The Virtual Scaling Function of AdS/CFT

Stefan Zieme

Max-Planck-Institut für Gravitationsphysik
Albert Einstein Institut
Potsdam

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Outlook

- Gluon Scattering/Wilson Loops
- Integrability
- String Theory

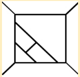
Review: Cusp anomalous dimension Γ_{cusp}

- integral equation
- weak $\longrightarrow$ strong coupling

New: Virtual Scaling Function

- What?
- Pig in the Poke
- weak $\longrightarrow$ strong coupling

Solve conformal $\mathcal{N} = 4$ super Yang-Mills

- Spectrum of Anomalous Dimension 
- Scattering Amplitudes of on-shell states 

Use anomalous dimension to "solve" gluon amplitudes

Lance Dixon: *"...gluon scattering amplitudes ... "1/4" solved."*

Virtual Scaling Function

gluon scattering amplitudes "3/8" solved

X-tra

Repeat at strong coupling and compare with string spectrum!

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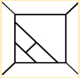

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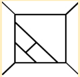

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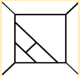

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Gluon amplitudes

- [Bern,Dixon,Smirnov] proposed the following form for n gluon planar, MHV scattering amplitude

$$\log\left(\frac{\mathcal{A}_n}{\mathcal{A}_n^{\mathrm{tree}}}\right) = \mathrm{Div}_n + \frac{f(\lambda)}{4} F_n(0) + nk(\lambda) + C(\lambda)$$

- in dimensional regularization $d = 4 - 2\epsilon$

$$\mathrm{Div}_n = -\sum_{i=1}^{n}\left[\frac{1}{8\epsilon^2} f^{(-2)}\left(\frac{\lambda\mu^{2\epsilon}}{s_{i,i+1}^{\epsilon}}\right) + \frac{1}{4\epsilon} g^{(-1)}\left(\frac{\lambda\mu^{2\epsilon}}{s_{i,i+1}^{\epsilon}}\right)\right]$$

- with the cusp anomalous dimension $f(\lambda)$ and collinear ad $g(\lambda)$

$$\left(\lambda\frac{d}{d\lambda}\right)^2 f^{(-2)}(\lambda) = f(\lambda), \quad \lambda\frac{d}{d\lambda} g^{(-1)}(\lambda) = g(\lambda)$$

Wilson loops

- specified by light-like segments proportional to gluon-momenta with expectation value

$$< W_{\{k_i\}} >= 1 + \frac{\lambda}{8\pi}[\mathrm{Div}' + \omega_n + c + nc']$$

- divergent terms from UV divergencies at the cusps of the Wilson lines

Gluon Amp./Wilson loop

- finite parts are identical $\omega_n = F_n$ [Brandhuber,Heslop,Travaglini]
- The leading poles $\sim 1/\epsilon^2$ agree. They are related to the anomalous dimension of twist-two operators $f \sim \Delta(\mathrm{Tr}\,(\mathcal{Z}\mathcal{D}^M\mathcal{Z}))$
- the functions $g(\lambda)$ are different! $\rightarrow$ virtual scaling function

for completeness...

BDS conjecture is not valid for $n \geq 6$ beyond one-loop by/in/at

- strong coupling, approximation by classical surfaces in AdS_5 space [Alday,Maldacena]
- conformal Ward identities [Drummond,Henn,Korchemsky,Sokatchev]
- analytic properties at high energies [Bartels,Lipatov,Sabio Vera]
- two-loop six-gluon MHV [Bern,Dixon,Kosower,Roiban,Spradlin,Vergu,Volovich]

Unfortunately...

this implies '$3/8 \longrightarrow 3/10$' solved

- leading poles are given by cusp anomalous dimension $f(g)$, which in turn is equal to the anomalous dimension of twist-two operators

Twist-two operators

- can be represented by doping covariant light-cone derivatives $\mathcal{D}$ into protected 1/2-BPS state $\mathrm{Tr}\,(\mathcal{Z}^L)$
- spin-M, twist-L operator takes the form $\mathrm{Tr}\,(\mathcal{D}^{s_1}\mathcal{Z}\mathcal{D}^{s_2}\mathcal{Z}\cdots\mathcal{D}^{s_L}\mathcal{Z})$ with $s_1 + s_2 + \ldots + s_L = M$,
- anomalous dimension are given by energy eigenvalues of a length-L non-compact $\mathfrak{sl}(2)$ spin chain with M magnons underlying a factorized scattering
- use Bethe ansatz to diagonalize spin chain Hamiltonian and compute anomalous dimension

Higher-loops

- extended $\mathfrak{psu}(2|2)$ determines asymptotic S-Matrix up to a phase [Beisert]
- crossing-like equation to constraint the phase [Janik]
- of the solutions [Beisert,Hernandez,Lopez] one gets singled out [Beisert,Eden,Staudacher] and reproduces the expected result [Bern et al]

- wrapping effects at order g^{2L+4} not captured by Bethe ansatz
- How can we compute $f(g)$ for $L = 2$ to all orders?

[Belitksy,Gorsky,Korchemsky]

- anomalous dimension of twist operators grow logarithmic with spin M, $\gamma_L(g, M) = f(g) \log M + \ldots$
- scaling function $f(g)$ is independent of the twist L and universal

- that implies $f_2 = f_3 = \ldots = f_\infty$, which can be determined by asymptotic Bethe ansatz
- $f(g)$ is an all loop result

Twist-two and Bethe ansatz [Beisert,Eden,Staudacher]

From the Bethe ansatz for twist-two operators $\mathrm{Tr}\,(\mathcal{Z}\mathcal{D}^M\mathcal{Z})$

$$\left(\frac{x_k^+}{x_k^-}\right)^2 = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\,\frac{1-g^2/x_k^+x_j^-}{1-g^2/x_k^-x_j^+}\,\exp(2i\theta(u_k,u_j))\,, \qquad \prod_{k=1}^{M}\frac{x_k^+}{x_k^-} = 1\,.$$

one can derive an integral equation for $f(g)$ in the large spin limit $M \to \infty$

- the scaling function is given by $f(g) = 16g^2\hat{\sigma}(0)$ where

$$\hat{\sigma}(t) = \frac{t}{e^t - 1}\left[K(2gt, 0) - 4g^2 \int_0^\infty dt' K(2gt, 2gt')\hat{\sigma}(t')\right],$$

- the integral kernel $K(t,t') = K_0(t,t') + K_1(t,t') + K_d(t,t')$ is given by

$$K_0(t,t') = \frac{2}{tt'}\sum_{n=1}^{\infty}(2n-1)J_{2n-1}(t)J_{2n-1}(t'),$$

$$K_1(t,t') = \frac{2}{tt'}\sum_{n=1}^{\infty}(2n)J_{2n}(t)J_{2n}(t'),$$

$$K_d(t,t) = 8g^2 \int_0^\infty dt'' K_1(t, 2gt'')\frac{t''}{e^{t''} - 1}K_0(2gt'', t').$$

- Fredholm integral equation of 2nd kind, at weak coupling the equation can be solved iteratively by expanding in g
- first few orders are obtained very fast

$$f(g) = 8\,g^2 - \frac{8\pi^2}{3}\,g^4 + \frac{88\pi^4}{45}\,g^6 - \left(\frac{584\pi^6}{315} + 64\zeta_3^2\right)g^8 + \dots$$

- it agrees to four-loops with the leading singularities of the gluon amplitudes [Bern,Dixon,Smirnov] [Chachazo,Spradlin,Volovich]
- at five-loops it agrees with Padé approximation estimates [Bern,Czakon,Dixon,Kosower,Smirnov]

weak coupling

f has finite radius of convergence
$r_c = 1/4$

strong coupling and string theory

f becomes an asymptotic series
$r_c = 0$

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much afford was spend to analyze the integral equation at strong coupling [Lipatov,Kotikov], [Benna,Benvenuti,Klebanov,Scardicchio], [Alday,Arutyunov,Benna,Eden,Klebanov], [Kostov,Serban,Volin], [Basso,Korchemsky,Kotanski], [Kostov,Serban,Volin], ...

strong coupling limit [Basso,Korchemsky,Kotanski]

- split density in parity even/odd parts [Eden]

$$\frac{e^t - 1}{t}\sigma(t) = \frac{\gamma_+(2gt)}{2gt} + \frac{\gamma_-(2gt)}{2gt}\,,$$

  where $\gamma_\pm$ are expanded into even/odd Bessel function Neumann series

- BES equation is equivalent to (infinite) system of equations $n \geq 1$

$$\int_0^\infty \frac{dt}{t}\left[\frac{\gamma_+(t)}{1-e^{-t/2g}} - \frac{\gamma_-(t)}{e^{t/2g}-1}\right] J_{2n}(t) = 0\,,$$
$$\int_0^\infty \frac{dt}{t}\left[\frac{\gamma_-(t)}{1-e^{-t/2g}} + \frac{\gamma_+(t)}{e^{t/2g}-1}\right] J_{2n-1}(t) = \frac{1}{2}\delta_{n,1}$$

strong coupling limit [Basso,Korchemsky,Kotanski]

- dependence on g can be put into $\Gamma_\pm$ with change of variables

$$2\gamma_\pm(t) = \big(1 - \mathrm{sech}(t/2g)\big)\Gamma_\pm(t) \pm \tanh(t/2g)\Gamma_\mp(t)\,,$$

- anomalous dimension is given by $f(g) = -4g\Gamma_+(0)$ with

$$\Gamma_{\frac{+}{-}}(t) = \sum_{k=0}^{\infty}(-1)^{(k+1)} J_{\frac{2k}{2k-1}}(t)\Gamma_{\frac{2k}{2k-1}}\,,$$

$$\Gamma_k = -\frac{1}{2}\Gamma_k^{(0)} + \sum_{p=1}^{\infty}\frac{1}{g^p}\left(c_p^-\Gamma_k^{(2p-1)} + c_p^+\Gamma_k^{(2p)}\right),$$

with the coefficients $c_p^\pm$ given by $c_p^\pm = \sum_{r\geq 0} g^{-r} c_{p,r}^\pm$ and

$$\Gamma_{2m}^{(p)} = \frac{\Gamma(m+p-\frac{1}{2})}{\Gamma(m+1)\Gamma(\frac{1}{2})}, \quad \Gamma_{2m-1}^{(p)} = \frac{(-1)^p\Gamma(m-\frac{1}{2})}{\Gamma(m+1-p)\Gamma(\frac{1}{2})}\,.$$

strong coupling limit [Basso,Korchemsky,Kotanski]

- cusp anomalous dimension is given by

$$f(g) = 4g + \sum_{p=1}^{\infty} \frac{2}{g^{p-1}} \left[\frac{2c_p^-}{\sqrt{\pi}} \Gamma(2p - \tfrac{3}{2}) + \frac{2c_p^+}{\sqrt{\pi}} \Gamma(2p - \tfrac{1}{2}) \right]$$

- the coefficients $c_p^\pm$ can be determined to all orders in g by examining the asymptotic behavior of series coefficients of $\gamma_\pm$ i.e. $\gamma_{2m-1} \pm \gamma_{2m}$ in the limit $m, g \to \infty$, $x = (m - 1/4)^2/g = \mathrm{fixed}$
- strong coupling expansion is given by

$$f(g) = 4g - \frac{3\log 2}{\pi} - \frac{1}{g}\frac{\mathrm{K}}{4\pi^2} - \frac{1}{g^2}\left(\frac{3\mathrm{K}\log 2}{16\pi^3} + \frac{27\zeta_3}{512\pi^3} \right) + \dots$$

- agrees with known string data [Gubser,Klebanov,Polyakov], [Frolov,Tseytlin], [Roiban,Tirziu,Tseytlin], [Roiban,Tseytlin], first interpolating function

What?

- twist L operators can be represented by $\mathrm{Tr}\,(\mathcal{D}^M \mathcal{Z}^L)$
- the large spin scaling behavior of anomalous dimension is given by

$$\gamma_L(g, M) = f(g)(\log M) + B_L(g) + \ldots .$$

- in analogy with the QCD splitting function we call B_L the virtual part
- for finite values of the spin, anomalous dim. can be computed for twist-two and -three operators in terms of harmonic sums analytical by solving the Baxter equation [Kotikov,Rej,Zieme]
- for twist L operators wrapping effects are expected to enter the an. dim. at order g^{2L+4}
- A twist depended quantity can not be computed from the Bethe ansatz correctly to all orders! (?)

Wrapping Contributions

the four-loop anomalous dimension of the Konishi operator, $L = M = 2$, has been computed from

- the asymptotic dilatation generator [Fiamberti,Santambrogio,Sieg,Zanon]
- finite-size corrections to the ABA using Lüscher formulas [Bajnok,Janik]
- 131.015 Feynman diagrams [Velizhanin]

they all agree!

Twist-two general spin M

- using the Lüscher formulas the four-loop anomalous dimension has been computed for general values of the spin M [Bajnok,Janik,Lukowski]
- the result agrees with pole structure predicted by the BFKL equation
- leading transcendental part agrees with *full direct* computation [Velizhanin]

The Pig in the Poke

- the result is given by

$$\gamma_4^{\text{wrap}} = 128 S_1^2 (2S_{-5} - 2S_5 + 4S_{-2,-3} - 4S_{3,-2} + 4S_{4,1} - 8S_{-2,-2,1} - 5\zeta_5 - 3S_{-2}\zeta_3)$$

- for $M \to \infty$ it behaves like $\gamma_4^{\text{wrap}} \sim \log^2 M/M^2$ [Beccaria,Forini]

Twist-three at five loops [Beccaria,Forini,Lukowski,Zieme]

- γ_5 computed with ABA, max. transcendentality, reciprocity, Lüscher, agrees with Y-System, ▸ result
- not related to BFKL, but poles at $M = -2$ can be resumed and agree with NLO conjecture [Kotikov,Lipatov,Rej, Staudacher,Velizhanin]
- again for large spin $\gamma_5^{\text{wrap}} \sim \log^2 M/M^2$

The finite order corrections $\mathcal{O}(M^0)$ can be computed to all orders from the ABA. $B_L(g) = L\, h_1(g) + h_2(g)$

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How do we compute these corrections?

- tricky from the Bethe ansatz
- we start from the non-linear integral equation (NLIE) of $\mathfrak{sl}(2)$ for the counting function that takes into account 'hidden' degrees of freedom [Freyhult,Rej,Staudacher] (roots of the transfer matrix)
- 'off-shell' version of one-loop Bethe equations

$$(u + \tfrac{i}{2})Q(u+i) + (u - \tfrac{i}{2})Q(u-i) = t(u)Q(u)$$

where $t(u) = 2u^L + \sum_{i\geq 2} q_i u^{L-i}$

- Bethe equations also recovered for $t(u) = 2\prod_{i\geq 1}(u - u_i^{(h)})$

From the NLIE one can derive an integral equation including $\mathcal{O}(M^0)$ corrections

$$\hat{\sigma}(t) = \frac{t}{e^t - 1}\Big[K(2gt,0)(\log M + \gamma_{\mathrm{E}} - (L-2)\log 2) - \frac{L}{8g^2 t}(J_0(2gt) - 1)$$
$$+\frac{1}{2}\int_0^\infty dt' \left(\frac{2}{e^{t'} - 1} - \frac{L-2}{e^{t'/2} + 1}\right)(K(2gt,2gt') - K(2gt,0))$$
$$-4g^2 \int_0^\infty dt' K(2gt,2gt')\hat{\sigma}(t')\Big],$$

- anomalous dimension given by $\gamma_L(g,M) = 16g^2\,\hat{\sigma}(0)$
- $\gamma_L(g,M) = f(g)(\log M + \gamma_{\mathrm{E}} - (L-2)\log 2) + B_L(g) + \dots$
- at weak coupling easy to expand and to solve iteratively
- to solve at strong coupling face the same trouble as before, try to map it to functions that determine $f(g)$
- drop the part $\sim f(g)$, which has already been analyzed

- decompose into parity even/odd parts, introduce a dummy index,

$$\int_0^\infty \frac{dt}{t}\left[\frac{\gamma_+(t,j)}{1-e^{-t/2g}} - \frac{\gamma_-(t,j)}{e^{t/2g}-1}\right] J_{2n}(t) = \frac{jL}{8ng} + jh_{2n}\,,$$
$$\int_0^\infty \frac{dt}{t}\left[\frac{\gamma_-(t,j)}{1-e^{-t/2g}} + \frac{\gamma_+(t,j)}{e^{t/2g}-1}\right] J_{2n-1}(t) = \frac{1-j}{2}\delta_{n,1} + jh_{2n-1}$$

with $h_n = h_n(g)$ given by

$$h_n(g) = \int_0^\infty \frac{dt}{4}\left(\frac{2}{e^t-1} - \frac{L-2}{e^{t/2}+1}\right)\left(\frac{J_n(2gt)}{gt} - \delta_{n,1}\right).$$

For $j=0$ one recovers the solution of the BES equation, while $j=1$ leads to the system of equations that determines $B_L(g)$.

how to solve...

- choose some reference j' and multiply both sides of the system with $(2n)\gamma_{2n}(g,j')$ and $(2n-1)\gamma_{2n-1}(g,j')$, respectively
- sum over all $n \geq 1$, make use of the expansion formulas of the even/odd parts $\gamma_+(t) = 2\sum_{n\geq 1}(2n)J_{2n}(t)\gamma_{2n}$
- two equations for the even/odd parts $\gamma_\pm(t,j')$, subtract the even from odd part $\rightarrow$ integral kernel invariant under $j \leftrightarrow j'$ as such should be its solution, can be used to obtain $B_L(g) = 16g^2\gamma_1(g,1)$

$$B_L(g) = 4g^2 \int_0^\infty dt \Big[\frac{2}{e^t - 1} - \frac{L-2}{e^{t/2}+1}\Big] \times \Big[\frac{\gamma_-^{(0)}(2gt) - \gamma_+^{(0)}(2gt)}{gt} - 2\gamma_1^{(0)}(g)\Big] - 4gL\int_0^\infty \frac{dt}{t}\gamma_+^{(0)}(2gt)$$

- $B_L(g) = -8g^4(7-2L)\zeta(3) + \mathcal{O}(g^6)$ agrees with max. transc. part of QCD splitting function [Moch,Vermaseren,Voigt]

Strong Coupling Expansion

- make the very same change of variables $\gamma_\pm \to \Gamma_\pm$
- perform the integrals to find for $B_L(g) = 16g^2\gamma_1^{(1)}(g)$ ▸ B_2

$$\gamma_1^{(1)}(g) = \frac{1}{16g^2}(L-2)\epsilon_1(g) + \gamma_1^{(0)}(L-2)\log 2 - \gamma_1^{(0)}(\gamma_{\mathrm{E}} + \log g)$$
$$- \frac{1}{4g}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k}\Gamma_{2k} - \frac{1}{4g}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2k-1}\Gamma_{2k-1}$$

- 1st generalized scaling function $\epsilon_1(g) = -1 + \mathcal{O}(e^{-\pi g})$
  [Basso,Korchemsky], [Fioravanti,Grinza,Rossi]
- solve quantization conditions for $c_p^\pm$, plug in, perform sums
- remember $\Gamma_k = -\frac{1}{2}\Gamma_k^{(0)} + \sum_{p=1}^{\infty}\frac{1}{g^p}\left(c_p^-\Gamma_k^{(2p-1)} + c_p^+\Gamma_k^{(2p)}\right)$

$$\Gamma_{2m}^{(p)} = \frac{\Gamma(m+p-\frac{1}{2})}{\Gamma(m+1)\Gamma(\frac{1}{2})}, \quad \Gamma_{2m-1}^{(p)} = \frac{(-1)^p\Gamma(m-\frac{1}{2})}{\Gamma(m+1-p)\Gamma(\frac{1}{2})}.$$

two-cusp spinning string $L = 2$

- with strong coupling $f(g) = 4g - 3\log 2/\pi$ and $g = \sqrt{\lambda}/4\pi$ we predict string energy up to one-loop

$$E - S = L + \gamma_L\left(\frac{\sqrt{\lambda}}{4\pi}, S\right)\Big|_{L=2} = \left(\frac{\sqrt{\lambda}}{\pi} - \frac{3\log 2}{\pi}\right)\log\frac{4\pi\,S}{\sqrt{\lambda}}$$
$$+ \frac{\sqrt{\lambda}}{\pi}(\log 2 - 1) + 1 + \frac{6\log 2}{\pi} - \frac{3(\log 2)^2}{\pi}\,,$$

- and determine the constant c of [Beccaria,Forini,Tirziu,Tseytlin] to be $c = 6\log 2 + \pi$ in agreement with algebraic curve approach [Gromov]

Proof of $B_L(g) = L\,h_1(g) + h_2(g)$

To take the strong coupling limit, we had to resum *all* orders at weak coupling!

more strong coupling, where $c_1 = \frac{3\log 2}{4\pi}$

$$B_2(g+c_1) = (\log\tfrac{2}{g} - \gamma_{\rm E}) f(g+c_1) - 4g - 1 + \frac{1}{g}\frac{\mathrm{K}}{2\pi^2} - \frac{1}{g^2}\frac{9\zeta(3)}{2^8\pi^3} + \frac{1}{g^3}\left(\frac{9\,\beta(4)}{2^7\pi^4} - \frac{\mathrm{K}^2}{2^7\pi^4}\right) - \frac{1}{g^4}\left(\frac{6831\,\zeta(5)}{2^{18}\pi^5} - \frac{423\,\mathrm{K}\,\zeta(3)}{2^{13}\pi^5}\right) + \mathcal{O}(1/g^5)$$

more spin

use explicit holes (roots of transfer matrix) to derive from NLIE

$$\gamma_2(g,M) = f(g)\left(\log M + \gamma_{\rm E} + \frac{f(g)}{2}\frac{\log M + \gamma_{\rm E}}{M} + \frac{1+B_2(g)}{2\,M}\right) + B_2(g) + \dots,$$

agrees with string theory [Beccaria,Forini,Tirziu,Tseytlin] and reciprocity [Basso,Korchemsky]

more strong coupling, where $c_1 = \frac{3\log 2}{4\pi}$

$$B_2(g+c_1) = (\log\tfrac{2}{g} - \gamma_{\mathrm{E}})f(g+c_1) - 4g - 1 + \frac{1}{g}\frac{\mathrm{K}}{2\pi^2} - \frac{1}{g^2}\frac{9\zeta(3)}{2^8\pi^3}$$
$$+\frac{1}{g^3}\left(\frac{9\,\beta(4)}{2^7\pi^4} - \frac{\mathrm{K}^2}{2^7\pi^4}\right) - \frac{1}{g^4}\left(\frac{6831\,\zeta(5)}{2^{18}\pi^5} - \frac{423\,\mathrm{K}\,\zeta(3)}{2^{13}\pi^5}\right) + \mathcal{O}(1/g^5)$$

more spin

use explicit holes (roots of transfer matrix) to derive from NLIE

$$\gamma_2(g,M) = f(g)\Big(\log M + \gamma_{\mathrm{E}} + \frac{f(g)}{2}\frac{\log M + \gamma_{\mathrm{E}}}{M}$$
$$+\frac{1+B_2(g)}{2\,M}\Big) + B_2(g) + \ldots\,,$$

agrees with string theory [Beccaria,Forini,Tirziu,Tseytlin] and reciprocity [Basso,Korchemsky]

Gluon Amplitudes/Wilson loops

- leading poles identical, subleading singularities are different
- they differ by $B_2(g)$

[Dixon,Magnea,Sterman]

$$G = G_{eik} + B_2$$

- consistent with available data [Dixon et al], [Korchemsky et al], two-loop $n = 4, 5$ cusp Wilson loop
- gluon amplitudes '$\frac{3}{8}$-solved'

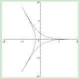
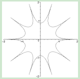
Summary

- determined finite order correction to large spin asymptotic of anomalous dimension of twist operators at weak and strong coupling
- at strong coupling and twist $L = 2$ we reproduce string theory result
  - also reproduce $\log M$, $1/M$ and $\log M/M$ string results by the depend on $f(g)$ and $B_2(g)$ in agreement with reciprocity
  - obtained a further interpolating function of AdS/CFT
  - reassurance of the dressing phase
- determined the difference between subleading poles of gluon amplitudes and Wilson loops
- gluon amplitudes '$\frac{3}{8}$-solved'

Outlook

String Theory

- compute higher orders in $1/g$?
- compute subleading spin correction of spiky string configurations

Gluon Amplitudes/Wilson loops

- Which operator hides behind G_{eik} in $\mathcal{N} = 4$ sYM?
- Can we solve gluon amplitudes at least to '1/2'?

Outlook

String Theory

- compute higher orders in $1/g$?
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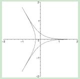

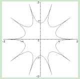

Gluon Amplitudes/Wilson loops

- Which operator hides behind G_{eik} in $\mathcal{N} = 4$ sYM?
- Can we solve gluon amplitudes at least to '1/2'?

$$
\begin{aligned}
\gamma_{10}^{\mathrm{ABA}} = {} & 136 S_{9} + 368 S_{1,8} + 2832 S_{2,7} + 4272 S_{3,6} + 848 S_{4,5} - 3024 S_{5,4} - 2736 S_{6,3} - 1168 S_{7,2} \\
& - 496 S_{8,1} - 5376 S_{1,1,7} - 12352 S_{1,2,6} - 8832 S_{1,3,5} + 1600 S_{1,4,4} + 3968 S_{1,5,3} - 64 S_{1,6,2} \\
& - 1344 S_{1,7,1} - 12352 S_{2,1,6} - 13760 S_{2,2,5} - 2112 S_{2,3,4} + 4288 S_{2,4,3} - 960 S_{2,5,2} - 5440 S_{2,6,1} \\
& - 9088 S_{3,1,5} - 2432 S_{3,2,4} + 5120 S_{3,3,3} + 2688 S_{3,4,2} - 4160 S_{3,5,1} + 1280 S_{4,1,4} + 5824 S_{4,2,3} \\
& + 6400 S_{4,3,2} + 2112 S_{4,4,1} + 5120 S_{5,1,3} + 6208 S_{5,2,2} + 5312 S_{5,3,1} + 3904 S_{6,1,2} + 3904 S_{6,2,1} \\
& + 1728 S_{7,1,1} + 21504 S_{1,1,1,6} + 22784 S_{1,1,2,5} + 5632 S_{1,1,3,4} - 1280 S_{1,1,4,3} + 6912 S_{1,1,5,2} \\
& + 11520 S_{1,1,6,1} + 22784 S_{1,2,1,5} + 9088 S_{1,2,2,4} - 1024 S_{1,2,3,3} + 6784 S_{1,2,4,2} + 17152 S_{1,2,5,1} \\
& + 5504 S_{1,3,1,4} - 3456 S_{1,3,2,3} - 1536 S_{1,3,3,2} + 7680 S_{1,3,4,1} - 4480 S_{1,4,1,3} - 6272 S_{1,4,2,2} \\
& - 3584 S_{1,4,3,1} - 3840 S_{1,5,1,2} - 3840 S_{1,5,2,1} + 768 S_{1,6,1,1} + 22784 S_{2,1,1,5} + 9088 S_{2,1,2,4} \\
& - 1024 S_{2,1,3,3} + 6784 S_{2,1,4,2} + 17152 S_{2,1,5,1} + 9088 S_{2,2,1,4} - 2688 S_{2,2,2,3} + 640 S_{2,2,3,2} \\
& + 13440 S_{2,2,4,1} - 3456 S_{2,3,1,3} - 7040 S_{2,3,2,2} - 768 S_{2,3,3,1} - 4480 S_{2,4,1,2} - 4480 S_{2,4,2,1} \\
& + 2816 S_{2,5,1,1} + 6272 S_{3,1,1,4} - 2944 S_{3,1,2,3} - 1536 S_{3,1,3,2} + 7936 S_{3,1,4,1} - 2944 S_{3,2,1,3} \\
& - 7296 S_{3,2,2,2} - 768 S_{3,2,3,1} - 6656 S_{3,3,1,2} - 6656 S_{3,3,2,1} - 1024 S_{3,4,1,1} - 3968 S_{4,1,1,3} \\
& - 6528 S_{4,1,2,2} - 3584 S_{4,1,3,1} - 6528 S_{4,2,1,2} - 6528 S_{4,2,2,1} - 4864 S_{4,3,1,1} - 5376 S_{5,1,1,2} \\
& - 5376 S_{5,1,2,1} - 5376 S_{5,2,1,1} - 4608 S_{6,1,1,1} - 32768 S_{1,1,1,1,5} - 10240 S_{1,1,1,2,4} - 3072 S_{1,1,1,3,3} \\
& - 17920 S_{1,1,1,4,2} - 30720 S_{1,1,1,5,1} - 10240 S_{1,1,2,1,4} - 8704 S_{1,1,2,3,2} - 24064 S_{1,1,2,4,1}
\end{aligned}
$$

$$
\begin{aligned}
&+1024S_{1,1,3,1,3} + 2560S_{1,1,3,2,2} - 4096S_{1,1,3,3,1} - 512S_{1,1,4,1,2} - 512S_{1,1,4,2,1} - 10240S_{1,1,5,1} \\
&-10240S_{1,2,1,1,4} - 8704S_{1,2,1,3,2} - 24064S_{1,2,1,4,1} + 3072S_{1,2,2,2,2} - 6656S_{1,2,2,3,1} \\
&+512S_{1,2,3,1,2} + 512S_{1,2,3,2,1} - 10752S_{1,2,4,1,1} + 1024S_{1,3,1,1,3} + 3072S_{1,3,1,2,2} - 3584S_{1,3,1,3} \\
&+3072S_{1,3,2,1,2} + 3072S_{1,3,2,2,1} - 2560S_{1,3,3,1,1} + 3072S_{1,4,1,1,2} + 3072S_{1,4,1,2,1} + 3072S_{1,4,2,} \\
&+3072S_{1,5,1,1,1} - 10240S_{2,1,1,1,4} - 8704S_{2,1,1,3,2} - 24064S_{2,1,1,4,1} + 3072S_{2,1,2,2,2} \\
&-6656S_{2,1,2,3,1} + 512S_{2,1,3,1,2} + 512S_{2,1,3,2,1} - 10752S_{2,1,4,1,1} + 3072S_{2,2,1,2,2} - 6656S_{2,2,1,3} \\
&+3072S_{2,2,2,1,2} + 3072S_{2,2,2,2,1} - 5632S_{2,2,3,1,1} + 3072S_{2,3,1,1,2} + 3072S_{2,3,1,2,1} + 3072S_{2,3,2,} \\
&+3072S_{2,4,1,1,1} + 3072S_{3,1,1,2,2} - 4096S_{3,1,1,3,1} + 3072S_{3,1,2,1,2} + 3072S_{3,1,2,2,1} - 2560S_{3,1,3,} \\
&+3072S_{3,2,1,1,2} + 3072S_{3,2,1,2,1} + 3072S_{3,2,2,1,1} + 4608S_{3,3,1,1,1} + 3072S_{4,1,1,1,2} + 3072S_{4,1,1,} \\
&+3072S_{4,1,2,1,1} + 3072S_{4,2,1,1,1} + 3072S_{5,1,1,1,1} + 16384S_{1,1,1,1,3,2} + 32768S_{1,1,1,1,4,1} \\
&+8192S_{1,1,1,2,3,1} + 4096S_{1,1,1,3,1,2} + 4096S_{1,1,1,3,2,1} + 20480S_{1,1,1,4,1,1} + 8192\,S_{1,1,2,1,3,1} \\
&+12288S_{1,1,2,3,1,1} + 8192S_{1,2,1,1,3,1} + 12288S_{1,2,1,3,1,1} + 8192S_{2,1,1,1,3,1} + 12288S_{2,1,1,3,1,1} \\
&-16384S_{1,1,1,1,3,1,1} + \zeta_3\,(896S_6 - 2304S_{1,5} - 1792S_{2,4} - 768S_{3,3} - 1792S_{4,2} - 2304S_{5,1} \\
&+2560S_{1,1,4} + 512S_{1,2,3} + 1536S_{1,3,2} + 3584S_{1,4,1} + 512S_{2,1,3} + 1536S_{2,3,1} + 512S_{3,1,2} \\
&+512S_{3,2,1} + 2560S_{4,1,1} - 2048S_{1,1,3,1} - 2048S_{1,3,1,1}) + 1280\,\zeta_5\,(S_{1,3} + S_{3,1} - S_4)
\end{aligned}
$$

$$
\begin{aligned}
\gamma^{wrap} = &-64\,S_1^2\,(35\zeta_7 - 40S_2\zeta_5 + (-8S_4 + 16S_{2,2})\zeta_3 + 2S_7 - 4S_{2,5} - 2S_{3,4} - 4S_{4,3} \\
&-2S_{6,1} + 8S_{2,2,3} + 4S_{3,3,1})\,.
\end{aligned}
$$

◂ ENOUGH

for twist L=2 one finds

$$B_2(g) = (-\gamma_{\mathrm{E}} - \log g) f(g) - 4g(1 - \log 2) - (1 - \frac{6 \log 2}{\pi} + \frac{3(\log 2)^2}{\pi}) + \mathcal{O}(1/g)\,.$$

B_L