Projective superspace

WHO?

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- [arXiv:hep-th/0211184] A new holographic limit of $AdS_5 \times S^5$
- [arXiv:0709.4605] Superconformal spaces and implications for superstrings

⇒→ [arXiv:0812.4569] First-quantized N=4 Yang-Mills ←⇐

WHY?

- AdS/CFT, or some new version or string that's more manageable
- string-inspired 1st-quantized Feynman diagrams, with faster results (& off shell)
- field theory in extended superspace, 2nd-quantization

WHAT?

The simplest way to write (super)conformal transformations is as projective transformations: Start with a rectangle $\bar{z}_{A'}{}^{\mathcal{M}}$ of (super)coordinates, where $\mathcal{M}$ is a defining index of the global (superconformal) symmetry U(N|2,2) (for the case of D=4), ignoring P's & S's, and A' transforms under a local subgroup U(n|2). Then we can separate the global index as $\mathcal{M} = (M, M')$, where M' has the same range as A', and write

$$\bar{z}_{A'}{}^{\mathcal{M}} = \bar{u}_{A'}{}^{N'}(w_{N'}{}^{M}, \delta_{N'}^{M'})$$

So w is the ratio ("projection") between the 2 parts of z (square part in the denominator) that's gauge invariant. ($\bar{u}$ is pure gauge.) For various purposes, it's also useful to define the orthogonal $z_{\mathcal{M}}{}^{A}$, with gauge group U(N−n|2),

$$\bar{z}_{A'}{}^{\mathcal{M}} z_{\mathcal{M}}{}^{B} = 0 \quad \Rightarrow \quad z_{\mathcal{M}}{}^{A} = \begin{pmatrix} \delta_M^N \\ -w_{M'}{}^{N} \end{pmatrix} u_N{}^A$$

Dividing up the symmetry group element, now using matrix notation, as

$$g = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \qquad g^{-1} = \begin{pmatrix} \tilde{d} & -\tilde{c} \\ -\tilde{b} & \tilde{a} \end{pmatrix}$$

where $z_{A'}{}^{\mathcal{M}}$ transforms with g and thus $z_{\mathcal{M}}{}^{A}$ with g^{-1}, we have 2 forms of the symmetry transformation as fractional linear transformations:

$$w' = (wc + d)^{-1}(wa + b) = (\tilde{a}w + \tilde{b})(\tilde{c}w + \tilde{d})^{-1}$$

or we can write the infinitesimal form

$$\delta w = \alpha + \beta w + w\gamma + w\epsilon w$$

These should be familiar as SL(2) (the scale part cancels) for the projective space RP(1) (or CP(1) for complex) when g is 2×2 and w is 1×1 (just a number). Much less familiar is conformal transformations (N=0) in D=4, where g is 4×4 and w is 2×2, describing spacetime as HP(1) (a quaternion). Then this is the simplest way to write conformal transformations, finite or infinitesimal (again proving spinor indices are good for more than just fermions). A useful application is the ADHM construction for instantons (and its supersymmetric generalization, for n=0 & N>0, on HP(1|$\frac{1}{2}$N)).

Superconformal invariants are easiest to derive in this approach: We 1st construct superconformal, but not gauge, invariant objects of the form (in matrix notation)

$$z_{12} \equiv \bar{z}_1 z_2 = \bar{u}_1(w_1 - w_2)u_2$$

The u's transform linearly under the gauge group, so it's then easy to construct invariants by canceling them (e.g., $str(z_{12}z_{32}^{-1}z_{34}z_{14}^{-1})$).

For N>0, w has the usual 4 spacetime coordinates x, half the full set of anticommuting coordinates θ (i.e., 2N), and varying numbers of internal coordinates y: Writing $M = (m, \mu)$, $M' = (m', \dot{\mu})$, where $\mu, \dot{\mu}$ are Weyl spinor indices and m, m' are R-indices labeling which supersymmetry,

$$w_{M'}{}^{M} = \begin{array}{c} \\ m' \\ \dot{\mu} \end{array} \!\!\begin{array}{c} \begin{array}{cc} m & \mu \end{array} \\ \begin{pmatrix} y_{m'}{}^{m} & \theta_{m'}{}^{\mu} \\ \bar{\theta}_{\dot{\mu}}{}^{m} & x_{\dot{\mu}}{}^{\mu} \end{pmatrix} \end{array}$$

For example, n=0 & N describe chiral and antichiral superspace, with no y's. Note that the rest of the θ's of the usual full superspace are contained in u & $\bar{u}$; they are treated as nondynamical, but appear in such things as covariant derivatives. (Consider again the example of chiral superspace, for N=1.)

We'll mostly be interested in n=N/2, so w is a square matrix, because these cases allow the definition of a reality condition. This follows from the modified unitarity of the superconformal group elements,

$$g^\dagger \Upsilon g = \Upsilon, \quad \Upsilon^2 = 1, \quad \Upsilon^\dagger = \Upsilon$$

$$\Upsilon^{\dot{\mathcal{M}}\mathcal{N}} = \begin{array}{c} \\ \dot{\mu} \\ \dot{m} \\ \mu \\ \dot{m}' \end{array} \!\!\begin{array}{c} \begin{array}{cccc} \nu & n & \dot{\nu} & n' \end{array} \\ \begin{pmatrix} 0 & 0 & -iC & 0 \\ 0 & I & 0 & 0 \\ iC & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \end{array}, \qquad C^{\mu\nu} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad (I)^{\dot{m}n} \equiv \delta_m^n$$

which leads us to define charge conjugation as

$$\mathcal{C}\bar{z} \equiv z^\dagger \Upsilon; \quad \bar{z}' = \bar{z}g, \quad z' = g^{-1}z \quad \Rightarrow \quad (\mathcal{C}\bar{z})' = (\mathcal{C}\bar{z})g$$

(although the local transformations of $\bar{z}$ and $\mathcal{C}\bar{z}$ are different). Then $\mathcal{C}w$ is obtained from $\mathcal{C}\bar{z}$ in the same way as w from $\bar{z}$:

$$(\mathcal{C}w)^\dagger = \begin{array}{c} \\ m \\ \dot{\mu} \end{array} \begin{array}{c} \begin{array}{cc} \quad m' \quad & \quad \mu \quad \end{array} \\ \begin{pmatrix} -y^{-1}{}_m{}^{m'} & iy^{-1}{}_m{}^{n'}\theta_{n'}{}^\nu C_{\nu\mu} \\ -iC^{\dot{\mu}\dot{\nu}}\bar{\theta}_{\dot{\nu}}{}^b y^{-1}{}_n{}^{m'} & C^{\dot{\mu}\dot{\nu}}(x_{\dot{\nu}}{}^\nu - \bar{\theta}_{\dot{\nu}}{}^n y^{-1}{}_n{}^{n'}\theta_{n'}{}^\nu)C_{\nu\mu} \end{pmatrix} \end{array}$$

This allows definition of a superconformally invariant (4D extension of the Hilbert space) inner product as

$$\langle A|B\rangle \equiv \int dw \ (\mathcal{C}A)(w)B(w) = \langle B|A\rangle^*$$

for any A and B that transform as half-densities

$$dw'[A'(w')]^2 = dw[A(w)]^2$$

where $\mathcal{C}A$, which transforms in the same way as A, is defined by the relation of complex conjugation to charge conjugation in the above inner product,

$$(\mathcal{C}A)(w) \equiv [det(y)]^{-str(I)}[A(\mathcal{C}w)]^\dagger, \quad str(I) = \tfrac{1}{2}N - 2$$

General off-shell theories have been written in projective superspace for the case of N=2 (Lindström, Roček, ..., '84-).

WHERE? WHEN? (by Lorentz) SWHERE? (supersymmetric)

Unfortunately, gauge fixing breaks conformal invariance. (The field strength is the curl of the gauge field, which is thus treated as a 1-form, but the Lorenz gauge-fixing function is the divergence, treating it as a 3-form, which would like a different conformal weight.) Furthermore, we need some kind of (supergroup) metric to perform 1st-quantization conveniently. But 4D N=4 Yang-Mills is superconformal, so it's the same in any conformally flat space, like Minkowski or anti-de Sitter. (Note: CP(1) is *not* the sphere; it's anything conformal to the sphere; similarly for HP(1) vs. Minkowski or AdS.)

So, rather then the projective space above (which makes superconformal symmetry nice), we use the coset space

$$\frac{\mathrm{OSp(N|4)}}{\mathrm{OSp(n|2)OSp(N-n|2)}}$$

which leads to exactly the same coordinates w, but different "u". We can also use the contraction (to a different superconformal subgroup)

$$\frac{\mathrm{I[OSp(n|2)OSp(N-n|2)]}}{\mathrm{OSp(n|2)OSp(N-n|2)}}$$

which has a flat & torsion-free coordinate space (w), but a "curved" tangent space (u): Translations of w include $\frac{1}{2}$ the supersymmetries, while rotations include the sum of the other $\frac{1}{2}$ plus $\frac{1}{2}$ the S-supersymmetries (not the direct sum).

Also, the general set of 1st-class constraints (field equations) for supersymmetric theories has too many free indices, so the ghosts have more and more indices at each ghost level, making a mess for quantization. The superconformal constraints $\hat{\mathcal{G}}$ can be written in terms of its generators $\hat{G}$ as

$$\hat{\mathcal{G}}_{\mathcal{MN}}{}^{\mathcal{PQ}} \equiv \hat{G}_{(\mathcal{M}}{}^{(\mathcal{P}}\hat{G}_{\mathcal{N}]}{}^{\mathcal{Q}]} - \tfrac{1}{2}\delta_{(\mathcal{M}}{}^{\mathcal{P}}\delta_{\mathcal{N}]}{}^{\mathcal{Q}} = 0$$

(Our grading is defined so R-indices are treated as bosonic, Weyl as fermionic.) However, only a small subset of these can be written in terms of the super-AdS generators G,

$$G_{\mathcal{MN}} \equiv \hat{G}_{[\mathcal{M}|}{}^{\mathcal{P}}\eta_{\mathcal{P}|\mathcal{N})}$$

(where η is the OSp(N|4) metric), namely

$$\mathcal{G}_{\mathcal{MN}} \equiv G_{(\mathcal{M}|}{}^{\mathcal{P}}G_{\mathcal{P}|\mathcal{N}]} + str(I)\eta_{\mathcal{MN}} = 0$$

These constraints can be derived by a general group-theory analysis, solved in a lightcone gauge, & found sufficient. Note that the constraints (and thus the Faddeev-Popov ghosts) carry 2 indices, just as the generators do; this pattern will persist to all generations of ghosts.

The net result is that the complete minimal BRST operator can be written in the simple form (matrix multiplication with metric, and trace, implied)

$$Q = \sum_{m,n=0}^{\infty} c_{m+n+1}b_m b_n + f...$$

where the indices label the ghost generation, and

$$b_0 = d$$

where we have replaced the constraints with the "dual" ones in terms of covariant derivatives d. Here "f..." denotes structure-constant terms. (We won't need those for the contracted projective case, where we can replace d with $\partial/\partial w$.)

In terms of the graded transpose "^{T}", we have

$$d^T = -d$$

$$(d\eta d)^T = +(d\eta d)$$

for the covariant derivatives d and constraints $\mathcal{G}_1 \equiv d\eta d = 0$. Then the reducibility conditions are

$$\mathcal{G}_2 \equiv d\eta\mathcal{G}_1 - \mathcal{G}_1\eta d = +\mathcal{G}_2^T = 0$$

$$\mathcal{G}_3 \equiv d\eta\mathcal{G}_2 + \mathcal{G}_2\eta d = -\mathcal{G}_3^T = 0$$

etc., where the sign for the symmetry of $\mathcal{G}_n$ alternates as $-++--++--...$. Explicitly, with $\mathcal{G}_0 \equiv d$,

$$\mathcal{G}_{n+1} \equiv d\eta\mathcal{G}_n + (-1)^n\mathcal{G}_n\eta d = (-1)^{n(n-1)/2}\mathcal{G}_{n+1}^T = 0$$

Using this construction for the BRST operator, and including terms for closure ($Q^2 = 0$) leads to the above expression for the BRST operator, where

$$c_n = -(-1)^{n(n+1)/2}c_n^T$$

and similarly for b.

We now specialize to our coset representation by setting to 0 the derivatives that are isotropy constraints, so we can drop some terms, & some constraints altogether. The above result for Q can then be applied directly by dividing the ranges of the indices in half and dropping irrelevant blocks. The result is as above for odd n, except both indices are primed or both unprimed, while for even n we have mixed indices:

$$\begin{cases} c_{2n+1,AB}, \ c_{2n+1,A'B'} & \text{where } c_{2n+1} = (-1)^n c_{2n+1}^T \\ c_{2n,AB'} \end{cases}$$

Thus the symmetry has a cycle of 4, going as asymmetric, (twice) graded symmetric, asymmetric, (twice) graded antisymmetric.

Note that for N=4 the number of bosons & fermions is equal at each ghost level; this suggests ghost 0-modes could cancel each other without any type of insertion. Furthermore, the use of super-AdS or its contraction suggests all momenta should appear in the propagator (because of OSp symmetry), preventing momentum 0-modes.

HOW?

Consider now the gauge superfield theory for N=4 Yang-Mills. We start with the "full" OSp(4|4) superspace of w and the u's, which includes x, all 4N θ's, a bunch of y's (for SO(4)), and even some Lorentz coordinates. We introduce gauge-covariant derivatives for all these coordinates. We then want to reduce to the projective superspace coset OSp(4|4)/OSp(2|2)^2.

N=4 is the only projective case where the ∇_u algebra has field strengths, but these can be absorbed by (the gauge fields of) the SO(2) derivatives. (A similar procedure works for the N=2 chiral case, but not for N=4 chiral.) Examining the relations

$$\{\nabla_{a\alpha}, \nabla_{b\beta}\} = -C_{\alpha\beta}(\nabla_{ab} + \phi_{ab}) - \eta_{ab}\nabla_{\alpha\beta}$$
$$\{\nabla_{a'\dot\alpha}, \nabla_{b'\dot\beta}\} = -C_{\dot\alpha\dot\beta}(\nabla_{a'b'} + \bar\phi_{a'b'}) - \eta_{a'b'}\nabla_{\dot\alpha\dot\beta}$$
$$\phi_{ab} = C_{ab}\phi, \qquad \bar\phi_{a'b'} = C_{a'b'}\phi, \qquad [\nabla_u, \phi] = 0$$

(for $\nabla_u = (\nabla_{ab}, \nabla_{a'b'}, \nabla_{a\alpha}, \nabla_{a'\dot\alpha}, \nabla_{\alpha\beta}, \nabla_{\dot\alpha\dot\beta})$), we see that we can consistently impose the isotropy constraints

$$\nabla_{\alpha\beta} = \nabla_{\dot\alpha\dot\beta} = \nabla_{a\alpha} = \nabla_{a'\dot\alpha} = \nabla_{ab} + \phi_{ab} = \nabla_{a'b'} + \bar\phi_{a'b'} = 0$$

as a closed algebra. (This is equivalent to a redefinition of the SO(2) derivatives.) In particular, we can choose the gauge where the above isotropy constraints reduce to just $d_u = 0$. In this gauge, there is a residual gauge invariance with $d_u\lambda = 0$; i.e., the gauge parameter λ is projective. At that point we can work exclusively in terms of ∇_w. Here ϕ is the projective field strength, which contains all component field strengths in its expansion.

Some interesting features of this required modification are: (1) It involves only the SO(n)SO(N−n) isotropy derivatives, and hence requires the super anti-de Sitter construction. (The analogous derivatives in flat superspace would be central charges, which would break superconformal invariance. However, we can still use our contracted coset, since the isotropy group is unchanged.) (2) The modifications must involve only a single field strength (ϕ) to avoid generation of field-strength commutator terms (and hence nonclosure) in the algebra of isotropy constraints, and hence both n and N−n $\leq$ 2. This shows that chiral superspace does not exist for N=4 Yang-Mills.

As an aside, note that the apparently excessive Lorentz derivatives can have their uses: For example, even in the N=0 case, these derivatives are useful for selfdual Yang-Mills. In the lightcone gauge for this theory, we separate the + and − components of the undotted spinor index (but not the dotted one) to solve some of the selfduality conditions as

$$\{\nabla_{+\dot{\alpha}}, \nabla_{+\dot{\beta}}\} = 0 \quad \Rightarrow \quad A_{+\dot{\alpha}} = 0$$
$$\{\nabla_{+[\dot{\alpha}}, \nabla_{-\dot{\beta}]}\} = 0 \quad \Rightarrow \quad A_{-\dot{\alpha}} = \partial_{+\dot{\alpha}} A_{--}$$

(where $\nabla = d + iA$) in terms of the "prepotential" A_{--}. But the solution of the second equation automatically follows from the first because of Lorentz invariance, when it is gauged; the prepotential appears already as a potential: Introducing $\nabla_{\alpha\beta}$,

$$\{\nabla_{+\dot{\alpha}}, \nabla_{--}\} = \nabla_{-\dot{\alpha}} \quad \Rightarrow \quad A_{-\dot{\alpha}} = \partial_{+\dot{\alpha}} A_{--}$$

Pure spinors are also related to (coset) Lorentz coordinates.

The usefulness of projective superspace for scattering amplitudes can be seen in the simple example of the 4-point (although there is not yet an off-shell N=4 covariant derivation). It has the same kinematic factor at all loops, which is multiplied by a purely x-space expression. This factor has a very simple form (Kallosh, '07), especially because the projective field strength ϕ introduced above is a single scalar. For example, for the tree graph, the θ dependence is the local product, and the y dependence evaluates at $y = 0$:

$$\hat{\mathcal{A}}_{4\Pi} = \int d^{16}x_i \; d^8\theta \; \phi(x_1, \theta, 0)\phi(x_2, \theta, 0)\phi(x_3, \theta, 0)\phi(x_4, \theta, 0) \; \frac{\delta^4(x_1 - x_2 + x_3 - x_4)}{x_{12}^2 x_{23}^2}$$

Neither superconformal nor even OSp(4|4) invariance is manifest, possibly due to the gauge choice breaking some of the symmetry (off shell), but maybe further simplifications are possible.

On shell this amplitude also has a simple supertwistor expression: Besides conservation δ-functions for conjugate momenta for both x and (the 2N projective) θ's (but not redundant ones for y), it's simply $1/st$. Unlike the chiral or antichiral (MHV or $\overline{\text{MHV}}$) cases, this depends only on momenta, without twistor phases. This is related to the fact that there are no chiral scalars in N=4 Yang-Mills.