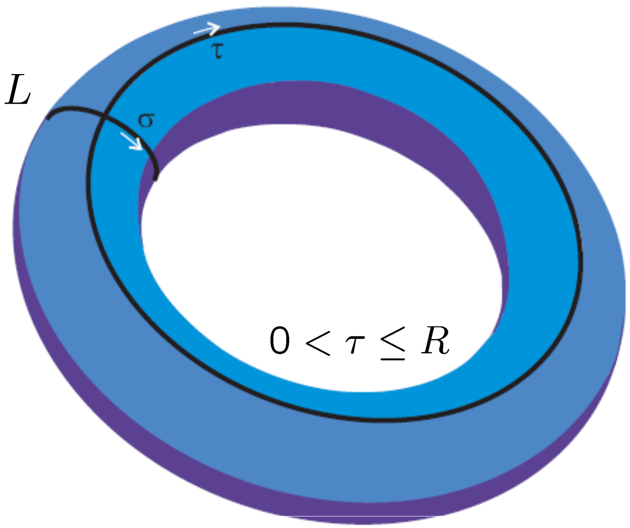
- Our inspiration: TBA in relativistic massive integrable sigma models at "finite size" . Important example –
  $SU(2)_L \times SU(2)_R$ principle chiral field (PCF).

- Superstring in the light cone gauge: $SU(2|2)_L \times SU(2|2)_R$ "PCF"

- Integrable discrete Hirota dynamics (=Y-system) and (super)-group representation theory are very important.

- We conjecture a Y-system describing planar AdS/CFT at any coupling. Passes all the known tests!

TBA: Free energy = ground state E



"mirror" original

$$\mathcal{Z} = \sum_n e^{-E_n(R)L} = \sum_n e^{-E_n(L)R}$$

R=∞ ↓

$$e^{-E_0(L)R}$$

- I.e. from the asymptotic spectrum (R=∞) we can compute the ground state energy for ANY finite volume L!

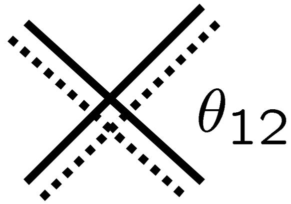
- The excited states can be included by a certain analytical continuation

"Toy" model: SU(2)xSU(2) principal chiral field

$$\mathcal{L} = \frac{\sqrt{\lambda}}{4\pi} \int d^2x \left(g^{-1}\partial_\mu g(x)\right)^2, \qquad g \in SU(2).$$

- Asymptotically free theory with dynamically generated mass $m = \Lambda\, e^{-\frac{\sqrt{\lambda}}{4\pi}}$

- S-matrix: $\hat{S}(\theta_{12}) = \hat{S}_L(\theta_{12}) \times \hat{S}_R(\theta_{12}) \quad \Rightarrow$



$$\hat{S}_{L,R}(\theta) = S_0(\theta)\left(P^+_{L,R} + \tfrac{\theta+i}{\theta-i} P^-_{L,R}\right)$$

$$E = m \cosh \pi\theta$$
$$P = m \sinh \pi\theta$$

- Scalar (dressing) factor: $S_0(\theta) = \dfrac{\Gamma\left(-\frac{\theta}{2i}\right)\Gamma\left(\frac{1}{2}+\frac{\theta}{2i}\right)}{\Gamma\left(\frac{\theta}{2i}\right)\Gamma\left(\frac{1}{2}-\frac{\theta}{2i}\right)}$

Satisfies Yang-Baxter, unitarity, analyticity and crossing :

$$S_0(\theta + i/2) S_0(\theta - i/2) = \frac{\theta - \frac{1}{2}}{\theta + \frac{1}{2}}$$

- Footnote: Compare to AdS/CFT:

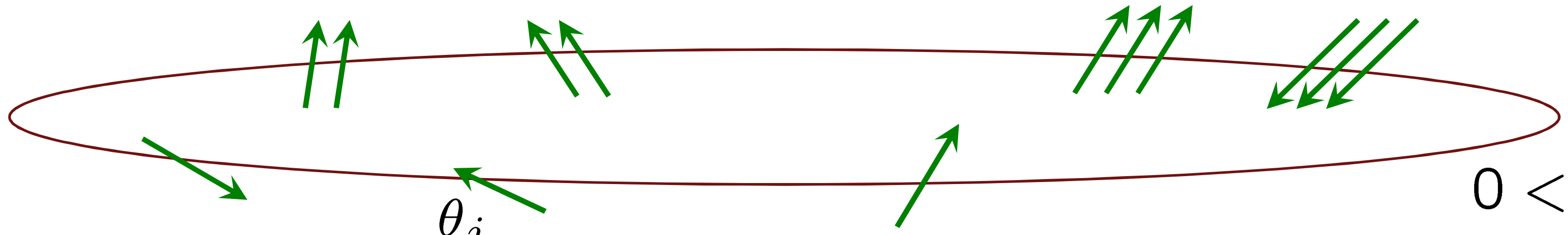
$$S_{PSU(2,2|4)}(p_1,p_2) = S_0^2(p_1,p_2)\; S_{SU(2|2)}(p_1,p_2) \times S_{SU(2|2)}(p_1,p_2)$$

*

Complex formation in (almost) infinite volume

- Periodicity: $$\left[e^{-iRm\,\sinh\pi\theta_k} - \prod_j{}' \widehat{S}(\theta_k-\theta_j)\right]|\Psi\rangle = 0$$

- Magnon bound states for R-wing and L-wing, in full analogy with Heisenberg chain



- Exact thermodynamic equations for densities of bound states and their holes w.r.t. θ s>0(s<0) for R(L)

$$\rho_s + \bar{\rho}_s = \delta_{s,0} mR\cosh\pi\theta + K_{sm} * \rho_m, \qquad -\infty < s < \infty$$

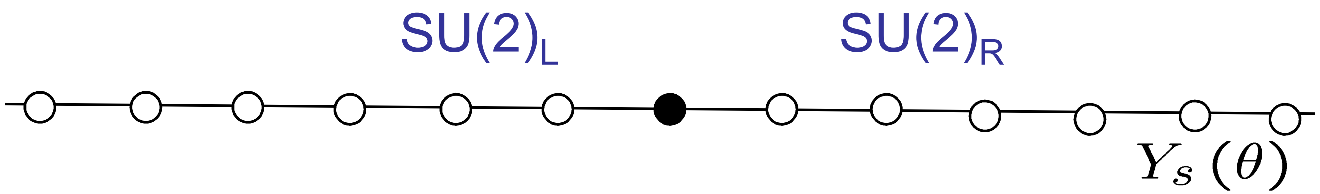
- Minimization of the free energy at finite temperature T=1/L

$$f = \int d\theta \left(Rm\,\rho_0\ \cosh\pi\theta - T\sum_{k=-\infty}^{\infty}\left[\rho_k \log\left(1+\frac{\bar{\rho}_k}{\rho_k}\right) + \bar{\rho}_k \log\left(1+\frac{\rho_k}{\bar{\rho}_k}\right)\right]\right)$$

*

SU(2)×SU(2) Principal Chiral Field in finite volume

- Thermodynamics of complexes → TBA → Y-system



$$Y_s(\theta) = e^{-\delta_{s,0} mL \cosh \pi\theta} \left(\left[1 + Y_{s+1}\right] \left[1 + Y_{s-1}\right] \right)^{*\hat{h}},$$

$$\hat{h}^{-1}(\theta) \;=\; e^{\frac{i}{2}\partial_\theta} + e^{-\frac{i}{2}\partial_\theta}$$

$$Y_s = \frac{\bar{\rho}_s}{\rho_s}, \qquad s = \pm 1, \pm 2, \pm 3, \ldots;$$ (densities of magnon holes/complexes)

$$Y_0 = \frac{\rho_0}{\bar{\rho}_0}$$ (densities of particles/holes)

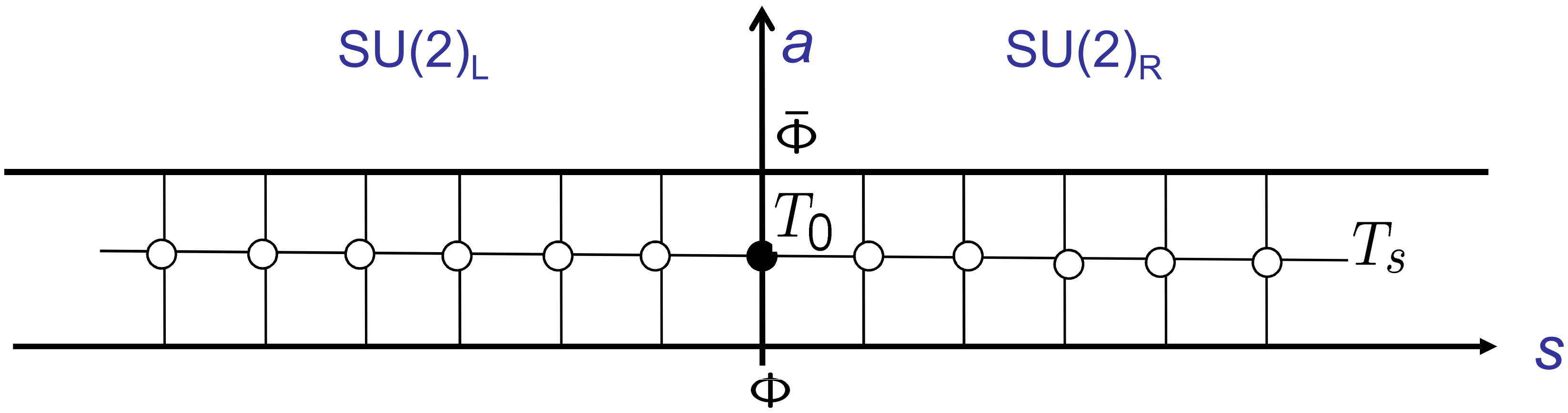
- Equivalent to Y-system: $$Y_s^+ Y_s^- = \left[1 + Y_{s+1}\right] \; \left[1 + Y_{s-1}\right]$$

- Energy: $$E_{state} = -\frac{m}{2} \int d\theta \cosh \pi\theta \log(1+Y_0) + \sum_{j=1}^{N} m \cosh(\pi\theta_j) \; ,$$

- Bethe eq. $$Y_0(\theta_j \pm i/2) = -1$$

Y-system and Hirota relation

$$Y_s^+ Y_s^- = \left[1 + Y_{s+1}\right] \; \left[1 + Y_{s-1}\right]$$



Parametrize: $$Y_s = \frac{T_{s+1} T_{s-1}}{\Phi^{[+s]} \bar{\Phi}^{[-s]}}$$

Hirota equation:

$$T_s^+ T_s^- - T_{s-1} \, T_{s+1} = \Phi^{[+s]} \, \bar{\Phi}^{[-s]}$$

- Solution: linear Lax pair (discrete integrable dynamics! Baxter eq. etc):

$$\begin{cases} T_{s+1}(\theta) Q(\theta + is/2) - T_s(\theta - i/2) Q(\theta + i(s+2)/2) \; = \; \Phi(\theta + i(s+1)/2) \bar{Q}(\theta - i(s+2)/2) \\ \text{Complex conjugate} \end{cases}$$

Determinant solution of Hirota eq.

$$\Phi(x) = h(x+i/2)\begin{vmatrix} R(x) & Q(x) \\ R(x+i) & Q(x+i) \end{vmatrix}$$

$$h(x+i) = h(x)$$

Wronskian relation

$$T_k(x) = h(x+ik/2)\begin{vmatrix} Q(x+i\frac{k+1}{2}) & R(x+i\frac{k+1}{2}) \\ \bar{Q}(x-i\frac{k+1}{2}) & \bar{R}(x-i\frac{k+1}{2}) \end{vmatrix}$$

Gauge transformation $g(x)$

$$\begin{aligned} T_k(x) &\rightarrow g\left(x+i\frac{k}{2}\right)\bar{g}\left(x-i\frac{k}{2}\right)T_k(x) \\ \Phi(x) &\rightarrow g(x-i/2)g(x+i/2)\Phi(x) \\ Q(x) &\rightarrow g(x-i/2)Q(x) \end{aligned}$$

Leaves Y's and Lax pair invariant!

Large Volume Limit $L \to \infty$

(helps to see zeroes/poles for excited states)

$$Y_0 = \frac{T_1 T_{-1}}{\Phi \bar{\Phi}} \sim e^{-mL\cosh(\pi\theta)} \to 0$$

- It is a spin chain limit: $T_{-1} \simeq 0$

$$T_0 \simeq \Phi^+ \simeq \Phi^- \simeq \phi(\theta) = \prod_{k=1}^{N} (\theta - \theta_k)$$

- T-system splits into two wings with $T_{k>0} = T_k^R, \quad T_{k<0} = T_{-k}^L$
- Y-system in this limit trivially gives

$$Y_0(\theta) \simeq e^{-mL\cosh(\pi\theta)} T_1^R(\theta)\; T_{-1}^L(\theta)\; \frac{\prod_j S_0^2(\theta - \theta_j + i/2)}{\phi^2(\theta - i/2)}$$

- Main BAE at large L: $Y_0(\theta_j \pm i/2) = -1$

$$-1 \simeq e^{-imL\sinh(\pi\theta_k)} \frac{Q_R(\theta_k + i/2) Q_L(\theta_k + i/2)}{Q_R(\theta_k - i/2) Q_L(\theta_k - i/2)} \prod_j S_0^2(\theta_k - \theta_j)$$

- Auxiliary BAE – from polynomiality of $T_1^R(x),\; T_{-1}^L(x)$ (defined by Lax eq)

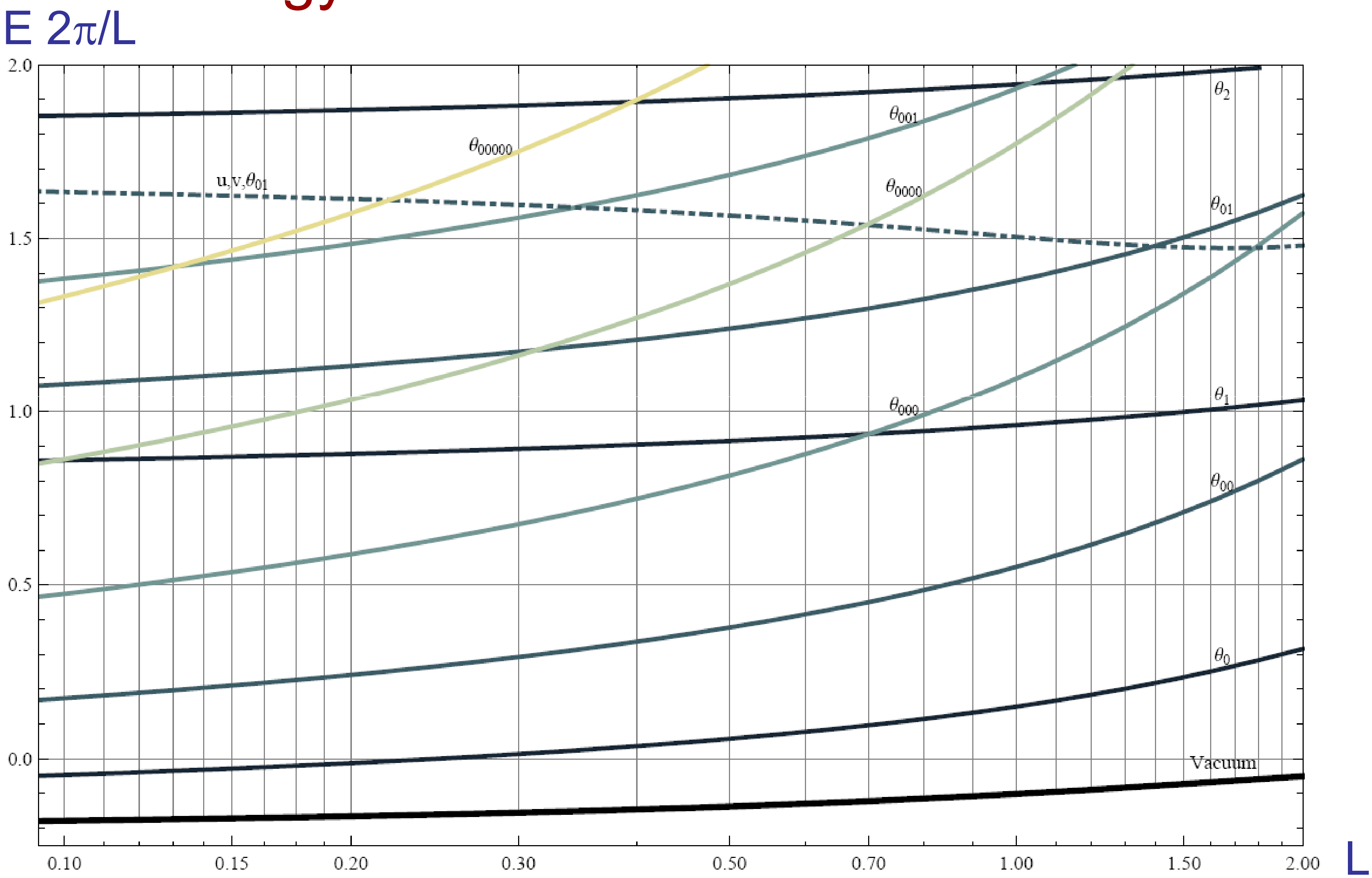
In our first work, using analyticity arguments, we transform Hirota equations into a single integral equation and get...

(may be in the future Pedro's talk...)

*

Gromov,V.K.,Vieira'08

Energy versus size for various states



Now AdS/CFT...

*

Definitions

- g is the AdS/CFT coupling.
- Zhukovski map: $u = g(x + 1/x),$ $x(u) = \frac{1}{2g}\left[u + \sqrt{u^2 - 2g^2}\right]$
- Shift operation: $f^{\pm} = f(u \pm i/2),$ $f^{[\pm k]} = f(u \pm ik/2)$
- Star operation (*), for a pair $x^{[\pm a]}$:

$$x^{[+a]} \to x^{[+a]}$$

$$x^{[-a]} \to 1/x^{[-a]}$$

(mirror kinematics, passing to other u-sheet)

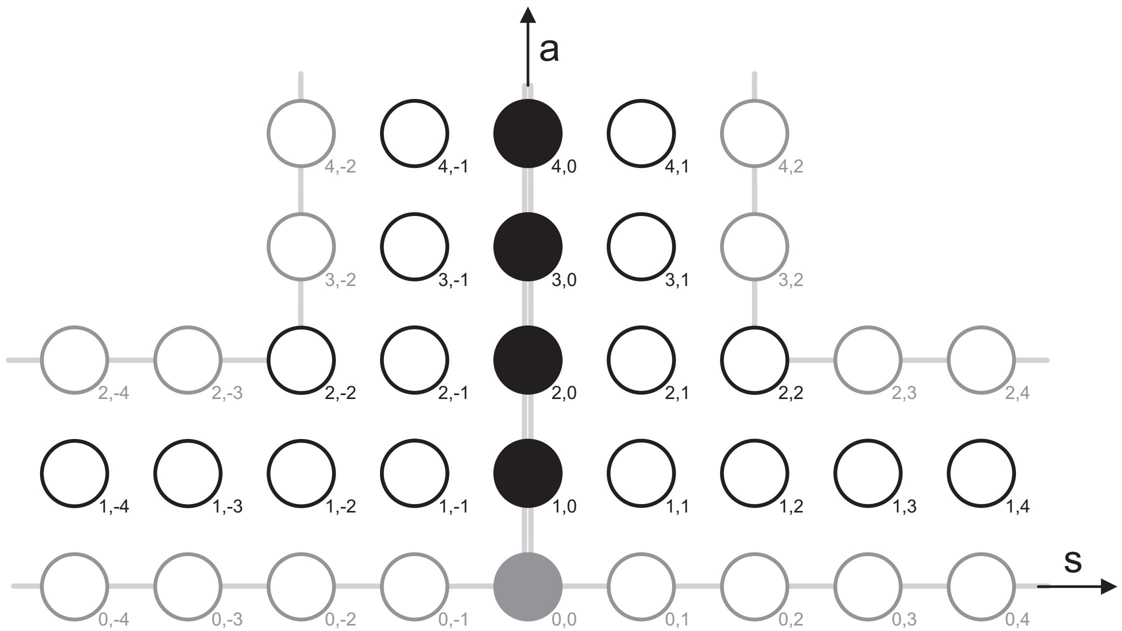
*

Result:

- Y-system

$$\frac{Y_{a,s}^{+}\,Y_{a,s}^{-}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{\left[1+Y_{a,s+1}\right]\,\left[1+Y_{a,s-1}\right]}{\left[1+Y_{a+1,s}\right]\,\left[1+Y_{a+1,s}\right]}$$

$Y_{a,s}\,(u)$



- Asymptotics

$$Y_{a,s\neq 0}\,(u\to\infty)\;\to\;\mathrm{const}_{a,s}$$

$$Y_{a,0}\,(u\to\infty)\;\to\;\left(\frac{x^{[-a]}}{x^{[+a]}}\right)^{L}\times\mathrm{const}_{a}$$

Energy (Anomalous Dimension)

$$E_{state} - J = \int \frac{du}{2\pi i}\, \partial_u \epsilon_a^*(u) \log\left(1 + Y_{a,0}^*(u)\right) + \sum_{j=1}^{M} \epsilon_1(u_j)$$

$$Y_{1,0}^*(u_j) + 1 = 0, \qquad j = 1, 2, \ldots, M$$

- 1-particle dispersion

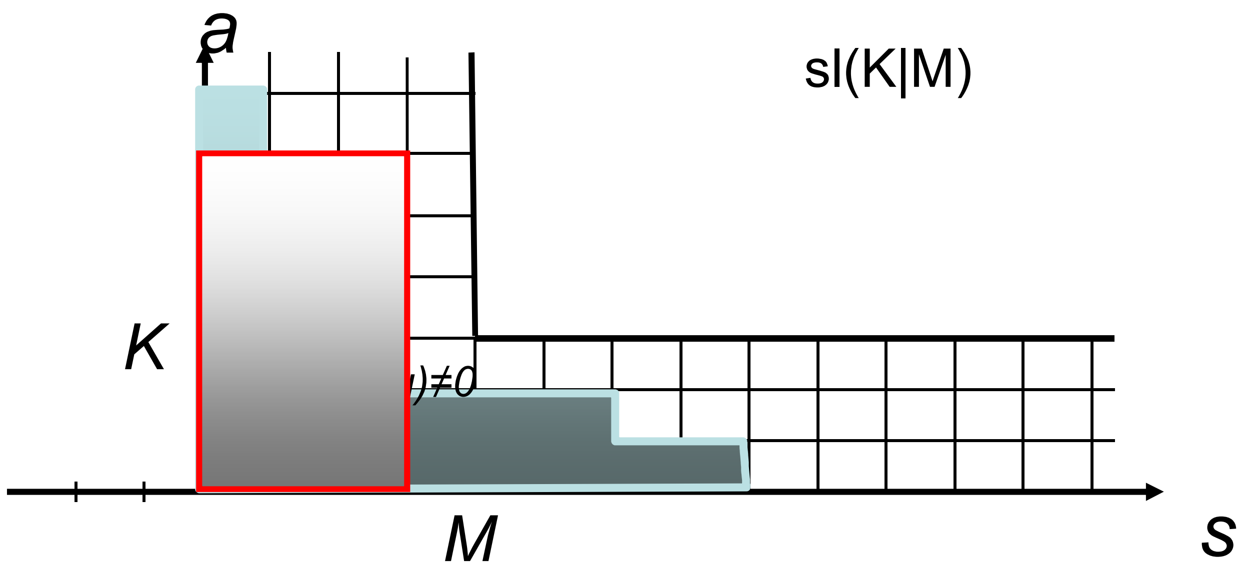
$$\begin{cases} \epsilon_a(u) = \dfrac{2gi}{x^{[+a]}} - \dfrac{2gi}{x^{[-a]}} + a \\[2ex] p_a(u) = -i \log \dfrac{x^{[+a]}}{x^{[-a]}} \end{cases}$$

$$\epsilon = \sqrt{a^2 + 16 g^2 \sin \frac{p^2}{2}}$$

- What are the arguments in favor of all this?

*

Quantum integrable super-spin chains
SUSY Boundary Conditions: Fat Hook



- All super Young tableaux of sl(K|M) live within this fat hook
- All transfer matrices T(a,s,u) of super-spin chains as well
- For (super)-characters for rectangular Young tableaux (a,s)

$$\chi_{a,s}(g) = \det_{1\le i,j\le a} \chi_{s+i-j}(g)\,.$$

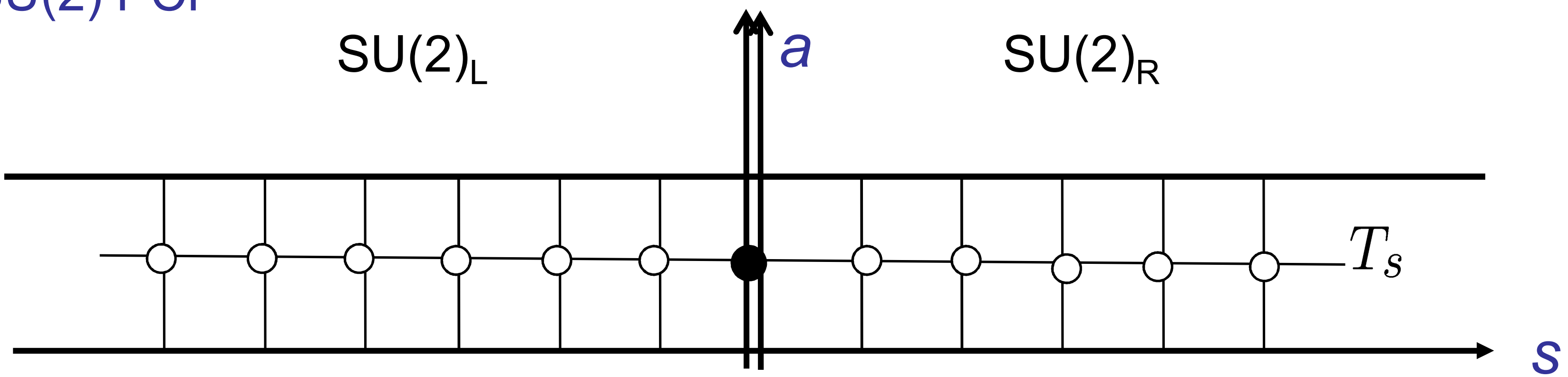
$$w(z) \equiv \mathrm{sdet}\,(1-zg)^{-1} = \sum_{s=1}^{\infty} \chi_s(g)\, z^s, \qquad g \in gl(K|M)$$

- Hirota eq. with fat hook b.c.: $\chi_{a,s}^2 = \chi_{a+1,s}\chi_{a-1,s} + \chi_{a,s+1}\chi_{a,s-1}$

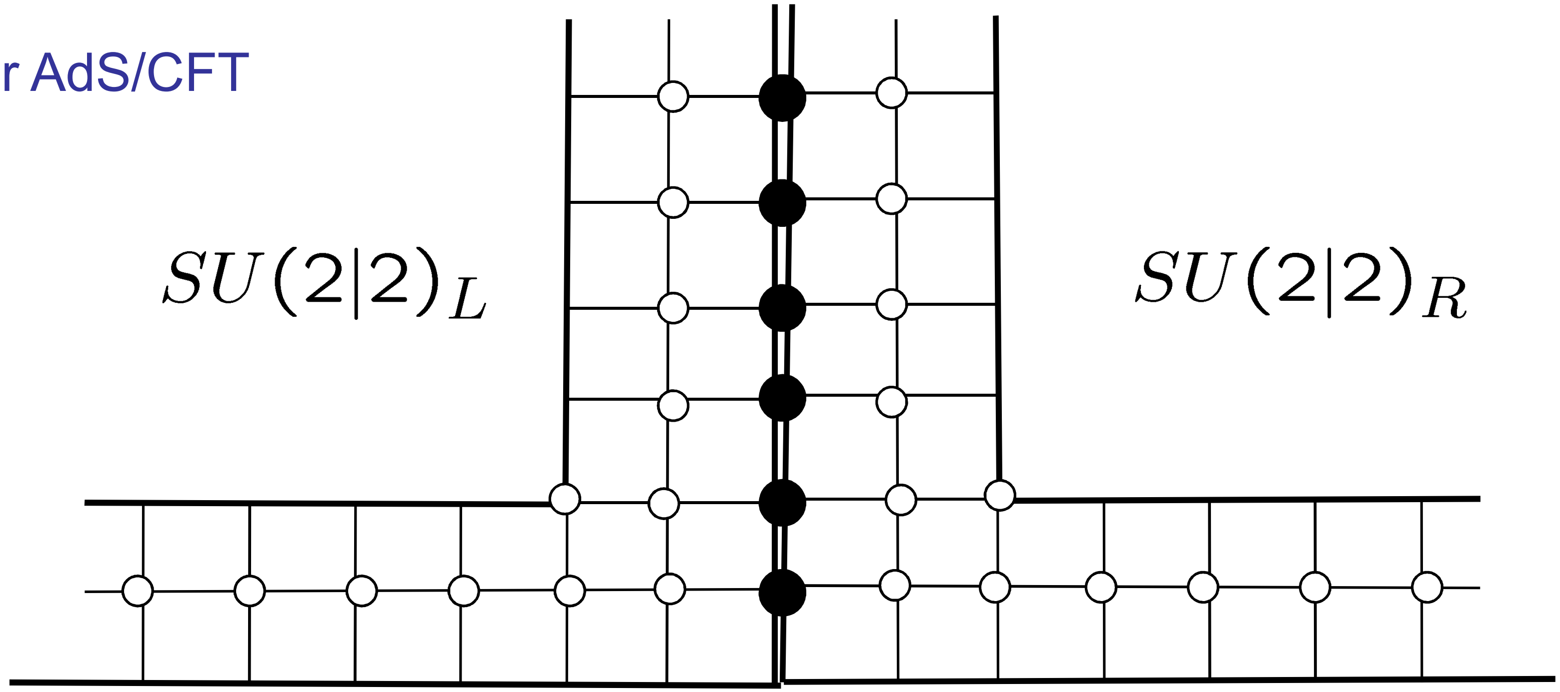
Gluing T-hook out of two SU(2|2) fat hooks.

Integrability=Global Hirota eq., and presumably PSU(2,2|4)

For SU(2) PCF



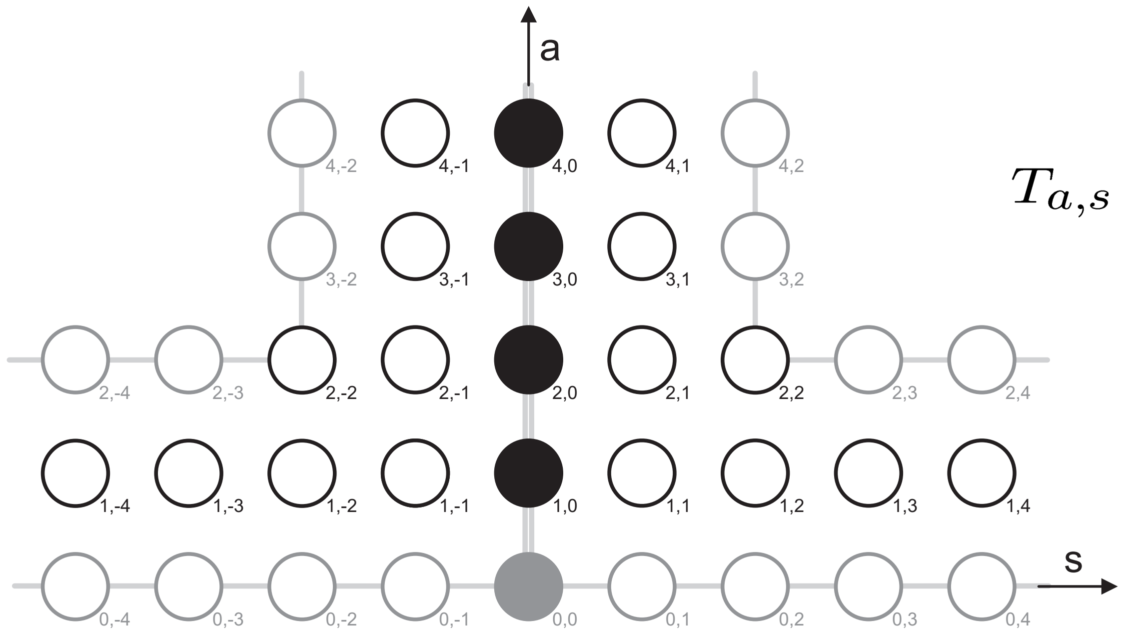
For AdS/CFT



*

Equivalently to Y-system, global Hirota equation...

$T_{a,s}\,(u)$



$T_{0,s} = 1$

$$Y_{a,s} = \frac{T_{a,s+1}\,T_{a,s-1}}{T_{a+1,s}\,T_{a-1,s}}$$

$$T^{+}_{a,s} T^{-}_{a,s} = T_{a,s+1}\,T_{a,s-1} + T_{a+1,s}\,T_{a-1,s}$$

- Gauge invariance of Hirota eq.

$$T_{a,s} \to g_1^{[a+s]} g_2^{[a-s]} g_3^{[s-a]} g_4^{[-a-s]} T_{a,s}$$

*

Back to AdS/CFT: L→∞

$$Y_{a\geq 1,0} \sim \left(\frac{x^{[-a]}}{x^{[+a]}}\right)^L \to 0$$

$$1 + Y_{a,s} = \frac{T_{a,s}^+ T_{a,s}^-}{T_{a+1,s} T_{a-1,s}}$$

- It is a spin chain limit:

$$\frac{Y_{a,0}^+ Y_{a,0}^-}{Y_{a-1,0} Y_{a+1,0}} \simeq \left(\frac{T_{a,1}^+ T_{a,1}^-}{T_{a-1,1} T_{a+1,1}}\right)\left(\frac{T_{a,-1}^+ T_{a,-1}^-}{T_{a-1,-1} T_{a+1,-1}}\right)$$

- T-system splits into two SU(2|2) wings: $T_{a,s>0} = T_{a,s}^R, \qquad T_{a,s<0} = T_{-k}^L$

- Solving this D'Alambert eq. we get

$$Y_{a,0}(u) \simeq \left(\frac{x^{[-a]}}{x^{[+a]}}\right)^L \frac{\phi^{[-a]}}{\phi^{[+a]}} T_{a,-1}^L T_{a,1}^R$$

transfer matrices of SU(2|2) in antisymmetric irreps

*

Asymptotic Bethe Ansatz (ABA)

- Fundamental transfer matrix for $SU(2|2)_{L,R}$

$$T_{1,1} = \frac{R^{-(+)}}{R^{-(-)}}\left[-\frac{R^{-(-)}Q_3^{+}}{R^{-(+)}Q_3^{-}} + \frac{Q_2^{--}Q_3^{+}}{Q_2 Q_3^{-}} + \frac{Q_2^{++}Q_1^{-}}{Q_2 Q_1^{+}} - \frac{B^{+(+)}Q_1^{-}}{B^{+(-)}Q_1^{+}}\right]$$

$$R_l^{(\pm)}(u) \equiv \prod_{j=1}^{K_l} \frac{x(u) - x_{l,j}^{\mp}}{(x_{l,j}^{\mp})^{1/2}}\ ,\ B_l^{(\pm)}(u) \equiv \prod_{j=1}^{K_l} \frac{\frac{1}{x(u)} - x_{l,j}^{\mp}}{(x_{l,j}^{\mp})^{1/2}}$$

$$Q_l(u) = \prod_{j=1}^{J_l}(u - u_{l,j}) = -R_l(u)B_l(u), \qquad l = 1,2,3(5,6,7)$$

- Ansatz, to fit ABA of Beisert-Staudacher eq. $Y_{1,0}(u_{4,j}) = -1$

$$\frac{\phi^-}{\phi^+} = S^2\,\frac{B^{+(+)}R^{-(-)}}{B^{-(-)}R^{+(+)}}\frac{B_{1L}^{+}B_{3L}^{-}}{B_{1L}^{-}B_{3L}^{+}}\frac{B_{1R}^{+}B_{3R}^{-}}{B_{1R}^{-}B_{3R}^{+}}$$

- Solution of Hirota eq. generates explicitly the rest of $T_{a,s\neq 0}(u)$

*

Crossing

- Crossing: particle-antiparticle transformation: $x^{\pm} \to 1/x^{\pm}$
- Y's should stay invariant!

- For each dressing factor in $S(u) = \prod_j \sigma(x(u), x_{4,j})$
  it gives Janik's crossing
  for each $SU(2|2)_{L,R}$

$$\sigma_{12}\sigma_{\bar{1}2} = \frac{x_2^-}{x_2^+}\,\frac{x_1^- - x_2^-}{x_1^+ - x_2^-}\,\frac{1/x_1^- - x_2^+}{1/x_1^+ - x_2^+}$$

*

Checks and Questions

- Crossing is an important check!
- We restored all known results for one wrapping corrections in week coupling (4-loop Konishi, one wrapping in twist two operators,…) for both SL(2) and SU(2) settings.
- Hirota and Y-system – a natural mathematical scheme to incorporate integrability at any g and L

****

- Numerics for $\Delta_{Konishi}(g)$?
- Destri-DeVega type eq.?
- Direct derivation of dressing factor?
- BFKL?

*

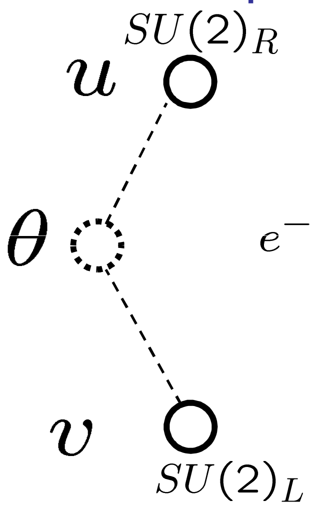
END

Asymptotic Bethe Ansatz eqs. (L → ∞)

- Periodicity:

$$e^{-iLm\,\sinh \pi\theta_k} = \prod_j{}' \widehat{S}(\theta_k - \theta_j)$$

- Bethe equations from periodicity



$$\begin{aligned}
1 &= \prod_{\beta}^{J_u} \frac{u_j - \theta_\beta - i/2}{u_j - \theta_\beta + i/2} \prod_{i \neq j}^{J_u} \frac{u_j - u_i + i}{u_j - u_i - i}\,, \\
e^{-imL\,\sinh \pi\theta_\alpha} &= \prod_{\beta \neq \alpha}^{L} S_0^2\left(\theta_\alpha - \theta_\beta\right) \prod_j^{J_u} \frac{\theta_\alpha - u_j + i/2}{\theta_\alpha - u_j - i/2} \prod_k^{J_v} \frac{\theta_\alpha - v_k + i/2}{\theta_\alpha - v_k - i/2}\,, \\
1 &= \prod_{\beta}^{J_v} \frac{v_k - \theta_\beta - i/2}{v_k - \theta_\beta + i/2} \prod_{l \neq k}^{J_v} \frac{v_k - v_l + i}{v_k - v_l - i}\,,
\end{aligned}$$

- θ -variables describe U(1)-sector (main circle of S^3 in O(4) model),

u, v -“magnon” variables – the transverse excitations on S^3, or SU(2)xSU(2)

- Energy and momentum of a state:

$$E = \sum_{k=1}^{L} m \cosh \pi\theta_k\,, \qquad P = \sum_{k=1}^{L} m \sinh \pi\theta_k$$

*

Integrable discrete Hirota dynamics

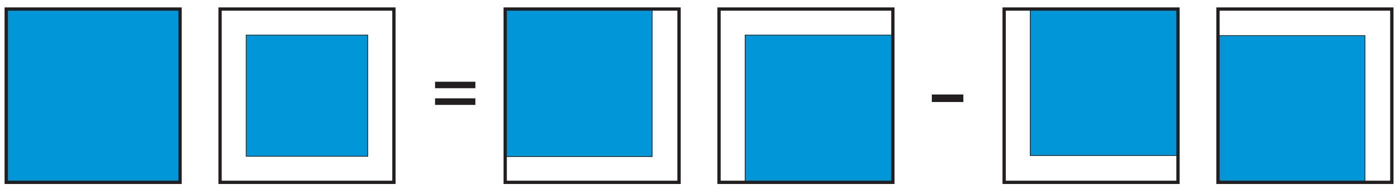
- Hirota classical integrable hierarchy for tau-function $T(n_1, n_2, n_3, n_4, \cdots)$:
  for any ordered triple $(n_\alpha, n_\beta, n_\gamma)$ of arguments Hirota equation is true

$$T(\cdots, n_\alpha+1, \cdots)\, T(\cdots, n_\alpha-1, \cdots) =$$
$$T(\cdots, n_\beta+1, \cdots)\, T(\cdots, n_\beta-1, \cdots) - T(\cdots, n_\gamma+1, \cdots)\, T(\cdots, n_\gamma-1, \cdots)$$

- General solution:

$$T(a, s, u) = \det_{1 \geq i,j \geq a} t\,(s + i - j, u + a - i - j)$$

- Check: by Jacobi relation:



*

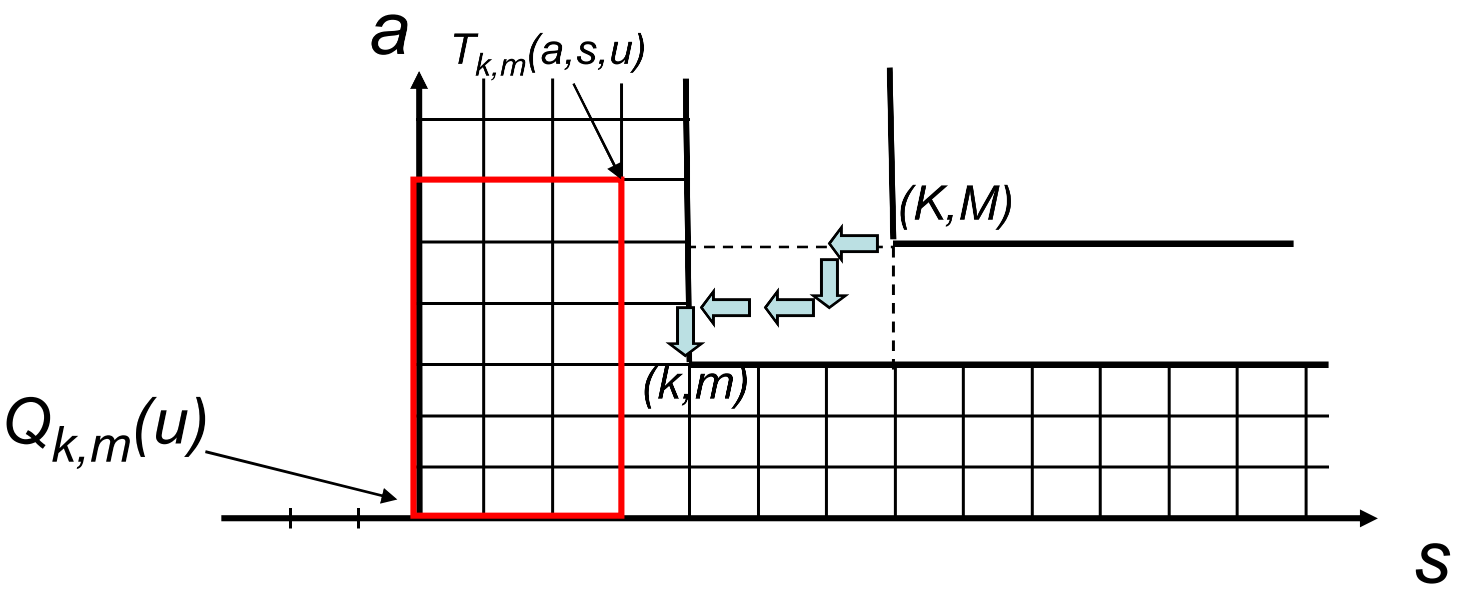
SUSY nesting (Backlund reduction)

[V.K.,Sorin,Zabrodin'07]

$$gl(K|M) \supset gl(K\text{-}1|M) \;\ldots\ldots \supset gl(k|m) \ldots \supset 0$$

Gelfand-Zeitlin scheme

- All super Young tableaux of $gl(k|m) \subset gl(K|M)$ live within (k,m) fat hook



Baxter's Q-functions: $Q_{k,m}(u)=\Pi_j\,(u-u_j)$ k=1,…,K m=1,…M

- By construction $T_{k,m}(u,a,s)$ and $Q_{k,m}(u)$ are polynomials of same power.

Asymptotic Bethe Ansatz (ABA)

- Fundamental transfer matrix for $SU(2|2)_{L,R}$

$$T_{1,1} = \frac{R^{-(+)}}{R^{-(-)}}\left[-\frac{R^{-(-)}Q_3^{+}}{R^{-(+)}Q_3^{-}} + \frac{Q_2^{--}Q_3^{+}}{Q_2 Q_3^{-}} + \frac{Q_2^{++}Q_1^{-}}{Q_2 Q_1^{+}} - \frac{B^{+(+)}Q_1^{-}}{B^{+(-)}Q_1^{+}}\right]$$

- Solution of Hirota eq. generates the rest of $T_{a,s\neq 0}(u)$

$$\mathcal{W} = \sum_{s=0}^{\infty} T_{1,s} D^s \ , \qquad \mathcal{W}^{-1} = \sum_{a=0}^{\infty} (-1)^a T_{a,1} D^a \qquad D = e^{-i\partial_u}$$

$$\mathcal{W} = \left[1 - \frac{Q_1^{-} S^{+(+)} R^{-(+)}}{Q_1^{+} S^{+(-)} R^{-(-)}} D\right]\left[1 - \frac{Q_1^{-} Q_2^{++} R^{-(+)}}{Q_1^{+} Q_2 R^{-(-)}} D\right]^{-1} \times \left[1 - \frac{Q_2^{--} Q_3^{+} R^{-(+)}}{Q_2 Q_3^{-} R^{-(-)}} D\right]^{-1}\left[1 - \frac{Q_3^{+}}{Q_3^{-}} D\right]$$

*