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Cosmological Singularities in String Theory

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Motivation

(1) Big Bang
- beginning of time?
- bounce (pre-big bang)?

(2) Observables?
- initial conditions at $t=0$?
- asymptotic regions, "S-matrix"?

(3) Fluctuation spectra in Big Crunch / Big Bang cosmologies

(4) $\Lambda > 0$ in string theory

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Outline

A) String theory model

B) Effective 4D FRW cosmology

C) Global wavefunctions for scalar fluctuations

D) Quantization, particle production

E) Fluctuation spectra

F) Information loss

G) Backreaction

H) Conclusions

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A) String theory model

$$S = \frac{1}{2\kappa^2}\int d^{d+1}x \sqrt{-g}\, e^{-2\Phi}\left[R + 4 g^{\mu\nu}\partial_\mu\Phi\,\partial_\nu\Phi - 2\Lambda\right]$$

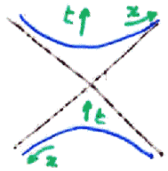
$$\Lambda = \frac{d-9}{2\alpha'} \equiv 2Q^2$$

Solution: $\begin{cases} ds^2 = -dt^2 + \tanh^2(Qt)\,dx^2 + (d\vec{X})^2 \\ \Phi = -\log\cosh(Qt) + \Phi_0 \end{cases}$

Limiting behavior:

$Q|t| \to \infty$: $\begin{cases} ds^2 = -dt^2 + dx^2 + (d\vec{X})^2 \\ \Phi = -Q|t| + \Phi_0 \end{cases}$ TLD

$Q|t| \to 0$: $\begin{cases} ds^2 = -dt^2 + Q^2t^2\,dx^2 + (d\vec{X})^2 \\ \Phi = \Phi_0 \end{cases}$ Milne



New coordinates

$$\begin{cases} u = \sinh(Qt)\, e^{-Qx} \\ v = -\sinh(Qt)\, e^{Qx} \end{cases}$$

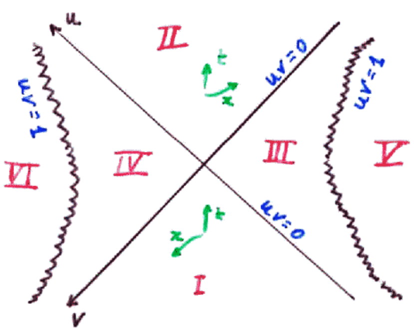
and extend to

$$-\infty < u, v < \infty$$

Then

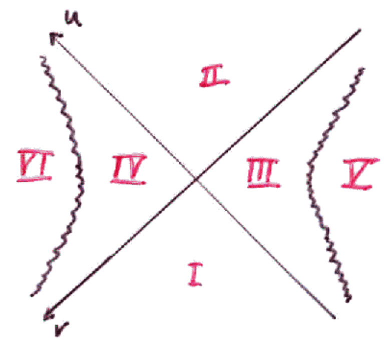
$$ds^2 = \frac{1}{Q^2} \frac{du\, dv}{1 - uv} + (d\vec{x})^2$$

$$\Phi = -\frac{1}{4} \log\left[(1-uv)^2\right]$$



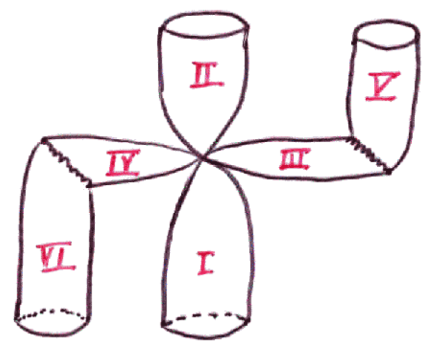
"Generalized Milne space"

cf. 2D black hole



Identify $(u, v) \sim (u\, e^{-2\pi r_0 Q}, v\, e^{2\pi r_0 Q})$

(i.e. $x \sim x + 2\pi r_0$)



"Generalized Milne orbifold"

(near $u = v = 0$: Milne orbifold)

Generalized Milne orbifold in string theory

(a) $PSL(2,\mathbb{R})$ group manifold

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad , \quad ad - bc = 1 \quad , \quad g \sim -g$$

Strings on $PSL(2,\mathbb{R})$: WZW model

(b) Gauge $U(1)$: $\quad g \mapsto e^{\rho\sigma_3}\, g\, e^{\rho\sigma_3}$

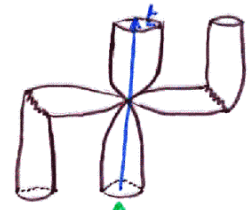
$\Rightarrow$ sigma model with

$$\begin{cases} ds^2 = -\alpha' k \, \dfrac{du\, dv}{1-uv} \\ \Phi = -\frac{1}{4} \log\left[(1-uv)^2\right] \end{cases}$$

- exact CFT : no α' corrections (for worldsheet susy version)
- $k = -\dfrac{1}{\alpha' Q^2} < 0$
- $\hat{c} = 2 + \dfrac{4}{k} < 2$ — supercritical / cigar CFT

(c) $\mathbb{Z}$ orbifold : $\quad g \sim e^{\pi r_0 \sigma_3}\, g\, e^{-\pi r_0 \sigma_3}$

B) Effective 4D FRW cosmology



$\times \; \mathbb{R}^3 \; \times \; T^{d-4}$

radius $\sim e^{\sigma_R}$ radius $\sim e^{\sigma_T}$

ϕ : 4D dilaton

4D effective action :

$$S = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-g}\, \Big[R - \tfrac{1}{2} g^{\mu\nu} \partial_\mu \phi\, \partial_\nu \phi - \tfrac{1}{2} g^{\mu\nu} \partial_\mu \sigma_T\, \partial_\nu \sigma_T - \tfrac{1}{2} g^{\mu\nu} \partial_\mu \sigma_R\, \partial_\nu \sigma_R - 4Q^2 e^{\phi} \Big]$$

(breaks down near $t = 0$)

Background solution :

$$\sigma_T = 0$$
$$\sigma_R = \sqrt{2} \log |\tanh (Qt)|$$
$$\phi = 2\phi_0 - \log |\sinh (2Qt)|$$
$$g_{\mu\nu} = e^{-2\phi_0} |\sinh (2Qt)|\, \eta_{\mu\nu}$$

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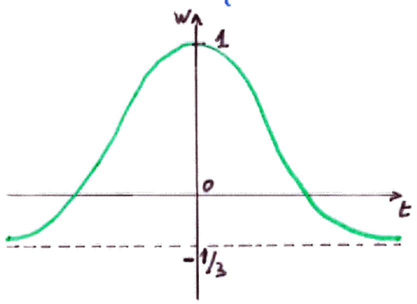
Equation of state $w = \frac{p}{\rho}$



Hubble radius $|a^2/a'| = R_H$

$$R_H \sim \frac{1}{Q} \frac{|\sinh(2Qt)|^{3/2}}{\cosh(2Qt)}$$

$t \to \infty$: $\vec{k}^2 > Q^2$: inside horizon; $\vec{k}^2 < Q^2$: outside horizon

$t = 0$: all modes outside horizon

$t \to -\infty$: $\vec{k}^2 > Q^2$: inside horizon; $\vec{k}^2 < Q^2$: outside horizon

($\vec{k}$: comoving momentum)

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Scalar fluctuations $\delta\phi$, $\delta\sigma_R$, $\delta\sigma_T$

e.g. $\delta\sigma_T = \delta T(t)\, e^{i\vec{k}.\vec{x}}$

$$\delta T'' + 2Q \coth(2Qt)\, \delta T' + \vec{k}^2 \delta T = 0$$

Question: what is $\langle 0 | \delta\sigma_T(t,\vec{x})\, \delta\sigma_T(t,\vec{x}') | 0 \rangle$ for $t<0$ and $t>0$?

Use string theory to address this question

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c) Global wavefunctions for scalar fluctuations

$$S_T = \frac{1}{2\kappa^2}\int d^{d+1}x \sqrt{-g}\, e^{-2\Phi}\left(-g^{\mu\nu}\partial_\mu T\,\partial_\nu T - M^2 T^2\right)$$

Warm-up: solve wave equation in regions I and II

$$T = T(t)\, e^{ipx}\, e^{i\vec{k}.\vec{x}}$$

$$\ddot{T} + 2Q\coth(2Qt)\,\dot{T} + \left(\frac{p^2}{\tanh^2(Qt)} + \vec{k}^2 + M^2\right)T = 0$$

Solutions: hypergeometric functions (in each region)

$t \to \pm\infty$: $\ddot{T} + 2Q\dot{T} + (p^2 + \vec{k}^2 + M^2)\,T = 0$

$$T \sim e^{-Q|t|}\, e^{\pm iEt}$$

with $E \equiv \sqrt{p^2 + \vec{k}^2 + M^2 - Q^2}$

- $E^2 < 0$: "tachyons", instabilities (?)
- $E^2 > 0$: modes we will focus on

$t \to 0$: $T \sim \begin{cases} |t|^{\pm ip} & (p \neq 0) \\ \log|t| + \text{const} & (p = 0) \end{cases}$

singular: $\dot{T}^2 \sim 1$...

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Global wavefunctions on $\frac{PSL(2,\mathbb{R})}{U(1)} / \mathbb{Z} \times \mathbb{R}^{d-1}$

$e^{i\vec{k}.\vec{x}}$

(1) Wavefunctions on $PSL(2,\mathbb{R})$

$$\phi_{j,\alpha,\beta}(g) = \langle j,\alpha|\, g\, |j,\beta\rangle$$

$g \in PSL(2,\mathbb{R})$

j: $PSL(2,\mathbb{R})$ representation

α,β: states in representation j

Will consider $j = -\frac{1}{2} + is$ with $s \in \mathbb{R}$

(corresponds to $E^2 \geq 0$)

Note: $PSL(2,\mathbb{R})_L \times PSL(2,\mathbb{R})_R$ symmetry

$$g \mapsto g_L\, g\, g_R$$

Laplacian = Casimir of $PSL(2,\mathbb{R})$ = $j(j+1)$

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(2) Choose basis of wavefunctions ($\leftrightarrow$ basis of states α, β) such that $U(1)_L \times U(1)_R$ is diagonalized, where both factors are generated by σ_3

$$e^{\rho\sigma_3} | j, m, \pm \rangle = e^{2im\rho} | j, m, \pm \rangle$$

2 states $\forall m \Rightarrow$ 4 wavefunctions $\forall j, m, \bar{m}$

$$K_{\pm\pm}(j, m, \bar{m}; g) = \langle j, m, \pm | g | j, \bar{m}, \pm \rangle$$

$PSL(2,\mathbb{R})$ splits in 6 regions such that $K_{\pm\pm}$ are smooth within each region but not at the boundaries between the regions

$\Rightarrow K_{\pm\pm}$ are not smooth, but still globally defined !

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(3) Wavefunctions on $\dfrac{PSL(2,\mathbb{R})}{U(1)}$

$$U(1): \quad g \mapsto e^{\rho\sigma_3} g \, e^{\rho\sigma_3}$$

Restrict to the $U(1)$-invariant $K_{\pm\pm}$, i.e. $m + \bar{m} = 0$

(4) $\mathbb{Z}$ orbifold

Quantize momentum $m - \bar{m}$

Introduce winding $m + \bar{m}$

$$m = \frac{1}{2}\left(\frac{n}{Q r_0} - \frac{w r_0}{Q \alpha'}\right)$$

$$\bar{m} = \frac{1}{2}\left(-\frac{n}{Q r_0} - \frac{w r_0}{Q \alpha'}\right)$$

$$s = \frac{1}{2Q}\sqrt{M^2 + \vec{k}^2 + \left(\frac{n}{r_0}\right)^2 + \left(\frac{w r_0}{\alpha'}\right)^2 - Q^2} = \frac{E}{2Q}$$

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Milne orbifold as a limit

$uv \to 0$ with $\left\{ \begin{array}{ll} \frac{\vec{k}^2 uv}{Q^2} & \text{fixed} \\ Q r_0 & \text{fixed} \\ \vec{k}^2 \alpha' & \text{fixed} \end{array} \right.$

Generalized Milne orbifold $\to$ Milne orbifold

- wave equation for winding modes
- $PSL(2,\mathbb{R})$ prescription for continuing wavefunctions $\to$ analytic continuation

(at least if $m = \bar{m} = 0$)

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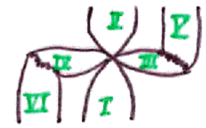
D) Quantization, particle production



(1) $T = a_{II} K_{--}^* + a_{II}^\dagger K_{--} + a_V K_{-+}^* + a_V^\dagger K_{-+}$

K_{--}^* : $\left\{ \begin{array}{l} \text{positive frequency in region II} \\ \text{vanishes in region V} \end{array} \right.$

K_{-+}^* : $\left\{ \begin{array}{l} \text{positive frequency in region V} \\ \text{vanishes in region II} \end{array} \right.$

$a_{II} |0\rangle_{out} = a_V |0\rangle_{out} = 0$

"empty" in regions II and V

(2) $T = a_I K_{++} + a_I^\dagger K_{++}^* + a_{VI} K_{+-} + a_{VI}^\dagger K_{+-}^*$

$a_I |0\rangle_{in} = a_{VI} |0\rangle_{in} = 0$

"empty" in regions I and VI

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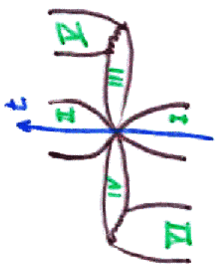
(3) Generalization: $|0\rangle_{\gamma\tilde{\gamma}}$

"empty" in region I, not in region VI



Question

Given a natural initial state, what does this state look like at late times?

For instance, what does $|0\rangle_{in}$ look like in region II?

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$$\begin{pmatrix} a_I^+ \\ a_{VI}^+ \\ a_I \\ a_{VI} \end{pmatrix} = \begin{pmatrix} A & C & 0 & B^* \\ C & A & B^* & 0 \\ 0 & B & A^* & C^* \\ B & 0 & C^* & A^* \end{pmatrix} \begin{pmatrix} a_{II}^+ \\ a_V^+ \\ a_{II} \\ a_V \end{pmatrix}$$

A, B, C: known functions of $s, m, \tilde{m}$

$${}_{in}\langle 0|\, a_{II}^+ a_{II}\, |0\rangle_{in} = {}_{in}\langle 0|\, a_V^+ a_V\, |0\rangle_{in} = |B|^2 = \frac{\cosh^2 \pi m}{\sinh^2 \pi s}$$

$$m = \frac{1}{2}\left(\frac{n}{Q r_0} - \frac{w r_0}{Q \alpha'} \right)$$

$$s = \frac{1}{2Q}\sqrt{M^2 + \vec{k}^2 + \left(\frac{n}{r_0}\right)^2 + \left(\frac{w r_0}{\alpha'}\right)^2 - Q^2}$$

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E) Fluctuation spectra

Use global wavefunctions and natural vacuum from string theory to compute fluctuation spectra in effective 4D FRW cosmology

$$ {}_{in}\langle 0 | \, \delta\sigma_T(t,\vec{x}) \; \delta\sigma_T(t,\vec{x}') \, | 0 \rangle_{in} \sim $$

$$ \begin{cases} \displaystyle\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\sqrt{\vec{k}^2 - Q^2}} \frac{\sin \vec{k}\cdot(\vec{x}-\vec{x}')}{\vec{k}\cdot(\vec{x}-\vec{x}')} & \left(t \ll \frac{-1}{Q}\right) \\ \displaystyle\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\sqrt{\vec{k}^2 - Q^2}} \frac{\sin \vec{k}\cdot(\vec{x}-\vec{x}')}{\vec{k}\cdot(\vec{x}-\vec{x}')} \Delta(\vec{k},t) & \left(t \gg \frac{1}{Q}\right) \end{cases} $$

with

$$ \Delta(\vec{k},t) = 1 + 2|B|^2 + 2|B|^2 \cos(2E_{\vec{k}}t) $$

$$ E_{\vec{k}} = \sqrt{\vec{k}^2 - Q^2} $$

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Comments:

$$ \left[\Delta(\vec{k},t) = 1 + 2|B|^2 + 2|B|^2 \cos(2E_{\vec{k}}t) \right] $$

(1) Fluctuation spectrum is modified by the Big Crunch / Big Bang transition

(2) $B \to 0$ in Milne orbifold limit (for the modes under consideration : $M = m = \tilde{m} = 0$)
$\Rightarrow$ no modification in Milne orbifold limit

(3) Constant term $1 + 2|B|^2$ signals creation of $|B|^2$ particles. Oscillating term contains additional information.

(4) $\Delta(\vec{k},t)$ cannot be reproduced by a conventional Bogolubov transformation between regions I and II

(5) In some of the generalized in-vacua $|0\rangle_{\gamma\tilde{\gamma}}$, particle production is suppressed

F) Information loss

$$|0\rangle_{in} = \mathcal{N} \exp\left[\theta\,(a_{II}^{+})^2 + \lambda\,(a_V^{+})^2 + \mu\, a_{II}^{+}\, a_V^{+}\right] |0\rangle_{out}$$

$$\begin{cases} \theta = \lambda = \dfrac{1}{2}\,\dfrac{BC^*}{(A^*)^2 - (C^*)^2} \\ \mu = -\dfrac{BA^*}{(A^*)^2 - (C^*)^2} \end{cases}$$

How would observer in region II describe $|0\rangle_{in}$?

$$\mathcal{H} = \mathcal{H}_{II} \otimes \mathcal{H}_V$$

$\uparrow$ a_{II}, a_{II}^{+} $\qquad$ $\uparrow$ a_V, a_V^{+}

$$|0\rangle_{out} = |0\rangle_{II} \otimes |0\rangle_V$$

$|0\rangle_{in}$ is entangled state in $\mathcal{H}_{II} \otimes \mathcal{H}_V$

Density matrix $\quad \rho = |0\rangle_{in}\, {}_{in}\langle 0|$

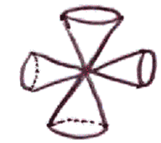
To compute correlation functions in region II, use

$$\rho^{II} = \sum_i {}_V\langle i|\,\rho\,|i\rangle_V$$

describing a non-trivial density matrix (mixed state) in region II.

G) Backreaction

Milne orbifold $\quad \mathbb{R}^{1,1}/\mathbb{Z}$



$\uparrow$

$$X^{\pm} \mapsto e^{\pm 2\pi} X^{\pm}$$

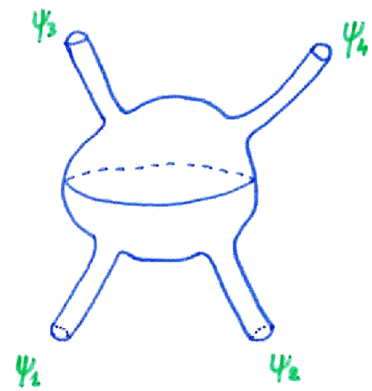
Vertex operators

$$\psi(x^+, x^-, \vec{x})_{p^+, p^-, \ell, \vec{p}} \sim e^{i\vec{p}\cdot\vec{x}} \int_{\mathbb{R}} dw\; e^{i(p^+ x^- e^{-w} + p^- x^+ e^{w} + \ell w)}$$

orbifold invariant if $\ell \in \mathbb{Z}$

$\Lsh$ momentum along Milne circle

$2 \to 2$ string scattering amplitude



$$\langle \psi_3^* \, \psi_4^* \, \psi_1 \, \psi_2 \rangle$$

$$\sim \delta^{(24)}\Big(\sum_i \epsilon_i \vec{p}_i\Big) \, \delta\Big(\sum_i \epsilon_i \ell_i\Big) \int_0^\infty dv_4 \; G(s) G(t) G(u) \, f(v_4)$$

$$G(s) = \frac{\Gamma(-1 - \frac{s}{4})}{\Gamma(2 + \frac{s}{4})}$$

Contribution from $v_4 \to \infty$: (Regge limit)

$$\begin{cases} s \sim v_4 \\ t \approx -(\vec{p}_1 - \vec{p}_3)^2 \end{cases}$$

$$\int^\infty dv_4 \; G(s) G(t) G(u) \, f(v_4)$$

$$\sim \int^\infty dv_4 \quad v_4^{\,i(\ell_2 - \ell_4) - \frac{\alpha'}{2}(\vec{p}_1 - \vec{p}_3)^2}$$

Divergence if $(\vec{p}_1 - \vec{p}_3)^2 \leq \frac{2}{\alpha'}$

In $\alpha' t \to 0$ field theory limit :

$$\sim \frac{1}{(\vec{p}_1 - \vec{p}_3)^2} \int^\infty dv_4 \quad v_4^{\,i(\ell_2 - \ell_4)}$$

Claim : this divergence comes from (tree-level) graviton exchange near the singularity

$$S = \int d^D x \sqrt{-g} \; e^{-2\Phi} \left(R + 4 g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - g^{\mu\nu} \partial_\mu T \partial_\nu T - m^2 T^2 \right)$$

$$\tilde{g}_{\mu\nu} = e^{-\frac{4\Phi}{D-2}} \, g_{\mu\nu}$$

$$S = \int d^D x \sqrt{-\tilde{g}} \left(\tilde{R} - \frac{4}{D-2} \tilde{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \tilde{g}^{\mu\nu} \partial_\mu T \partial_\nu T - m^2 T^2 e^{\frac{4\Phi}{D-2}} \right)$$

$$\begin{cases} \Box \Phi = \frac{1}{2} m^2 T^2 \\ \tilde{R}_{\mu\nu} = \partial_\mu T \, \partial_\nu T + \frac{1}{D-2} \eta_{\mu\nu} \, m^2 T^2 \end{cases}$$

Near $X^\pm = 0$: $\quad T \sim \# (X^+)^{i\ell} + \# (X^-)^{i\ell}$

$\Rightarrow$ $R_{\mu\nu}$ diverges , Φ stays finite

If $T \sim (X^+)^{i\ell}$, then only R_{++} diverges

$\Rightarrow$ milder backreaction

Integrate out $h_{\mu\nu}$, Φ classically :

$$S_4 = \int d^D x \left\{ \partial_\mu T \, \partial_\nu T \, \frac{1}{\Box} \, \partial^\mu T \, \partial^\nu T + \text{finite} \right\}$$

reproduces $v_4 \to \infty$ divergence if behavior of T near $X^\pm = 0$ is plugged in

H) Conclusions

(1) Simple string theory models with a cosmological singularity tend to have various asymptotic and intermediate regions

(2) In the limit of free string propagation, globally defined wavefunctions and fluctuation spectra are known

(3) The additional regions give rise to a mixed state after the Big Bang

(4) Large tree-level gravitational backreaction makes string perturbation theory problematic

- role of winding modes ?
- $\mathbb{Z}_2$, branes $\Rightarrow$ more interesting spectra ?
- AdS/CFT, ... ?