Kavli Institute for Theoretical Physics
University of California, Santa Barbara

Designer Quantum Systems Out of Equilibrium
Nov 14, 2016 – Nov 18, 2016

Design and Characterization of Topological Boundary Modes: from Floquet engineering to a generalized Bloch Ansatz

Lorenza Viola
Dept. Physics & Astronomy
Dartmouth College

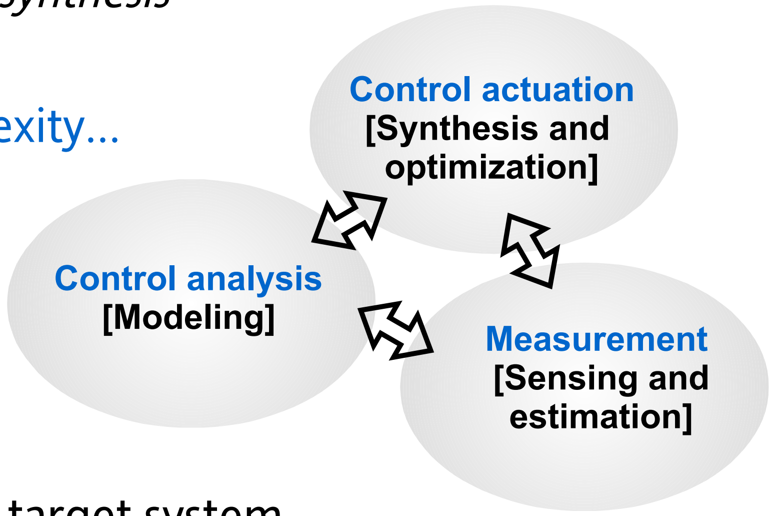
Broad context

Many-body quantum physics meets [quantum] control engineering...

- Explore new possibilities [and limitations] of control methodology...
  - Identify dynamical model for target system ⇒ *Control analysis*
  - Design controller in order to modify dynamics ⇒ *Control synthesis*
  - Validate performance ⇒ *Optimization*

  ...in concert with the whole gamut of many-body complexity...
  - Highly entangled quantum states
  - Competing interactions
  - Non-conventional [topological] orders

  ...to uncover and realize new Physics...



- Out-of-equilibrium phenomena entail coupling between target system and external 'controller' or 'environment' – [some] pathways:

  ➞ Switched Hamiltonian dynamics: *Quantum quenches*

  ➞ Time-dependent Hamiltonian dynamics: *Coherently driven* quantum systems

  ➞ Open-quantum system dynamics: *Uncontrolled* and *controlled dissipation*

  ⋮

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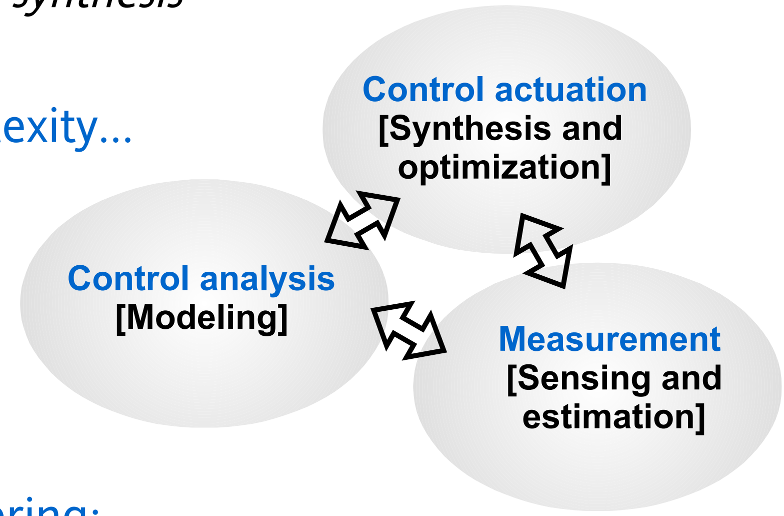


- Dissipative [Kraus or Lindblad] quantum control engineering:

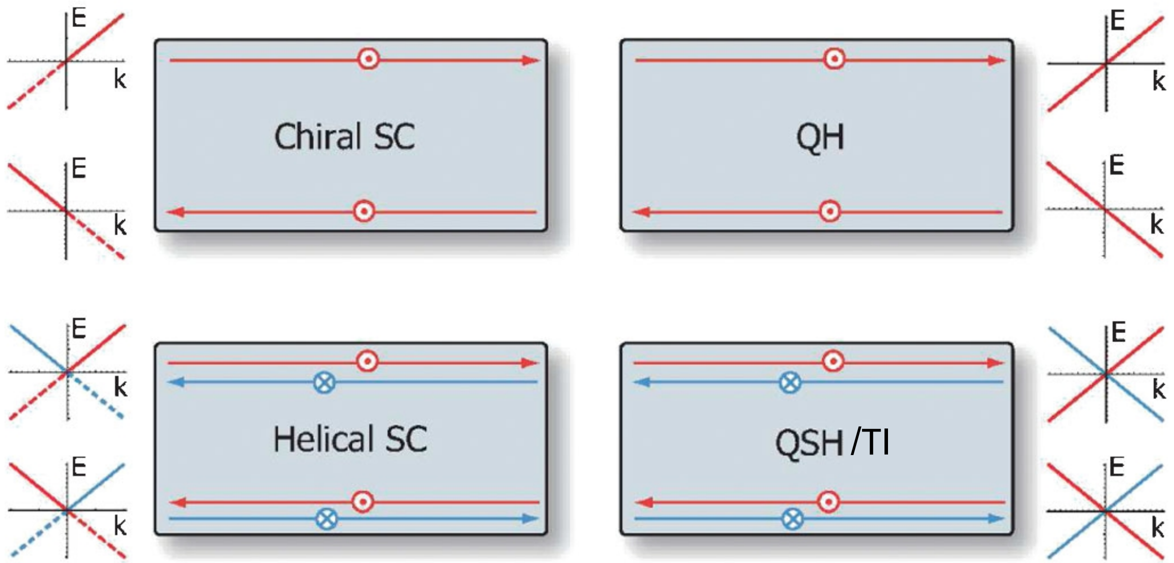
LV & Lloyd, PRA 65 (2001).

➞ Leverage *engineered dissipation* towards tasks not (or not robustly) achievable by unitary control alone e.g., open-system simulators, fixed-point tuning, steady-state phase transitions...

P.D. Johnson, F. Ticozzi & LV, *General fixed points of quasi-local frustration-free quantum semigroups: from invariance to stabilization*, QIC 16, 0657 (2016).

Focus: Topological fermionic matter

Topological insulators/superconductors are gapped phases of fermionic matter which support *'symmetry protected' mid-gap states localized on the boundary.*



Broken TR invariance ⇒ *Chiral* boundary states

TR invariance preserved ⇒ *Helical* boundary states

$$\gamma(\epsilon) = \gamma^\dagger(-\epsilon) \Rightarrow \gamma(0) = \gamma^\dagger(0)$$

This talk: Closed-system, Hamiltonian dynamics of non-interacting fermionic matter

I. Time-translation symmetry – Floquet engineering of Majorana *flat bands* in *s*-wave TSs...

II. Space-translation symmetry *up to boundaries* – Generalizing Bloch theorem, witnessing the bulk-boundary correspondence, and all that...

Part I:
Floquet engineering of topological boundary modes [non-equilibrium Majorana flat bands]

Shusa Deng, Gerardo Ortiz, Amrit Poudel & LV
Majorana flat bands in s-wave gapless topological superconductors
Phys. Rev. B 89, 140507(R) (2014).

Amrit Poudel, Gerardo Ortiz & LV
Dynamical generation of Floquet Majorana flat bands in s-wave superconductors ⇦
EPL 110, 17004 (2015).

Why Majorana flat bands?...

Gapless *s*-wave superconductors* provide a different route to topological superconductivity ⇒ Emergence of protected boundary *Majorana flat bands* (MFB).

*Abrikosov & Gor'kov, Sov. Phys. JEPT 12 (1961).

- Case study: Two-band, TR-invariant [mean-field] model on square lattice

$$H_0 = \frac{1}{2}\sum_{\mathbf{k}} \psi^\dagger_{\mathbf{k}} \hat{H}_0(\mathbf{k})\psi_{\mathbf{k}} \qquad \psi^\dagger_{\mathbf{k}} \equiv (c^\dagger_{\mathbf{k},\uparrow}, c^\dagger_{\mathbf{k},\downarrow}, d^\dagger_{\mathbf{k},\uparrow}, d^\dagger_{\mathbf{k},\downarrow}, c_{-\mathbf{k},\uparrow}, c_{-\mathbf{k},\downarrow}, d_{-\mathbf{k},\uparrow}, d_{-\mathbf{k},\downarrow})$$

$$\hat{H}_0(\mathbf{k}) = s_z(m_{\mathbf{k}}\tau_z - \mu) + \tau_x(\lambda_{k_x}\sigma_x + \lambda_{k_z}\sigma_z) - \Delta\, s_x\tau_y\sigma_x$$

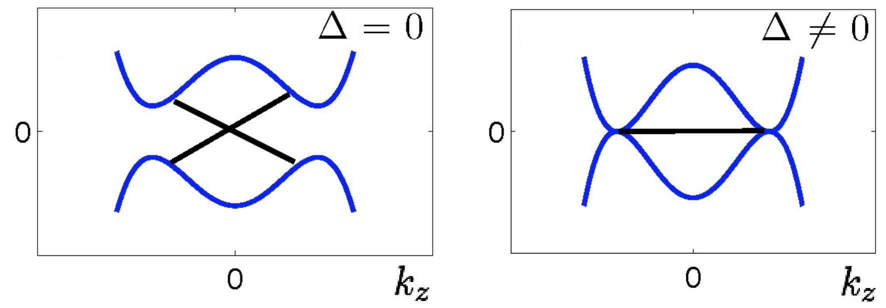
On-site potential + intra-band pairing | Spin-orbit inter-band interaction | Inter-band [spin-triplet] *s*-wave pairing

$$m_{\mathbf{k}} \equiv u_{cd} - 2\mathrm{w}(\cos k_x + \cos k_z) \quad \lambda_{\mathbf{k}} \equiv (\lambda_{k_x}, \lambda_{k_z}) = -2\lambda(\sin k_x, \sin k_z)$$

➡ The bulk excitation spectrum can *close* at [a *finite* set of] special momentum values, e.g.

$$\mu = 0: \ (k_x, k_z) \equiv (k_{x,c}, \pm k_*), \ k_{x,c} \in \{0, \pi\}$$

➡ A *continuum* of zero-energy Majorana modes may emerge in the thermodynamic limit.



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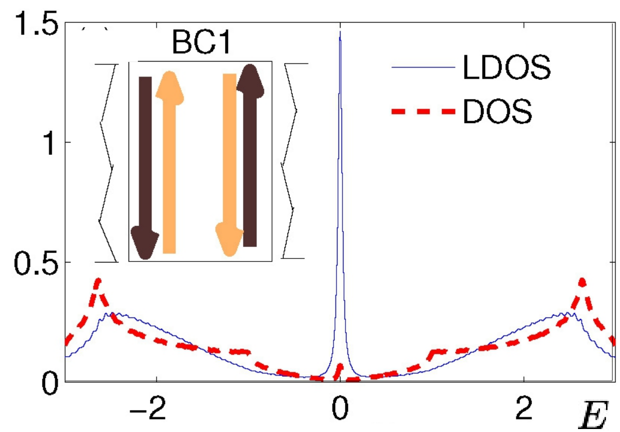
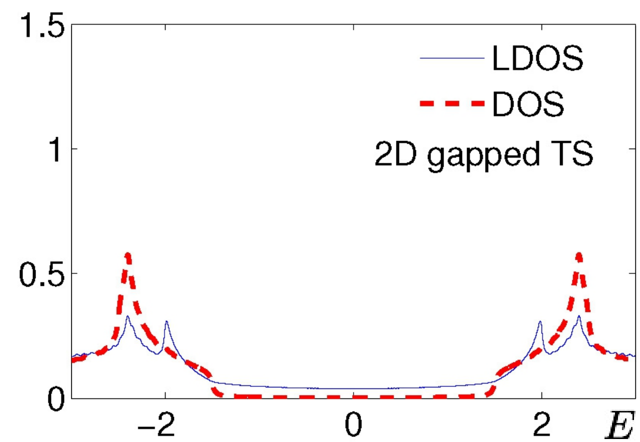
On-site potential + intra-band pairing — Spin-orbit inter-band interaction — Inter-band [spin-triplet] *s*-wave pairing

➞ Conceptual significance: Anomalous bulk-boundary correspondence – Emergence of MFB depends on *how* boundary conditions are imposed [for *same* bulk].

➞ Experimental significance: A MFB implies a *large* peak in the LDOS at the surface...





Control framework and Floquet formalism

- Objective: Use external *periodic* control to engineer *non-equilibrium MFB* in *s*-wave superconductors where they do not exist at equilibrium ⇒ Floquet MFB

$$H(t) = H_0 + H_c(t), \quad U_c(t) \equiv \mathcal{T} \exp\left\{-i\int_0^t H_c(t')dt'\right\} = U_c(t+T_c)$$

Necessary symmetry requirement: Design time-independent *effective Hamiltonian* H_{eff} such that appropriate *chiral symmetry* is in place

$$[H_{\text{eff}}, \mathcal{K}]_+ = 0 \Leftrightarrow [\hat{H}_{\text{eff}}(\mathbf{k}), U_{\mathcal{K}}]_+ = 0$$

- Floquet formalism leverages *translational invariance in time* to obtain exact Ansatz for time-dependent basis states:

$$\Psi_\alpha(t) \equiv e^{-i\varepsilon_\alpha t}\Phi_\alpha(t), \quad \Phi_\alpha(t) = \Phi_\alpha(t+nT_c),\ n \in \mathbb{Z}$$

Floquet quasi-energies [Time-periodic] Floquet eigenstates

➡ Map to a formally time-independent problem on extended space $\mathcal{H}_F \equiv \mathcal{H} \otimes \mathcal{F}$,

$$\left[H(t) - i\frac{\partial}{\partial t}\right]\Phi_\alpha(t) = \varepsilon_\alpha \Phi_\alpha(t) \equiv H^{(F)}\Phi_\alpha(t), \quad \mathcal{F} \equiv \text{span}\{e^{i\omega n t}\},\ \omega = 2\pi/T_c$$

➡ Restriction to first Brillouin zone yields *physical* effective Hamiltonian:

$$H_{\text{eff}} = H^{(F)}|_{n=0} \equiv H_F$$

Floquet quasi-energy spectrum

- Effective Hamiltonian gives exact description of *stroboscopic* time-evolution under $H(t)$

$$U(t_M) = e^{-iH_{\text{eff}}t_M} = [e^{-iH_F T_c}]^M \equiv [U(T_c)]^M$$

➞ If external control is spatially homogeneous, momentum is conserved [under PBC] ⇒

$$H(t) = \frac{1}{2}\sum_{\mathbf{k}} \psi^\dagger_{\mathbf{k}}[\hat{H}_0(\mathbf{k}) + \hat{H}_c(\mathbf{k},t)]\psi_{\mathbf{k}} \;\Rightarrow\; H_F = \frac{1}{2}\sum_{\mathbf{k}} \psi^\dagger_{\mathbf{k}}\hat{H}_F(\mathbf{k})\psi_{\mathbf{k}}$$

- Topological features are encoded in the Floquet quasi-energy spectrum $\{\varepsilon_0(\mathbf{k})\}$

➞ Depending on the applied control protocol, spectrum may be determined numerically from [block-]diagonalization of $\hat{H}^{(F)}(\mathbf{k})$ or from direct diagonalization of the *Floquet propagator*

$$\hat{U}_F(\mathbf{k}) \equiv \hat{U}(\mathbf{k},T_c) = \mathcal{T}\exp[-i\int_0^{T_c}(\hat{H}_0(\mathbf{k}) + \hat{H}_c(\mathbf{k},t'))dt']$$

$$\hat{U}_F(\mathbf{k})\Phi_\alpha(\mathbf{k},T_c) = e^{-i\varepsilon_\alpha(\mathbf{k})\,T_c}\Phi_\alpha(\mathbf{k},T_c)$$

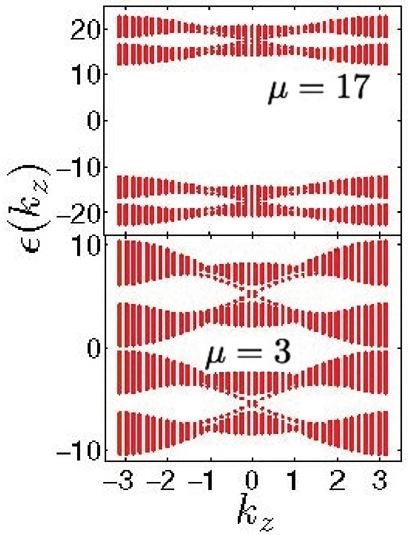
➞ The necessary symmetry requirement for H_{eff} may be met in two different ways:

(1) Use control to 'activate' a desired chiral symmetry already present at equilibrium... ☜

(2) Use control to 'generate' a desired chiral symmetry that is broken at equilibrium...

MFB via dynamical chiral-symmetry activation

- Target system: *s*-wave gapless spin-triplet SC in a *topologically trivial phase.*
  - ➞ Periodic modulation of chemical potential:
  $$\hat{H}_c(\mathbf{k},t) = \mu_d \cos(\omega t)\, s_z \otimes I \otimes I$$
  - ➞ Can show that chiral symmetry is obeyed,
  $$[\hat{H}_0(\mathbf{k}), U_{\mathcal{K}}]_+ = 0 = [\hat{H}_F(\mathbf{k}), U_{\mathcal{K}}]_+ = 0$$
  and the *instantaneous Hamiltonian remains in a topologically trivial phase throughout...*



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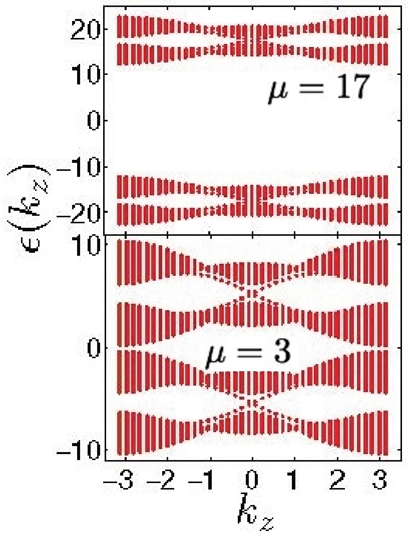

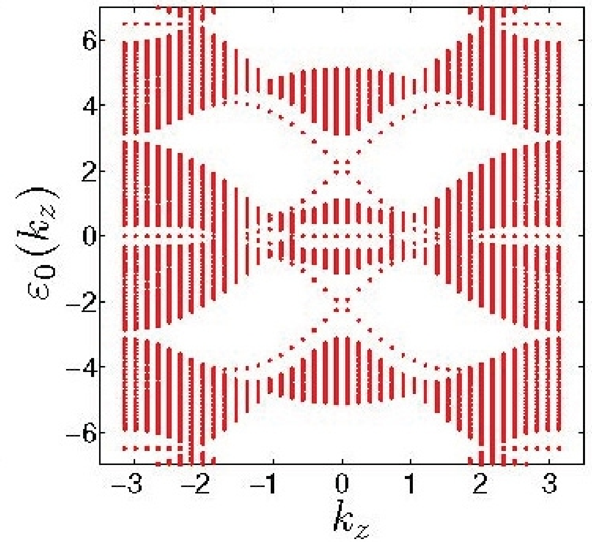

- Floquet MFB emerge at zero energy as well as non-zero, driving-dependent energies – *no* equilibrium counterpart.

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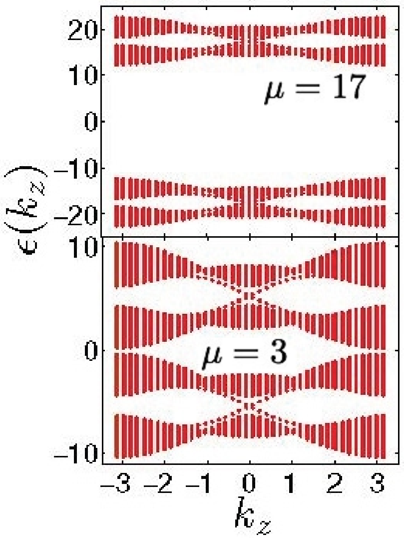
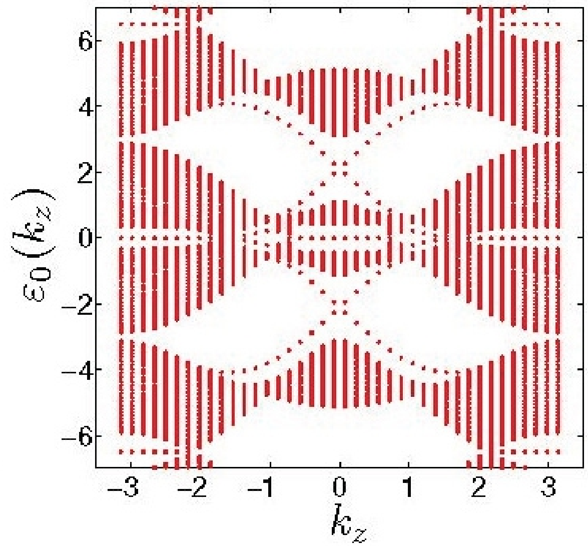


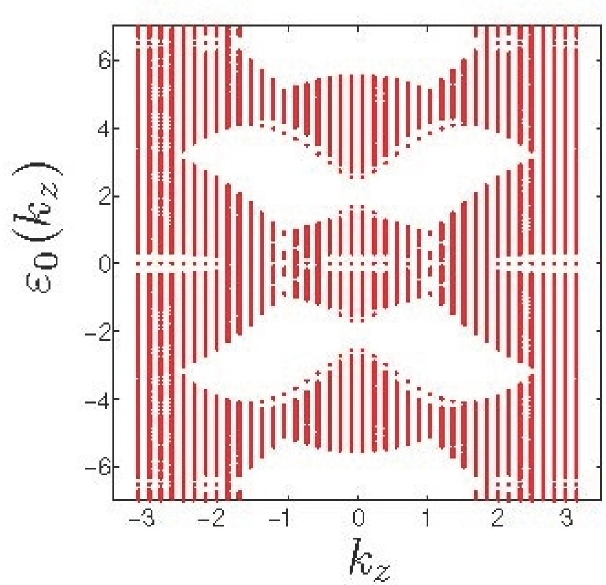


- Floquet MFB emerge at zero energy as well as non-zero, driving-dependent energies – *no* equilibrium counterpart.

  ➞ All MFB are *robust* against external perturbations that do not break the activated chiral symmetry – e.g., in-plane (x, z) magnetic field:

  $$\hat{H}_{F,\mathrm{tot}} = \hat{H}_F(\mathbf{k}) + \hat{H}_{\mathrm{pert}}(\mathbf{k})$$

  $$\hat{H}_{\mathrm{pert}}(\mathbf{k}) = \hat{H}_x(\mathbf{k}) = h_x\, s_z \otimes I \otimes \sigma_x$$



MFB via dynamical chiral-symmetry activation

- Target system: *s*-wave gapless spin-triplet SC in a *topologically trivial phase.*

  ➞ Periodic in-plane magnetic field:

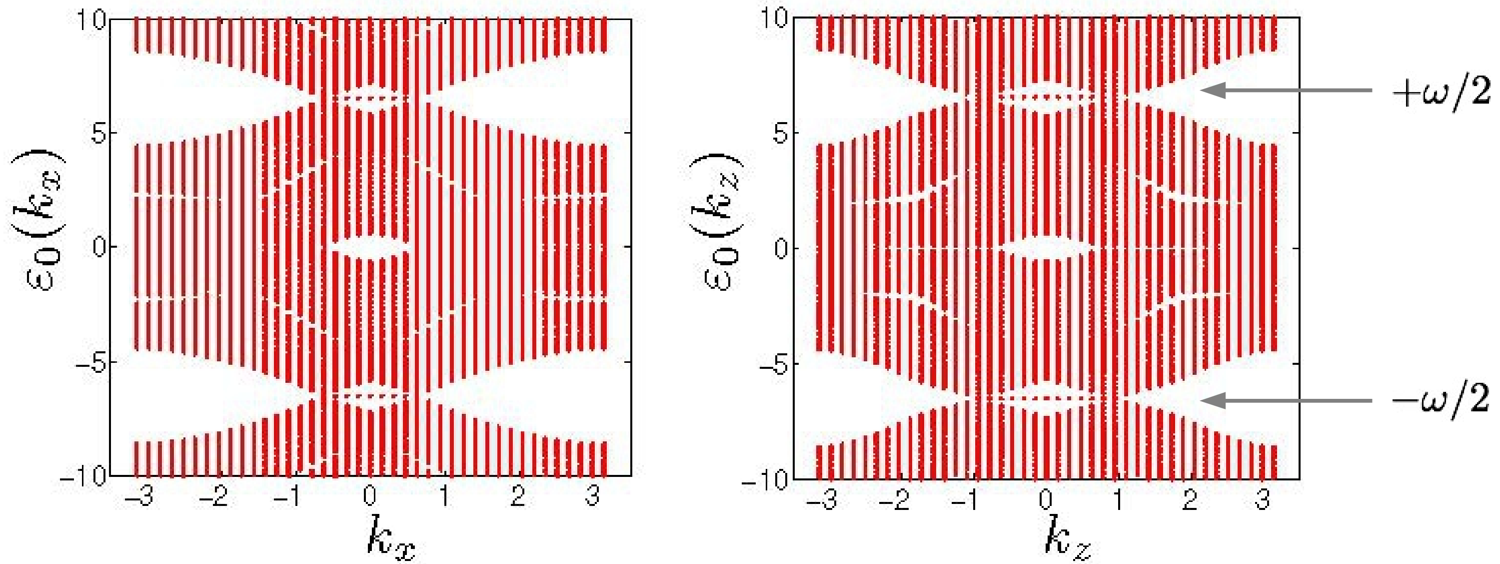
  $$\hat{H}_c(\mathbf{k},t) = [h_{x_0} + h_x \cos(\omega t)]\, s_x \otimes I \otimes \sigma_z$$

  ➞ Can show that chiral symmetry is preserved at all times.

- Unlike the equilibrium case [or when chemical potential is modulated], Floquet MFB may emerge independently of the choice of OBC vs. PBC – albeit only at *non-zero energies*...

  ➞ 'Standard' bulk-boundary correspondence is restored.

  ➞ Only [known] example of *s-wave* topological SC hosting MFB *along both boundaries*!



MFB via dynamical chiral-symmetry generation

- Target system: *s*-wave gapped spin-singlet SC in *a topologically non-trivial phase, but hosting only one Majorana pair per boundary.*

$$\hat{H}_0(\mathbf{k}) = s_z(m_{\mathbf{k}}\tau_z - \mu) + \tau_x(\lambda_{k_x}\sigma_x + \lambda_{k_z}\sigma_z) - \Delta\, s_y\tau_x\sigma_y$$

On-site potential + intra-band pairing | Spin-orbit inter-band interaction | Inter-band [spin-singlet] *s*-wave pairing

→ At equilibrium, MFB may exist if *z*-component of SO coupling vanishes, $\lambda_{k_z} = \lambda_z \sin k_z \equiv 0$.
Any *non-zero* λ_z causes the relevant chiral symmetry to be broken...

- Strategy: Suppress unwanted S-O contribution via repeated sign-flips [dynamical-decoupling]

Viola & Lloyd, PRA 58 (1998); Viola, Knill & Lloyd, PRL 85 (2000).

→ Design a 'parity-kick' operator that selectively maps $\hat{H}_0(\mathbf{k}, +\lambda_z) \equiv H_+$ to $\hat{H}_0(\mathbf{k}, -\lambda_z) \equiv H_-$:

$$\mathcal{P}^{-1} H_+ \mathcal{P} = H_-, \quad \mathcal{P}^2 = I \qquad \mathcal{P} \equiv e^{-i(\pi/2)H_{\rm kick}}, \; H_c(t) = H_{\rm kick} \sum_{p=1}^{\infty} \delta(t - pT_c/2)$$

→ Single-cycle controlled propagator:

$$U(T_c) = \mathcal{P}\, U_+(T_c/2)\, \mathcal{P}\, U_+(T_c/2) = e^{-iH_- T_c/2} e^{-iH_+ T_c/2} \equiv e^{-iH_{\rm eff}T_c}$$

MFB via dynamical chiral-symmetry generation

- The effective Hamiltonian may be computed [perturbatively] via Magnus expansion:

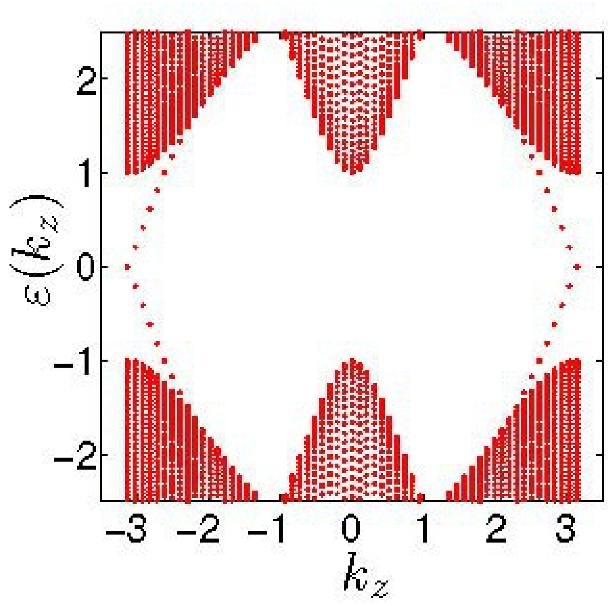
$$H_{\text{eff}} \approx \overline{H}^{(2)} = \frac{1}{2}(H_- + H_+) - \frac{iT_c}{8}[H_-, H_+] - \frac{T_c^2}{96}\Big([H_-, [H_-, H_+]] + [[H_-, H_+], H_+]\Big) \dots$$

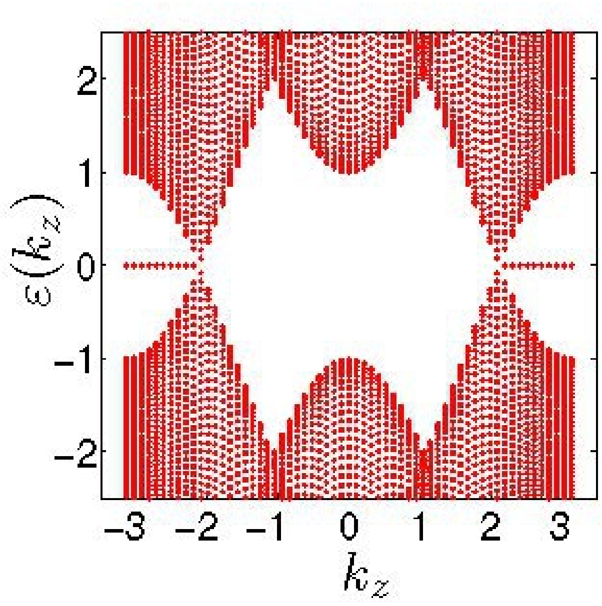
➞ Sufficient convergence condition: $2\lambda_z t_M \lesssim \pi$.

- Zero-energy MFB emerge from equilibrium Majorana pairs in the presence of periodic kicks – under the appropriate boundary condition.

➞ Can show that effective Hamiltonian preserves chiral symmetry *up to arbitrary order*.

➞ Control dynamically generates *at once* Floquet MFB and their protecting symmetry...





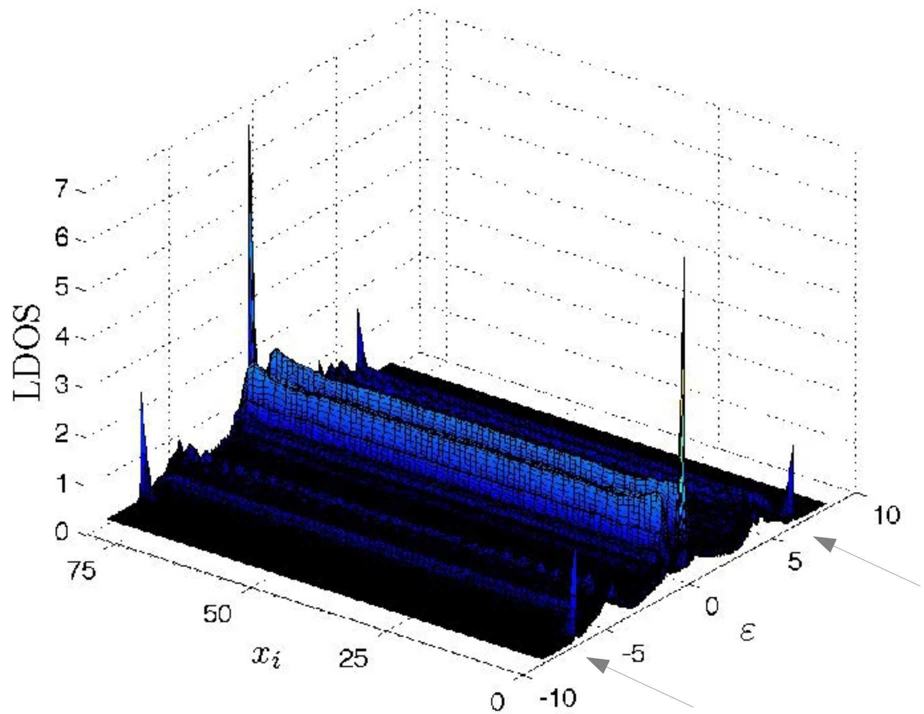
Summary – thus far…

- Driven quantum matter may access a broader range of possibilities – including non-equilibrium topological quantum phases without known equilibrium counterpart.

➞ *Symmetry-protected MFB* may be engineered in two-band *s*-wave superconductors starting from equilibrium conditions where *none* or *at most a pair* of Majorana modes exist.

➞ Floquet MFB maintain their advantage in terms of *enhanced transport signatures*, and need *not* depend on how boundary conditions are applied as sensitively as equilibrium MFB do.

➞ Control techniques may be *portable to other designer platforms* [AMO systems].



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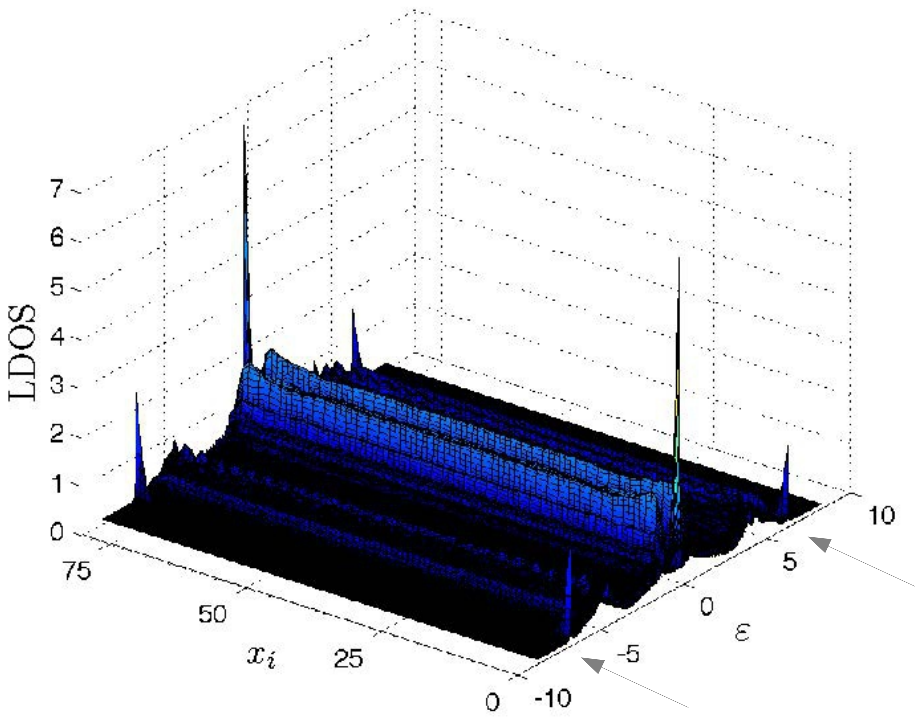

- But... How to identify the bulk-boundary combinations that do support protected boundary modes?... How to tune parameters at 'sweet spots' where they are maximally robust?...

Part II:
Exact characterization of topological boundary modes [generalized Bloch theorem]

Abhijeet Alase, Emilio Cobanera, Gerardo Ortiz & LV
Exact solution of quadratic fermionic Hamiltonians for arbitrary boundary conditions
Phys. Rev. Lett. 117, 076804 (2016).

Emilio Cobanera, Abhijeet Alase, Gerardo Ortiz & LV
Exact solution of corner-modified banded block-Toeplitz eigensystems
J. Phys. A: Math. & Theor., Forthcoming (2016).

Why 'arbitrary' boundaries?...

Bulk-boundary correspondence (BBC): Joining two systems in distinct phases mandates emergence of states localized on the boundary – irrespective of how the systems are joined.

- BBC is a powerful principle but… beyond 1D quantum walks, *no general analytic* insight nor rigorous theory is available as yet.

Kitagawa, QIP 11 (2012); Cedzich et al, JPA **49** (2016); ArXiv:1611.04439.

➡ *Genesis* of boundary modes: Exactly, how does it happen?...

➡ *Robustness* of boundary modes: Exactly, what is the interplay between bulk/ boundary?...

- Response to boundary perturbations is key to topological robustness… in turn…
- Robustness against changes of BCs may influence bulk symmetries at equilibrium.

Isaev, Moon, Ortiz, PRB 84 (2011), Fagotti, J. Stat. Mech. (2016)...

➡ Exactly, what does this all mean at the basic *system-theoretic* level?...

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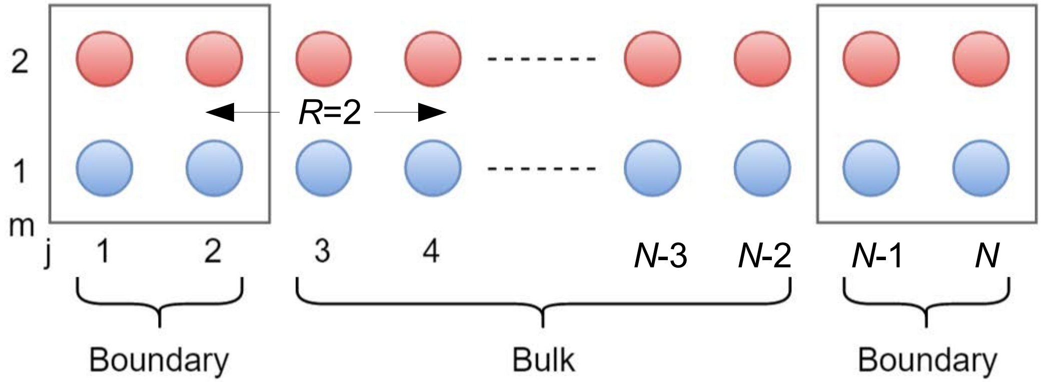
  → Exactly, what does this all mean at the basic *system-theoretic* level?...

- Goal: Develop an analytic approach to the BBC, starting from the simplest setting of clean systems – *translational invariance broken only by boundary conditions*.

↓

Necessary consistency requirement: Allow for arbitrary BCs from the outset...

Tight-binding models with boundaries



$R \ll N$

PBC: $h_r = g_r$
OBC: $g_r \equiv 0$
...

- Case study: Finite-range disorder-free quadratic fermionic Hamiltonians on $D=1$ lattice

$$\widehat{H} = \sum_{r=0}^{R} \Big(\sum_{j=1}^{N-r} \psi_j^\dagger h_r \psi_{j+r} + \sum_{j=N-r+1}^{N} \psi_j^\dagger g_r \psi_{j+r-N} + \text{H.c.} \Big) \equiv \frac{1}{2} \Psi^\dagger H \Psi$$

Hopping/pairing among fermions located r cells apart:
in the bulk — at the boundary

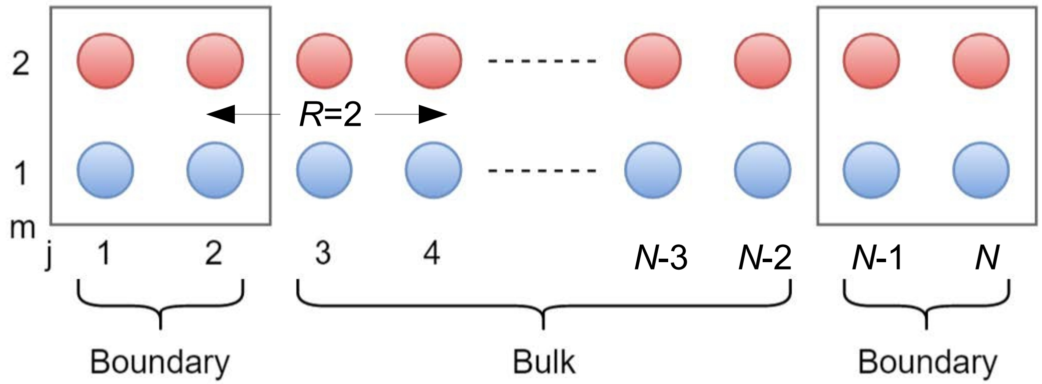
- Strategy: Try to mimic the success story of Fourier transform by making it explicit that a translation-invariant Hamiltonian may *still* be constructed 'away from the boundary'...

➞ Introduce subsystem decomposition on single-particle space:

$$\mathcal{H} \simeq \mathbb{C}^N \otimes \mathbb{C}^{2d} \equiv \text{span}\{|j\rangle|m\rangle \,|\, 1 \le j \le N;\, 1 \le m \le 2d\}$$

➞ Introduce 'translation-like' left shift operator: $T \equiv \sum_{j=1}^{N-1} |j\rangle\langle j+1|$

Bulk-boundary separation



$$P_B \equiv \sum_{j=R+1}^{N-R} |j\rangle\langle j| \otimes \mathbb{I}$$

$$P_\partial \equiv \mathbb{I} - P_B$$

- Single-particle Hamiltonian rewrites as a 'corner-modified' banded block-Toeplitz matrix:

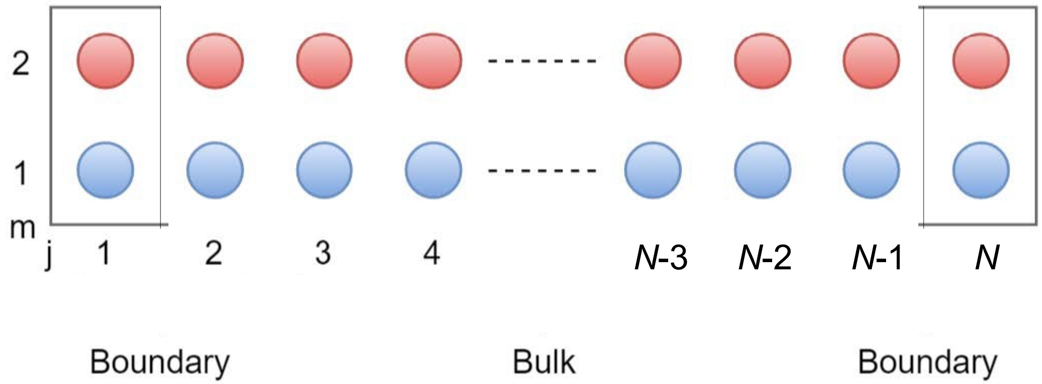
$$H = \sum_{r=0}^{R} [T^r \otimes h_r + (T^\dagger)^{L-r} \otimes g_r + \text{H.c.}] \equiv \sum_{r=0}^{R} [T^r \otimes h_r + \text{H.c.}] + W \equiv H_N + W$$

such that W enforces BCs via $P_B W = 0$ and H_N may be naturally associated to an *infinite* [banded block-Laurent] *translation-invariant Hamiltonian*

$$\mathbf{H} = \mathbb{I} \otimes h_0 + \sum_{r=1}^{R} [\mathbf{T}^r \otimes h_r + (\mathbf{T}^{-1})^r \otimes h_r^\dagger], \quad \mathbf{T} \equiv \sum_{j=-\infty}^{\infty} |j\rangle\langle j+1|$$

Bulk-boundary separation [intermezzo]

$R = 1$



$$P_B \equiv \sum_{j=R+1}^{N-R} |j\rangle\langle j| \otimes \mathbb{I}$$

$$P_\partial = |1\rangle\langle 1| + |N\rangle\langle N|$$

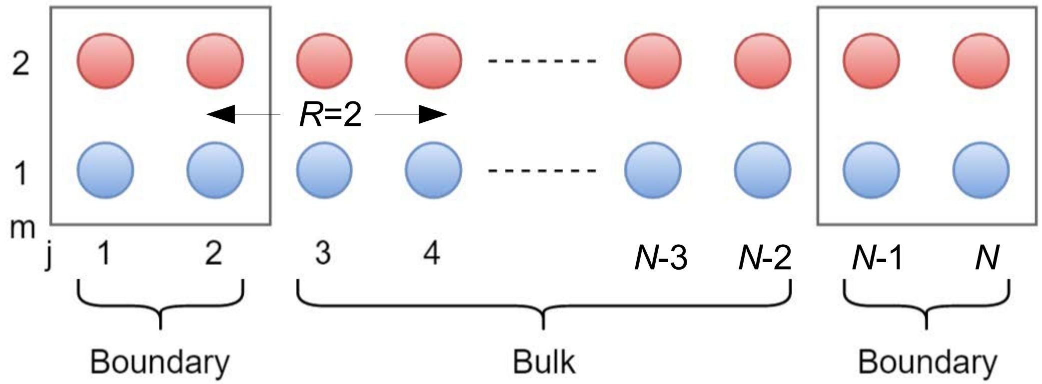
$$H_N + W =$$

Corner modif cation

$$\begin{bmatrix} h_0 & h_1 & 0 & 0 & 0 & 0 \\ h_1^\dagger & h_0 & h_1 & 0 & 0 & 0 \\ 0 & h_1^\dagger & h_0 & h_1 & 0 & 0 \\ 0 & 0 & h_1^\dagger & h_0 & h_1 & 0 \\ 0 & 0 & 0 & h_1^\dagger & h_0 & h_1 \\ 0 & 0 & 0 & 0 & h_1^\dagger & h_0 \end{bmatrix} + \begin{bmatrix} w_{1,1} & 0 & 0 & 0 & 0 & w_{1,N} \\ 0 & 0 & \dots & & & 0 \\ 0 & & \dots & & 0 & 0 \\ 0 & 0 & & & & 0 \\ 0 & 0 & 0 & & \dots & 0 \\ w_{1,N}^\dagger & 0 & 0 & 0 & 0 & w_{N,N} \end{bmatrix}$$

$\mathbf{H}$ is obtained as infinite extension of the BBT matrix $P_B H_N$ with boundary rows removed…

Bulk-boundary separation



$$P_B \equiv \sum_{j=R+1}^{N-R} |j\rangle\langle j| \otimes \mathbb{I}$$

$$P_\partial \equiv \mathbb{I} - P_B$$

- Single-particle Hamiltonian rewrites as a 'corner-modified' banded block-Toeplitz matrix:

$$H = \sum_{r=0}^{R} [T^r \otimes h_r + (T^\dagger)^{L-r} \otimes g_r + \text{H.c.}] \equiv \sum_{r=0}^{R} [T^r \otimes h_r + \text{H.c.}] + W \equiv H_N + W$$

such that W enforces BCs via $P_B W = 0$ and H_N may be naturally associated to an *infinite* [banded block-Laurent] *translation-invariant Hamiltonian*

$$\mathbf{H} = \mathbb{I} \otimes h_0 + \sum_{r=1}^{R} [\mathbf{T}^r \otimes h_r + (\mathbf{T}^{-1})^r \otimes h_r^\dagger], \quad \mathbf{T} \equiv \sum_{j=-\infty}^{\infty} |j\rangle\langle j+1|$$

- Diagonalization problem for H may be *exactly* recast into the simultaneous solution of

$$\begin{cases} P_B H_N |\epsilon\rangle = \epsilon\, P_B |\epsilon\rangle & \text{BULK EQUATION} \\ (P_\partial H_N + W)|\epsilon\rangle = \epsilon\, P_\partial |\epsilon\rangle & \text{BOUNDARY EQUATION} \end{cases}$$

Solution approach (in words...)

- Step 1: Obtain *eigenvalue-dependent* Ansatz for the solutions to the bulk equation.

$$P_B H_N|\epsilon\rangle = \epsilon\, P_B|\epsilon\rangle \Leftrightarrow |\epsilon\rangle \in \text{Ker}\, P_B(H_N - \epsilon)$$

→ Key observation: For arbitrary ϵ, it is *easy* to compute and store a basis of the kernel of a corner-modified BBT matrix – *complexity is independent of* N.

→ *Generically*, all solutions may be obtained as solutions to the associated infinite BBL system – which is translation-invariant: Kernel determination entails solving a *polynomial equation of small degree*, at most $4dR$.

- Generic case: $\mathcal{M}_N = \mathbf{P}_N \mathcal{M}_\infty, \quad \det h_R \neq 0$ ⇒ quasi-invariant solutions: extended support

- Non-invertible case: Additional solutions may emerge because of projection from infinite-to-finite system, $\mathbf{H} \mapsto H_N$, $\det h_R = 0$ ⇒ emergent solutions: finite support (localized)

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- Step 2: Impose BCs, by using Ansatz to select solutions that *also* solve boundary equation.

$$(P_\partial H_N + W)|\epsilon\rangle = \epsilon\, P_\partial|\epsilon\rangle \Leftrightarrow P_\partial(H - \epsilon)|\epsilon\rangle = 0$$

➞ Using the Ansatz for $|\epsilon\rangle$, recast boundary equation as the kernel equation of a $4dR \times 4dR$ boundary matrix B, so that if $|\epsilon\rangle$ is an eigenvector $\Rightarrow$ $\det B = 0$.

A generalized Bloch Ansatz

- Exact solution yields a *structural characterization* of [single-particle] energy eigenstates – effectively generalizing Bloch theorem from periodic to arbitrary BCs:

$$|\epsilon\rangle = \sum_{\ell=1}^{n}\sum_{s=1}^{s_\ell} \alpha_{\ell s}|\psi_{\ell s}\rangle + \sum_{s=1}^{s_+}\alpha_s^+|\psi_s^+\rangle + \sum_{s=1}^{s_-}\alpha_s^-|\psi_s^-\rangle, \quad \alpha_{\ell s}, \alpha_s^+, \alpha_s^- \in \mathbb{C}$$

Translation-invariant — Emergent

➡ Translation-invariant basis states are built out of eigenvectors of 'reduced bulk Hamiltonian':

$$H_B(z) \equiv h_0 + \sum_{r=1}^{R}(z^r h_r + z^{-r}h_r^\dagger) \Rightarrow P(z,\epsilon) \equiv \det[z^R(H_B(z)-\epsilon)] = 0$$

➡ For *generic parameter values and generic* ϵ, no emergent solution exists, and Ansatz can be expressed entirely in terms of 'exponential solutions' [*Bloch waves with complex momentum*...]

$$|\psi_{\ell s}\rangle \equiv |z_\ell\rangle|u_\ell\rangle \propto \sum_{j=1}^{N} z_\ell^j\,|j\rangle|u(\epsilon,z_\ell)\rangle \qquad z \leftrightarrow e^{ik}$$

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- Ansatz allows to completely characterize *all* possible eigenstates for specified BCs that *can naturally exist or be engineered* via parameter tuning...

Example: The Kitaev chain (new surprises)

- Three 'exceptional' but relevant scenarios:

  (1) Non-invertible case, $\det h_R = 0$. Or, *regardless* of invertibility:

  (2) Characteristic polynomial $P(z,\epsilon) \equiv 0$, for *some* $\epsilon \Rightarrow$ Dispersionless band, [bulk-]localized

  (3) Two or more roots coincide, for *some* $\epsilon \Rightarrow$ Power-law solutions [despite short range!]

$$|\psi_{\ell s}\rangle \propto \sum_{j=1}^{N} [z_\ell^j \, |j\rangle |u_1(\epsilon, z_\ell)\rangle + j z_\ell^j \, |j\rangle |u_2(\epsilon, z_\ell)\rangle]$$

...all represented in the Kitaev chain at $t = \Delta$:

$$H_N = T \otimes h_1 + \mathbb{I} \otimes h_0 + T^\dagger \otimes h_1^\dagger, \;\; h_0 = -\begin{bmatrix} \mu & 0 \\ 0 & -\mu \end{bmatrix}, \; h_1 = -\begin{bmatrix} t & -\Delta \\ \Delta & -t \end{bmatrix}$$

$$\det h_R = \det h_1 = 0$$

$$P(z,\epsilon) = 2\mu t (z^3 + z) + (\mu^2 + 4t^2 - \epsilon^2) z^2$$

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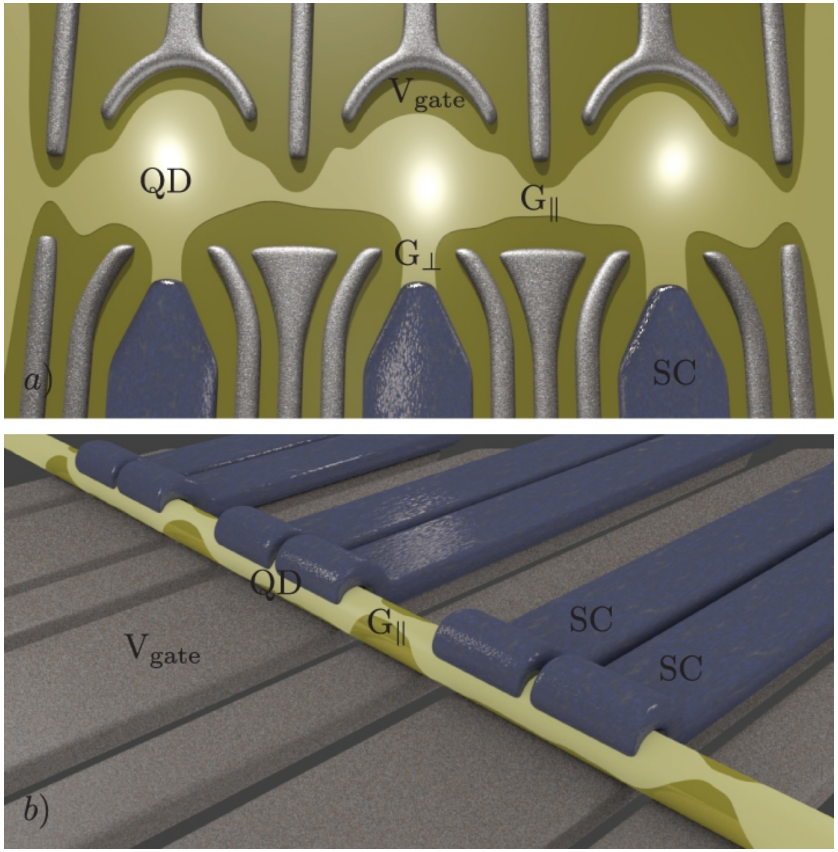
Fulga et al, NJP 15 (2013).



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(3) Doubly degenerate roots [power-law Majoranas...]

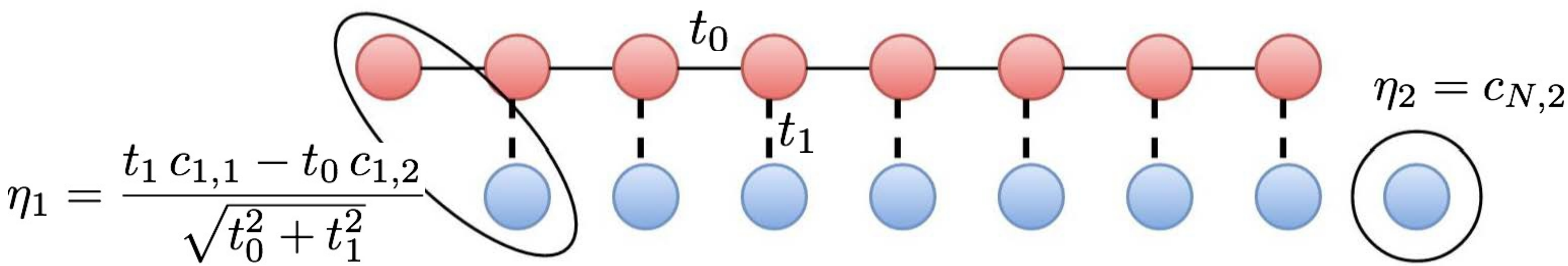
$$\epsilon \in \{\pm(\mu \pm 2t)\}$$

(3) Perfectly localized bulk solutions at 'sweet spot'

$$\mu = 0, \;\; \epsilon = \pm 2t$$

Example: A topological comb

- Ansatz may be used to gain analytic insight and design 'exotic' zero-energy boundary modes...



<u>Case study</u>: A fermionic ladder with intra- and inter-ladder NN hopping

$$H_N = T \otimes h_1 + T^\dagger \otimes h_1^\dagger \quad \Rightarrow \quad H_N(|1\rangle|u^-\rangle) = 0 = H_N(|N\rangle|u^+\rangle)$$

$$h_1 = \begin{bmatrix} t_0 & 0 \\ t_1 & 0 \end{bmatrix},\ |u^-\rangle \equiv \begin{bmatrix} t_1 \\ -t_0 \end{bmatrix},\ |u^+\rangle \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

➞ A non-trivial *perfectly localized* zero-energy mode exists, *split over two boundary sites* with weights controlled by ratio t_0/t_1 [independently of N].

➞ Full solution shows that model is gapped, and *no dispersionless bulk-localized band exists.*

➞ Zero-energy mode is robust, despite lack of obvious protecting chiral symmetry.

Further implications…

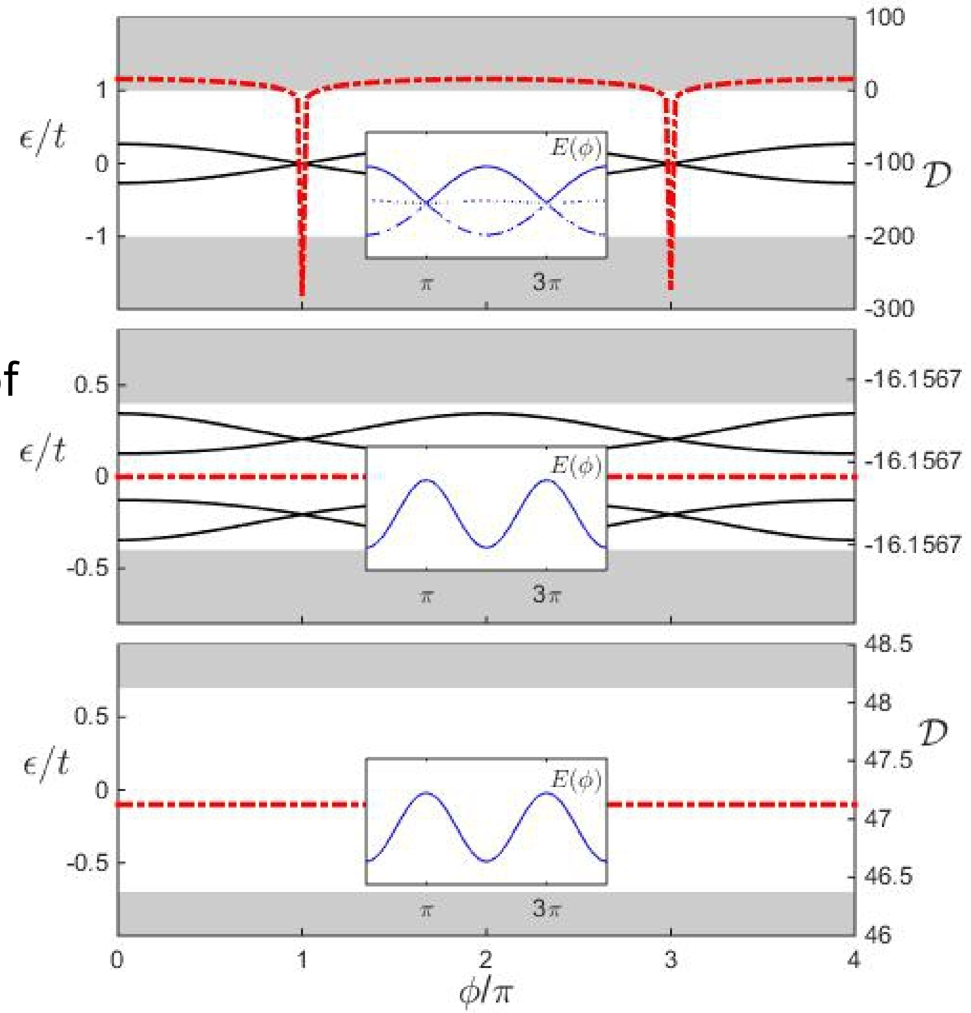
- The boundary matrix may used to construct useful [computationally tractable] *indicators of bulk-boundary correspondence that include both bulk and boundary information:*

$$\mathcal{D} \equiv \log\{\det[B_\infty(0)^\dagger B_\infty(0)]\}$$

  ➞ If either reduced bulk Hamiltonian or BCs are changed, singularity develops iff system hosts bound zero-energy modes…

- Approach may be extended to diagonalization of *clean systems with internal/multiple boundaries*
  ➞ Impurity problems
  ➞ Bound states on SN, SNS junctions

- Approach may be extended to $D > 1$, as long as periodic BCs are imposed on D–1 directions
  ➞ Graphene with arbitrary BCs
  ➞ Gapless *s*-wave superconductors, MFB…



Alase et al, PRL 117 (2016).

Summary and outlook

- A natural generalization of Bloch theorem is possible for 'almost translationally invariant' finite-range quadratic fermionic Hamiltonians – based on exact separation of eigenvalue problem into translation-invariant bulk equation, and a boundary equation.

- Standard Bloch theorem is consistently recovered for translationally invariant systems.

➞ The generalized Bloch theorem offers an *analytic window into the bulk-boundary correspondence* – including origin of perfectly localized eigenstates and existence of both exponential and power-law solutions in short-range models.

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- Plenty of directions call for further investigation…

  ➞ Bloch Ansatz for *Floquet systems with boundaries* [back to Majorana flat bands]…
  ➞ Relationship between bulk-boundary separation and *entanglement spectrum*…
  ➞ Bloch Ansatz for *quadratic systems of bosons*…
  ➞ Diagonalization of *quadratic Lindblad dynamics with boundaries*…
  ⋮

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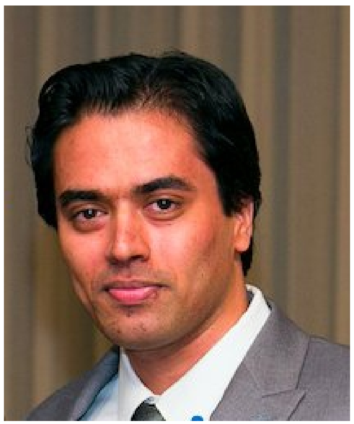

At Indiana U., Bloomington:
Gerardo Ortiz

…thanks for your attention!