Stress transmission and response in granular assemblies

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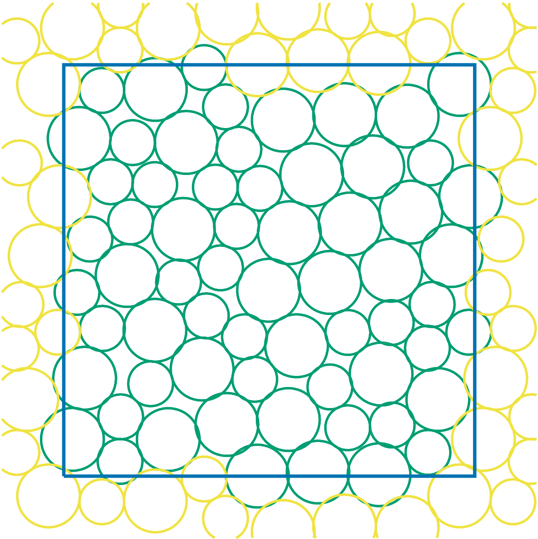
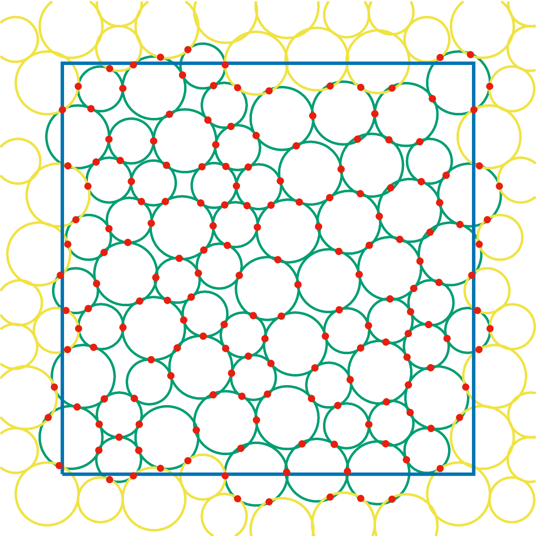
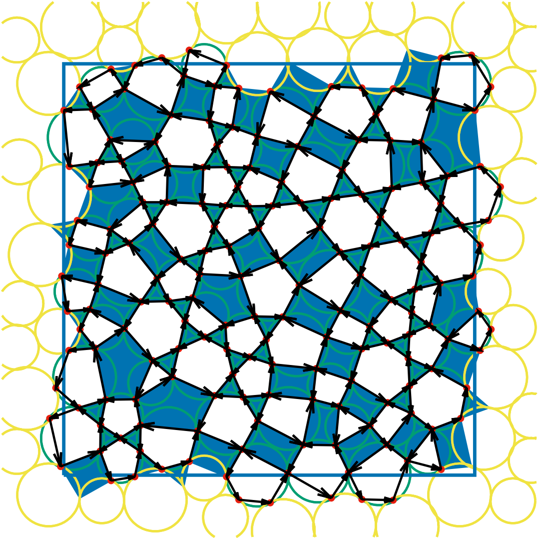
Jammed Packings



Jammed Packings: Contact Points



Jammed Packings: Grain Polygons and Void Polygons



Total Grain Area

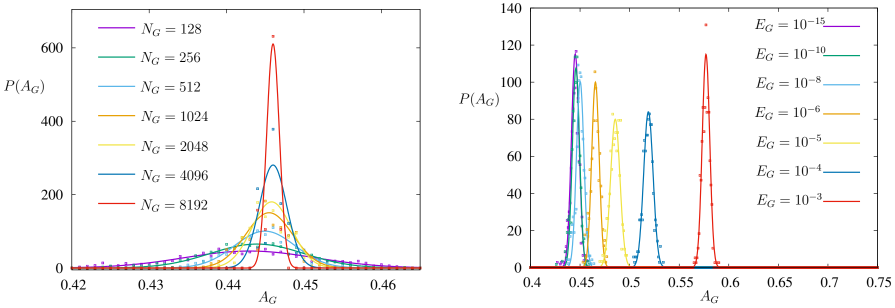


Figure: **(Left)** Distribution of the total area covered by the grain polygons A_G at $E_G = 10^{-15}$. Using finite-size scaling fits we find is $A_G^* = 0.446(1)$ as the number of grains $N_G \to \infty$ and $E_G \to 0^+$. **(Right)** Behaviour of the grain area distributions for different energies for packings of $N_G = 512$ disks.

Scaling with Energy

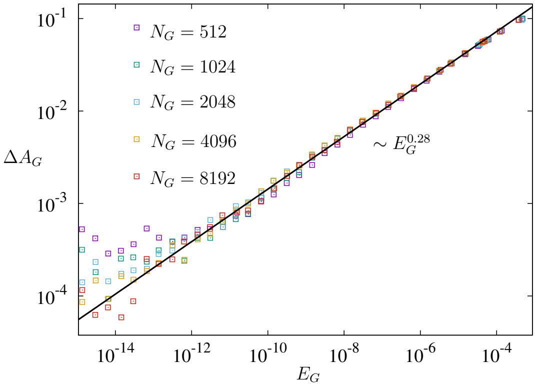


Figure: Scaling of the excess grain area $\Delta A_G = A_G - A_G^*$ with total energy per particle E_G. We find that the excess grain area scales as a power of the total energy in the system with exponent $\beta_E = 0.28(2)$.

Scaling with Coordination

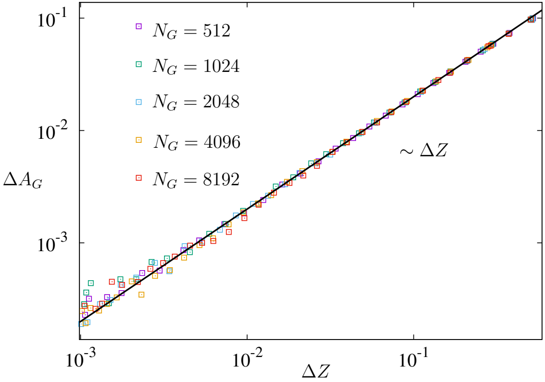


Figure: Scaling of ΔA_G with excess coordination in the system ΔZ. We find that the excess grain area scales as a power of ΔZ with exponent $\beta_Z = 1.00(1)$.

Stress Transmission in Granular Packings

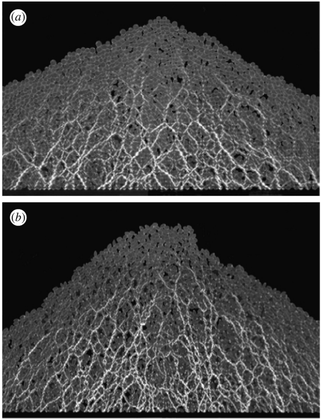


Figure: Inhomogeneous stress transmission in granular piles made with (a) disks and (b) elliptic cylinders. Ref: I. Zuriguel, T. Mullin, Proc. Royal Society A 464, 2089 (2008).

Stress Transmission in Granular Packings

- Depending on the underlying disorder, stress transmission can be either **wave-like** or **diffusive**. Ref: R. P. Behringer, "Forces in Static Packings.", *Handbook of Granular Materials* (CRC Press, NY, 2016).

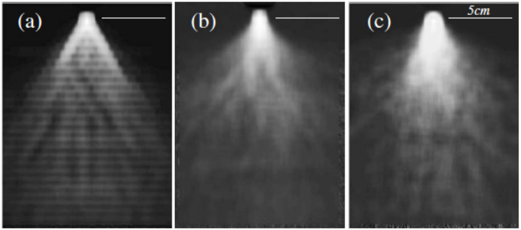


Figure: **Mean response** of a 50 g point force for (a) a uniform hexagonal packing of disks, (b) a bimodal packing of disks (c) pentagons. Ref: J. Geng, D. Howell, E. Longhi, R. P. Behringer, G. Reydellet, L. Vanel, E. Clément, and S. Luding, Phys. Rev. Lett. **87**, 035506 (2001).

C. H. Liu, S. R. Nagel, D. A. Schecter, S. N. Coppersmith, S. Majumdar, and T. A. Witten, Science 269, 513 (1995).

- Only the **vertical components of the forces** are considered.
- A **fraction** $q_{i,j}$ of the total weight $w(i, D)$ supported by the ith site in layer D, is **transmitted to particle** j **in layer** $D + 1$.

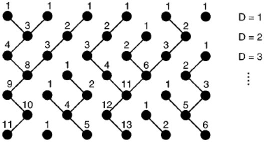


Figure: Schematic diagram showing the paths of weight support for a two-dimensional system in the $q_{0,1}$ limit where each site transmits its weight to exactly one neighbor below. The numbers at each site are the values of $w(i, D)$.

Models of Stress Transmission: The q-model

C. H. Liu, S. R. Nagel, D. A. Schecter, S. N. Coppersmith, S. Majumdar, and T. A. Witten, Science 269, 513 (1995).

- Force balance yields a **stochastic equation**

$$w(j, D+1) = 1 + \sum_i q_{i,j}(D) w(i, D).$$

- Steady state produces an **exponential distribution of forces**.

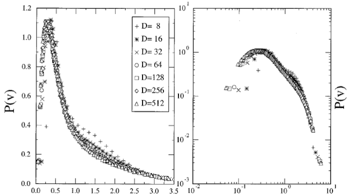


Figure: Linear-linear and log-log plots of the normalized weight distribution function $P_D(v)$ vs $v = w/D$.

Grains and Voids

- The two dimensional plane can be decomposed into regions belonging to **grains** and **voids**. These **two graphs are dual** to each other.

Ref: K. Ramola and B. Chakraborty, J. Stat. Mech. 114002 (2016).

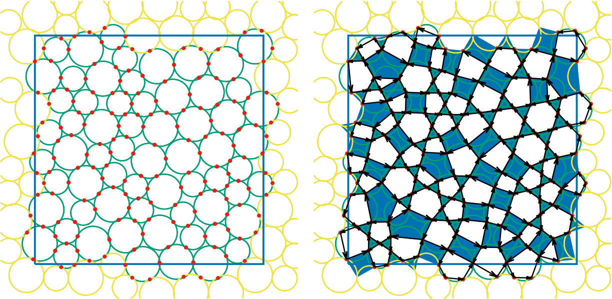


Figure: (**Left**) A jammed packing of bidispersed frictionless disks with periodic boundary conditions. (**Right**) The same configuration with the associated grain polygons (white) and void polygons (blue).

Stress Tensor and Continuum Descriptions

- The **stress tensor** for a given packing is defined as

$$\hat{\sigma} = \frac{1}{V} \sum_g \hat{\sigma}_g,$$
$$\hat{\sigma}_g = \sum_c \vec{r}_{g,c} \otimes \vec{f}_{g,c}.$$

where $\vec{r}_{g,c} = \vec{r}_c - \vec{r}_g$, with $\vec{r}_c$ being the position of the contact c, and $\vec{r}_g$ being the position of the grain g.

- The **continuum description** is

$$\nabla \cdot \hat{\sigma} = 0.$$

- In the **presence of external forces** we have

$$\nabla \cdot \hat{\sigma} = -\vec{f}_{\mathrm{ext}}.$$

Local Constraints in Granular Packings

- The **force balance constraint** for a given packing is

$$\sum_c \vec{f}_{g,c} = 0,$$

where $\vec{f}_{g,c}$ represents the force acting on the grain g, through the contact c.

- The **torque balance constraint** is

$$\sum_c \vec{r}_{g,c} \times \vec{f}_{g,c} = 0.$$

- The real space constraints can be parametrized as **loop constraints**

$$\sum \vec{r}_{g,g'} = 0,$$

where $\vec{r}_{g,g'} = \vec{r}_{g'} - \vec{r}_g$ is the inter-particle distance vector between two adjacent grains g and g'.

Height Fields

- **Mechanical equilibrium** ($\sum_c \vec{f}_{g,c} = 0$) leads to a **gauge representation** of the forces.

- The forces are given by the **difference of height variables**

$$\vec{f}_{g,c} = \vec{h}_{g,v} - \vec{h}_{g,v'}.$$

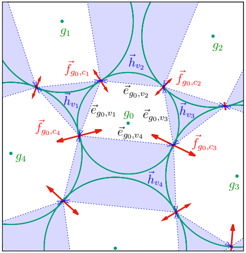
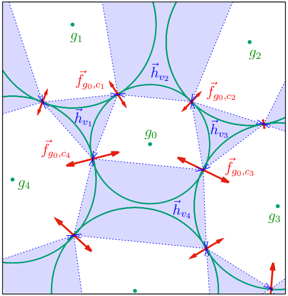


Figure: The height fields $\{\vec{h}\}$ are associated with the void polygons (shaded light blue). The forces are represented by (bidirectional) arrows.

Uniqueness of Heights

- Force balance **ensures the uniqueness** of heights.



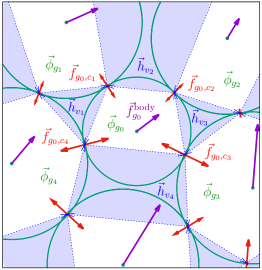
- For grain g_0 we have

$$\vec{f}_{g_0,c_1} = \vec{h}_{v_1} - \vec{h}_{v_2},$$
$$\vec{f}_{g_0,c_2} = \vec{h}_{v_2} - \vec{h}_{v_3},$$
$$\vec{f}_{g_0,c_3} = \vec{h}_{v_3} - \vec{h}_{v_4},$$
$$\underbrace{\vec{f}_{g_0,c_4}}_{0} = \underbrace{\vec{h}_{v_4} - \vec{h}_{v_1}}_{0}.$$

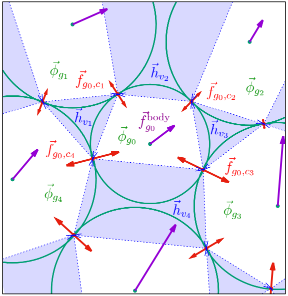
Generalization to Body Forces

- In the presence of **body forces** we have $\sum_c \vec{f}_{g,c} = -\vec{f}_g^{\mathrm{body}}$.
- We introduce **auxiliary fields on the grains** $\{\vec{\phi}_g\}$.
- The forces are given by the **difference of heights and** $\{\vec{\phi}\}$.

$$\vec{f}_{g,c} = \vec{h}_{v'} - \vec{h}_v + \vec{\phi}_{g'} - \vec{\phi}_g.$$



Generalization to Body Forces



- For grain g_0 we have

$$
\begin{aligned}
\vec{f}_{g_0,c_1} &= \vec{h}_{v_1} - \vec{h}_{v_2} + \vec{\phi}_{g_1} - \vec{\phi}_{g_0}, \\
\vec{f}_{g_0,c_2} &= \vec{h}_{v_2} - \vec{h}_{v_3} + \vec{\phi}_{g_2} - \vec{\phi}_{g_0}, \\
\vec{f}_{g_0,c_3} &= \vec{h}_{v_3} - \vec{h}_{v_4} + \vec{\phi}_{g_3} - \vec{\phi}_{g_0}, \\
\underbrace{\vec{f}_{g_0,c_4}}_{-\vec{f}^{\text{body}}_{g_0}} &= \underbrace{\vec{h}_{v_4} - \vec{h}_{v_1}}_{0} + \underbrace{\vec{\phi}_{g_4} - \vec{\phi}_{g_0}}_{\Box^2 \vec{\phi}_{g_0}}.
\end{aligned}
$$

- This is simply the **network laplacian** defined as

$$\Box^2 \vec{\phi}_{g_0} = \vec{\phi}_{g_1} + \vec{\phi}_{g_2} + \vec{\phi}_{g_3} + \vec{\phi}_{g_4} - 4\vec{\phi}_{g_0}.$$

Generalization to Body Forces (cont.)

- This is **valid for every grain**.
- We can represent this in vectorial notation as the **basic equation**

$$\Box^2|\vec{\phi}\rangle = -|\vec{f}^{\text{body}}\rangle.$$

- We can **invert this equation** to obtain the auxilliary fields $\{\vec{\phi}_g\}$.
- Given a set of body forces $\{\vec{f}_g^{\text{body}}\}$ and the contact network, the solution $\{\vec{\phi}_g\}$ is **unique**.

Properties of the Network Laplacian

- The network Laplacian is a $N_G \times N_G$ **real symmetric matrix**.
- $\Box^2$ has the eigenfunction expansion

$$\Box^2 = \sum_{i=1}^{N_G} \lambda_i |\lambda_i\rangle\langle\lambda_i|.$$

- $\Box^2$ has **one** zero eigenvalue, with eigenvector

$$\lambda_1 = 0, \quad |\lambda_1\rangle = (111...1).$$

- The rest of the eigenvalues are **all negative**.

Inverting the Body Forces

- We therefore have

$$\underbrace{\left(\sum_{i>1} \frac{1}{\lambda_i} |\lambda_i\rangle\langle\lambda_i|\right)}_{(\Box^2)^{-1}} \Box^2 = \mathbb{I} - |\lambda_1\rangle\langle\lambda_1|.$$

- Using this we have the inversion

$$\begin{aligned} -(\Box^2)^{-1}|\vec{f}^{body}\rangle &= |\vec{\phi}\rangle - |\lambda_1\rangle\langle\lambda_1|\vec{\phi}\rangle \\ &= |\vec{\phi} - \frac{1}{N}\sum_{i=1}^{N} \vec{\phi} \ \ \rangle. \end{aligned}$$

Response to a Body Force: Frictionless Systems

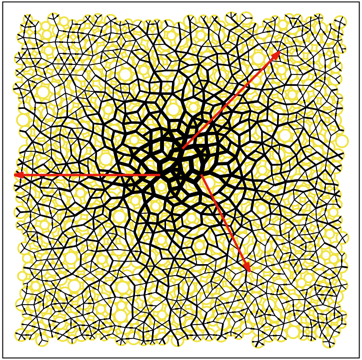


Figure: The response of a system of soft disks to applied body forces (represented by red arrows). The inhomogeneous nature of the stress response is clearly illustrated.

Response to a Body Force: Frictional Systems

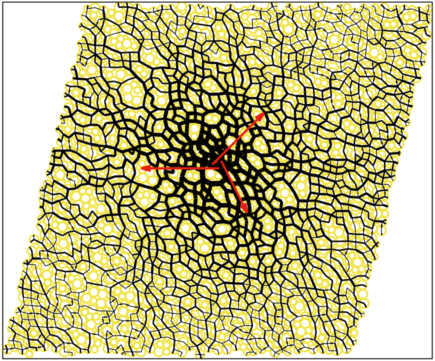


Figure: The response of a sheared system of soft frictional disks to applied body forces (represented by red arrows) with Lees-Edwards boundary conditions at global shear $\gamma = 0.43$. The response provides characteristic signatures of the emergence of "force chains" along the compressive direction.

Response to a Body Force

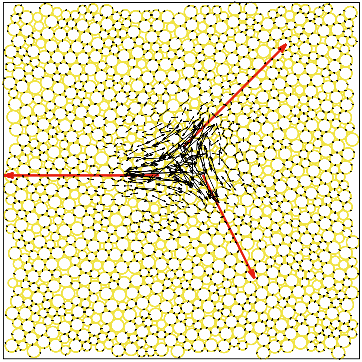


Figure: The response of a system of soft grains to applied body forces. The black arrows represent the changes in the contact force vectors in response to the imposed body forces (red arrows).

Response to a Body Force: Eigenvalue Expansion

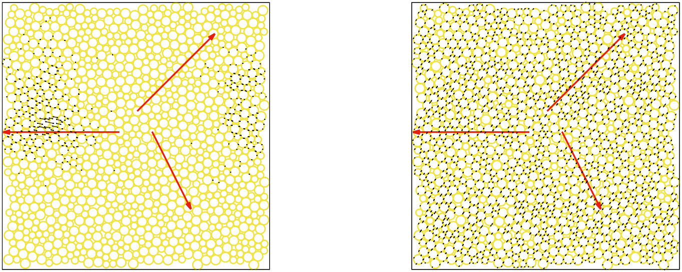


Figure: The stress response of the system (**left**) using only the largest negative eigenvector of the Laplacian matrix, illustrating a localized response, and (**right**) using only the smallest negative eigenvector of the Laplacian matrix, illustrating a delocalized response.

Density of States

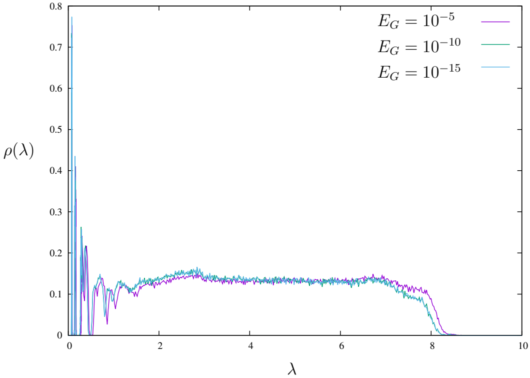


Figure: The density of states $\rho(\lambda)$ of the eigenvalues λ of the Laplacian matrix, for $N_G = 1024$ grains at different global energies (E_G). The data is averaged over 5000 configurations.

Measures of Localization: Inverse Participation Ratio

- The eigenvalues of the Laplacian $\lambda_i, i = 1, ..., N_G$ and corresponding normalized eigenvectors $\lambda \equiv \{e_{1,\lambda}, e_{1,\lambda}, ..., e_{N_G,\lambda}\}$.
- The **Inverse Participation Ratio** (IPR) corresponding to the eigenvector is defined as

$$q^{-1}(\lambda) = \sum_j e_{j,\lambda}^4$$

- For a **localized mode** the IPR would be of $O(1)$
- For a **delocalized mode** this quantity would be of $O(1/N_G)$.

Measures of Localization: Inverse Participation Ratio

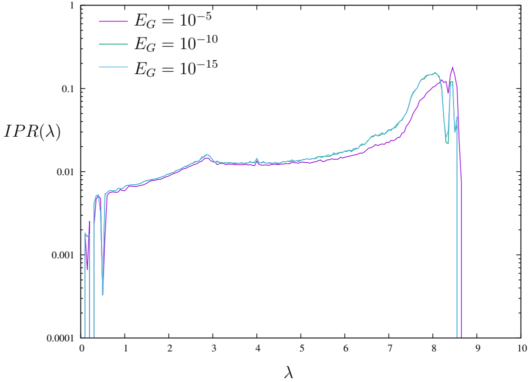


Figure: The inverse participation ratio (IPR) of the Laplacian eigenvectors, for $N_G = 1024$ grains at different global energies (E_G). The low modes are delocalized whereas a large part of the spectrum is localized. The data is averaged over 5000 configurations.

Stability of Networks

- Although force balance is satisfied at the grain level, other constraints such as the **Coulomb constraint** ($|f|_T \leq \mu|f|_N$) and **torque balance** would constrain the solutions.
- The network is stable to perturbations as long as **all the local constraints are respected**.
- Once the solutions fall outside these bounds, the **network must necessarily rearrange**.
- One can always find a torque balanced solution as long as **perturbation is small enough**.
- This construction therefore accurately describes systems in the **infinitely rigid limit**.

Thank You.