Force Chains in Granular Media

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Contact force measurements and stress-induced anisotropy in granular materials

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Interparticle forces in granular media form an inhomogeneous distribution of filamentary force chains. Understanding such forces and their spatial correlations, specifically in response to forces at the system boundaries[1,2], represents a fundamental goal of granular mechanics. The problem is of relevance to civil

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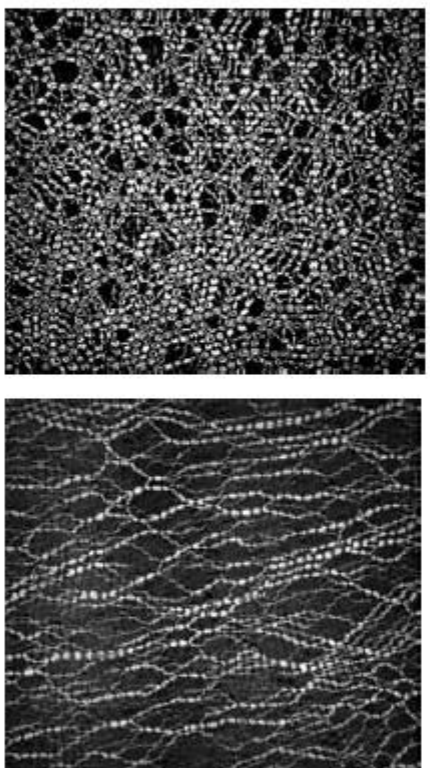


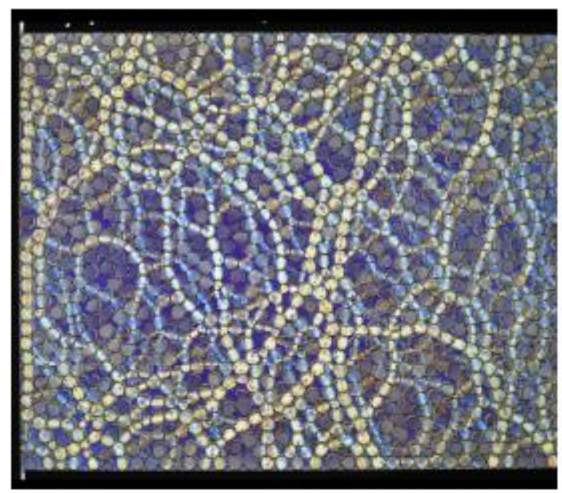


FIG. 1: Force chains in a binary system of plastic disks of two diameters, stressed uniaxially at the boundaries. The force chains are made visual by the optical birefringence of the stressed disks. The image is courtesy of V. Sathish Akella and Mahesh Bandi, Okinawa Institute of Science and Technology.

I will demonstrate the following:

- Given the external forces on the disks
- Given the directions connecting the centers of mass of the disks (but not the distance!)
- I can solve for all the forces, normal and tangential.
- I do not need to know the force law for the frictional forces.

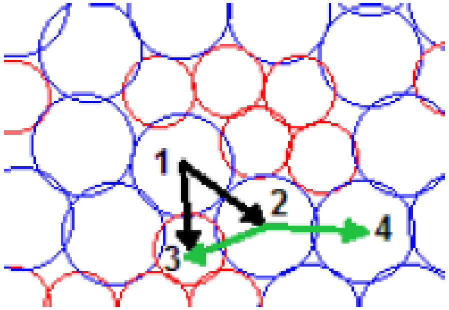
Why is the problem difficult?

If I have N disks and c contacts, there are N constraints on the c forces

$$\sum_j f_{i,j} + F_i^{\text{ext}} = 0 \ .$$

We can consider separately the force balance equation for the x-components and the y-components

$$M|f\rangle = -|F^{\text{ext}}\rangle \ ,$$



$$\begin{pmatrix} c = & 1 & 2 & \dots & 7 & 8 & \dots \\ & 1\text{with}2 & 1\text{with}3 & \dots & 2\text{with}3 & 2\text{with}4 & \dots \end{pmatrix}$$

$$M = \begin{pmatrix} -\hat{n}_{12}^{x} & -\hat{n}_{13}^{x} & ... & 0 & 0 & ...... & \hat{t}_{12}^{x} & \hat{t}_{13}^{x} & ... & 0 & 0 & ... \\ \hat{n}_{12}^{x} & 0 & ... & -\hat{n}_{23}^{x} & -\hat{n}_{24}^{x} & ...... & -\hat{t}_{12}^{x} & 0 & ... & \hat{t}_{23}^{x} & \hat{t}_{24}^{x} & ... \\ 0 & \hat{n}_{13}^{x} & ... & \hat{n}_{23}^{x} & 0 & ...... & 0 & -\hat{t}_{13}^{x} & ... & -\hat{t}_{23}^{x} & 0 & ... \\ 0 & 0 & ... & 0 & \hat{n}_{24}^{x} & ...... & 0 & 0 & ... & 0 & -\hat{t}_{24}^{x} & ... \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \\ -\hat{n}_{12}^{y} & -\hat{n}_{13}^{y} & ... & 0 & 0 & ...... & \hat{t}_{12}^{y} & \hat{t}_{13}^{y} & ... & 0 & 0 & ... \\ \hat{n}_{12}^{y} & 0 & ... & -\hat{n}_{23}^{y} & -\hat{n}_{24}^{y} & ...... & -\hat{t}_{12}^{y} & 0 & ... & \hat{t}_{23}^{y} & \hat{t}_{24}^{y} & ... \\ 0 & \hat{n}_{13}^{y} & ... & \hat{n}_{23}^{y} & 0 & ...... & 0 & -\hat{t}_{13}^{y} & ... & -\hat{t}_{23}^{y} & 0 & ... \\ 0 & 0 & ... & 0 & \hat{n}_{24}^{y} & ...... & 0 & 0 & ... & 0 & -\hat{t}_{24}^{y} & ... \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \end{pmatrix}$$

The matrix M has dimension 2Nx2c and it contains the “direction information”

The frictionless problem

Without friction the matrix M has dimension 2Nxc, and the system is isostatic when 2N=c

$$M^T M|f\rangle = -M^T|F^{\text{ext}}\rangle \ .$$

When the system is pressed c>>2N

$M^T M$ Has then c-2N zero modes and cannot be inverted!

Now you should say, but what about the torque balance?

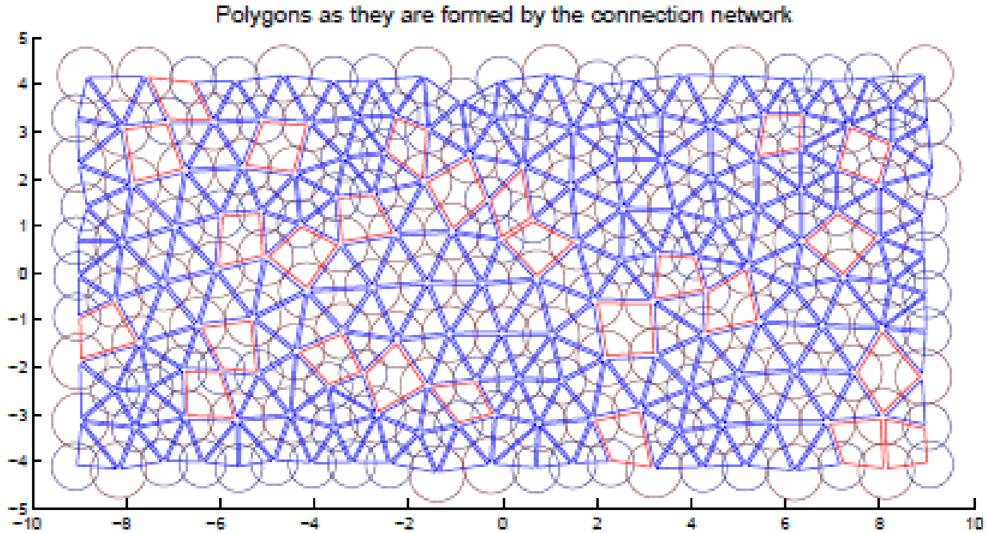
$$\sum_j r_i \times \bar{f}_{ij}^t + \Gamma_i^{\text{ext}} = 0$$

But now we have c new unknowns!

$$T = \begin{pmatrix} R_1 & R_1 & \dots & 0 & 0 & \dots \\ R_2 & 0 & \dots & R_2 & R_2 & \dots \\ 0 & R_3 & \dots & R_3 & 0 & \dots \\ 0 & 0 & \dots & 0 & R_4 & \dots \\ \vdots & \vdots & & \vdots & \vdots & \end{pmatrix}$$

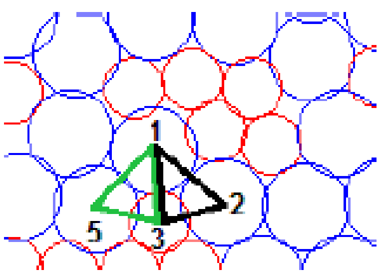
$$B|f\rangle \equiv \begin{pmatrix} & M & \\ 0 & & T \end{pmatrix} \left| \begin{pmatrix} f^n \\ f^t \end{pmatrix} \right\rangle = - \left| \begin{pmatrix} F_x^{\text{ext}} \\ F_y^{\text{ext}} \\ \Gamma^{\text{ext}} \end{pmatrix} \right\rangle .$$

The matrix B has dimension 3Nx2c and 2c>>3N



$$Q|r\rangle = 0 \ ,$$

The matrix Q has dimension 2Pxc



$$Q = \begin{pmatrix} \hat{n}_{12}{}^x & -\hat{n}_{13}{}^x & 0 & \dots & \hat{n}_{23}{}^x & \dots & 0 & \dots \\ 0 & \hat{n}_{13}{}^x & -\hat{n}_{15}{}^x & \dots & 0 & \dots & \hat{n}_{35}{}^x & \dots \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & \\ \hat{n}_{12}{}^y & -\hat{n}_{13}{}^y & 0 & \dots & \hat{n}_{23}{}^y & \dots & 0 & \dots \\ 0 & \hat{n}_{13}{}^y & -\hat{n}_{15}{}^y & \dots & 0 & \dots & \hat{n}_{35}{}^y & \dots \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & \end{pmatrix}$$

To proceed we need the normal force law. Assume Hookean.

$$f^n_{i,j} = \kappa[(\sigma_i + \sigma_j)\hat{n}_{ij}/2 - r_{ij}] \ .$$

$$\begin{aligned} Q|r\rangle &= Q|\sigma - f^n/\kappa\rangle = 0 \ , \\ Q|f^n\rangle &= Q|\kappa\sigma\rangle \ . \end{aligned}$$

$$G|f\rangle \equiv \begin{pmatrix} & M & \\ 0 & & T \\ Q & & 0 \end{pmatrix} \left| \begin{pmatrix} f^n \\ f^t \end{pmatrix} \right\rangle = \left| \begin{pmatrix} f_x^{\text{ext}} \\ f_y^{\text{ext}} \\ \Gamma^{\text{ext}} \\ Q|\kappa\sigma\rangle \end{pmatrix} \right\rangle$$

The dimension of the matrix G is (3N+2P)x2c

Are we home free or not?

Let us ask Euler:

$$N - c + (P + 1) = 2 \,, \quad \rightarrow 2c = 2N + 2P - 2 \ll 3N + 2P \,.$$

$$|f\rangle = (G^T G)^{-1} G^T |t\rangle = \sum_i \frac{\langle \Psi_i | G^T | t \rangle}{\lambda_i} |\Psi_i\rangle \,.$$

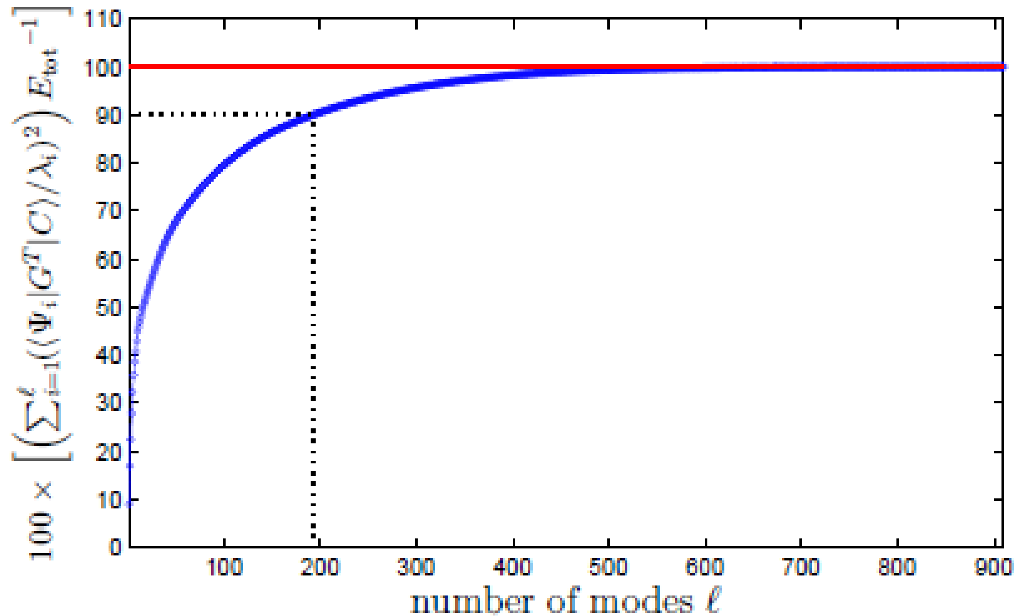
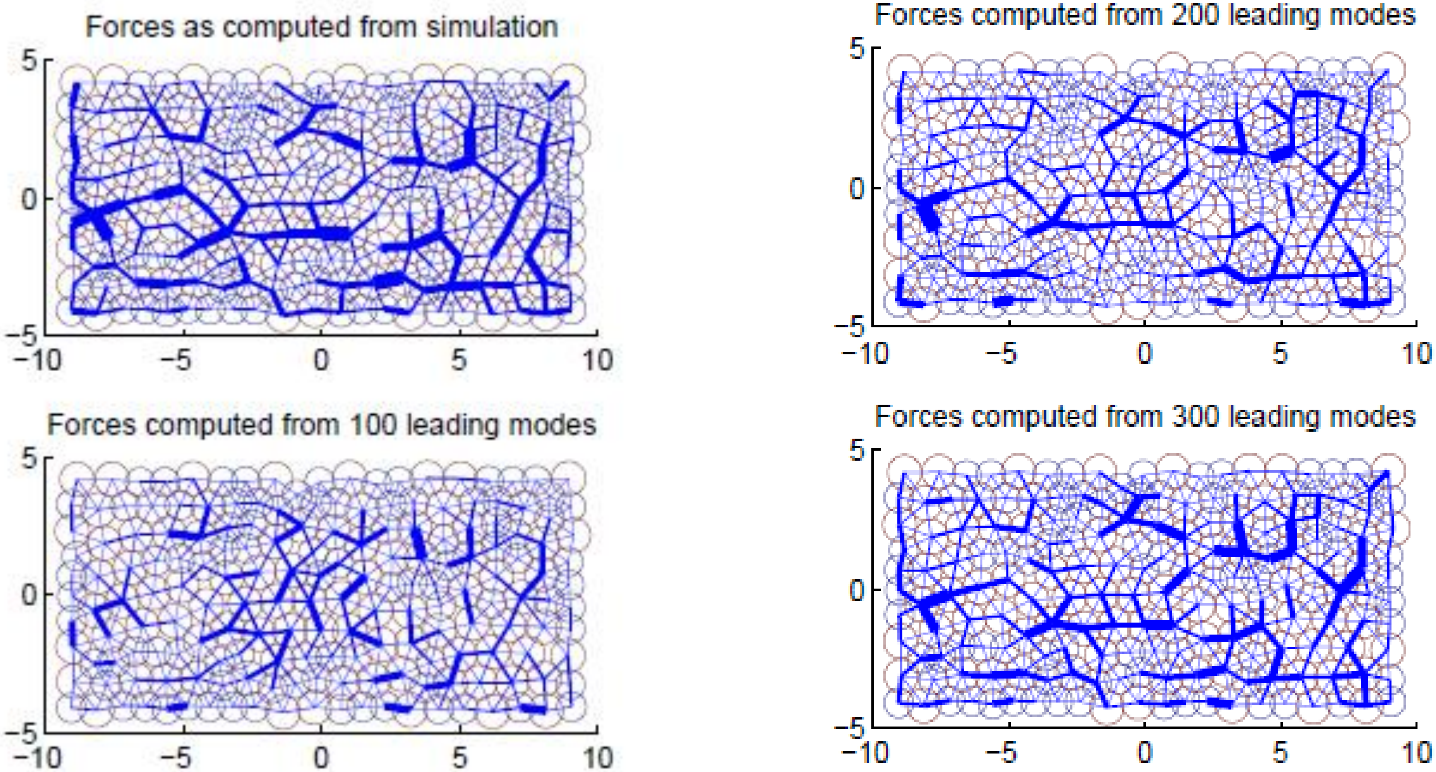


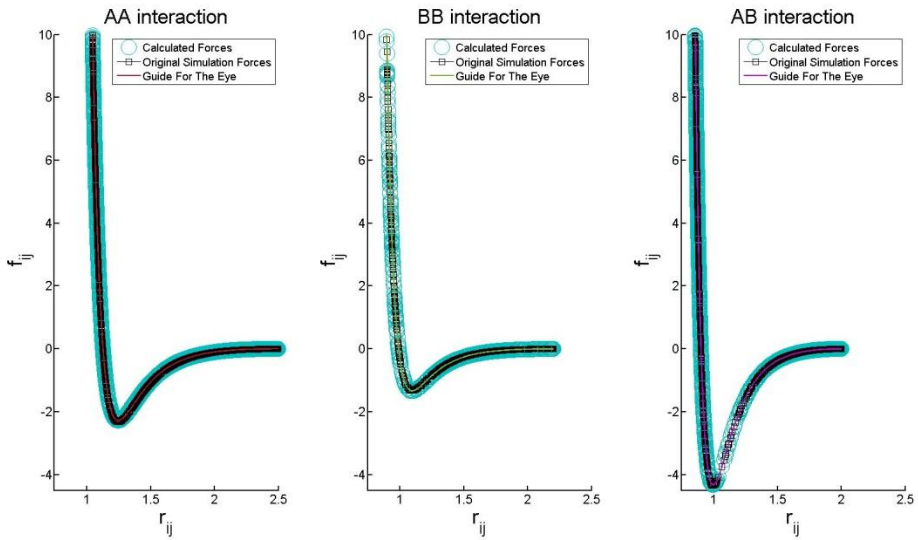
FIG. 3: The cumulative percentage contribution to the energy of the eigenfunctions Ψ_i of $G^T G$, ordered according to the magnitude of $\langle \Psi_i | G^T | t \rangle / \lambda_i$. The convergence is relatively fast with the first 192 leading eigenfunction (out of 910 modes) contributing 90% of the total energy.



Forces as computed from simulation
Forces computed from 200 leading modes
Forces computed from 100 leading modes
Forces computed from 300 leading modes
5
0
−5
−10
−5
0
5
10

What next?

- Generalize to 3-dimensions
- Compare to experiments
- Turn to a universal method
- Use to study the stability of force chains
- Use to invert to find the force law
- Etc. etc. etc.



Thank you!