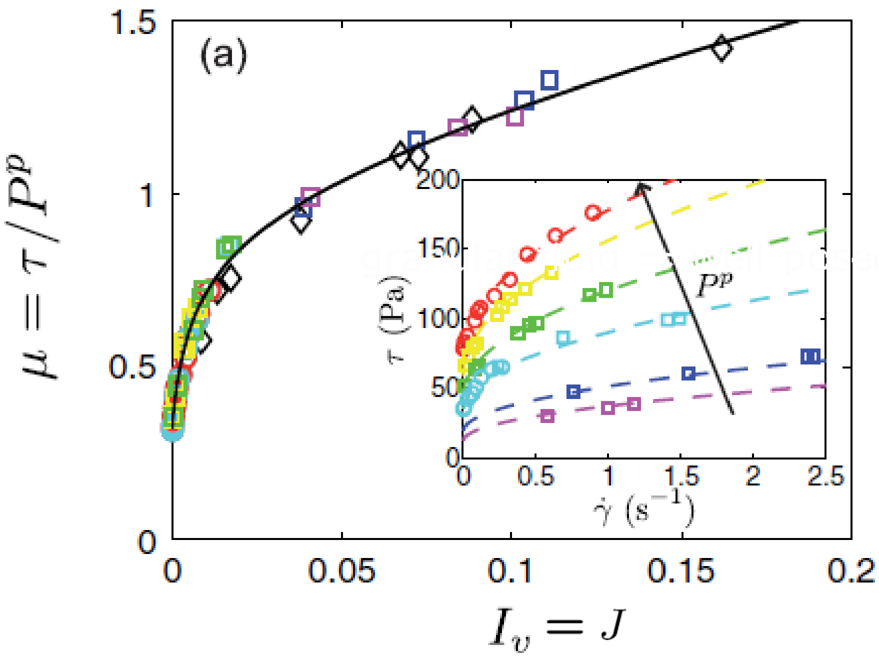
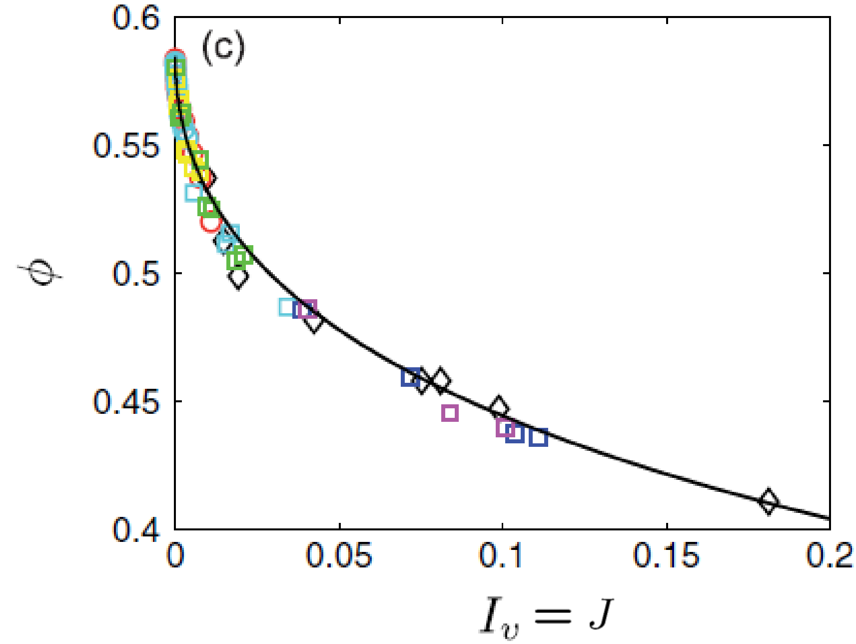
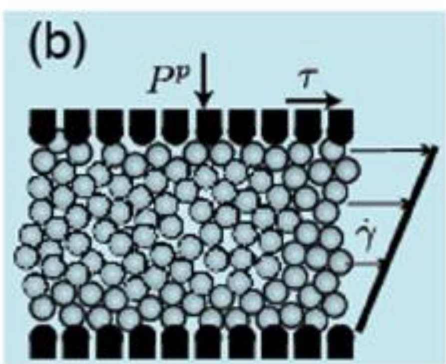
Discussion on *μ(J)* suspension rheology and links to *μ(I)*

Boyer, Guazzelli & Pouliquen (2011) *Phys. Rev. Lett.* **107**, 188301.







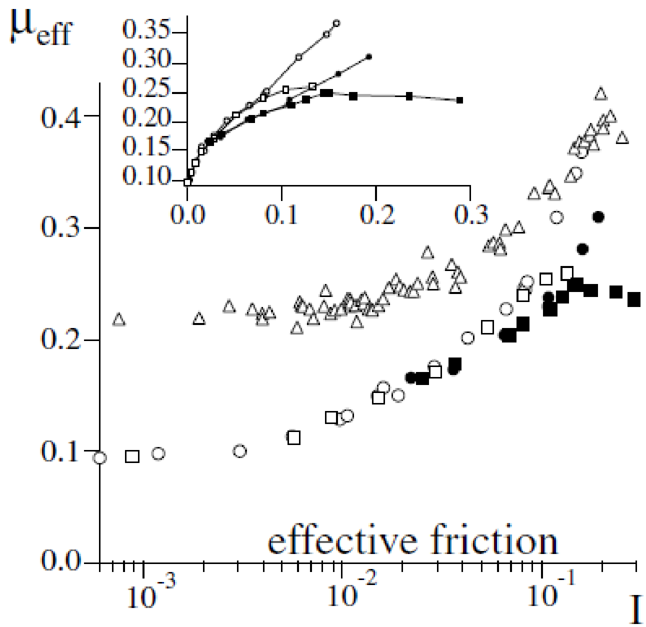
Pressure imposed simple shear experiment

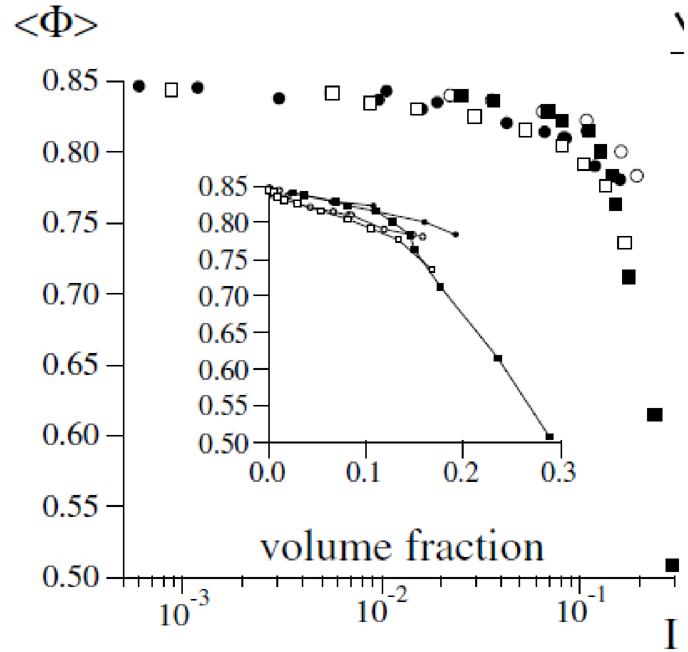
$$\tau = \mu(J) P^p, \quad \phi = \phi(J), \quad J = \frac{\eta_f \dot{\gamma}}{P^p}$$

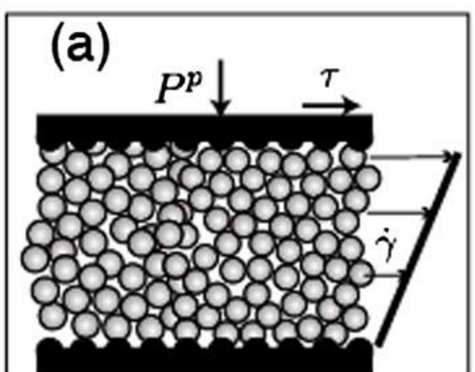
where τ is the shear stress, P^p is pressure supported by the grains, η_f is the viscosity of the suspending fluid, ϕ is the solids volume fraction, $\dot{\gamma}$ is the shear-rate and J is the dimensionless viscous number

The $\mu(I)$-rheology for dry grains is closely related

GDR MiDi (2004) *Eur. Phys. J. E* **14**, 341-365.





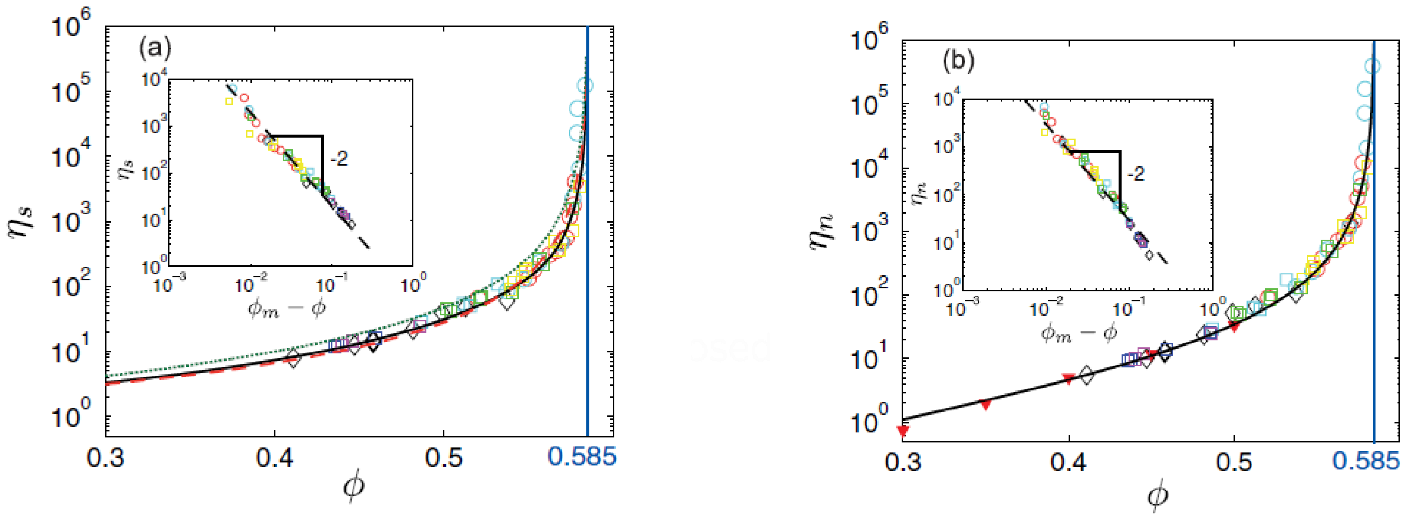


Simple shear experiment

$$\tau = \mu(I)P, \quad \phi = \phi(I), \quad I = \frac{\dot{\gamma} d}{\sqrt{P/\rho}}$$

where τ is the shear stress, P is pressure, ϕ is the solids volume fraction, $\dot{\gamma}$ is the shear-rate, d is the particle size, ρ is the density and I is the dimensionless inertial number

Link to standard suspensions formulations



At constant volume fraction suspensions scale viscously as

$$\tau = \eta_s(\phi)\eta_f\dot{\gamma}, \quad P^p = \eta_n(\phi)\eta_f\dot{\gamma}$$

where the shear and normal viscosities η_s and η_n diverge at ϕ_m

$$\eta_s = 1 + \frac{5}{2}\phi\left(1 - \frac{\phi}{\phi_m}\right)^{-1} + \mu_c(\phi)\left(\frac{\phi}{\phi_m - \phi}\right)^2, \qquad \eta_n = \left(\frac{\phi}{\phi_m - \phi}\right)^2$$

It is simple to show that $J(\phi) = 1/\eta_n(\phi)$ which can be inverted to determine $\phi(J)$. The standard suspension laws are equivalent to $\mu(J)$

$$\tau = \mu(J)P^p, \quad \phi = \phi(J), \quad J = \frac{\eta_f\dot{\gamma}}{P^p}$$

The $\mu(I)$-rheology has been generalized to tensorial form

Jop, Forterre & Pouliquen (2006) *Nature* **441**, 727-730.
Schaeffer (1987) *J. Differ. Equ.* **66**(1), 19-50.

Cauchy stress decomposed into a pressure p and deviatoric stress $\boldsymbol{\tau}$

$$\boldsymbol{\sigma} = -p\mathbf{1} + \boldsymbol{\tau}$$

The tensorial form of the incompressible $\mu(I)$-rheology is

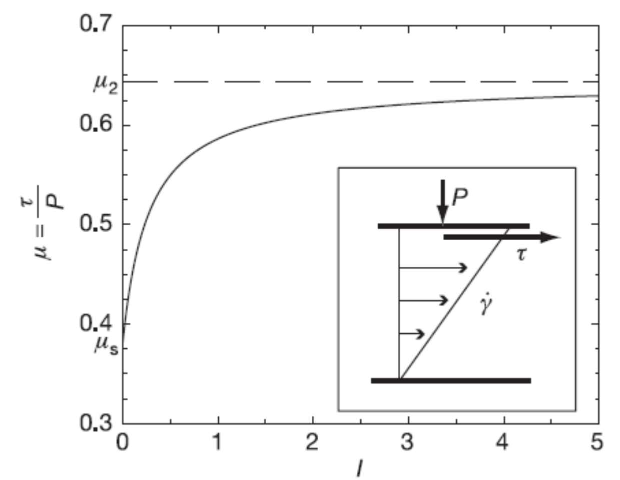
$$\boldsymbol{\tau} = \mu(I)p\frac{\boldsymbol{D}}{||\boldsymbol{D}||}$$

where $\boldsymbol{D}$ is the strain-rate, $||\boldsymbol{D}||$ is the second invariant

$$||\boldsymbol{D}|| = \sqrt{\frac{1}{2}\mathrm{tr}\boldsymbol{D}^2}, \quad \text{and} \qquad I = \frac{2||\boldsymbol{D}||d}{\sqrt{p/\rho}},$$

The $\mu(I)$-curve asymptotes to μ_2 as $I \longrightarrow \infty$

$$\mu(I) = \mu_1 + \frac{\mu_2 - \mu_1}{I_0/I + 1},$$



If $\mu =$ constant this reduces to Drucker-Prager plasticity, which is always ill-posed (Schaeffer 1987).

What is tensorial form for suspensions?
Are there any issues with ill-poseness?

Solver: We apply the Open-source Gerris (Popinet 2003)
http: //gfs.sourceforge.net
(incompressible Navier-Stokes equations using a VOF method) (Popinet 2003, 2009)

$$\begin{aligned}
\boldsymbol{\nabla}.\boldsymbol{u} &= 0 \\
\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}.\boldsymbol{\nabla}\boldsymbol{u}\right) &= -\boldsymbol{\nabla}p + \boldsymbol{\nabla}.(2\eta\boldsymbol{D}) + \rho\boldsymbol{g} \\
\frac{\partial c}{\partial t} + \boldsymbol{\nabla}.(c\boldsymbol{u}) &= 0
\end{aligned}$$

$$\rho = c\,\rho_{\text{air}} \;+\; (1-c)\,\rho_{\text{grains}}$$

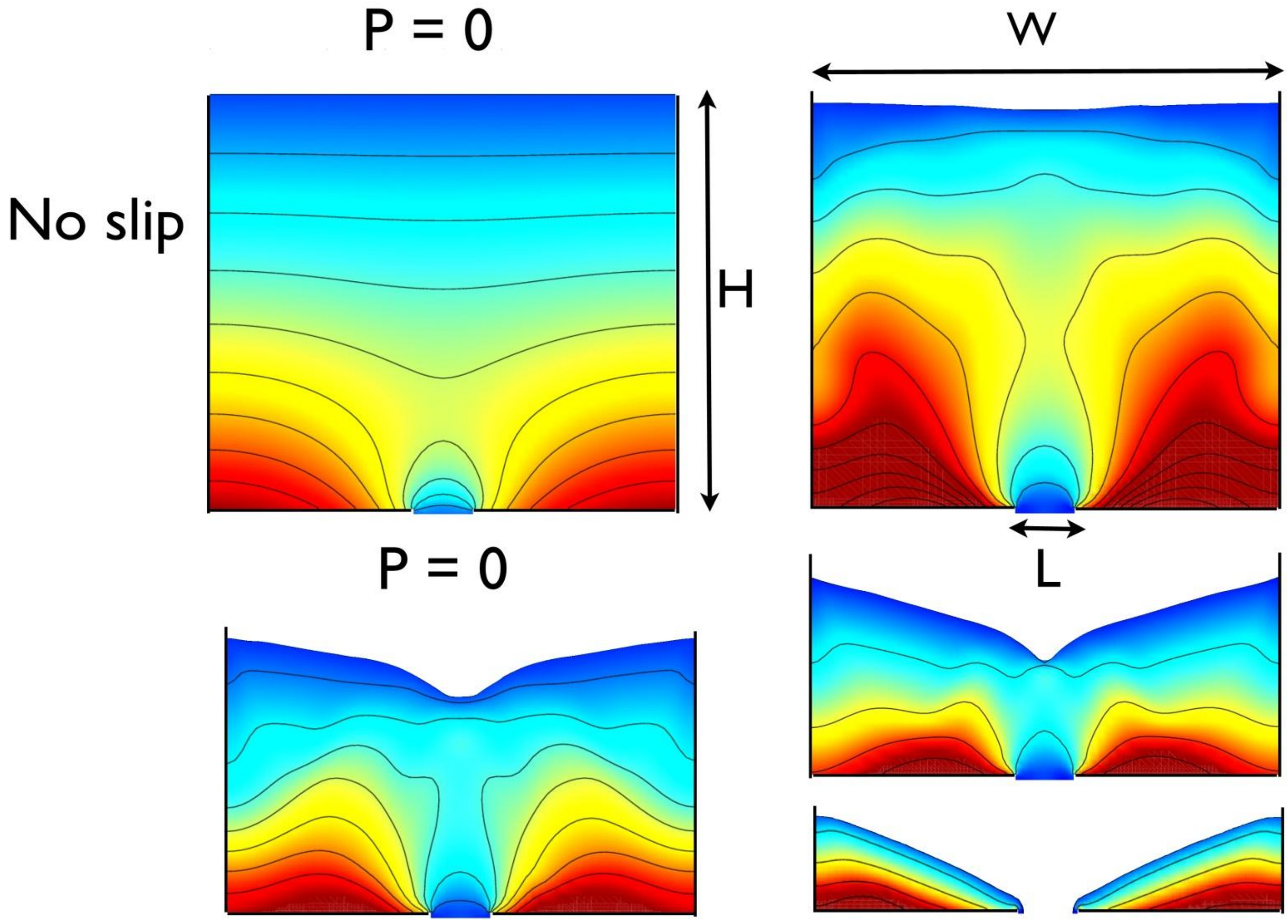
$$\eta = c\,\eta_{\text{air}} \;+\; (1-c)\,\eta_{\text{grains}}$$

⇒ We chose $\rho_{\text{air}} << \rho_{\text{grains}}$
⇒ The free surface is solved in the course of time
⇒ We implement the viscosity:

$$\eta_{\text{grains}} = \min\left(\frac{\mu P}{|\dot{\gamma}|}, \eta_{max}\right),$$

Lagree, Staron & Popinet 2011

The Continuum Sand-Glass

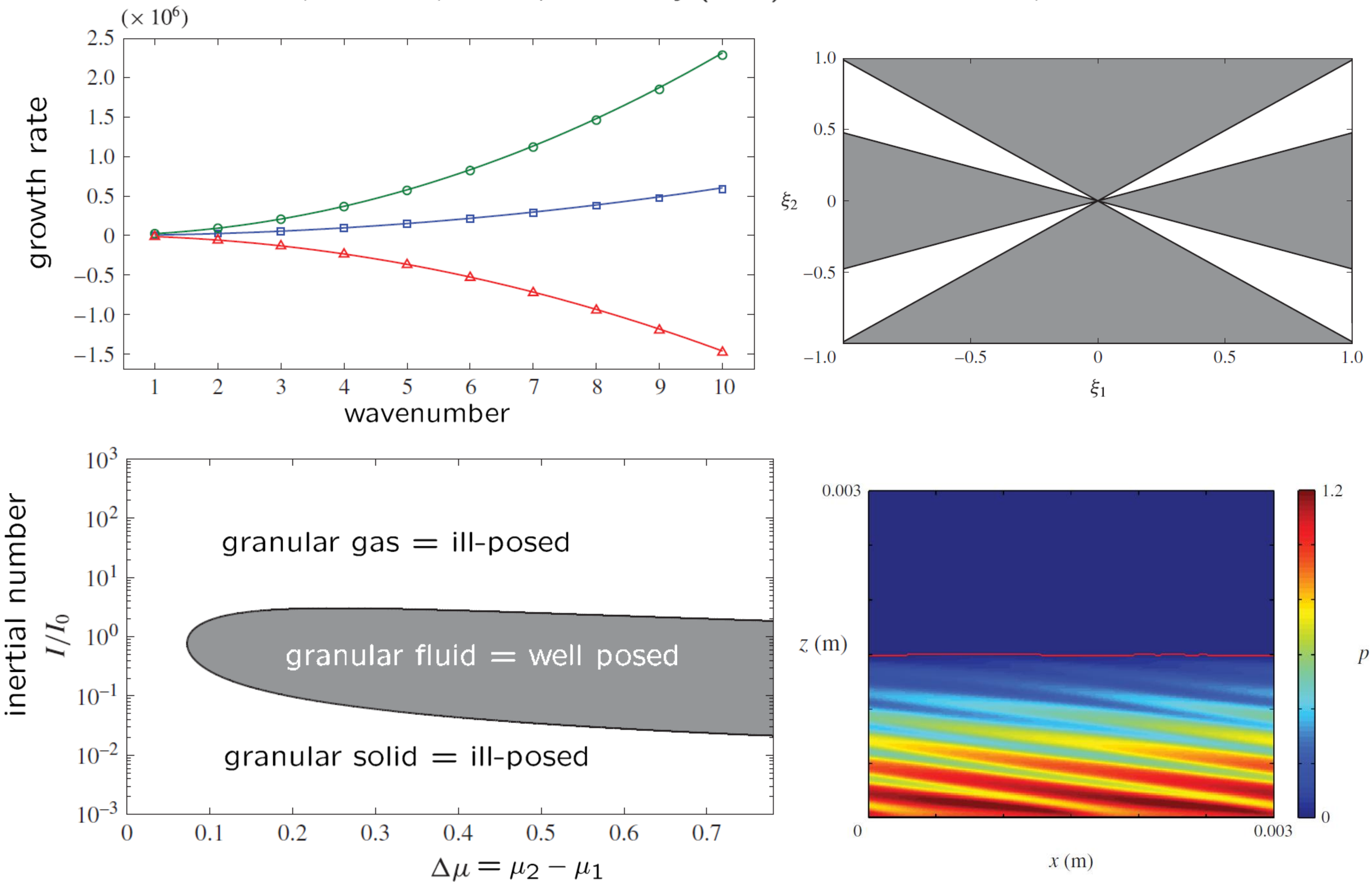


We chose the following value for the rheological parameters:
$\mu_s = 0.32$, $\mu_d = 0.60$, $I_0 = 0.4$

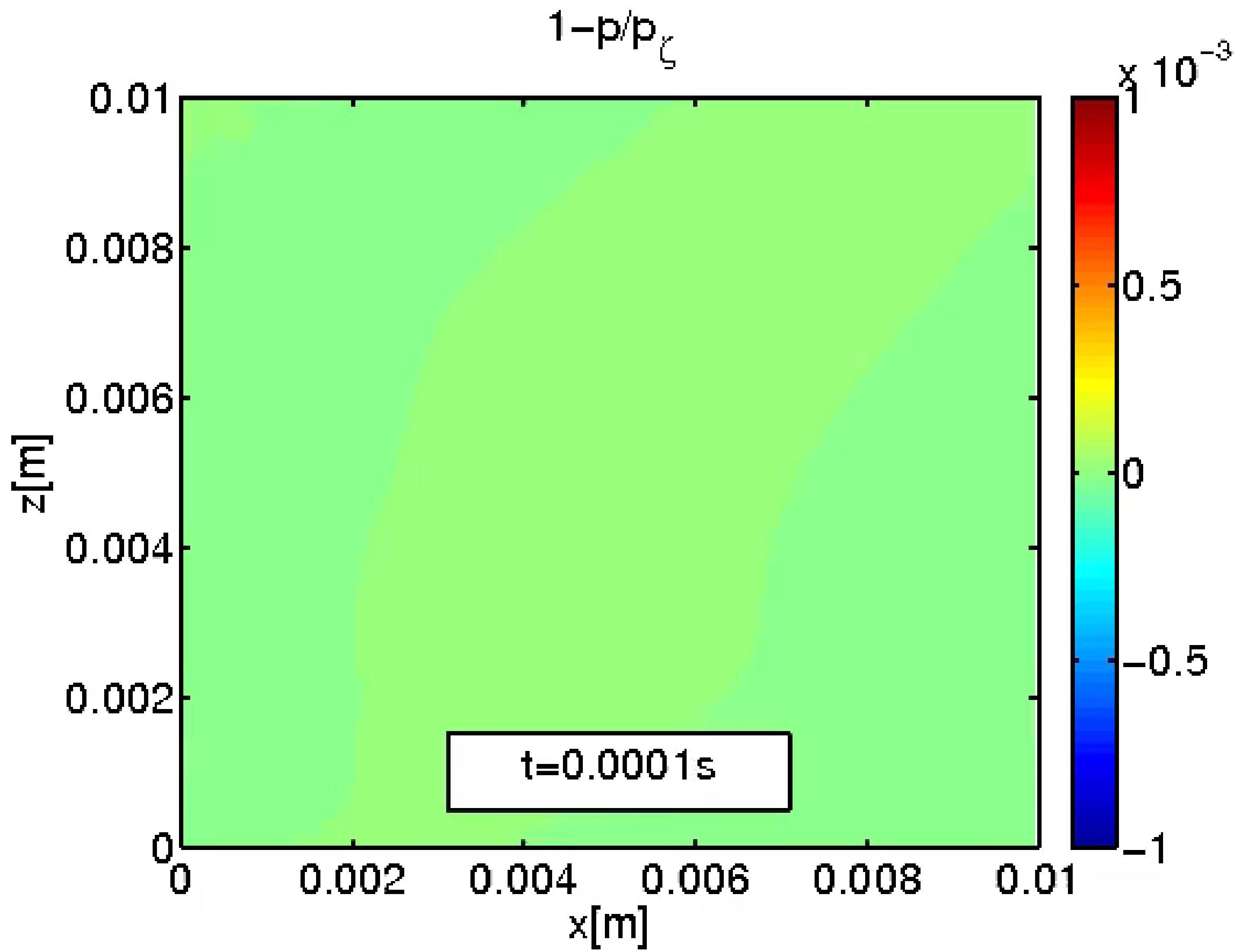
Staron, Lagree & Popinet 2011

Well-posed and ill-posed behaviour of the μ(I)-rheology

Barker, Schaeffer, Bohorquez & Gray (2015) *J. Fluid Mech.* *779*, 794-818.

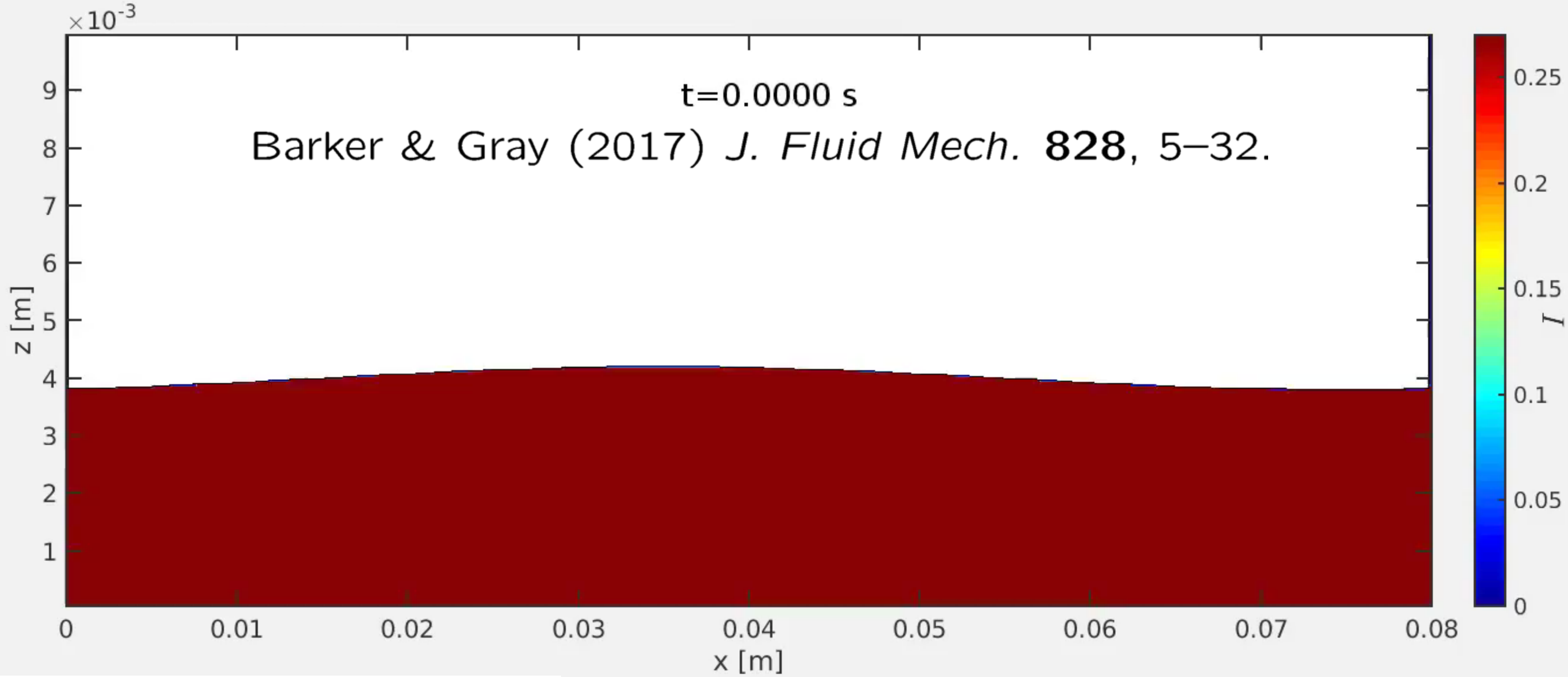


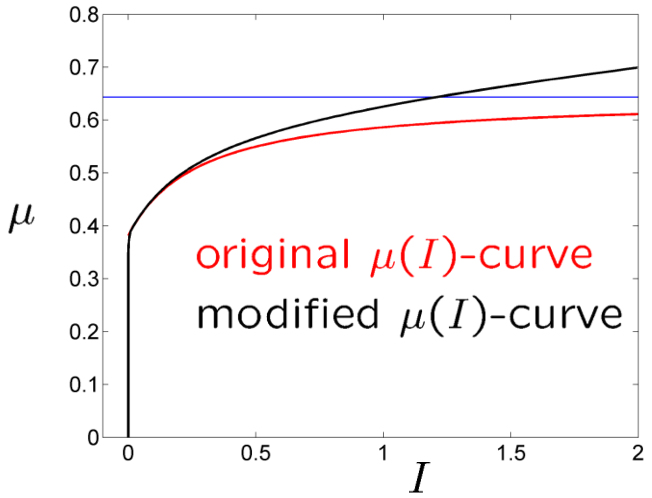
Outside well-posed region of parameter space solutions blow up



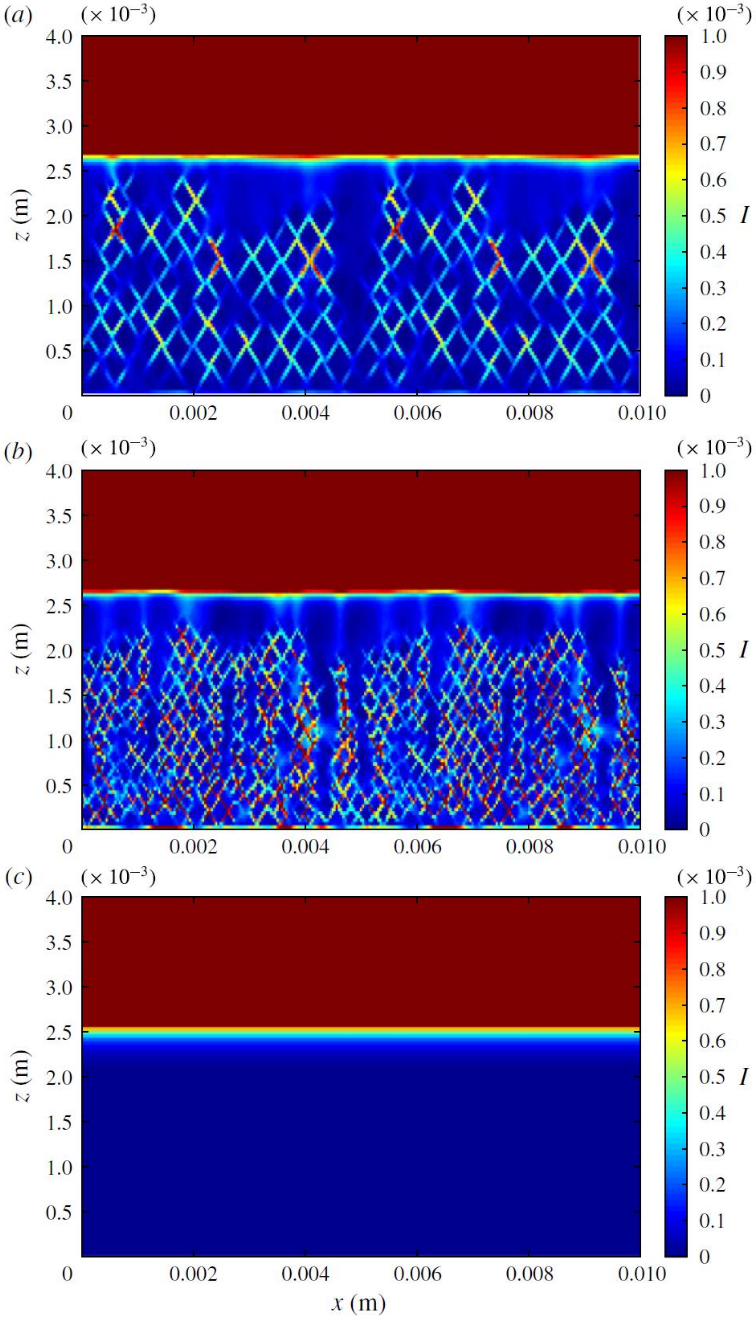
- Gerris rectangular mesh with no free-surface deformation

Partial regularization of the μ(I)-rheology

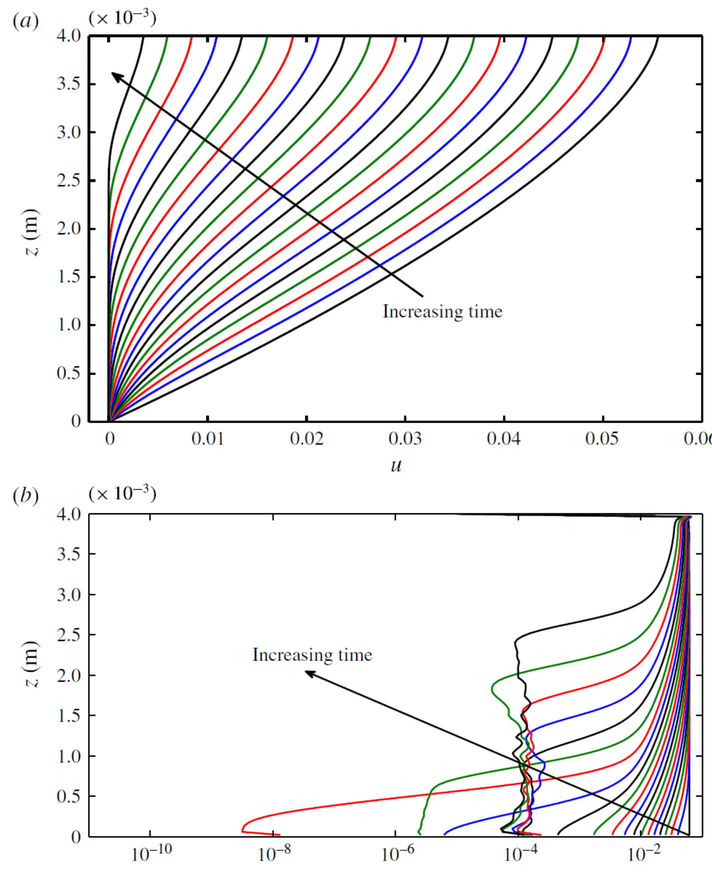




- The ill-posedness condition is used to maximize region of well-posedness
- For low I it must pass through zero and rise logarithmically to connect with the well-posed intermediate region
- For high I an approximately linear dependence extends the range of μ up to $\sqrt{2}$

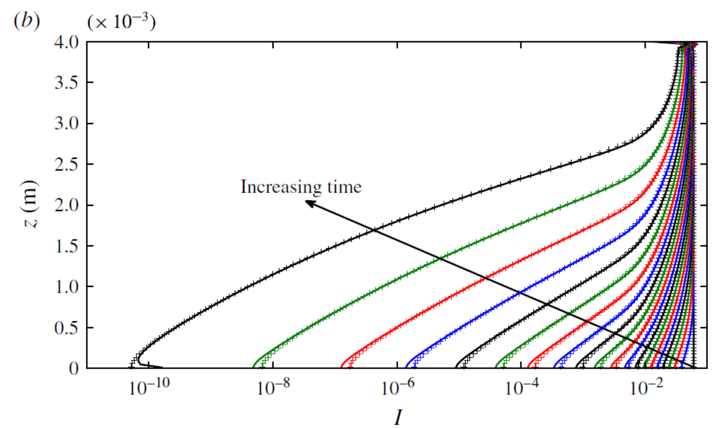


Slope reduced from 24° to 10°



original ill-posed

original ill-posed



partially regularized model is good

well-defined creep state

Compressible I Dependent Rheology (CIDR)

Barker, Schaeffer, Shearer & Gray (2017) *Proc. Roy. Soc. A* **473**, 20160846.

Assuming ϕ is the solids volume fraction mass and momentum are

$$(\partial_t + u_j \partial_j)\phi + \phi \text{ div } \boldsymbol{u} = 0$$

$$\rho_* \phi (\partial_t + u_j \partial_j) u_i = -\partial_i p + \partial_j \tau_{ij} + \rho_* \phi g_i$$

Deviatoric strain-rate tensor

$$D_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j) - \frac{1}{2}(\text{div } \boldsymbol{u})\delta_{ij} \qquad D_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$$

Alignment: $\dfrac{D_{ij}}{\|\boldsymbol{D}\|} = \dfrac{\tau_{ij}}{\|\boldsymbol{\tau}\|}$ where $\|\boldsymbol{D}\| = \sqrt{D_{ij} D_{ij}/2}$

Yield Condition: $\|\boldsymbol{\tau}\| = Y(p, \phi, I)$ $\qquad \|\boldsymbol{\tau}\| = \mu(I) p$

Flow Rule: $\text{div } \boldsymbol{u} = 2 f(p, \phi, I)\, \|\boldsymbol{D}\|$ $\qquad \text{div } \boldsymbol{u} = 0$

Compressible I Dependent Rheology (CIDR)

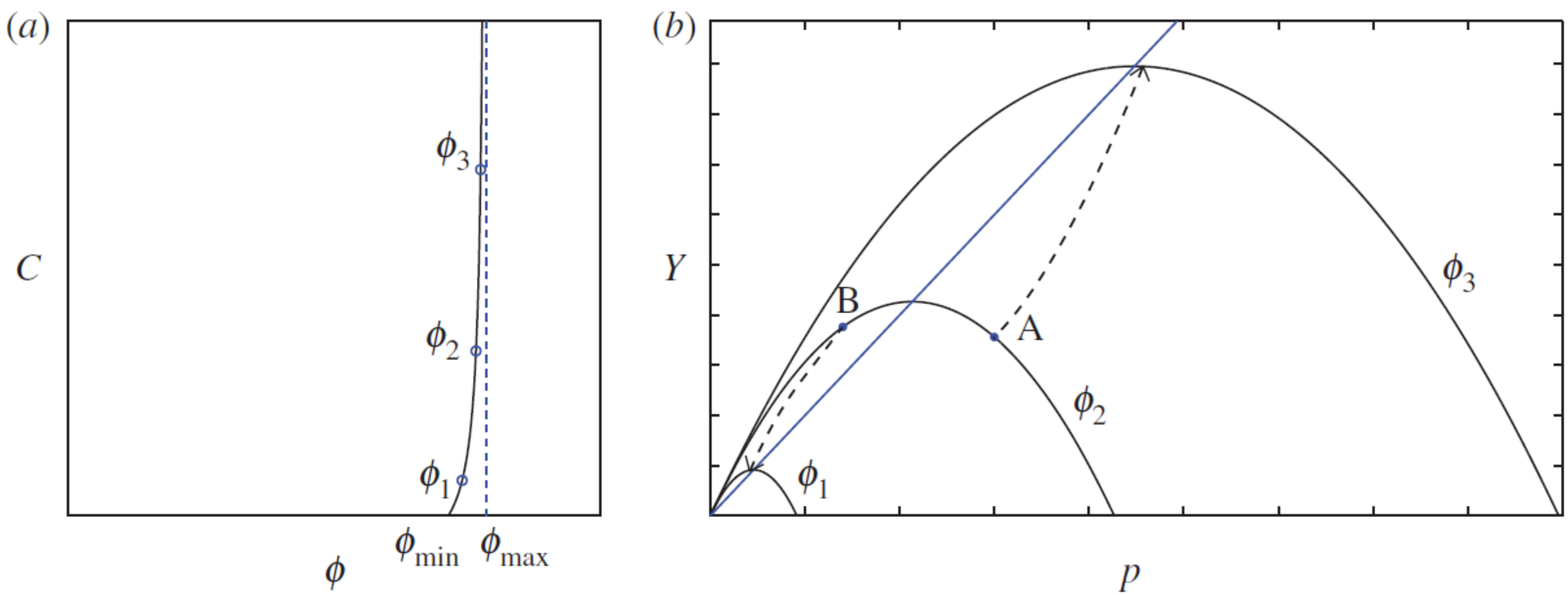
Barker, Schaeffer, Shearer & Gray (2017) *Proc. Roy. Soc. A* **473**, 20160846.

The model is ALWAYS WELL-POSED provided

$$\frac{\partial Y}{\partial p} - \frac{I}{2p}\frac{\partial Y}{\partial I} = f + I\frac{\partial f}{\partial I}$$

and that (a) $\partial_I Y > 0$ and $\partial_p f - I\partial_I f/(2p) < 0$. One example is

$$Y(p,\phi,I) = \alpha(I)p - p^2/C(\phi) \qquad f(p,\phi,I) = \beta(I) - 2p/C(\phi)$$



which is I dependent Critical State Soil Mechanics (CSSM).

Compressible I Dependent Rheology (CIDR)

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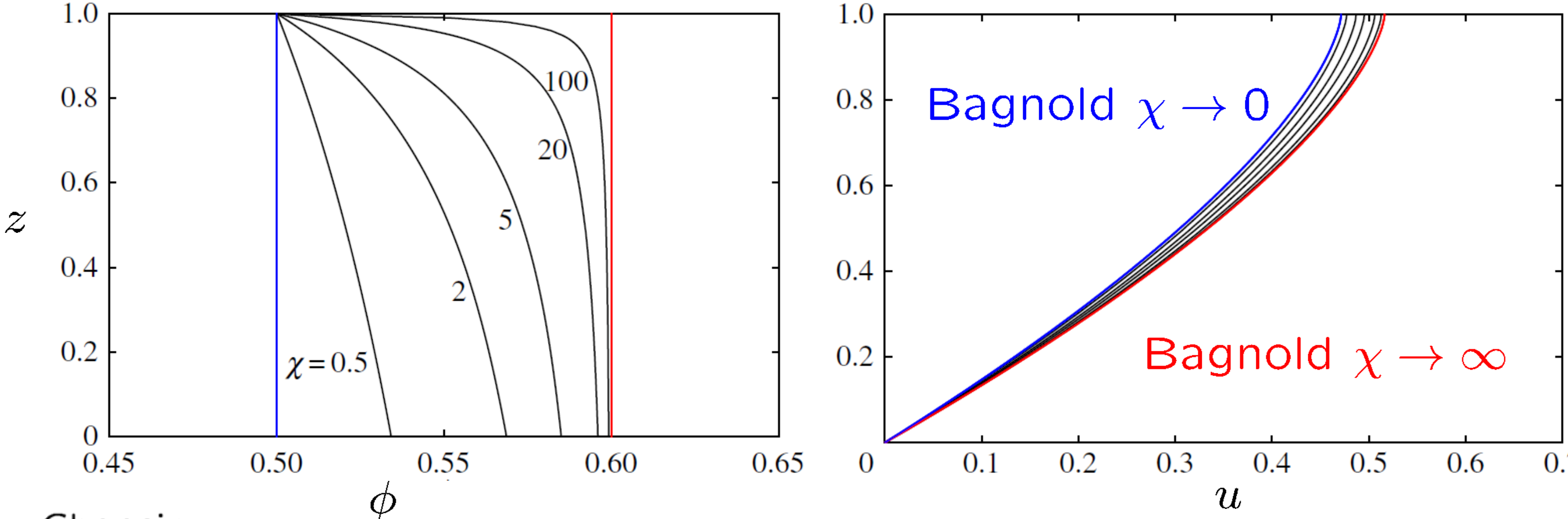

Choosing

$$\alpha(I) = \frac{4}{5}\mu(I) + \frac{12}{25} I^{-2/5} \int_0^I J^{-3/5}\mu(J)dJ, \quad \beta(I) = 2(\alpha(I) - \mu)$$

implies that for isochoric flow

$$\|\boldsymbol{\tau}\| = \mu(I)p$$

and steady-uniform CIDR flow is very close to Bagnold flow

But there are also non-local approaches (Bouzid *et al.* 2013, Kamrin & Henann 2015,) and higher gradient theories (Goddard 2017)