Some Lessons from Recent Progress in Quantum Field Theory in Curved Spacetime

Key references:

Linear fields: • RMW, "Quantum Field Theory in Curved Spacetime & Black Hole Thermodynamics" (U. of Chicago, 1994)

Interacting fields:

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- R. Brunetti & K. Fredenhagen, math-ph/9903028 (CMP 208, 623 (2000))
- R. Brunetti, K. Fredenhagen, & R. Verch math-ph/0112041 (CMP 237, 31 (2003))
- S. Hollands & RMW
  gr-qc/0103074 (CMP 223, 289 (2001))
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Quantum Field Theory in Curved Spacetime

Quantum field theory in curved spacetime is the theory of a quantum field propagating in a classical (globally hyperbolic) curved spacetime (M, g_{ab}). One can consider "back-reaction" effects within the context of this theory by imposing the semiclassical Einstein equation $G_{ab} = 8\pi \langle T_{ab} \rangle_\omega$, but these effects will not be considered here: This talk will focus exclusively on the formulation of interacting quantum field theory in curved spacetime (at the perturbative level). I will restrict consideration to a scalar field for simplicity, but the results should be generally applicable to all quantum fields.

Linear (Free) Quantum Fields in Curved Spacetime

The fundamental commutation relations for a free scalar field ϕ generalize straightforwardly to curved spacetime:

$$[\phi(f), \phi(g)] = -i\, \overbrace{\Delta}^{\text{adv - ret Greens fun.}}(f,g)\, 1$$

However, in the absence of time translation symmetry, have no preferred "vacuum state" nor any natural particle interpretation of the theory. Worse yet, in general, there is no preferred Hilbert space representation of the canonical commutation relations. Depending upon the asymptotics of the spacetime, the S-matrix cannot, in general, be naturally defined, and, when defined, need not exist.

Solution to all difficulties: Formulate the theory via the algebraic approach. Make all physical predictions in terms of probabilities for measuring field observables in given states.

Some Details of Linear QFTCS

Quantum fields make sense only distributionally, so wish to define an algebra, $\mathcal{A}$, of observables generated by expressions of form $\phi(f)$, where $f \in C_0^\infty$, i.e. f is smooth & of compact support.

Consider the "free algebra" $\mathcal{A}_0$ composed of all finite linear combinations of finite products of ϕ's & ϕ^*'s, e.g. expressions like

$$c_1\, \phi(f_1)\phi(f_2) + c_2\, \phi^*(f_3)\phi^*(f_4)\phi(f_5)$$

Impose: i) Linearity of $\phi(f)$ in f

ii) $\phi^*(f) = \phi(\bar{f})$ complex conjugate

iii) $\phi([\nabla_a \nabla^a - \xi R - m^2] f) = 0$

iv) $[\phi(f), \phi(g)] = -i\, \Delta(f,g)\, 1$ adv.-ret. solution

The desired algebra, $\mathcal{A}$, is defined to be $\mathcal{A}_0$ "factored" by the above relations.

States are maps $\omega : \mathcal{A} \to \mathbb{C}$ satisfying $\omega(A^*A) \geq 0$. For $A \in \mathcal{A}$, $\omega(A)$ is interpreted as the expectation value of A in the state ω. The probability distribution for A in the state ω can be obtained by going to any Hilbert space rep. that includes ω (such as the GNS rep. of ω) and computing by usual Hilbert space methods (provided that the observables are represented as self-adjoint ops.).

This yields a completely satisfactory theory of a linear quantum field in curved spacetime as far as observables in $\mathcal{A}$ are concerned, i.e., essentially the n-point functions of ϕ.

The GNS Construction

Every state, ω, in the algebraic sense corresponds to a vector $|\Psi\rangle$ in some Hilbert space $\mathcal{H}$:

Theorem (GNS): Let $\mathcal{A}$ be a $*$-algebra and let $\omega: \mathcal{A} \to \mathbb{C}$ be a state. Then there exists a Hilbert space $\mathcal{H}$, a representation π of $\mathcal{A}$ on $\mathcal{H}$, and a vector $|\Psi\rangle \in \mathcal{H}$ such that for all $A \in \mathcal{A}$ we have

$$\omega(A) = \langle \Psi | \pi(A) | \Psi \rangle$$

(Idea of proof: Start with the vector space $\mathcal{A}$, view ω as defining an "inner product" via $\langle A | B \rangle = \omega(A^* B)$, factor by the "zero norm" vectors, and complete to get a Hilbert space $\mathcal{H}$.)

Note, however, that the vectors $|\Psi_1\rangle$ & $|\Psi_2\rangle$ corresponding to the states ω_1 & ω_2 may live in different Hilbert spaces, $\mathcal{H}_1$ & $\mathcal{H}_2$, i.e. $|\Psi_1\rangle$ and $|\Psi_2\rangle$ belong to unitarily inequivalent representations.

Where Did the Possible Difficulties with Free QFT in Curved Spacetime Go?

- There are no "ultraviolet divergences" provided that one works with smeared field operators and considers only observables lying in $\mathcal{A}$.
- There do exist "infra-red divergences" but one can see clearly in the context of QFT in curved spacetime that these are not problems with the theory, but rather problems with existence of certain states or with auxilliary constructions, such as S-matrices.

Example

Incorrect statement: The theory of a free, massless scalar field in 1+1 dimensional Minkowski spacetime does not exist (on account of infra-red divergences occurring in a conventional Fock space construction of the theory).

Correct version: The theory of a free massless scalar field in Minkowski spacetime not only exists but suffers from no pathologies. However, it has the curious feature that it does not admit any Poincare invariant states.

Going Beyond the Theory of a Linear Field Based Upon $\mathcal{A}$

- Even if one were only interested in a linear field, there are many observables of interest (such as T_{ab}) than merely the ones found in $\mathcal{A}$ (i.e., the "n-point functions"). We need an enlarged algebra of observables.
- To make sense of nonlinear fields, one clearly must make sense of nonlinear functions of a quantum field. Perturbative rules for constructing interacting QFT require that one define "Wick powers" of a free field as well as the time-ordered-products of these Wick powers. Again, we need an enlarged algebra of observables.

The basic difficulties: (i) $\phi(x)$ is an algebra-valued distribution, so, a priori, $[\phi(x)]^2$ does not make sense. Attempts to define $\phi^2(f) = \lim \int \phi(x)\phi(y) f(x) F_n(x,y)$ as $F_n(x,y) \to \delta(x,y)$ yield divergent results, so some "regularization" is needed.

(ii) Once ϕ^k is defined as an algebra-valued distribution, $T(\phi^{k_1}(x_1) \cdots \phi^{k_n}(x_n))$ can be straightforwardly defined by "time ordering" when the support properties of $f_1, \dots, f_n$ have suitable causal relations. We must extend this distribution to act on all test functions. The difficult part is to extend the distribution to the "total diagonal" $x_1 = x_2 = \dots = x_n$.

"Microlocal Analysis"

[Hormander, Wightman, Radzikowski, Fredenhagen, ...]

Restrict attention to distributions, Ψ, of compact support. (If necessary, consider $f\Psi(g) \equiv \Psi(fg)$ where $f \in C_0^\infty$.) May pretend that $\mathrm{supp}\,\Psi$ is embedded in $\mathbb{R}^n$. (Must then later check the "coordinate invariance" of the various constructions & results.)

Define

$$\hat{\Psi}(k) = \frac{1}{(2\pi)^{n/2}} \Psi(e^{ik_\mu x^\mu})$$

Then $\hat{\Psi}(k)$ is a polynomially bounded analytic function of k.

Two further key results hold in the case where Ψ corresponds to a smooth function $\psi \in C_0^\infty$, i.e. $\Psi(f) = \int \psi f$:

1) $\hat{\Psi}(k) \to 0$ as $|k| \to \infty$ faster than any inverse power of $|k|$.

2) We have

$$\psi^2(x) = \frac{1}{(2\pi)^n} \int \hat{\psi}(k)\, \hat{\psi}(K-k)\, e^{iK\cdot x}\, dk\, dK$$

i.e. ψ^2 is the Fourier transform of the function $F(K) = \frac{1}{(2\pi)^{n/2}} \int \hat{\psi}(k)\hat{\psi}(K-k)\, dk$

The Wavefront Set

Let Ψ be a distribution of compact support. Let $(x, k) \in T_x^*(M)$, with $k \neq 0$. Call (x,k) a nonsingular point/direction of Ψ if there exists an $f \in C_0^\infty$ with $f(x) \neq 0$ and there exists an open nd. $\mathcal{O}$ of k s.t. for all $k' \in \mathcal{O}$, and all n, $\exists$ C_n s.t. $(\widehat{f\Psi})(\lambda k') \leq \frac{C_n}{\lambda^n + 1}$ for all $\lambda > 0$.

Define

$$WF(\Psi) = \{(x,k) \in T_x^*(M) \mid k \neq 0,\ (x,k) \text{ is not a nonsingular point/direction of } \Psi\}$$

This gives a refined characterization of the singularities of distributions. Its main advantage is that it allows one to define products of distributions if their wavefront set properties are such that the Fourier convolution integral converges.

Enlargement of the Algebra of Observables

Expect Wick powers to be defined only for a restricted class of states on $\mathcal{A}$.

Hadamard states:

$$<\phi(x)\phi(y)>_\omega = \frac{\Delta^{1/2}(x,y)}{\sigma} + V \ln\sigma + W$$

σ ← squared geodesic distance between x & y

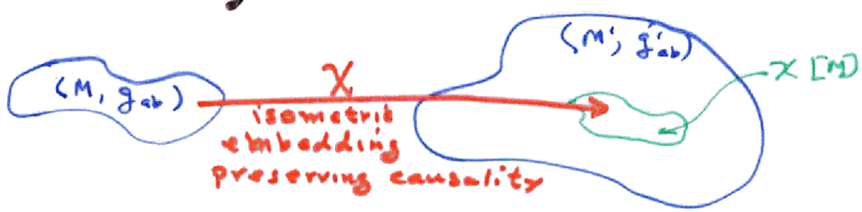
Radzikowski: Hadamard condition is equivalent to a simple condition on $WF[<\phi(x)\phi(y)>_\omega]$.

BFK Construction of enlarged algebra of observables $\mathcal{W}$: Choose a quasi-free ("vacuum") Hadamard state ω on $\mathcal{A}$. In the GNS rep. of ω ($\equiv$ Fock space based on the vacuum state ω), consider the normal ordered n-point functions $:\phi(x_1)\cdots\phi(x_n):_\omega$. The wavefront set properties of these operator-valued-distributions are such that, for a dense set of vectors, they continue to make sense as operators when smeared with a wide class of distributions, including $f(x)\,\delta(x_1, .., x_n)$. The resulting algebra of operators, $\mathcal{W}$, is independent of the choice of ω and defines a suitable enlarged algebra of observables. (Can also define $\mathcal{W}$ abstractly.)

However, the labeling of elements as $:\phi^n:_\omega$ does depend on ω. Which elements of the algebra should be viewed as representing the "true" $\phi^2(f)$ or $T(\phi^4(f)\phi^6(g))$, etc.?

Local, Covariant Fields

The algebra $\mathcal{W}$ depends upon the spacetime (M, g_{ab}); the algebras for different spacetimes cannot, in general, be "compared". However, suppose that one has the following situation:



Then χ gives rise to a natural injective $*$-homomorphism

$$i_\chi : \mathcal{W}[M, g_{ab}] \to \mathcal{W}[M', g'_{ab}]$$

A <u>quantum field</u>, Φ, is an assignment to every globally hyperbolic spacetime (M, g_{ab}) a distribution $\Phi_{(M, g_{ab})}$ valued in $\mathcal{W}[M, g_{ab}]$. Φ is said to be <u>local and covariant</u> if whenever $\chi : M \to M'$ is a causality preserving isometric embedding, we have for any test function f on $\chi[M]$,

$$i_\chi\left(\Phi_{(M,g)}(f \circ \chi)\right) = \Phi_{(M', g')}(f)$$

Axioms for Wick Powers

1) ϕ^n should be "local and covariant"

2) $[\phi^n(x), \phi(y)] = i n \Delta(x,y) \phi^{n-1}(x)$

3) $(\phi^n(f))^* = \phi^n(\bar{f})$

4) For any quasi-free Hadamard state ω, $\omega(\phi^n(x))$ is smooth.

5) ϕ^n varies analytically (smoothly) under an analytic (smooth) variation of the metric and coupling parameters. ← (Defined in terms of a wavefront set condition on $\phi^n[g_{ab}(s)](x)$, viewed as a distribution on $\mathbb{R} \times M$.)

6) Under scaling of the metric, $g_{ab} \to \lambda^2 g_{ab}$, have $\phi^n \to \phi^n(\lambda)$ where

$$\lambda^{-d} \phi^n(\lambda) = \phi^n + \sum_{i=1}^{m} \ln^i \lambda \, \psi_i$$

The axioms for time ordered products of Wick powers are similar, except that (4) is replaced by a much more complicated "microlocal spectral condition" and their are additional "unitarity" and "causal factorization" conditions.

Sketch of Uniqueness Argument

Consider ϕ^2. The commutation condition (2)

$$[\phi^2(x), \phi(y)] = 2i\,\Delta(x,y)\,\phi(x)$$

uniquely determines ϕ^2 up to a multiple of the identity, i.e., up to a term of the form $C(x)\,\mathbb{1}$. The local covariance condition $\Rightarrow$ C at x depends only on the spacetime geometry in an arbitrarily small nd. of x. Continuity, analyticity, & scaling $\Rightarrow$ C is a local curvature term of the "correct dimension", in this case $C = \alpha R$.

By induction, each higher power of ϕ leads to a new "multiple of the identity" ambiguity, given by curvature terms of the appropriate dimension.

A similar — but much more complicated — argument for time-ordered-products yields uniqueness up to certain specified ambiguities. These ambiguities in the def. of time ordered products give rise to corresponding ambiguities in the definition of ϕ_I. These latter ambiguities correspond precisely to adding appropriate "counterterms" to the Lagrangian, but these counterterms now include curvature couplings.

Existence of Wick Powers

It is well known how to define quantities quadratic in ϕ (such as ϕ^2 or $T_{\mu\nu}$) by an appropriate "point-splitting" prescription. Let

$$:\phi(x)\phi(y):_H = \phi(x)\phi(y) - H(x,y)\,\mathbb{1}$$

locally constructed Hadamard parametrix

Define $\phi^2(f)$ to be $:\phi(x)\phi(y):_H$ smeared with $f(x)\,\delta(x,y)$.

Can define $\phi^n(f)$ by a suitable "combinatorical generalization" of this prescription.

Note that the definition of Wick powers involves the subtraction of locally and covariantly constructed distributions, <u>not</u> the subtraction of expectation values in a "vacuum" or other state.

Existence of Time Ordered Products ("Renormalization")

Given the def. of Wick powers, the axioms themselves uniquely determine by induction (in the number of factors) the definition of

$$T\left(\phi^{k_1}(x_1)\cdots\phi^{k_n}(x_n)\right)$$

except on the "total diagonal" $x_1 = x_2 = \ldots = x_n$. The commutation condition implies a "local Wick expansion" in terms of $:\phi^{k_1-j_1}(x_1)\cdots\phi^{k_n-j_n}(x_n):_H$. Problem reduces to that of extending the c-number coefficients $t^0_{j_1\cdots j_n}(x_1,\ldots,x_n)$ to the total diagonal.

Key idea: We prove a "scaling expansion" for t^0 of the form,

$$t^0 = \sum_{k=0}^{m} \frac{1}{k!}\tau^0_k + r^0_m$$

where r^0_m has scaling behavior sufficient to guarantee that it can be uniquely extended to the total diagonal and each τ^0_k is of the form

$$\tau^0_k(x_1,\ldots,x_n) = C^{\mu_1\cdots\mu_k}(x_1)\; u^0(x_2,\ldots,x_n) \qquad (*)$$

where $C^{\mu_1\cdots\mu_k}$ is a local curvature expression involving a total of k derivatives of the metric, and u^0 corresponds to a Lorentz invariant distribution in the tangent space of x_1. (Eq. (*) may be viewed as proving a generalized form of the "local momentum space" expansion assumed by Bunch & Parker.)

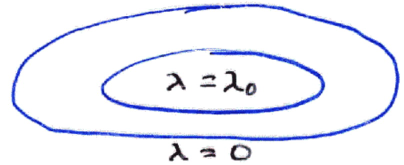
The distributions u^0 may then be extended to the origin (e.g. via the Minkowski spacetime methods of Epstein & Glaser) to yield a def. of time ordered products satisfying our axioms.

The Bogoliubov Formula

Consider

$$\mathcal{L} = \underbrace{\nabla_a\phi\nabla^a\phi + (\xi R + m^2)\phi^2}_{\mathcal{L}_0} + \underbrace{\lambda\phi^4}_{\mathcal{L}_1}$$

View λ as a smooth function of compact support:



so that action of $\mathcal{L}_1$ is $\phi^4(\lambda)$.

Define:

$$S[\mathcal{L}_1 = \lambda\phi^4] = \sum_n \frac{i^n}{n!} T(\underbrace{\phi^4(\lambda)\cdots\phi^4(\lambda)}_{n})$$

$$\boxed{\phi_I(x) = \frac{\delta}{i\delta f} S^{-1}[\mathcal{L}_1]\; S[\mathcal{L}_1 + f\phi]}$$

Bogoliubov formula for interacting field (to be interpreted as a formal power series expansion) Formula is adjusted so that $\phi_I = \phi$ before interaction is "turned on". Limit as $\lambda \to \lambda_0$ need not exist. However can modify formula to keep ϕ_I fixed "in the interior" of the spacetime & can then take limit as $\lambda \to \lambda_0$.

Upshot: The interacting field algebra for a quantum field in an arbitrary globally hyperbolic curved spacetime is well defined (as a formal power series).

Lesson 1: Hilbert spaces should be viewed as useful "auxilliary tools" for performing calculations in quantum field theory, but the set of all (physically reasonable) states does not comprise a (single) Hilbert space in general.

In many spacetimes there is no way of choosing a preferred Hilbert space representation of the theory from the various unitarily inequivalent representations. (In addition, every Hilbert space representation will contain many physically unreasonable states — namely those that do not lie in the domain of all the field observables.)

Lesson 2: Quantum field theory in curved spacetime is a theory of local quantum fields. It is not a theory of "particles", S-matrices, etc.

If a given spacetime possesses a time translation symmetry, a natural notion of "particles" can be introduced that is useful for describing the response of devices that are coupled to the field. If a given spacetime satisfies appropriate asymptotic conditions in the past and future, it may be possible (and useful) to define an S-matrix. But, at a fundamental level, all that one has are local quantum fields.

Lesson 3: "Preferred vacuum states" play absolutely no role in the formulation of quantum field theory in curved spacetime.

The question of "which vacuum state do we choose?" is as relevant to quantum field theory in curved spacetime as the question of "which coordinates do we choose?" is relevant to general relativity.

Comment on the "α-vacua": For the algebra $\mathcal{A}$, there exists a 1-parameter family of de Sitter invariant vacuum states on de Sitter spacetime. However, on the enlarged algebra $\mathcal{W}$, only the "Bunch-Davies" vacuum exists (except in the massless case, when no de Sitter invariant state whatsoever exists). In other words, the α-vacua do not define states on $\mathcal{W}$.

Lesson 4: Normal ordering (i.e., "vacuum subtraction") is fundamentally incorrect as a renormalization prescription.

The correct regularization of Wick powers is accomplished by subtraction of a locally and covariantly constructed Hadamard parametrix, $H(x, x')$. In Minkowski spacetime, it happens to be true that $H(x,x') = \langle 0 | \phi(x) \phi(x') | 0 \rangle$, so the regularization can be interpreted as a "vacuum subtraction". But it is not difficult to show that in a general curved spacetime, such an equality (and interpretation) cannot hold.

In QFT, every nonlinear function of a field displays the same behavior as T_{ab}. So, the "cosmological constant problem" is really: (1) Why is QFT so weird? and (2) What sets the scale of the (apparently) observed non-zero cosmological constant?

Lesson 5: Whenever a theory is extended into a new domain, it leads to new insights which overthrow prior viewpoints on the theory.

Therefore, further progress in the development of quantum gravity will surely invalidate all of the above "lessons".