BRST derivation of the pure spinor scattering amplitude prescription and anomalies

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In this talk I will discuss two topics:

1. BRST derivation of the minimal and non-minimal amplitude prescription for the pure spinor superstring.

2. Non-decoupling of BRST exact states in the minimal formalism.

1. Brief review of minimal pure spinor formalism

1. The worldsheet action is

$$S_\sigma = \int d^2z \left(\frac{1}{2} \partial x^m \bar{\partial} x_m + p_\alpha \bar{\partial} \theta^\alpha - w_\alpha \bar{\partial} \lambda^\alpha \right)$$

where we display only the left-moving sector for $p_\alpha, \theta^\alpha, w_\alpha, \lambda^\alpha$.

2. This model is invariant under a fermionic nilponent symmetry generated by

$$Q_S = \oint dz \lambda^\alpha(z) d_\alpha(z),$$

where $d_\alpha = p_\alpha - \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\beta \partial x_m - \frac{1}{8}\gamma^m_{\alpha\beta}\gamma_{m\,\gamma\delta}\theta^\beta\theta^\gamma\partial\theta^\delta$.

3. The cohomology of this operator at ghost number one reproduces the superstring spectrum.

4. The total central charge is zero, $c = 0$.

Scattering amplitude prescription [Berkovits, (2004)]

1. Vertex operators:

$$[Q_S, V^{(0)}] = 0, \qquad [Q_S, V^{(1)}] = \partial V^{(0)}$$

$V^{(0)}$ has dimension 0; $V^{(1)}$ has dimension 1.

2. "Picture raising and lowering" operators

$$\begin{aligned} Y_C &= C_\alpha \theta^\alpha \delta(C_\alpha \lambda^\alpha), \\ Z_B &= \frac{1}{2} B^{mn} d\gamma_{mn} \lambda \delta(B^{mn} N_{mn}), \; Z_J = (\lambda^\alpha d_\alpha)\delta(w_\alpha \lambda^\alpha) \end{aligned}$$

C_α is a constant spinor, B_{mn} is a constant antisymmetric tensor, $N_{mn} = \frac{1}{2} w_\alpha (\gamma_{mn})^\alpha{}_\beta \lambda^\beta$ is the Lorentz current.

3. Picture raised "b-field"

$$\{Q_S, b\} = Z_B T$$

where T is the stress energy tensor.

Amplitude prescription:

At tree-level:

$$A_0 = \langle V_1^{(0)}(z_1) V_2^{(0)}(z_2) V_3^{(0)}(z_3) \int dz_4 V_4^{(1)}(z_4) \cdots \int dz_N V_N^{(1)}(z_N) \\ Y_{C_1}(y_1) \cdots Y_{C_{11}}(y_{11}) \rangle$$

At g-loops ($g > 1$)

$$A_g = \int d^{3g-3}\tau \langle \prod_{j=1}^{3g-3} (b, \partial_j \hat{g}) \prod_{P=3g-2}^{10g} Z_{B_P}(w_P) \prod_{R=1}^{g} Z_J(u_R) \prod_{I=1}^{11} Y_{C_I}(y_I) \prod_{r=1}^{N} \int dz_r V_r^{(1)}(z_r) \rangle$$

where

$$(b, \partial_j \hat{g}) = \int d^2\sigma \sqrt{\hat{g}} b^{\alpha\beta} \frac{\partial \hat{g}_{\alpha\beta}(\sigma; \tau^k)}{\partial \tau^k}$$

and τ^k are the metric moduli.

Non-minimal formalism

We introduce non-minimal variables: the complex conjugate $\bar{\lambda}_\alpha$ of λ^α, a fermionic constrained spinor r_β satisfying

$$\bar{\lambda}_\alpha \gamma_m^{\alpha\beta} \bar{\lambda}_\beta = 0, \quad \bar{\lambda}_\alpha \gamma_m^{\alpha\beta} r_\beta = 0,$$

and their conjugate momenta, $\bar{w}^\alpha$ and s^a.

The action is now

$$S_\sigma \to S_\sigma + \int d^2 z \Big(-\bar{w}^\alpha \bar{\partial} \bar{\lambda}_\alpha + s^\alpha \bar{\partial} r_\alpha \Big),$$

and

$$Q_S \to Q_S + \oint dz \bar{w}^\alpha r_\alpha$$

This implies that the cohomology of Q_S is independent of the non-minimal variables.

Scattering prescription [Berkovits, (2005)]

1. Vertex operators: same as in minimal formulation.

2. Picture charging operators are replaced by the "regularization factor"

$$\mathcal{N} = \exp\Big[-\bar{\lambda}_\alpha\lambda^\alpha - r_\alpha\theta^\alpha - \sum_{I=1}^{g}\Big(N^{mnI}N^I_{mn} + \bar{J}^I J^I + S^{mnI}(d^I\gamma_{mn}\lambda)_0 + S^I(d^I_\alpha\lambda^\alpha)_0\Big)\Big]$$

$\bar{N}_{mn}, \bar{J}, S^{mn}, S$ are constructed from the non-mininal fields.

N^{mnI} is the zero mode part of N_{mn} etc.

3. Composite "$\tilde{b}$ field"

$$\{Q_S, \tilde{b}\} = T$$

Amplitude prescription

- At tree-level:

$$A_0 = \langle \mathcal{N} V_1^{(0)}(z_1) V_2^{(0)}(z_2) V_3^{(0)}(z_3) \int dz_4 V_4^{(1)}(z_4) \cdots \int dz_N V_N^{(1)}(z_N) \rangle$$

- At g-loops ($g > 1$)

$$A_g = \int d^{3g-3}\tau \langle \mathcal{N} \prod_{j=1}^{3g-3} (\tilde{b}, \partial_j \hat{g}) \prod_{r=1}^{N} \int dz_r V_r^{(1)}(z_r) \rangle$$

Plan

Our aim is to derive these scattering amplitude prescriptions from first principles.

Our point of view is:

1. We consider the original model as a "matter" sigma model with target space the 10d superspace times the pure spinor space.

2. To construct a string theory we couple this theory to 2d gravity and then BRST quantize this system.

Because the matter variables have $c = 0$ the $2d$ gravity has to be topological.

2. Coupling to topological gravity

The first step is to relax the conformal gauge.

The action of the minimal model coupled to gravity is given by

$$S_\sigma = \int d^2\sigma \sqrt{g} g^{ab} \left(\frac{1}{4} \partial_a x^m \partial_b x_m + \hat{p}_{a\alpha} \partial_b \theta^\alpha - \hat{w}_{a\alpha} \partial_b \lambda^\alpha \right)$$

where

$$\hat{p}_a = P^{(+)b}_{\ a} p_b$$

and $P^{(+)b}_{\ a}$ is a projection operator,

$$P^{(\pm)b}_{\ a} = \frac{1}{2} (\delta_a{}^b \mp i J_a{}^b) \,,$$

$J_a{}^b$ is the worldsheet complex structure.

For the $2d$ gravity to be topological the Q_S transformations should also act on g_{ab},

$$[Q_S, g_{ab}] = \hat{\psi}_{ab}$$

where ψ_{ab} is a new field.

To construct an invariant action we now add a new term to the action,

$$S_\sigma \longrightarrow S = S_\sigma + \frac{1}{2}\int d^2\sigma\sqrt{g}g^{ac}g^{bd}G_{ab}\hat{\psi}_{cd}$$

The new action would be invariant provided there exists a composite field G_{ab} transforming as

$$\{Q_S, G_{ab}\} = T_{ab}$$

Adding vertex operators

We will be interested in computing scattering amplitudes. For this aim, it is useful to introduce sources ρ^i with Weyl weight one that couple to vertex operators V_i

$$\mathcal{S} = S + \sum_{i=1}^{n} \rho^i V_i[\varphi](z_i, \zeta_i)$$

where φ denotes collectively all worldsheet fields.

The generating functional of scattering amplitudes is then

$$Z[\rho^i] = \int [d\mu] \exp(-\mathcal{S})$$

where the precise form of the path integral measure $[d\mu]$ will be derived presently.

The new action $\mathcal{S}$ depends on the positions of the vertex operators z_i and their Q_S partners, $[Q_S, z_i] = \zeta_i$.

Since ζ_i is fermionic

$$V_i[\varphi](z_i, \zeta_i) = V_i^{(0)}[\varphi](z_i) + \zeta_i V_i^{(1)}[\varphi](z_i).$$

Q_S invariance of $\mathcal{S}$ then requires

$$[Q_S, V^{(0)}] = 0, \qquad [Q_S, V^{(1)}] = \partial V^{(0)}$$

To summarize, we have obtained an action $\mathcal{S}$ which

1. depends on the original fields $\{X, p_\alpha, \theta^\alpha, \lambda^\alpha, w_\alpha\}$, the world-sheet metric g_{ab} and its Q_S partner $\hat{\psi}_{cd}$, the positions of the vertex operators $\{z_i, \zeta^i\}$, which are considered as constant "fields", and the sources ρ^i,

2. it is Q_S invariant,

3. it is invariant under local diffeomorphism and Weyl transformations.

This is our starting point for BRST quantization.

3. BRST quantization

• Now that we have an action with local invariance under worldsheet diffeomorphisms and Weyl transformations we can just follow the standard BRST rules to quantize this system.

• In our BRST treatment we include the ”gauge invariances” due to zero modes.

• Zero modes imply an invariance of the action, where the fields are shifted by their zero modes. This invariance must be fixed for the path integral to be well-defined.

• This can be done using the BV quantization scheme and amounts to introducing a gauge fixing condition, constant ghosts, ghosts-for-ghosts, extraghosts etc.

• In the case of the bosonic string, this leads straightforwardly to the usual scattering amplitude prescription with all the correct path integral insertions. The metric moduli themselves play the role of extraghosts. [Craps, KS (2005)]

• To maintain the Q_S symmetry in the process of quantization ghosts, antighost etc are introduced in Q_S-pairs.

• Gauge-fixing:

$$L_1 = Q_V Q_S(\tilde{\beta}^{ab}[g_{ab} - \hat{g}_{ab}(\tau)])$$

where Q_V is the BRST operators and $\hat{g}_{ab}(\tau)$ is a reference metric.

In addition when the Riemann surface has κ conformal Killing vectors we need to fix κ additional constant "gauge" symmetries.

$$L_2 = Q_V Q_S \left(\sum_f \beta^{\hat{j}}(z_{\hat{j}} - \hat{z}_{\hat{j}}) \right)$$

With $L_1 + L_2$ we deal with all gauge invariances except the ones due to zero modes of the original fields.

All fields introduced in the BRST quantization appear at most quadratically and can be explicitly integrated out leading to the following formula for the scattering amplitudes:

$$\int d\mu_\sigma e^{-S_\sigma} \prod_k d\tau^k (G, \partial_k \hat{g}) \prod_{\hat{j}=1}^{\kappa} V_{\hat{j}}^{(0)}(\hat{z}_{\hat{j}}) \prod_{i=\kappa+1}^{n} \int dz_i V_i^{(1)}(z_i)$$

where $(G, \partial_k \hat{g}) = \int_\Sigma d^2\sigma \sqrt{\hat{g}} G^{ab} \partial_k \hat{g}_{ab}$. Recall that G is defined by

$$\{Q_S, G\} = T$$

We still need to determine the path integral measure $d\mu_\sigma$ over the original fields $\{X, p_\alpha, \theta^\alpha, \lambda^\alpha, w_\alpha\}$.

Pure spinor measure

The pure spinor path integral measure is again determined by BRST-BV quantization treatment of the zero mode gauge invariances.

”Gauge” invariances due to fermionic zero modes are not very important in this context; the vertex operators provide the required zero modes so that the path integral is non-vanishing.

”Gauge” invariance due to bosonic non-compact zero modes must be gauge fixed, however, because otherwise the integration over them is divergent; the action S_σ does not contain a convergence factor because of the zero mode gauge invariance.

Recall that on a genus g surface:

A worldsheet scalar has one zero mode, and a worldsheet vector has g zero modes.

So in total we have the following bosonic zero modes:

10 zero modes x^m, 11 zero modes λ^α

$11g$ zero modes w^I_α. These we trade for:

$10g$ zero modes N^I_{mn} of the 10 independent components of N_{mn}

g zero modes J^I of J.

(N_{mn} is the (contribution of the pure spinors to the) Lorentz current and J is the "ghost" $U(1)$ generator).

Gauge fixing

The treatment of x-zero modes is standard.

To gauge fix the remaining invariances due to the pure spinor fields we introduce the following gauge fixing term:

$$L_3 = Q_V Q_S \left(b_\alpha \theta^\alpha + \sum_{I=1}^{g} (b^{mnI} N^I_{mn} + b^I J^I) \right).$$

where b_α, b^{mnI}, b^I are corresponding antighosts.

Integrating out ghost and antighosts leads to

$$
\begin{aligned}
L_3 &= \pi_\alpha \lambda^\alpha + \tilde{\pi}_\alpha \theta^\alpha \\
&+ \sum_{I=1}^{g} \Big(\tilde{\pi}^{mnI} N^I_{mn} + \tilde{\pi}^I J^I + \pi^{mnI} (d^I \gamma_{mn} \lambda)_0 + \pi^I (d^I_\alpha \lambda^\alpha)_0 \Big)
\end{aligned}
$$

where the π fields are the (constant) BRST auxiliary fields imposing the gauge fix conditions.

$\pi_\alpha, \tilde{\pi}_\alpha$ have 11 independent components each,

$\pi^{mnI}, \tilde{\pi}^{mnI}$ have $10g$ independent components each and

$\pi^I, \tilde{\pi}^I$ are g (constant) fields.

Non-minimal formulation

The auxiliary fields π_α and $\tilde{\pi}_\alpha$ have exactly the properties of the (zero modes) of the non-minimal variables $\bar{\lambda}_\alpha$ and r_α, namely the have the same number of components and the same Q_S transformations, so one may identify them:

$$\pi_\alpha = \bar{\lambda}^0_\alpha, \qquad \tilde{\pi}_\alpha = r^0_\alpha$$

Similarly one finds that one can identify:

$$\pi^{mnI} = \bar{N}^{mnI}, \quad \bar{\pi}^{mnI} = S^{mnI}, \quad \pi^I = S^I, \quad \tilde{\pi}^I = \bar{J}^I$$

With these identifications, the regularization factor is exactly equal to the gauge fixing Lagrangian:

$$\mathcal{N} = \exp(-L_3)$$

Furthermore, the equation

$$\{Q_S, G\} = T$$

has a solution

$$G_B = \frac{\bar\lambda_\alpha G^\alpha}{(\bar\lambda\lambda)} + \frac{\bar\lambda_\alpha r_\beta H^{[\alpha\beta]}}{(\bar\lambda\lambda)^2} - \frac{\bar\lambda_\alpha r_\beta r_\gamma K^{[\alpha\beta\gamma]}}{(\bar\lambda\lambda)^3} - \frac{\bar\lambda_\alpha r_\beta r_\gamma r_\delta L^{[\alpha\beta\gamma\delta]}}{(\bar\lambda\lambda)^4}$$

We thus arrive at the non-minimal scattering amplitude prescription.

The G_B field however has poles as $\bar\lambda\lambda \to 0$ so there are potentially problems.

Note that there exits a field [Berkovits, (2005)]

$$\xi_{nm} = \frac{\bar{\lambda}_\alpha \theta^\alpha}{\bar{\lambda}_\beta \lambda^\beta + r_\beta \theta^\beta}, \qquad \{Q_S, \xi_{nm}\} = 1.$$

This diverges as $(\bar{\lambda}\lambda)^{-11}$.

Had one allowed such singular behavior any closed operator V would also be exact,

$$\{Q_S, V\} = 0 \qquad \Rightarrow \qquad V = \{Q_S, (\xi_{nm} V)\}.$$

So one must ensure that no operators which diverge with this rate are allowed. A related issue is that the path integral will diverge if the insertions diverge as fast as $(\bar{\lambda}\lambda)^{-11}$. This can only happen for genus $g > 2$.

To avoid this problem one needs a different representative of $[G]$ that is less singular. A construction of such representative is discussed in [Berkovits, Nekrasov (2006)]. The actual construction however is very complicated and has not been used is actual computations to date.

Given that the issues with singularities are related to the $\bar{\lambda}\lambda \to 0$ limit, a different approach would be to modify the gauge fixing condition for the pure spinor zero modes such that they are fixed to a non-zero value.

If such gauge fixing can be consistently implemented it would lead to a simpler scattering amplitude prescription.

Minimal prescription

Let us go back to the gauged fixed action:

$$
\begin{aligned}
L_3 &= \pi_\alpha \lambda^\alpha + \tilde{\pi}_\alpha \theta^\alpha \\
&+ \sum_{I=1}^{g} \left(\tilde{\pi}^{mnI} N^I_{mn} + \tilde{\pi}^I J^I + \pi^{mnI} (d^I \gamma_{mn} \lambda)_0 + \pi^I (d^I_\alpha \lambda^\alpha)_0 \right)
\end{aligned}
$$

where the π fields are the (constant) BRST auxiliary fields imposing the gauge fix conditions.

$\pi_\alpha, \tilde{\pi}_\alpha$ have 11 independent components each,

$\pi^{mnI}, \tilde{\pi}^{mnI}$ have $10g$ independent components each and

$\pi^I, \tilde{\pi}^I$ are g (constant) fields.

Let us parametrize the 11 independent components of π_α and $\tilde{\pi}_\alpha$ as

$$\pi_\alpha = p_i C^i_\alpha, \qquad \tilde{\pi}_\alpha = \tilde{p}_i C^i_\alpha, \quad i = 1, \ldots, 11$$

where $p_i, \tilde{p}_i$ and the independent components and C^α_i is a constant matrix of rank 11. Integrating out $p_i, \tilde{p}_i$ leads to the insertion in the path integral

$$\prod_{i=1}^{11} \delta(C^i_\alpha \lambda^\alpha) C^i_\alpha \theta^\alpha$$

which is the same as the insertion of 11 “picture-lowering” operator Y_C.

Similarly, we parametrize the $10g$ independent components of π^{mnI} and $\tilde{\pi}^{mnI}$ as

$$\pi^{mnI} = p^{jI} B^{mn}_{jI}, \qquad \tilde{\pi}^{mnI} = \tilde{p}^{jI} B^{mn}_{jI}, \qquad j = 1, \ldots, 10$$

Integrating over $p_{jI}, \tilde{p}_{jI}$ and $\pi_I, \tilde{\pi}_I$ leads to $10g$ insertions of Z_B and g insertions of Z_J:

$$\prod_{i=1}^{g} Z_{Ji} \prod_{j=1}^{10g} Z_{B_j}$$

It remains to discuss the composite G field. A simple solution of the defining equation

$$\{Q_S, G\} = T$$

is

$$G_0 = \frac{C^\alpha G_\alpha}{C^\alpha \lambda_\alpha}$$

where G_α is a known expression. [Berkovits, (2001)] This solution is however not acceptable because had we allowed for operators with behavior $(C_\alpha\lambda^\alpha)^{-1}$ the Q_S-cohomology would be trivial. Indeed, since there exists

$$\xi = \frac{C_\alpha\theta^\alpha}{C_\alpha\lambda^\alpha}, \qquad \{Q_S, \xi\} = 1.$$

A related issue is that the positions of the poles of G are also the positions of the zeros of the path integral insertions thus making the expressions ill-defined.

The defining equation only defines a cohomology class $[G]$. The question is then whether there exists a different representative such that the poles in the new G would cancel against zeros in other path integration insertions. Indeed, such a representative G_1 exists,

$$G_1 = b/Z_B$$

where b is the “picture-raised b ghost”

$$\{Q_S, b\} = Z_B T.$$

G_1 indeed represents the same cohomology class as G_0 [Oda, Tonin (2005)].

Combing all ingredients we arrive at the scattering amplitude prescription in the minimal formulation.

In this case the b field is

- complicated but explicitly known.
- it has no poles as $\lambda \longrightarrow 0$.

So the minimal version appears to give in principle a well-defined prescription for any scattering amplitude.

However, as we will shortly see, the theory suffers from BRST anomalies, namely Q_S-exact states do not decouple.

• To define the minimal scattering amplitudes one has to supply the constant tensors C^α, B_{mn}. These enter through gauge fixing so physical quantities should be independent of them. In particular, the theory should still be Lorentz invariant.

• At tree-level we need to choose 11 independent tensors C^α. A Weyl spinor decomposed under $SU(5)$ as $\mathbf{16} \to \mathbf{1} \oplus \bar{\mathbf{10}} \oplus \mathbf{5}$, so a simple choice is:

$$C^1_\alpha = \delta^+_\alpha, \;\; (C^2)^{a_1 a_2} = \delta_1^{[a_1} \delta_2^{a_2]}, \cdots, (C^{11})^{a_1 a_2} = \delta_4^{[a_1} \delta_5^{a_2]},$$

all other $C^I_\alpha = 0$.

Let us compute a simple amplitude, the 3-point function of a gauge boson with two gluinos:

$$\langle \lambda^\alpha A_{1\alpha}(z_1) \lambda^\beta A_{2\beta}(z_2) \lambda^\gamma A_{3\gamma}(z_3) Y_{C_1}(\infty) \cdots Y_{C_{11}}(\infty) \rangle$$

where $A_\alpha(x,\theta) = (\frac{1}{2} a_m (\gamma^m \theta)_\alpha - \frac{1}{3}(\xi \gamma_m \theta)(\gamma^m \theta)_\alpha + \cdots)$,

Computing this amplitude with this choice of C's leads to a surprising answer:

$$\epsilon^{abcde} \xi^1_{ab} \xi^2_{cd} a^3_e$$

instead of the expected answer

$$\xi^1 \gamma^m \xi^2 a^3_m =$$

$$2(\xi^+_1 \xi^a_2 a^3_a + \xi^a_1 \xi^+_2 a^3_a - \frac{1}{4} \epsilon^{abcde} \xi^1_{ab} \xi^2_{cd} a^3_e + \xi^1_{ab} \xi^a_2 a^b_3 + \xi^a_1 \xi^2_{ab} a^b_3).$$

The amplitudes are formally independent of C^α but is this really true?

Let us shift

$$\delta C^{11}_\alpha = \delta^1_\alpha$$

This leads to

$$\delta Y_{C^{11}} = Q_S(\theta^1 \theta_{45} \delta'(\lambda_{45}))$$

Computing

$$\begin{aligned} \delta\mathcal{A} &= \langle V_1(z_1)V_2(z_2)V_3(z_3)Y_{C_1}(\infty)\cdots Y_{C_{10}}(\infty)\delta Y_{C_{11}}(\infty)\rangle \\ &= \int d^{16}\theta A^1_{(+}A^2_{+}A^3_{45)}\theta_{12}\cdots\theta_{45}. \end{aligned}$$

which is non-zero.

To understand the origin of the problem we need to know a little bit more about the pure spinor integration measure. This is given by [Berkovits (2004)]:

$$[d\lambda]\lambda^\alpha\lambda^\beta\lambda^\gamma = d\lambda^{\alpha_1}\wedge\cdots\wedge d\lambda^{\alpha_{11}}(\epsilon T)^{\alpha\beta\gamma}_{\alpha_1\cdots\alpha_{11}}$$

for a certain known invariant tensor $(\epsilon T)^{\alpha\beta\gamma}_{\alpha_1\cdots\alpha_{11}}$. In U(5) variables

$$[d\lambda] = \frac{d\lambda^+\wedge d\lambda_{12}\wedge\cdots\wedge d\lambda_{45}}{(\lambda^+)^3}$$

The problems originate from the poles in the measure.

The picture changing operators are formally Q_S-closed but because of the poles in the measure they are not closed inside correlators:

$$Q_S Y_C = C_\alpha \lambda^\alpha \delta(C_\alpha \lambda^\alpha) = \lambda^+ \delta(\lambda^+), \quad \text{for } C_\alpha^+ = \delta_\alpha^+$$

This would be zero, if the remaining expression was polynomial in λ^+ but the measure contains a factor of $(\lambda^+)^{-3}$.

Resolution

The minimal scattering amplitude prescription, as presented in the original paper [Berkovits (2004)], involved an additional step: since the amplitude is is (formally) independent of the C, one can integrate over C.

This step restores Lorentz invariance and furthermore one can show that now all BRST exact states decouple.

However, since the amplitudes do depend on the choice of C's, such procedure is questionable.

Lorentz invariant version of PCO

It turns out there is a Lorentz invariant version of the picture changing insertions that does not contain constant tensors:

$$\mathcal{A} = \langle V_1(z_1)V_2(z_2)V_3(z_3)\int dz_4 U_4(z_4)\cdots\int dz_N U_N(z_N)$$

$$\Lambda_{\alpha\beta\gamma}(\infty)(\epsilon T)^{\alpha\beta\gamma}_{\beta_1\cdots\beta_{11}}\theta^{\beta_1}(\infty)\cdots\theta^{\beta_{11}}(\infty)\rangle.$$

where $\Lambda_{\alpha\beta\gamma}$ is defined by

$$\int [d\lambda]\lambda^\alpha\lambda^\beta\lambda^\gamma\Lambda_{\alpha'\beta'\gamma'} = \delta^{(\alpha}_{\alpha'}\delta^{\beta}_{\beta'}\delta^{\gamma)}_{\gamma'} - \frac{1}{40}\gamma^{(\alpha\beta}_m\gamma^m_{(\alpha'\beta'}\delta^{\gamma)}_{\gamma')}.$$

• There is a unique such tensor. All components can be obtained from

$$\Lambda_{+++} = 6\delta(\lambda^{+})\delta(\lambda_{12})\cdots\delta(\lambda_{45}).$$

by acting with the Lorentz generators.

• One can show that Q_S-exact states decouple.

• This is equivalent with integrating over C's.

One-loop amplitudes

There are two new features. We need to insert

1. new PCO's: Z_B, Z_J.

2. composite b ghost.

Both of them involve the constant tensor B_{mn}. So let us choose some B_{mn}'s and see whether

- amplitudes are Lorentz invariant
- BRST exact states decouple.

We want to compute amplitudes with one insertion of a BRST exact state:

$$
\begin{aligned}
&\langle Q_S \Omega_1 \prod_{i=2}^{N} \int dz_i U_i(z_i) \int du \mu(u) \tilde{b}_{B^1}(u,w) \\
&(\lambda B^2 d)(y) \cdots (\lambda B^{10} d)(y)(\lambda d)(y) \\
&\delta(B^2 N(y)) \cdots \delta(B^{10} N(y)) \delta(J(y)) \\
&\Lambda_{\delta_1\delta_2\delta_3}(y)(\epsilon T)^{\delta_1\delta_2\delta_3}_{\beta_1 \cdots \beta_{11}} \theta^{\beta_1}(y) \cdots \theta^{\beta_{11}}(y) \rangle,
\end{aligned}
$$

"Integrating Q_S by parts" one may get a boundary contribution from Q_S acting on $\tilde{b}$. The anomaly we will discuss here originates instead from Q_S acting on the picture changing operators. These terms are formally zero since PCO are meant to be BRST closed.

We choose:

$$(B^1)_{ab} = \delta_a^{[1}\delta_b^{2]}, \cdots, (B^{10})_{ab} = \delta_a^{[4}\delta_b^{5]}, \quad (B^I)^{ab} = (B^I)^a{}_b = 0$$

It turns out that all one-loop amplitudes with one BRST exact state are proportional to the following integral:

$$I_{\beta_2 \cdots \beta_{11}} = \int [d\lambda] \frac{1}{(\lambda^+)^8} \lambda^{\beta_1} (\lambda\gamma^{13}d) \cdots (\lambda\gamma^{45}d)(\lambda d)\Lambda_{\alpha\beta\gamma}(\epsilon T)^{\alpha\beta\gamma}_{\beta_1 \cdots \beta_{11}}.$$

In this form it becomes apparent that the problems with factors of λ^+ in the denominator only become worse at one loop.

We have computed this integral and it is non-zero.

4- and 5-point 1-loop amplitudes

• For 4-point 1-loop amplitude with one BRST exact state, it turns out the amplitude vanishes after the d-integration is performed.

• This is special to 4-point functions, it does not happen for 5-point functions.

• The Lorentz variation of 4-point 1-loop amplitudes is also non-zero.

These computations were done for a specific choice of B_{mn}. What happens if we integrate over B_{mn}?

4-point functions: BRST exact states decouple. This follows from representation theory: the amplitude vanishes because a certain type of invariant tensor does not exist.

5-point functions: Group theory alone does not imply decoupling of BRST exact states.

BRST anomalies in non-minimal formalism?

Given that we have found BRST anomalies in the minimal formalism, one can ask whether there are similar problems in the non-minimal version. In the minimal formalism the problems originated from the Q_S variation of the PCO. These are replaced in the non-minimal version by

$$\mathcal{N} = \exp(-\lambda\bar{\lambda} - r\theta + \cdots)$$

The Q_S variation of this is

$$Q_S\mathcal{N} = -(\lambda r - r\lambda + (A - A))\exp(-\lambda\bar{\lambda} - r\theta + \cdots) = 0,$$

where $(A-A)$ comes from $Q_S(\cdots)$ and this formally implies decoupling of Q_S-exact states.

The question is then whether one gets other infinities that would invalidate decoupling.

To check this we split the amplitude into two parts and compute the piece with only the insertion $(r\lambda)$ (and its analogue for 1-loop) separately. If each piece is infinite one would have a $\infty - \infty$ situation and a potential anomaly.

We found no anomaly. However, the amplitudes behave differently depending whether the minimal computation is anomalous or not:

1. BRST anomaly in the minimal version $\leftrightarrow$ each piece is finite in the non-minimal version.

2. Group theory implies decoupling in the the minimal version $\leftrightarrow$ same group theory implies each piece vanishes separately.

Summary

1. I presented a first principles derivation of the scattering amplitude prescriptions for the pure spinor superstring.

2. Minimal and non-minimal formulations are formally equivalent.

3. Minimal formulation, however, suffers from BRST anomalies.

Outlook

There are indications that one can formulate a ”new minimal” version that does not suffer from BRST anomalies.

An analogy

One can describe the relation between the minimal and non-minimal formalisms using the following elementary example.

Consider complex x, p and consider the following integral along the real axis for both x, p,

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipx} f(x) = \int_{-\infty}^{\infty} dx \delta(x) f(x) = f(0) \tag{1}$$

Now rotate the contour so that $p = ix^*$ so the integral becomes

$$\frac{i}{2\pi} \int dx dx^* e^{-|x|^2} f(x) = \frac{1}{\pi} \int_0^{2\pi} d\theta \int_0^{\infty} r dr e^{-r^2} f(re^{i\theta}) \tag{2}$$

(1) and (2) would give exactly the same answer if $f(x)$ is non-singular but (1) is ill-defined for any choice of singular $f(x)$ whereas (2) may not be singular.

For example, choose

$$f(x) = \frac{1}{x} \tag{3}$$

(1) yields ∞ but (2) gives 0.