From the topological invariants to the characterization of the edge states

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Text

work with A. Essin

KITP, Oct 2011

Topological invariants

It is possible to go directly from topological invariants to edge states without studying Hamiltonians, Schrödinger equation or responses.

$$N_d = N_{d-2}(\Lambda) - N_{d-2}(-\Lambda)$$

Bulk invariant
in d dimensions

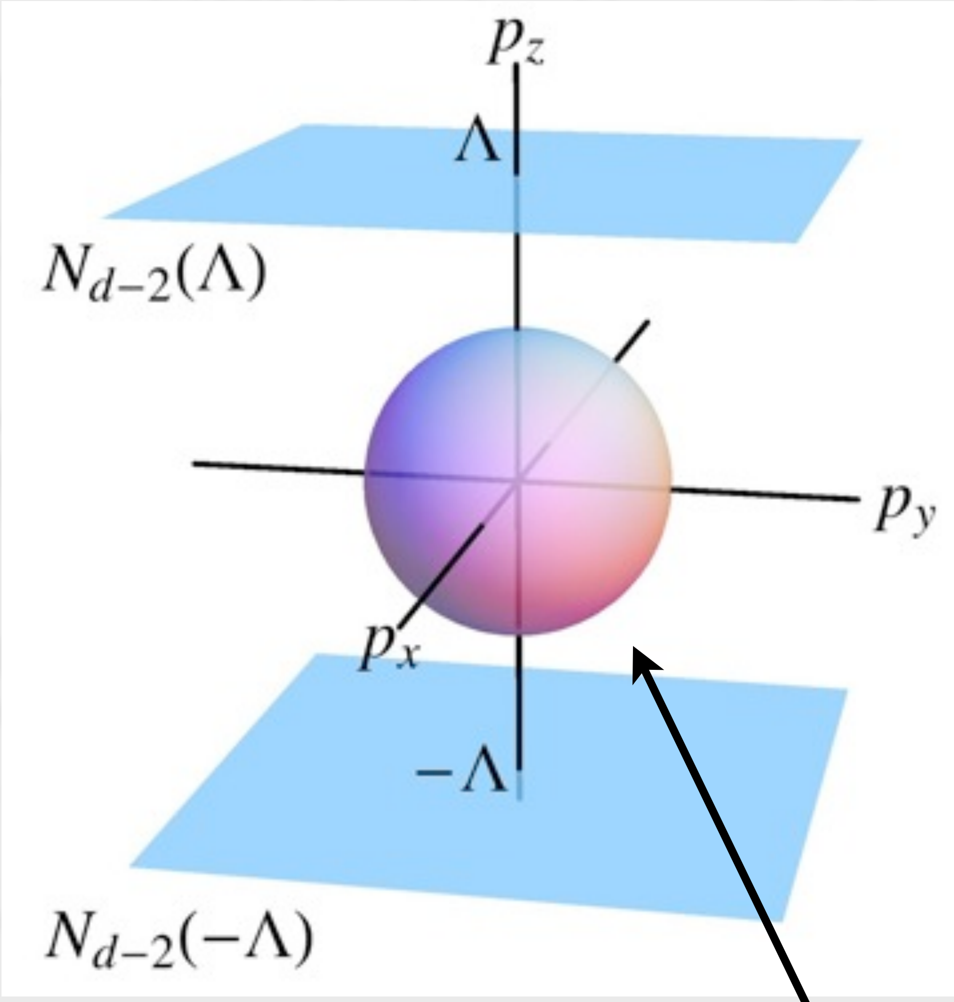
Edge invariant

G. Volovik, 1980s; VG, A. Essin, PRB 2011

Edge topological invariant

1. Bulk invariant N_d

Example: 3D edge of a 4D insulator



Fermi surface

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2. d-1 dimensional edge with d-1 momenta

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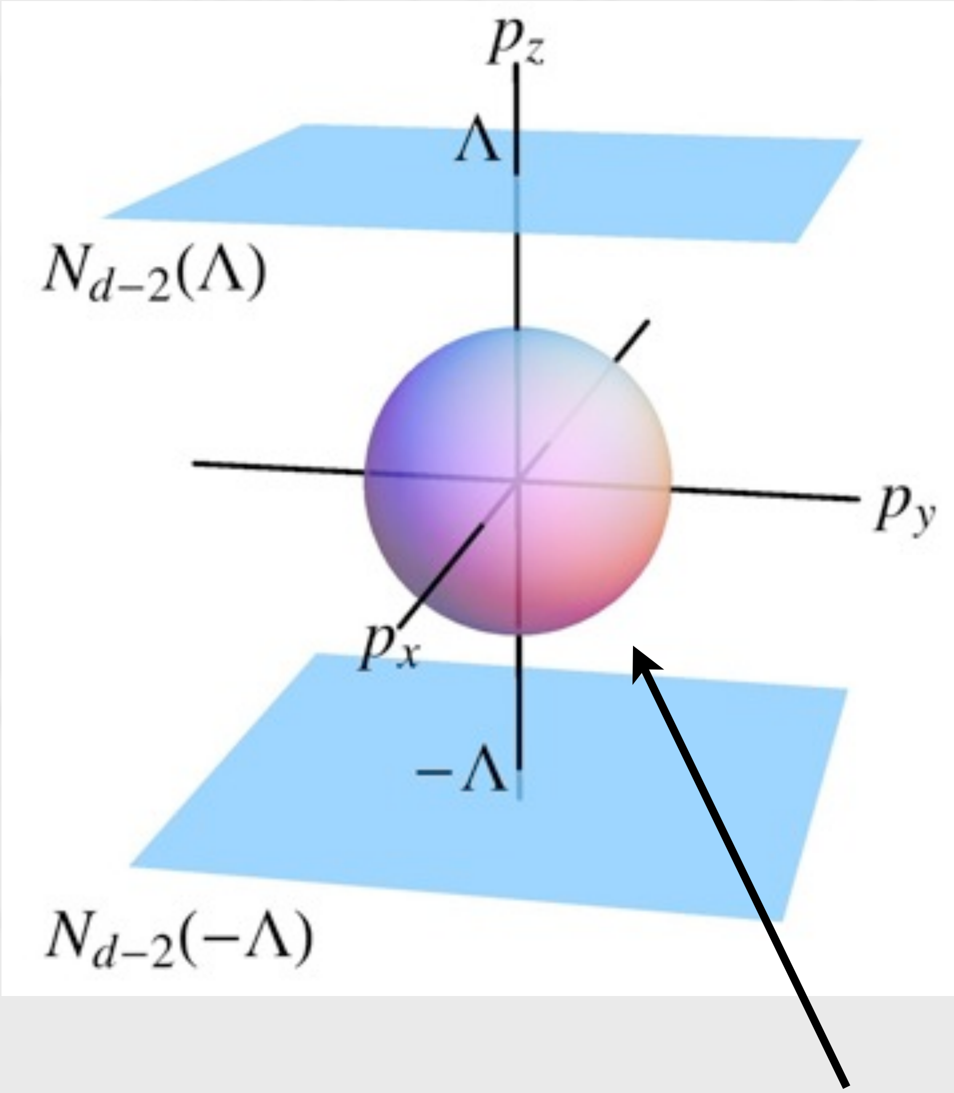

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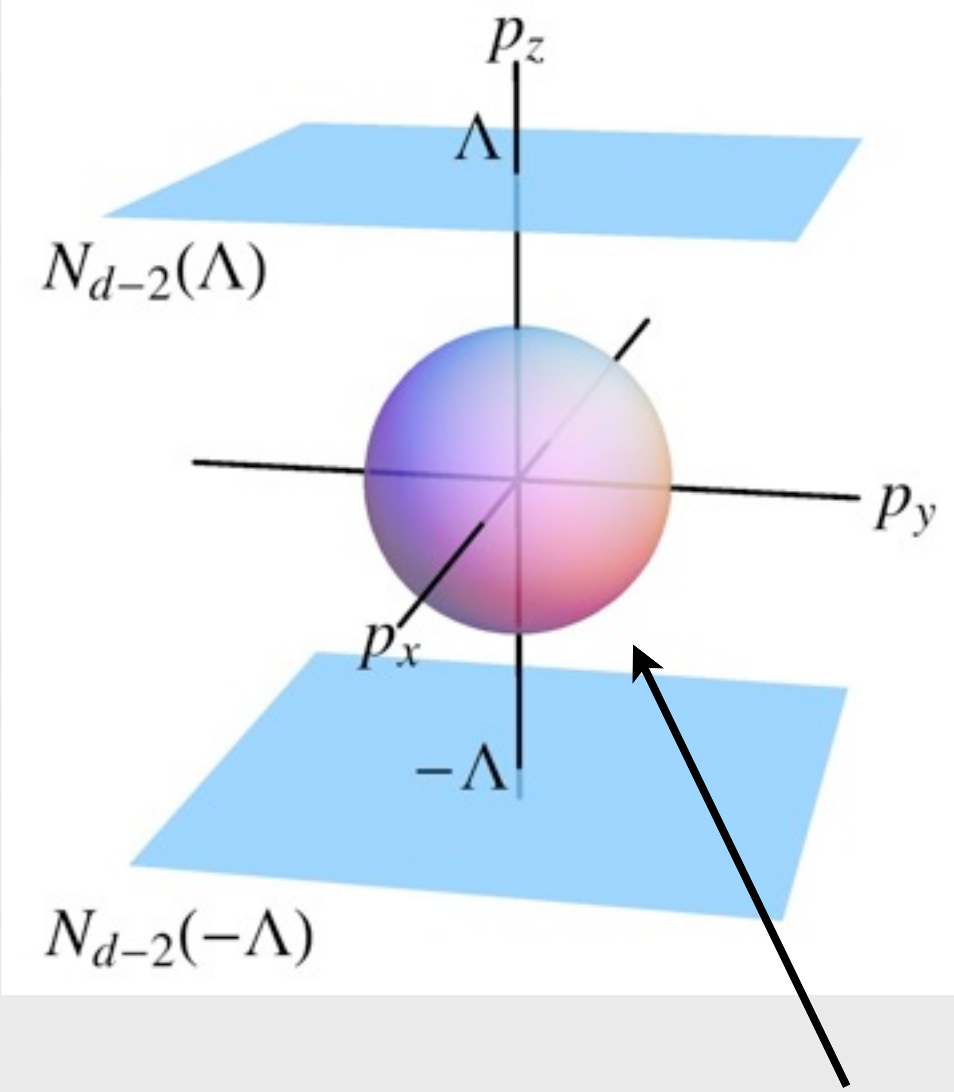

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5. Now the edge is an d-2 dim insulator

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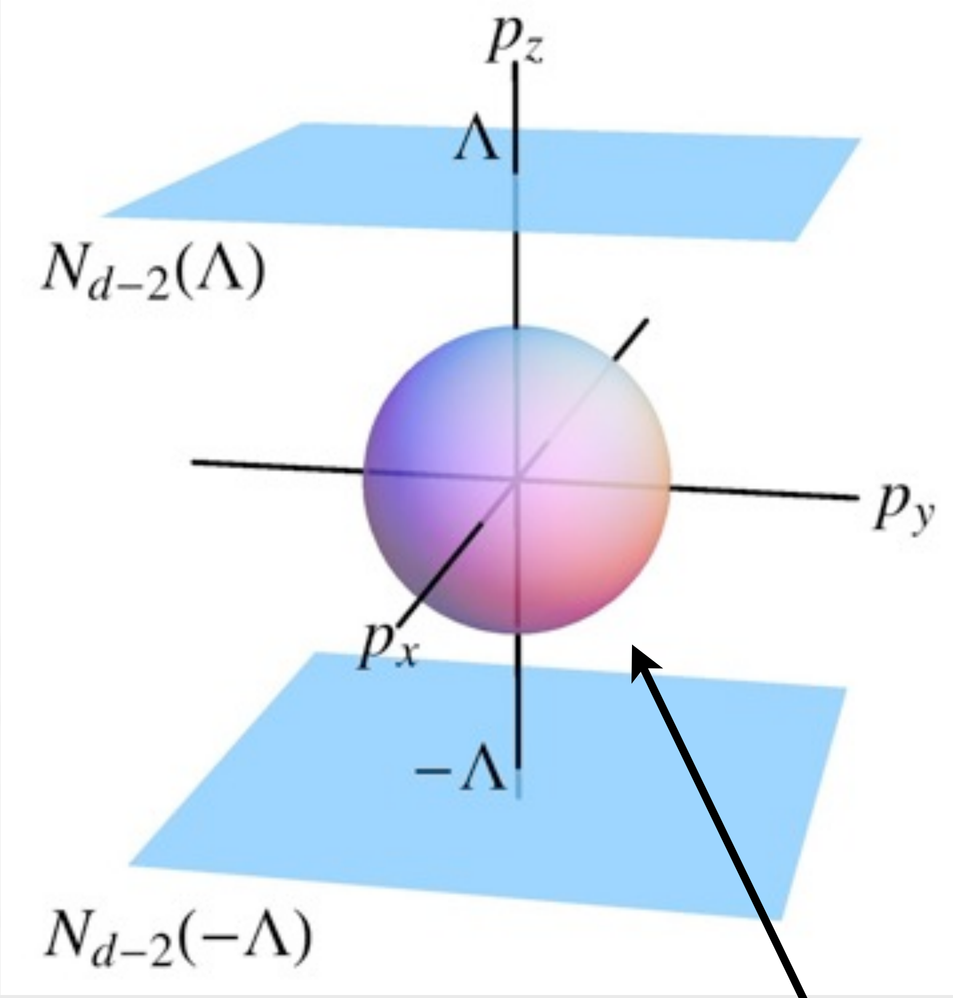

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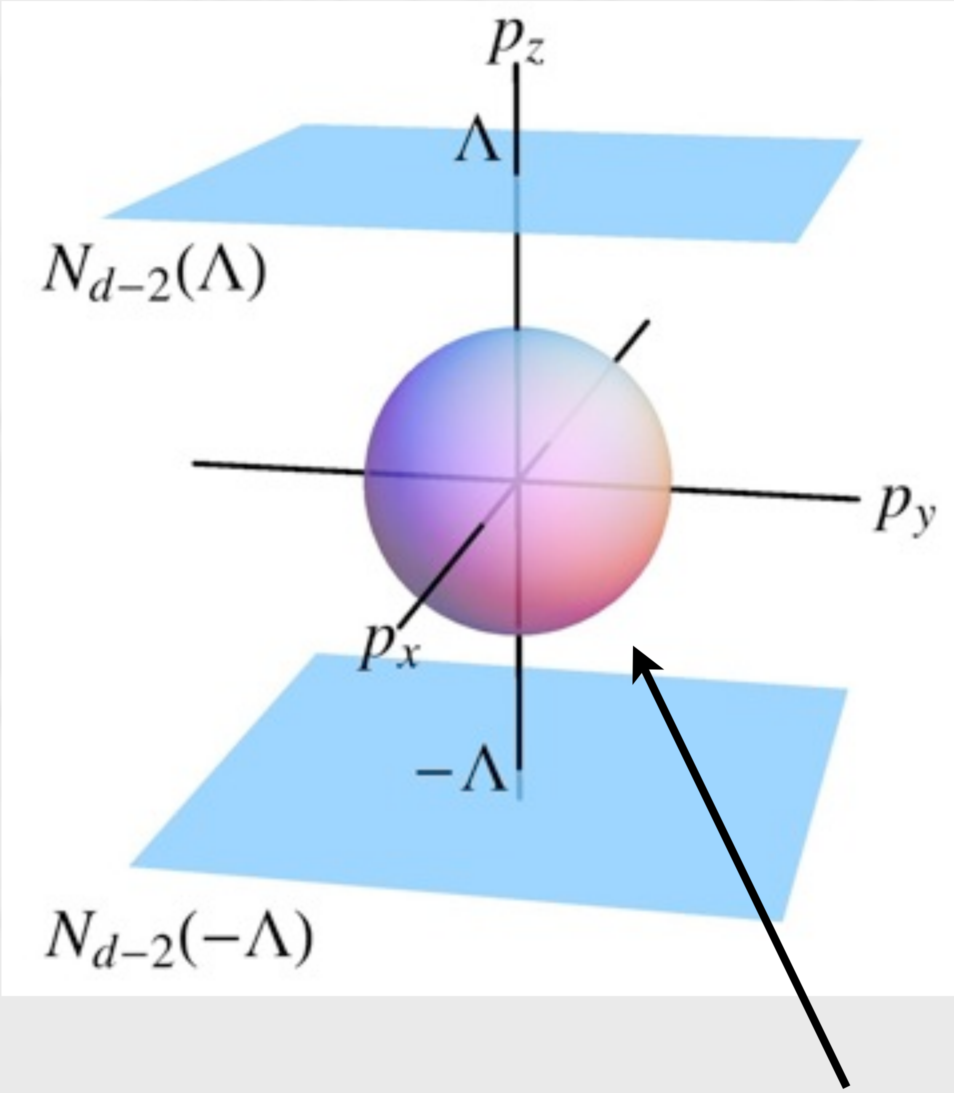

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7. Claim: $N_d = N_{d-2}(\Lambda) - N_{d-2}(-\Lambda)$

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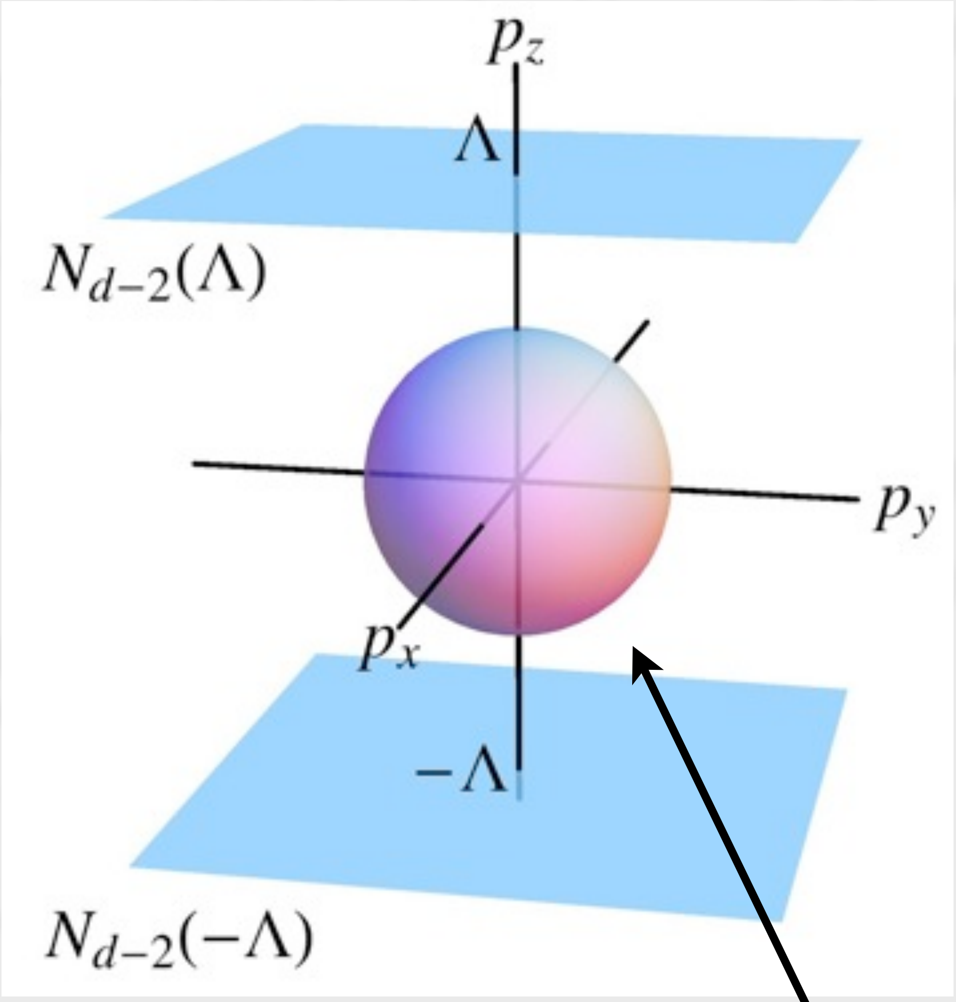


Fermi surface

Example: an edge of a 4D AII insulator

This edge is taken as

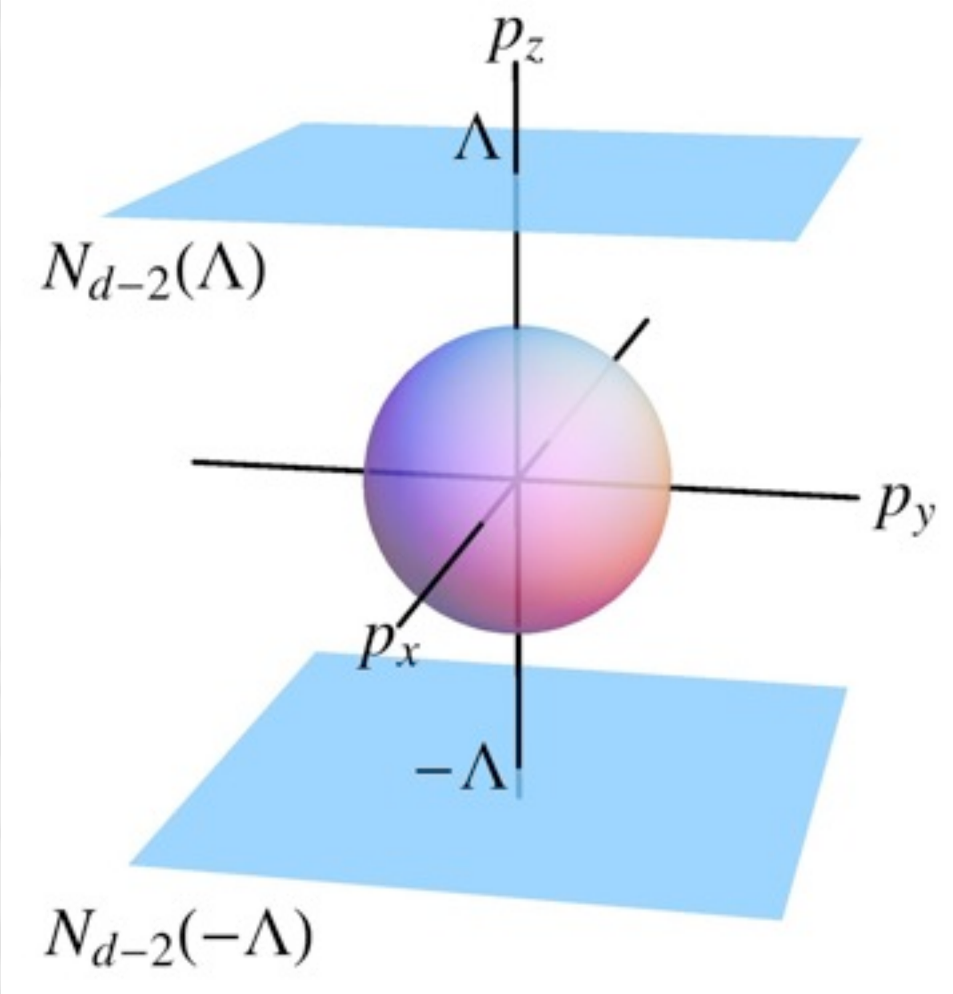
$$H = v \sum_{i=x,y,z} \sigma_i p_i - \mu$$

because it is

1. *linear in momenta*
2. *time-reversal invariant*

$$H(p) = \sigma_y H^*(-p)\sigma_y$$

But does it have the right edge invariant?



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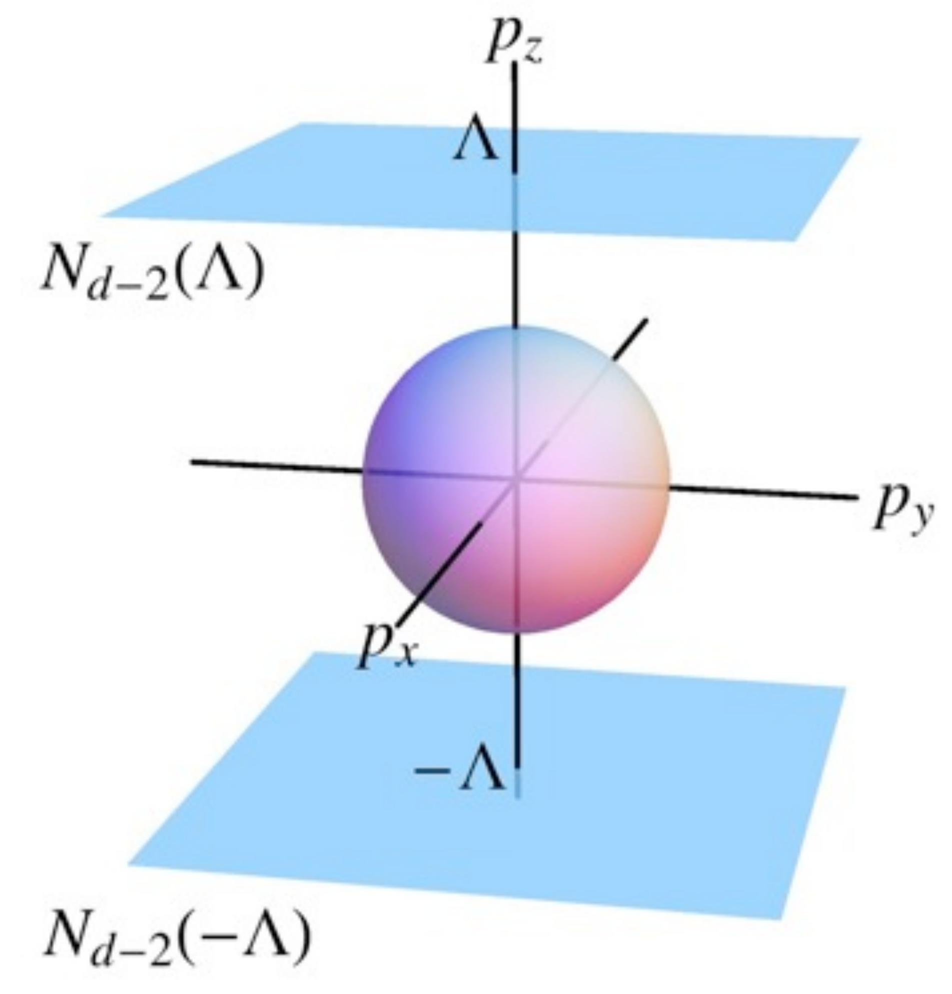

Fix $p_z=+\Lambda$ or $p_z=-\Lambda$

$$H = v\sigma_x p_x + v\sigma_y p_y \pm v\Lambda\sigma_z - \mu$$ Effectively 2D.

$$N_2(\Lambda) - N_2(-\Lambda) = 1$$ Well known relation.

LFSG, 1994

Yes, it is an edge.

Plan

$$N_d = N_{d-2}(\Lambda) - N_{d-2}(-\Lambda)$$

1. Derive this result

2. Use this result to study something useful

Topological invariant for even dimensions type $\mathbb{Z}$

Matsubara Green's function $G_{ab}(\omega, \mathbf{p})$

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topological invariant

known numerical coefficient, not particularly relevant

$$N_d = C_d \, \epsilon_{\alpha_0 \ldots \alpha_d} \, \mathrm{tr} \int d\omega d^d p \, G^{-1} \partial_{\alpha_0} G \ldots G^{-1} \partial_{\alpha_d} G$$

Summation over each $\alpha = \omega, p_1, \ldots, p_d$ is implied

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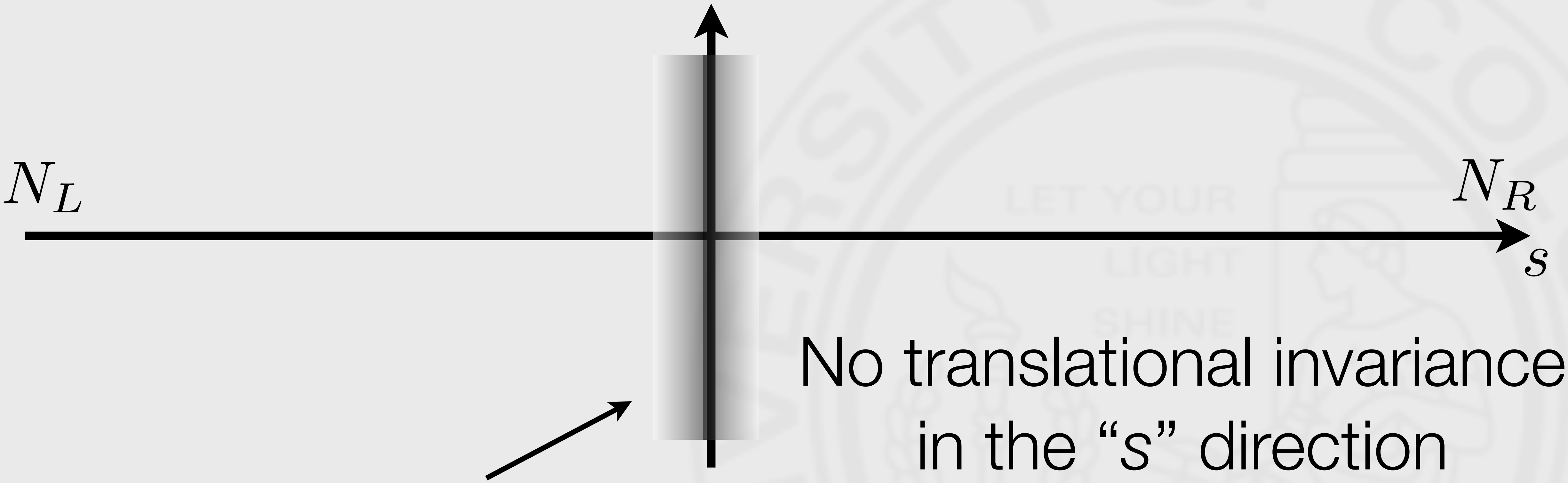
If d=2 this coincides with the TKNN invariant. Niu, Thouless, Wu (1985)

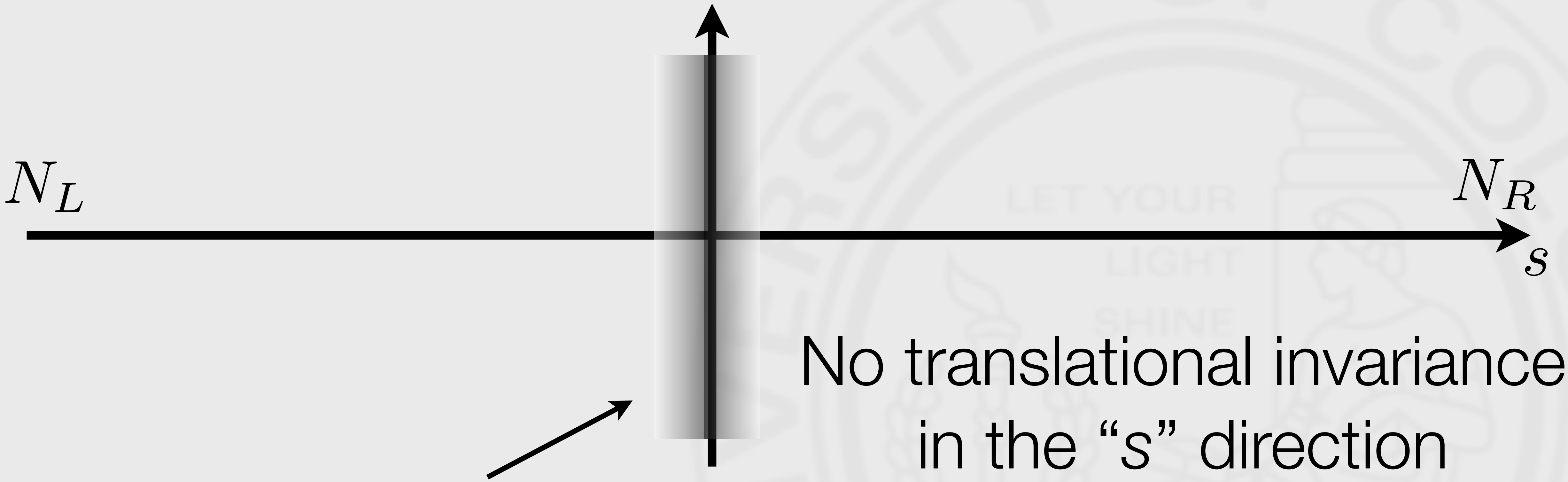
Domain walls



Domain wall (edge)

No translational invariance in the “s” direction

Domain walls



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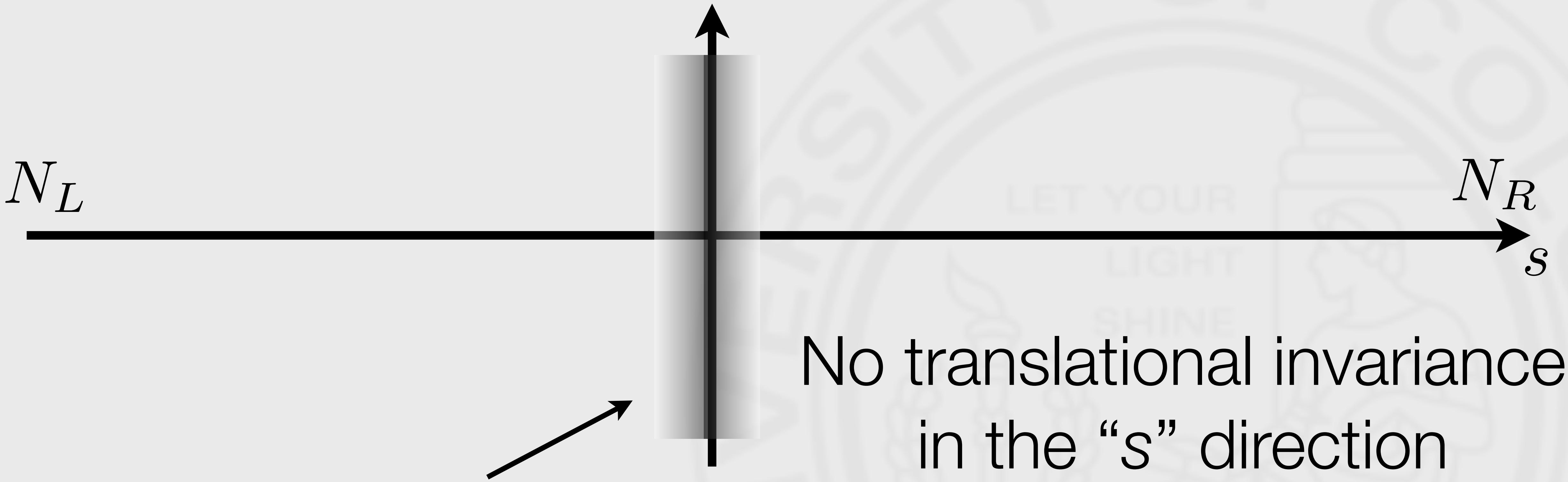

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2. Wigner transformed Green’s function

$$G_{ab}(\omega; p_1 \ldots p_d; s) = \int dr\, e^{ip_d r}\, G_{ab}(\omega; p_1 \ldots p_{d-1}; s + \frac{r}{2}, s - \frac{r}{2})$$

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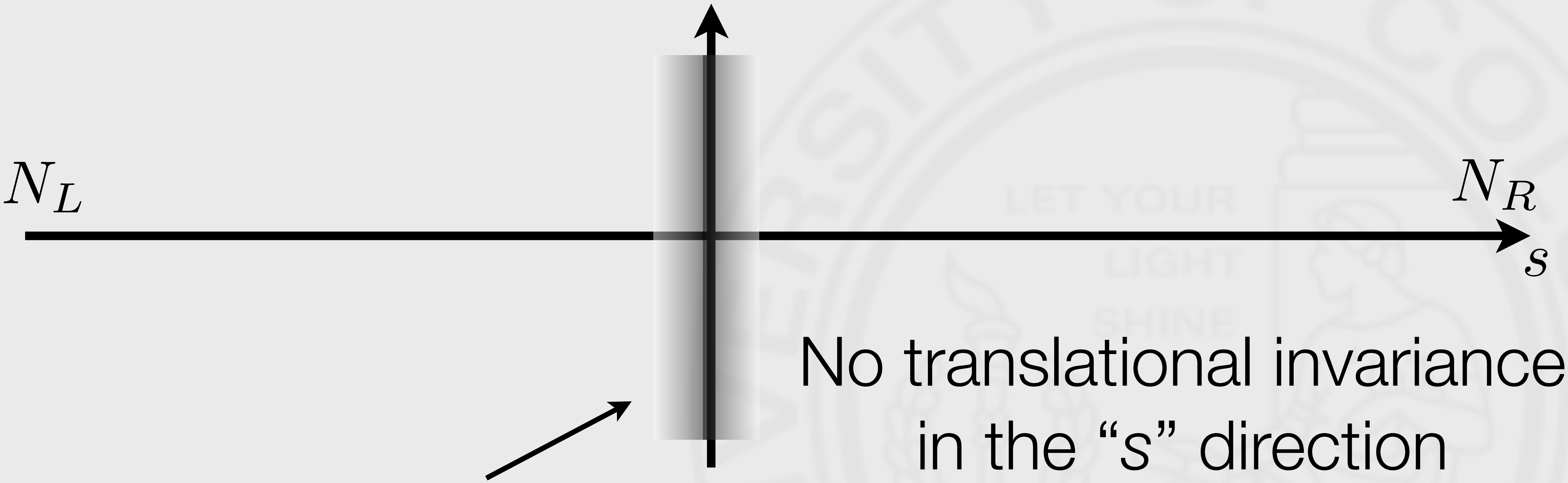

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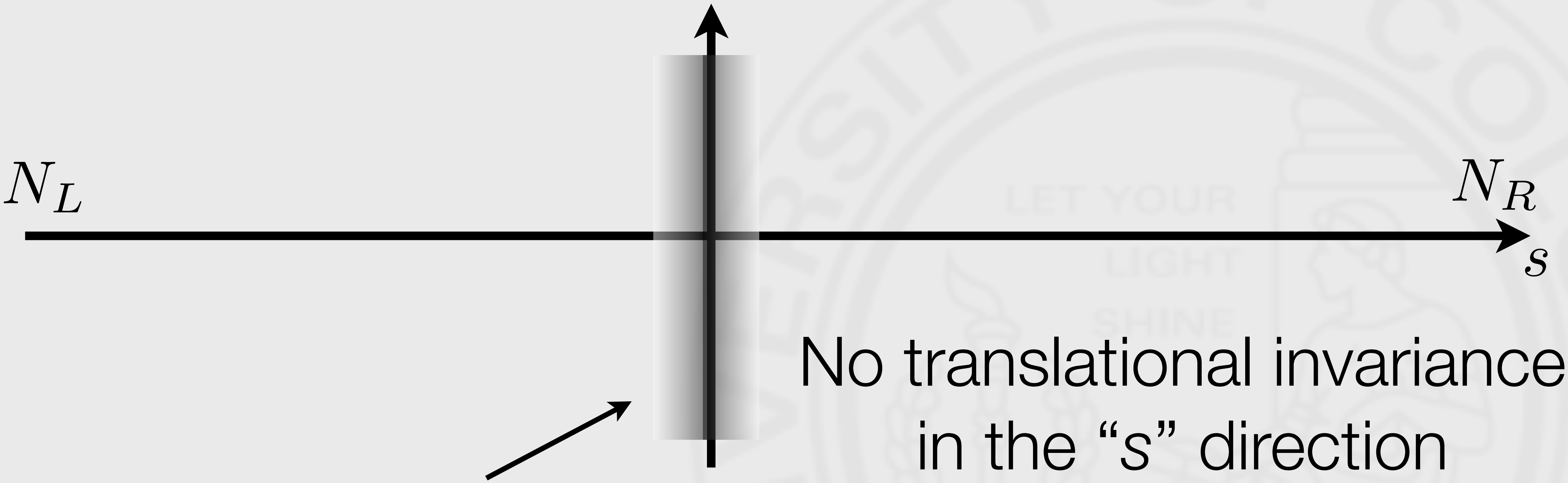
3. Inverse Green’s function K

$$\int ds'\, K_{ab}(\omega; p_1 \ldots p_{d-1}; s, s')\, G_{bc}(\omega; p_1 \ldots p_{d-1}; s', s'') = \delta_{ac}\delta(s - s'')$$

Domain walls



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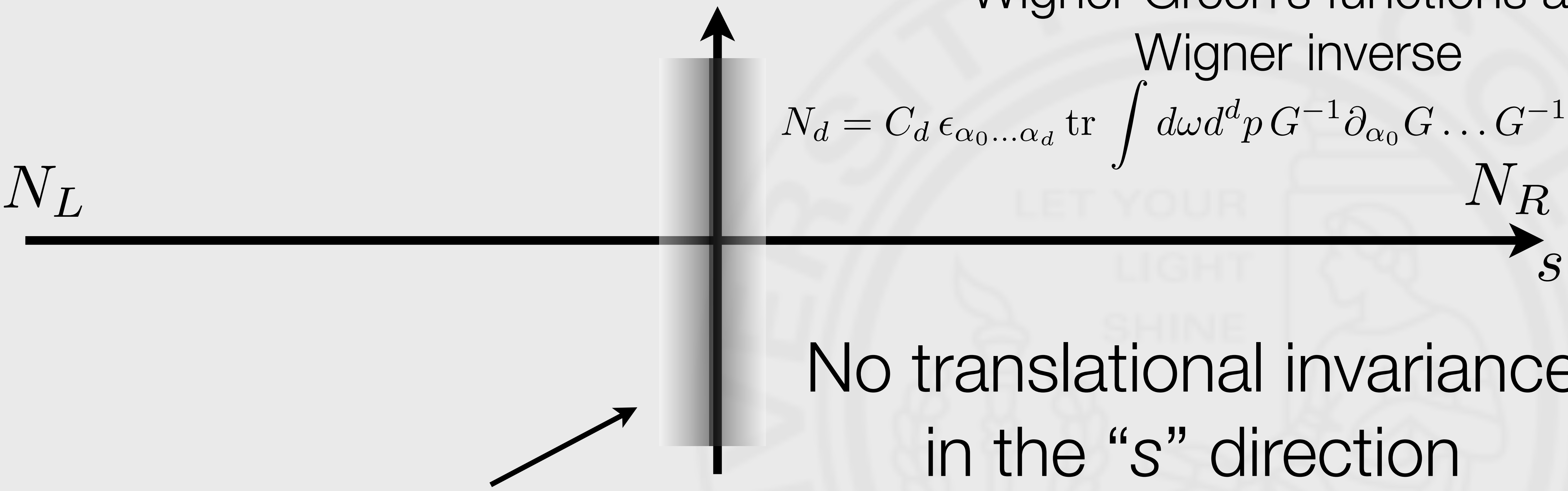
4. Local inverse $G^{-1}_{ab}(\omega; p_1 \ldots p_d; s)\, G_{bc}(\omega; p_1 \ldots p_d; s) = \delta_{ac}$

Domain walls

Local invariant, defined with Wigner Green's functions and Wigner inverse

$$N_d = C_d \, \epsilon_{\alpha_0 \ldots \alpha_d} \, \mathrm{tr} \int d\omega d^d p \, G^{-1} \partial_{\alpha_0} G \ldots G^{-1} \partial_{\alpha_d} G$$



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Topological invariant as a flux

$\omega; p_1 \ldots p_d; s$ d+2 dimensional space

$$\boxed{n_{\alpha_0} = C_d \, \epsilon_{\alpha_0 \ldots \alpha_{d+1}} \, \mathrm{tr} \, G^{-1} \partial_{\alpha_1} G \ldots G^{-1} \partial_{\alpha_{d+1}} G}$$

$\partial_\alpha n_\alpha = 0$ divergentless *d*+2 dimensional vector

$$N(s) = \int d\omega d^d p \, n_s$$

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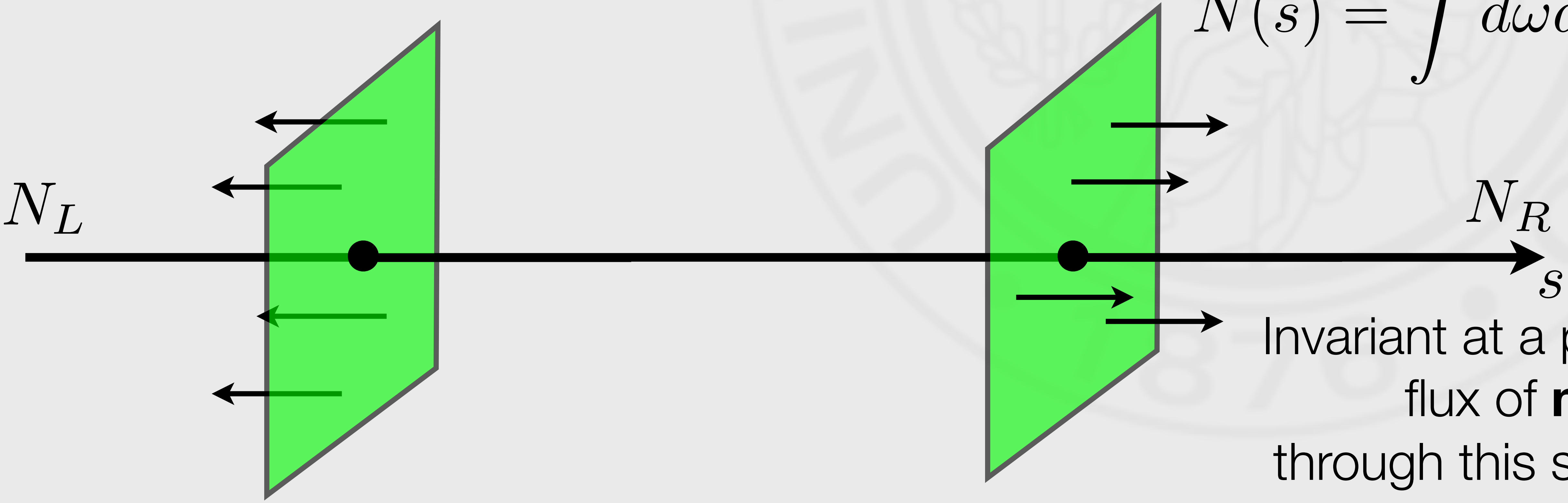

Invariant at a point s:
flux of **n**
through this surface

$$N_R - N_L = \int d\omega \, d^d p \, ds \, \partial_\alpha n_\alpha$$

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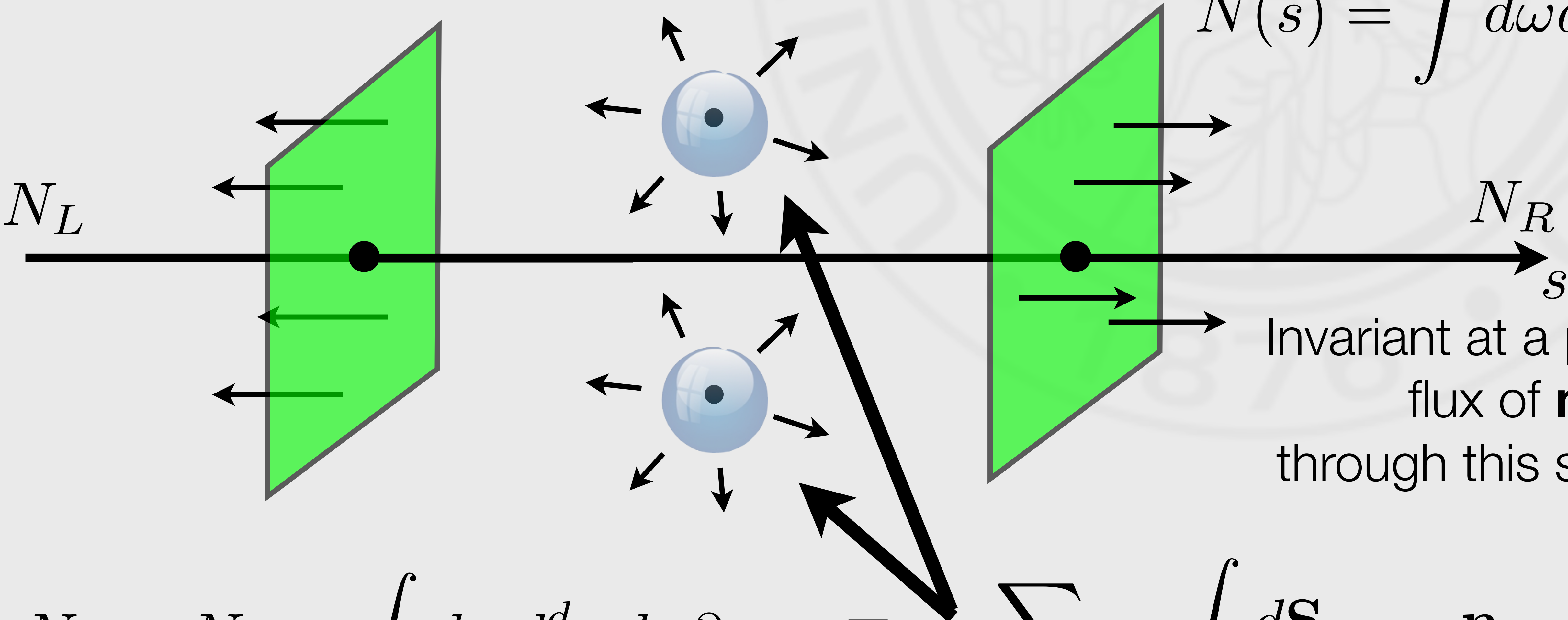

Invariant at a point s:
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$$N_R - N_L = \int d\omega\, d^d p\, ds\, \partial_\alpha n_\alpha = \sum_{\text{singularities}} \int d\mathbf{S}_{d+1} \cdot \mathbf{n}$$

Edge states

$\omega; p_1 \ldots p_{d-1}$ d-1 dimensional space spanning the edge

$$\boxed{r_{\alpha_0} = C_{d-2}\,\epsilon_{\alpha_0 \ldots \alpha_{d-1}} \mathrm{Tr}\,\left[K\,\partial_{\alpha_1} G \ldots K\,\partial_{\alpha_{d-1}} G\right]}$$

mixed Green's functions

$$G_{ab}\left(\omega; p_1 \ldots p_{d-1}; s, s'\right)$$

$$\mathrm{Tr}\, AB = \sum_{ab} \int ds ds'\, A_{ab}(\omega; p_1 \ldots p_{d-1}; s, s') B_{ba}(\omega; p_1 \ldots p_{d-1}; s', s)$$

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Gradient expansion shows

$$\int d\mathbf{S}^{d+1} \cdot \mathbf{n} = \int d\mathbf{S}^{d-1} \cdot \mathbf{r}$$

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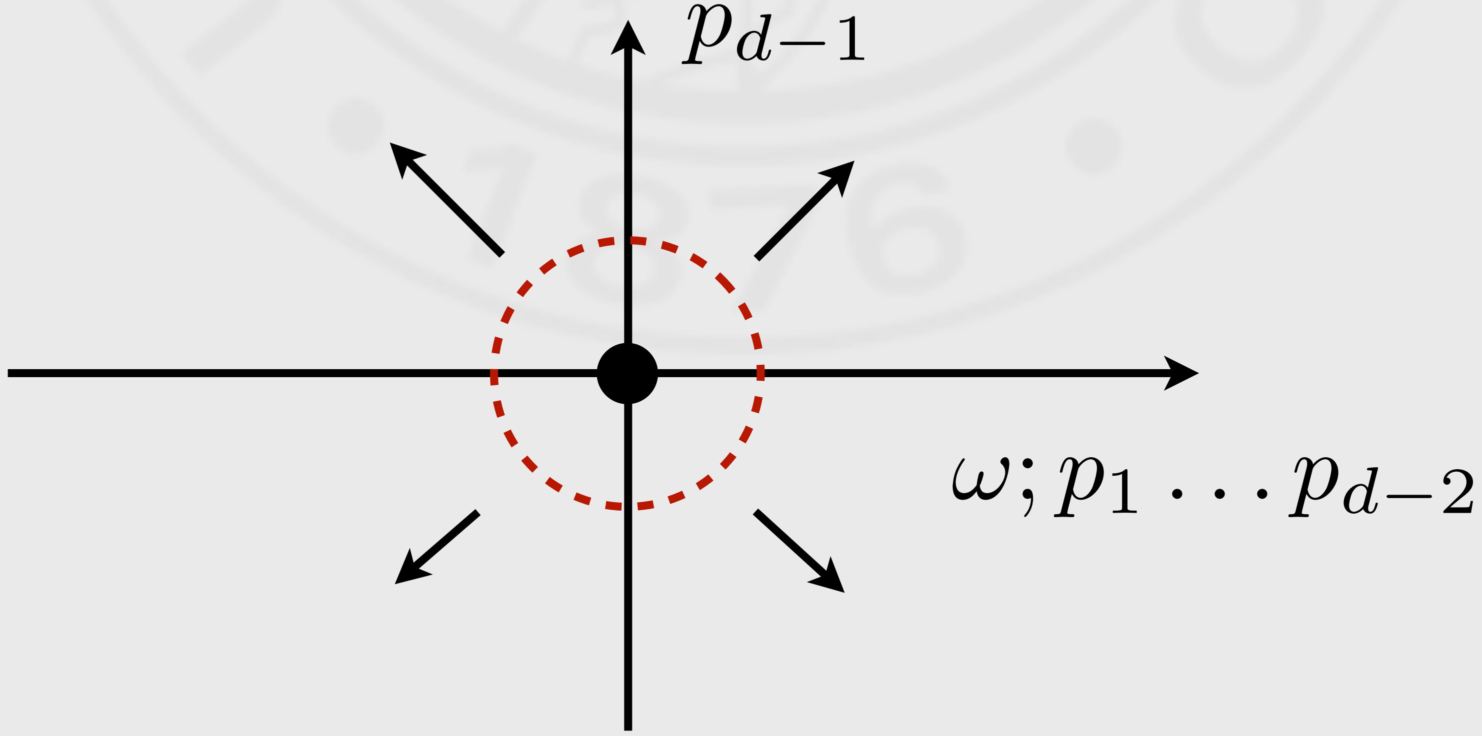

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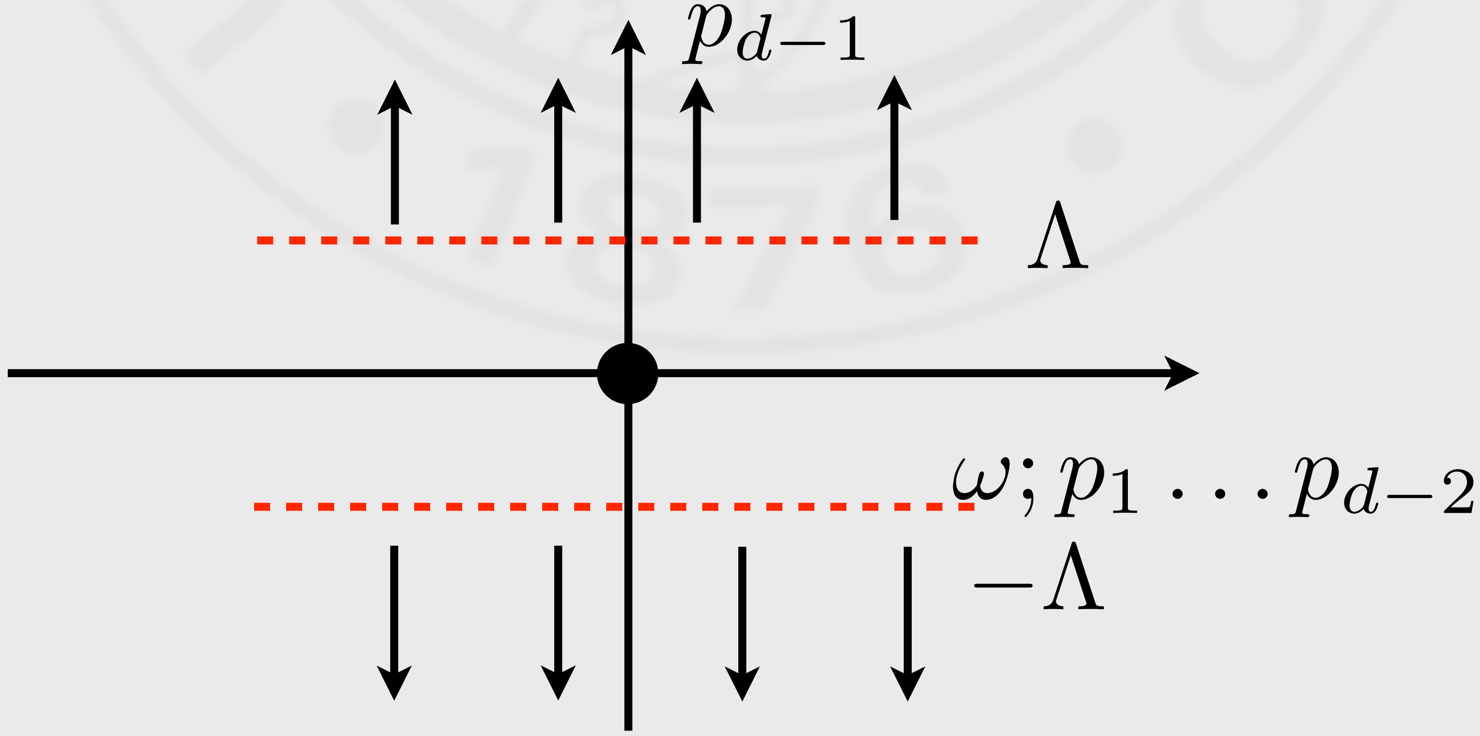

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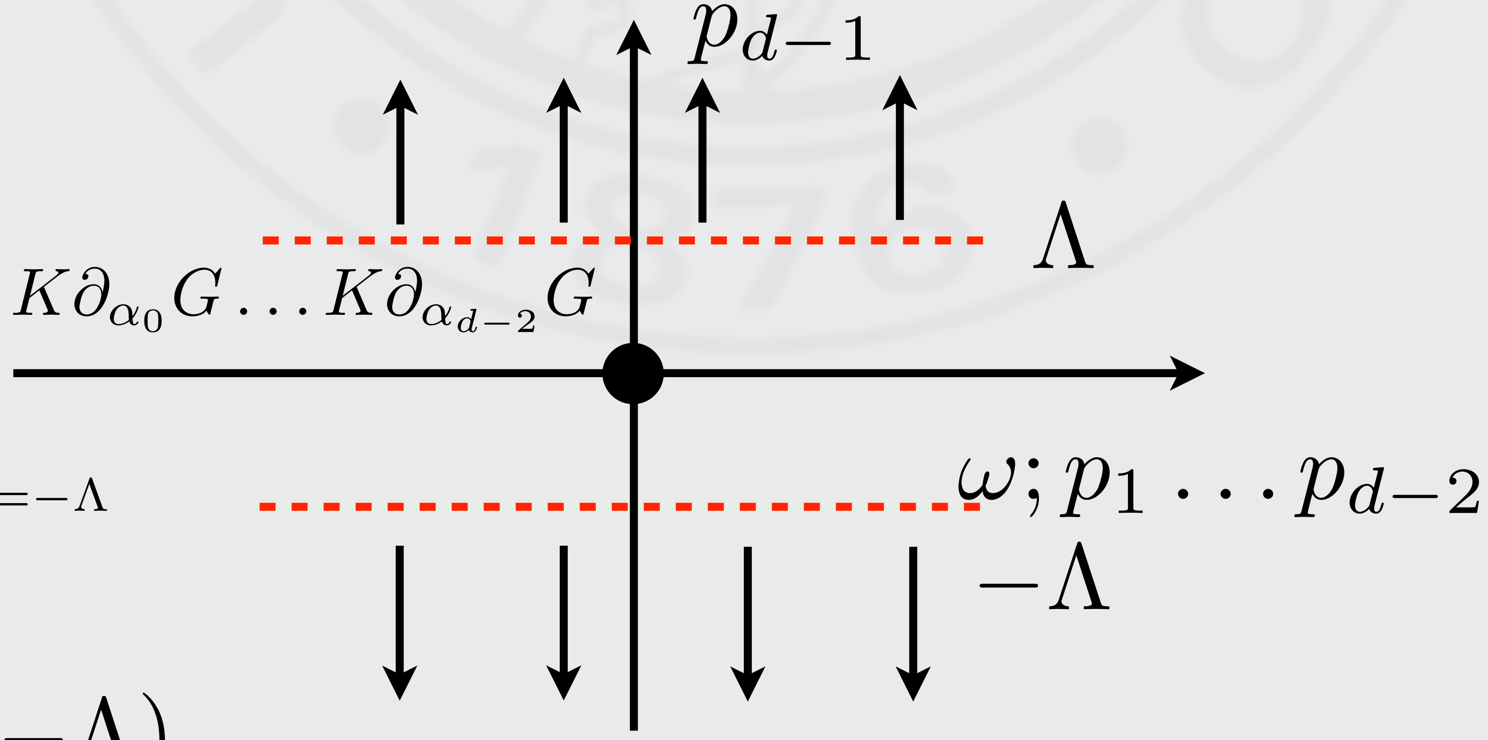
$$N_{d-2}(p_{d-1}) = C_{d-2} \epsilon_{\alpha_0 \dots \alpha_{d-2}} \int d\omega d^{d-2}p\, K \partial_{\alpha_0} G \dots K \partial_{\alpha_{d-2}} G$$

$$\int d\mathbf{S}^{d-1} \cdot \mathbf{r} = N_{d-2}|_{p_{d-1}=\Lambda} - N_{d-2}|_{p_{d-1}=-\Lambda}$$



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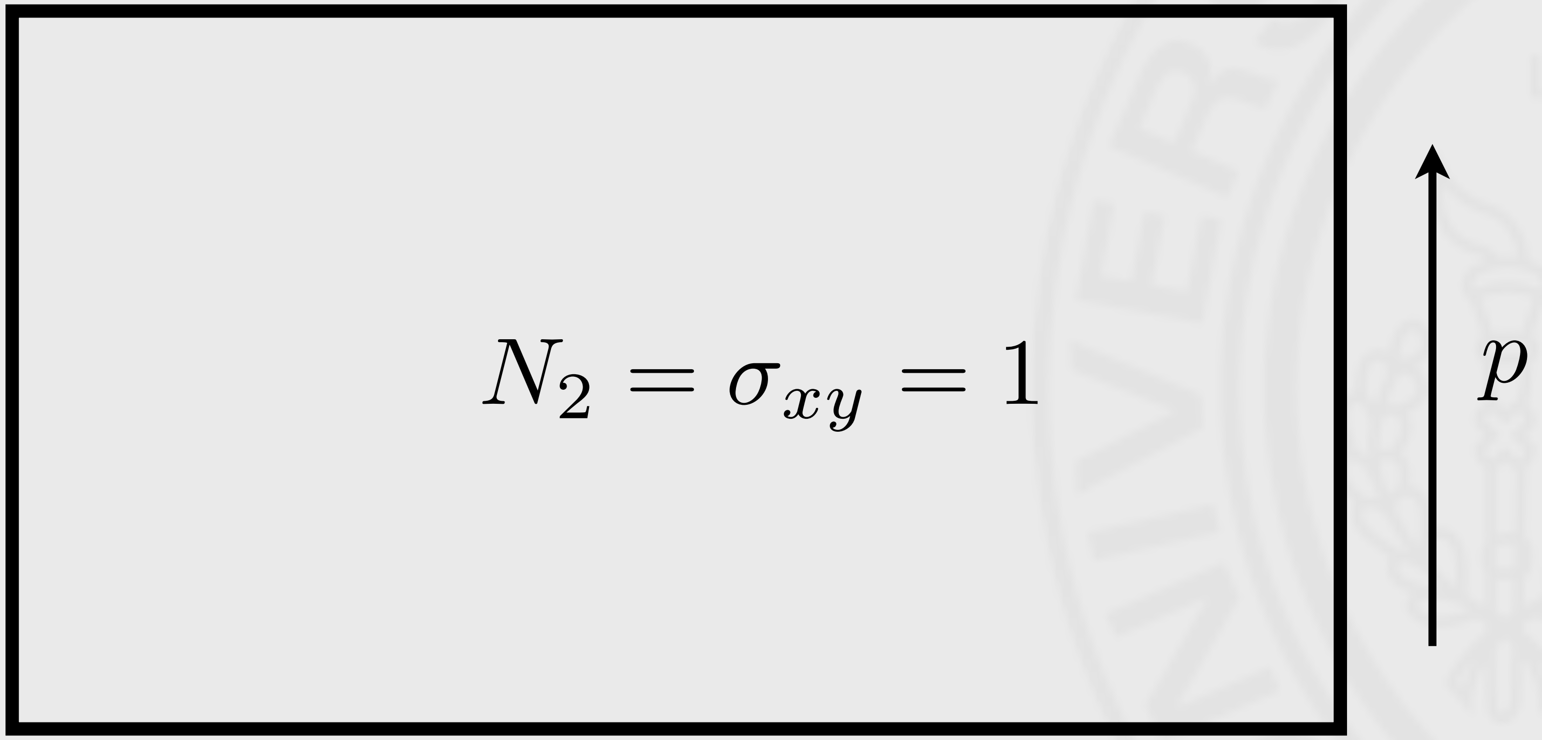
Application 1: IQHE



$$N_2 = \sigma_{xy} = 1$$

p

$$N_0(\Lambda) - N_0(-\Lambda) = 1$$

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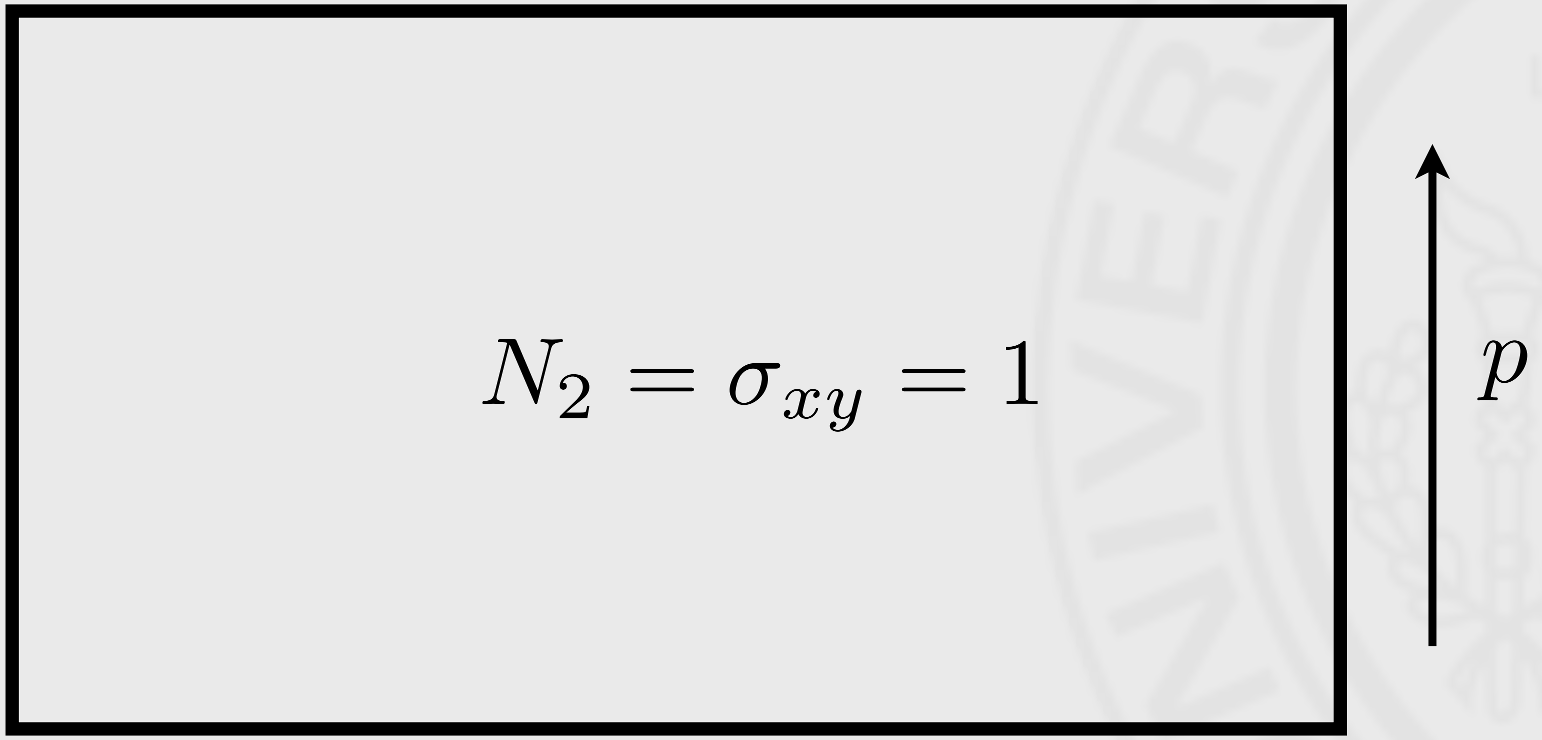

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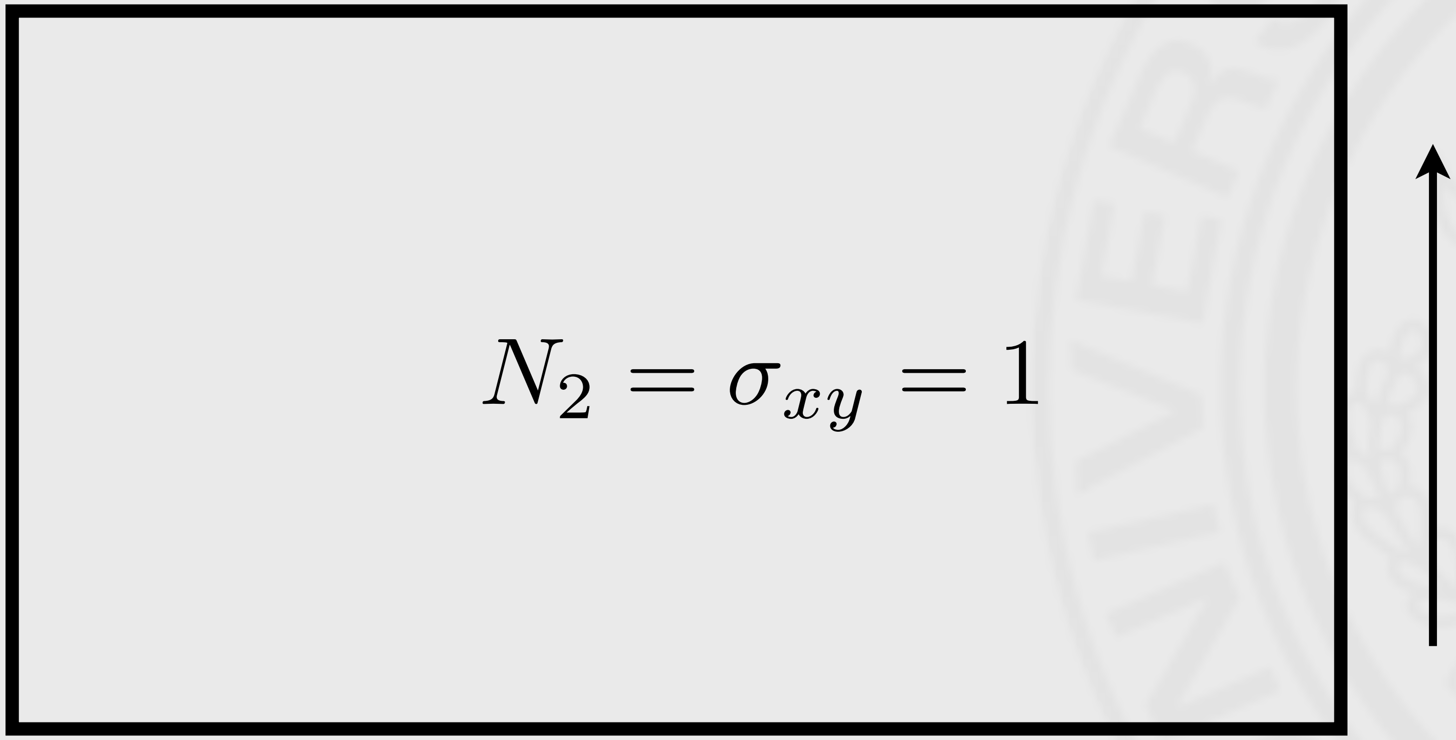

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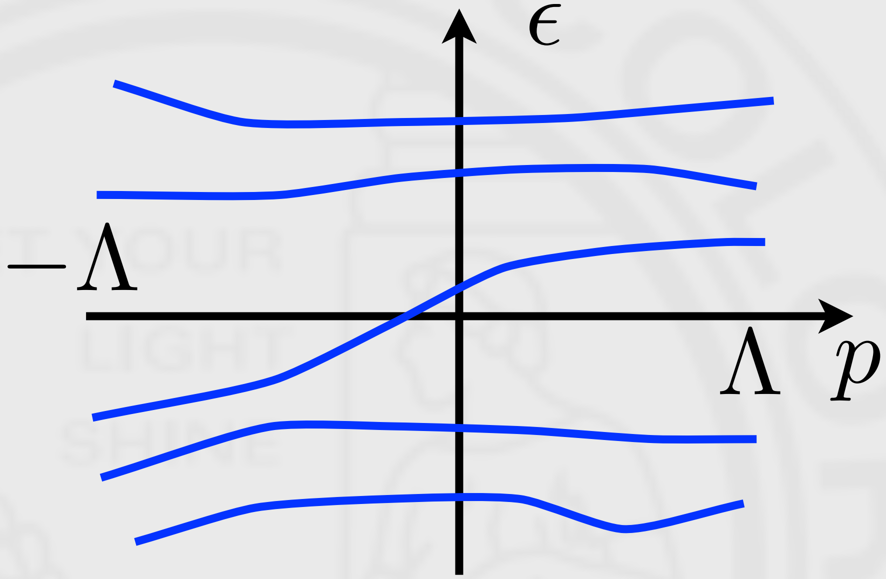

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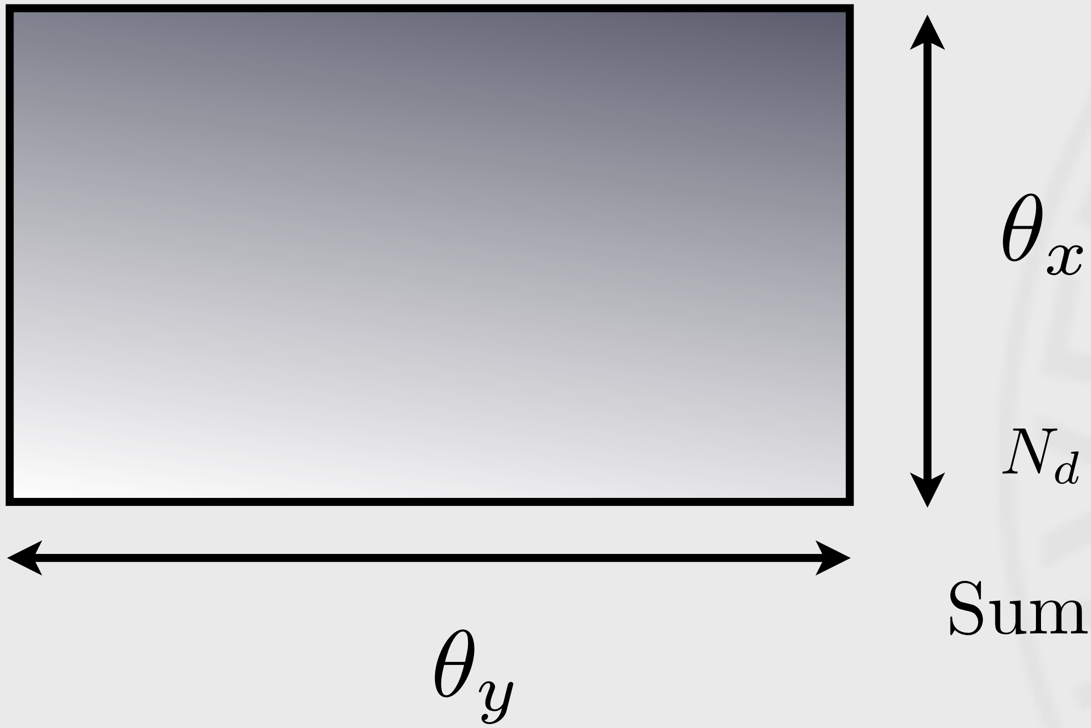
There has to be a level such that $\epsilon_m(\Lambda) > 0, \;\; \epsilon_m(-\Lambda) < 0$

This is the edge state!

Application 2: disorder

Old idea of Thouless, Wu, Niu: impose phases across the system



$$G_{ij}(\omega, \theta_x, \theta_y \dots)$$

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Summation over each $\alpha = \omega, \theta_1, \dots, \theta_d$ is implied

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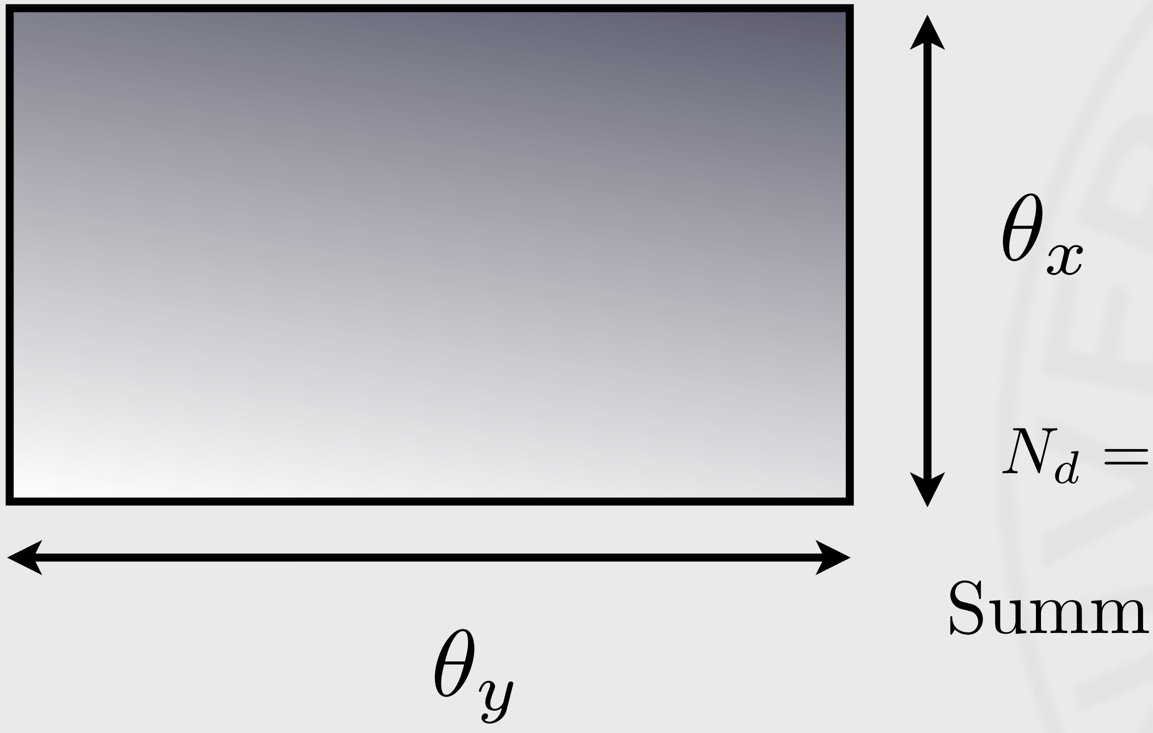

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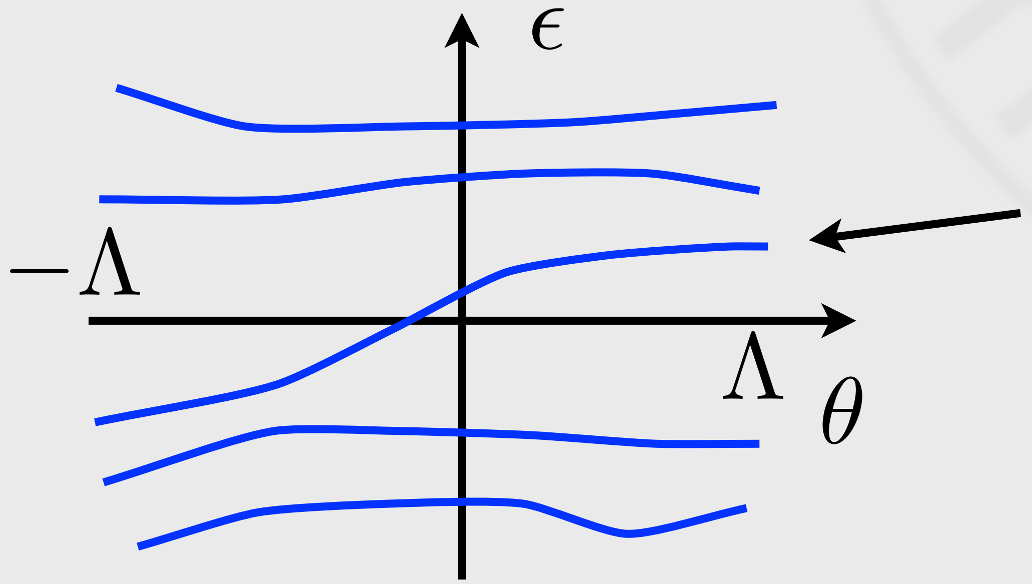

This edge level must be delocalized

$$N_0(\Lambda) - N_0(-\Lambda) = 1$$

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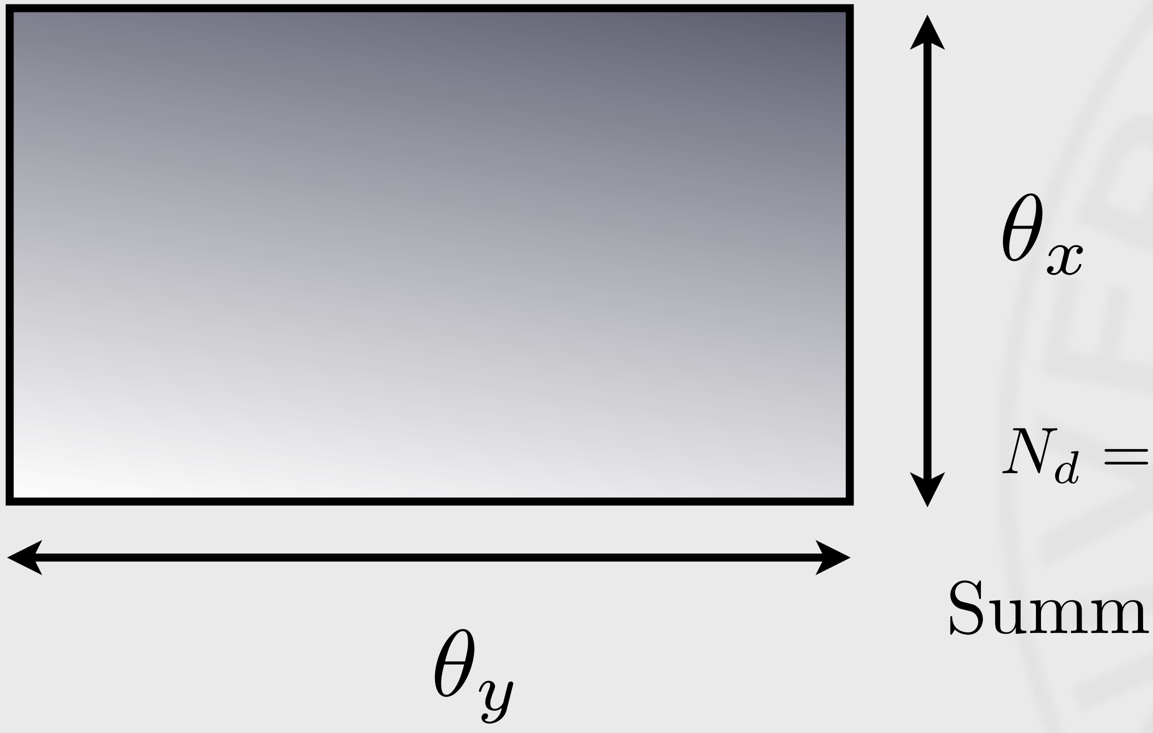

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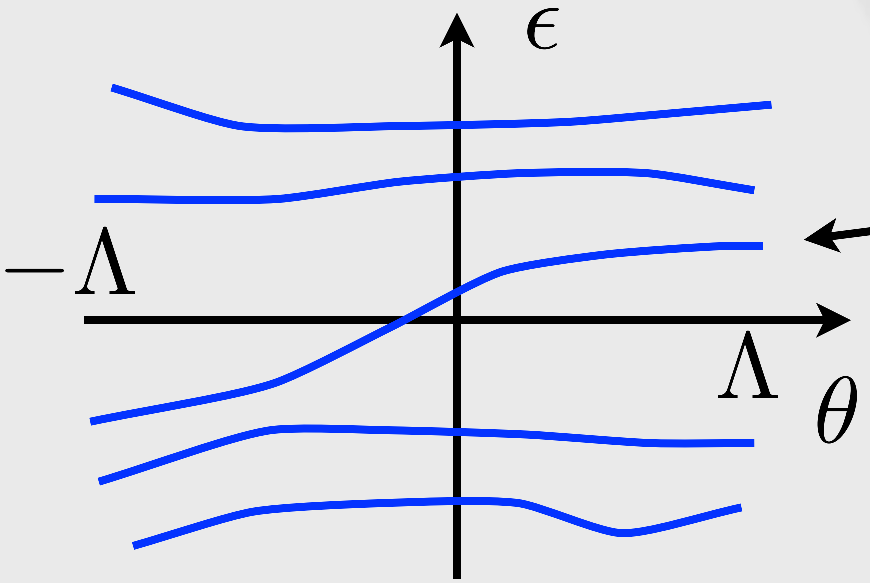

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New results:

A. Essin, A. Altland, M. Mueller, VG, in preparation

Application 3: 1D insulators

Need chiral symmetry: $G_{ij}(\omega) = -\sum_{kl} \Sigma_{ik} G_{kl}(-\omega) \Sigma_{lj}$

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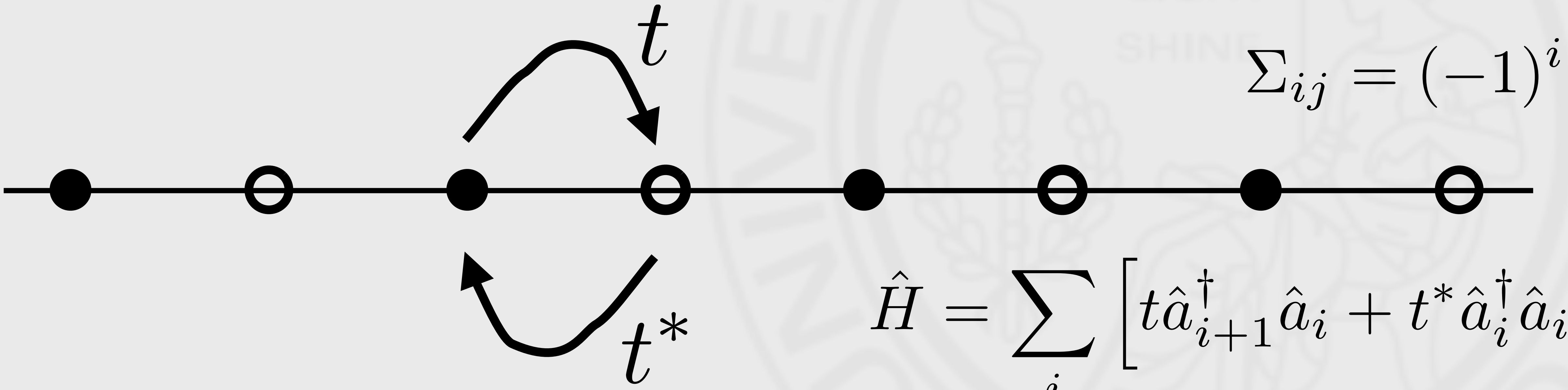
Often realized as hopping on a bipartite lattice



$$\Sigma_{ij} = (-1)^i \delta_{ij}$$

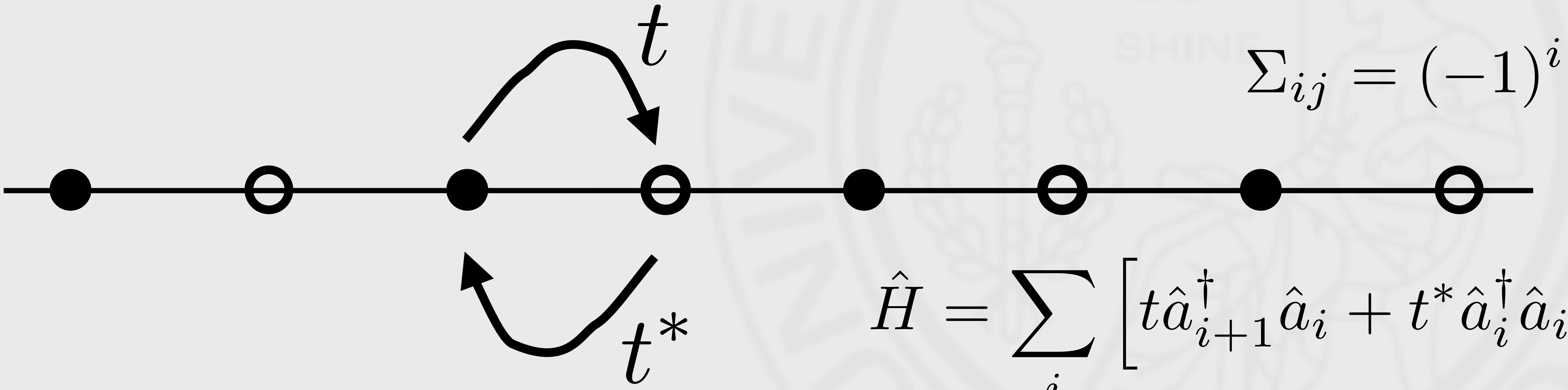
$$\hat{H} = \sum_i \left[t \hat{a}^\dagger_{i+1} \hat{a}_i + t^* \hat{a}^\dagger_i \hat{a}_{i+1} \right]$$

Topological invariant: $N_1 = \operatorname{tr} \int \frac{dp}{4\pi i} \Sigma G^{-1} \partial_p G \Big|_{\omega=0}$

Application 3: 1D insulators

Need chiral symmetry: $$G_{ij}(\omega) = - \sum_{kl} \Sigma_{ik} G_{kl}(-\omega) \Sigma_{lj}$$

Often realized as hopping on a bipartite lattice



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Topological invariant: $$N_1 = \mathrm{tr} \int \frac{dp}{4\pi i} \, \Sigma G^{-1} \partial_p G \Big|_{\omega=0}$$

$N_1 = N_{-1}(\Lambda) - N_{-1}(-\Lambda)?$ = # zero energy states at the boundary

$$\hat{H} = \sum_i \left[\left(t + (-1)^i \delta t \right) \hat{a}^\dagger_{i+1} \hat{a}_i + \left(t + (-1)^i \delta t \right) \hat{a}^\dagger_i \hat{a}_{i+1} \right] \qquad N_1 = \theta(\delta t)$$

Application 3: 1D interacting insulators

More generally, need a “particle-hole” symmetry:

$$\hat{\Sigma}^\dagger a_i^\dagger \hat{\Sigma} = \Sigma_{ij} a_j$$

$$\hat{\Sigma}^\dagger \hat{H} \hat{\Sigma} = \hat{H}^*$$

Example: particles hopping on a bipartite lattice with Hubbard interactions

$$G_{ij}(\omega) = -\sum_{kl} \Sigma_{ik} G_{kl}(-\omega) \Sigma_{lj}$$

$$N_1 = \operatorname{tr} \int \frac{dp}{4\pi i} \Sigma G^{-1} \partial_p G \Big|_{\omega=0}$$

$N_1 =$ # zero energy states at the boundary + # of zeros at the boundary

$$G_{ij}|_{\omega=0}\, \psi_j = 0.$$

this is a zero

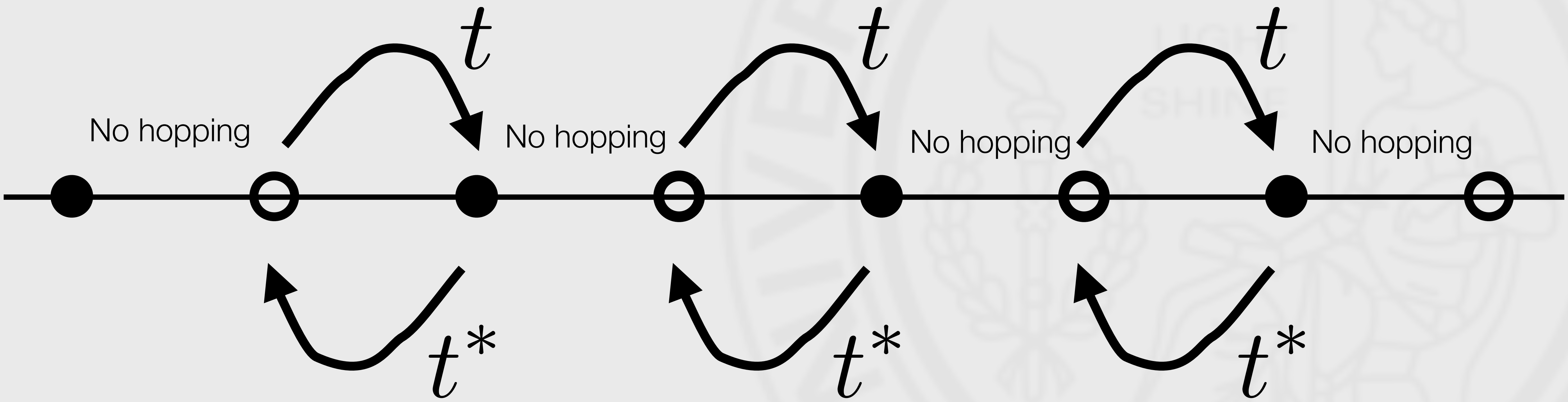
no interactions

$$G = [i\omega - H]^{-1}$$

no zeros

1D interacting model

Spin 1/2 fermion hopping on a 1D lattice with a large Hubbard repulsion ***U***, <u>one</u> fermion per site



Top invariant

$$N_1 = \mathrm{tr} \int \frac{dp}{4\pi i} \Sigma G^{-1} \partial_p G \Big|_{\omega=0} = 1$$

Yet large single particle gap ***U***

Where are the edge states?

S. Manmana, A. Essin, VG, work in progress

1D interacting model

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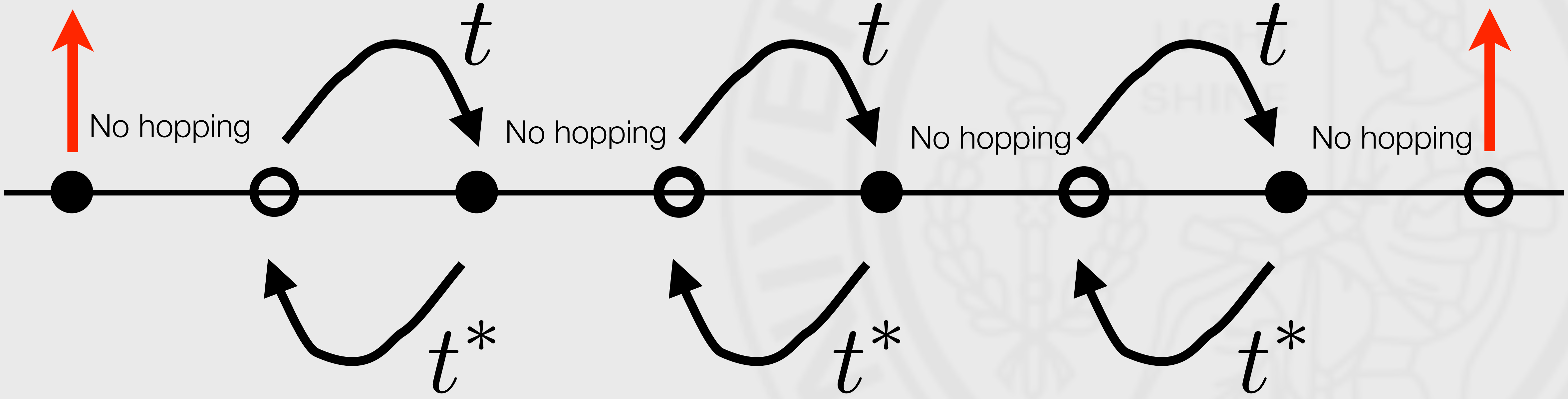

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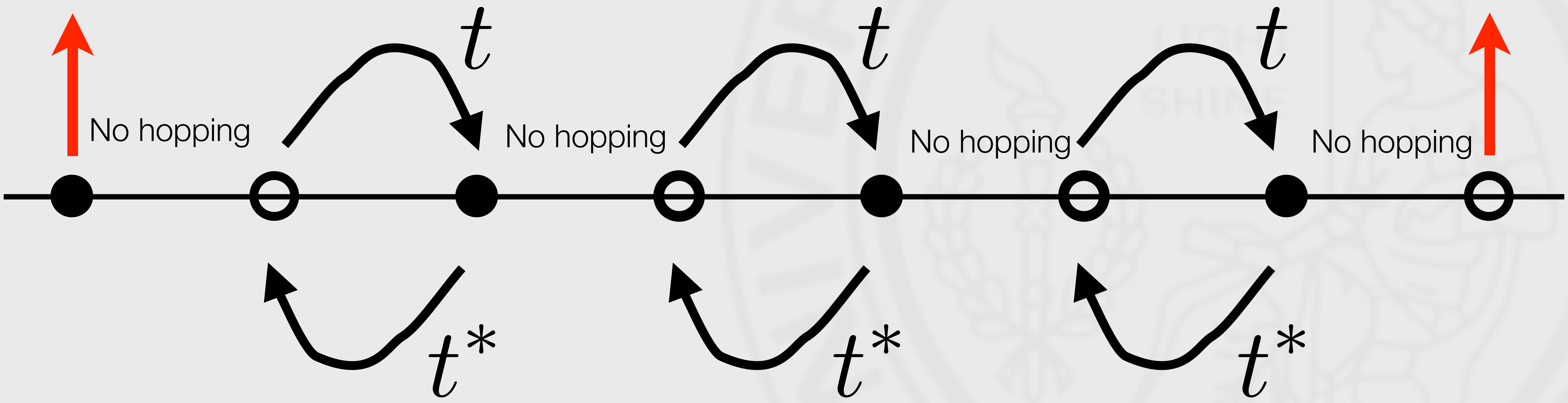

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Where are the edge states?

4 ground states which break particle-hole (chiral) symmetry

$$G(\omega) \neq -\Sigma G(-\omega) \Sigma$$

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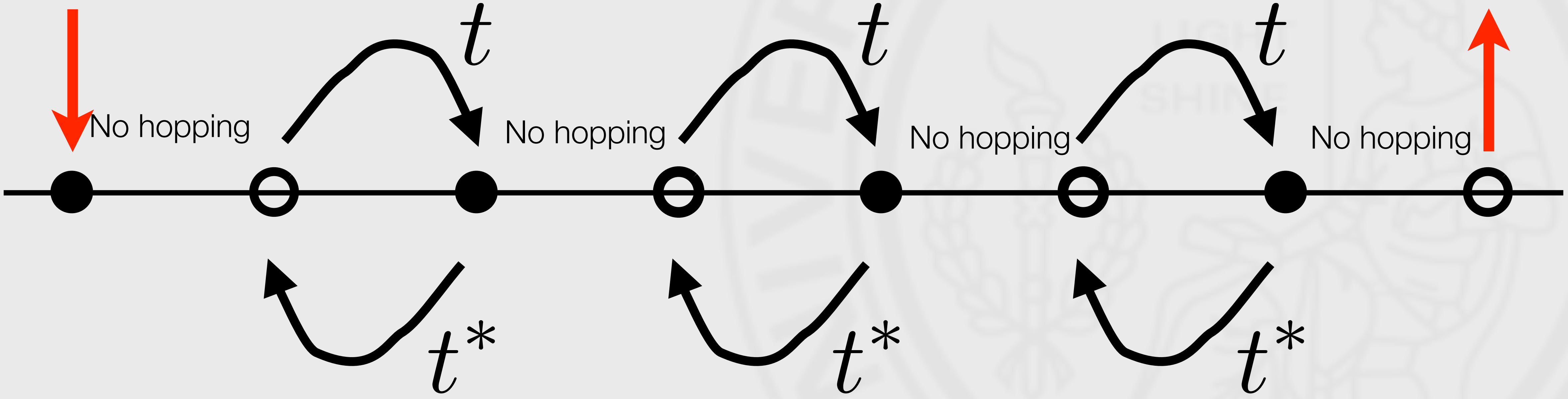

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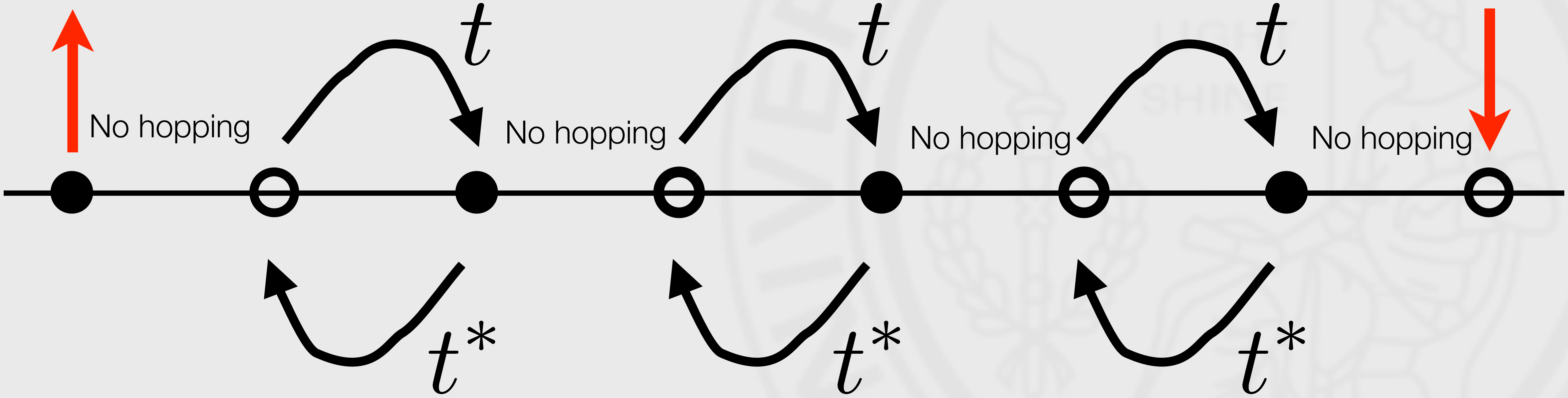

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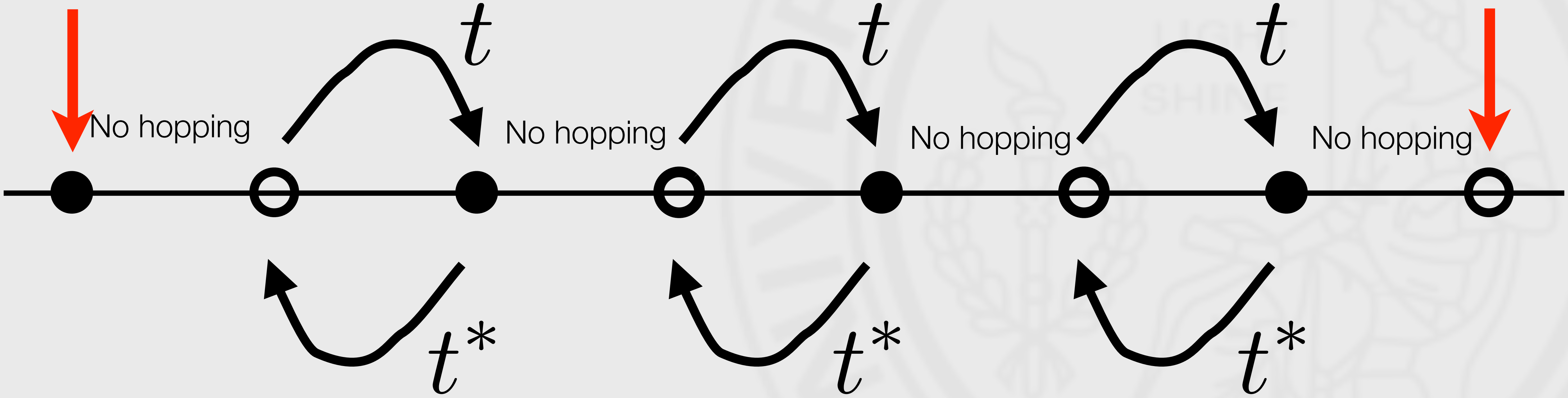

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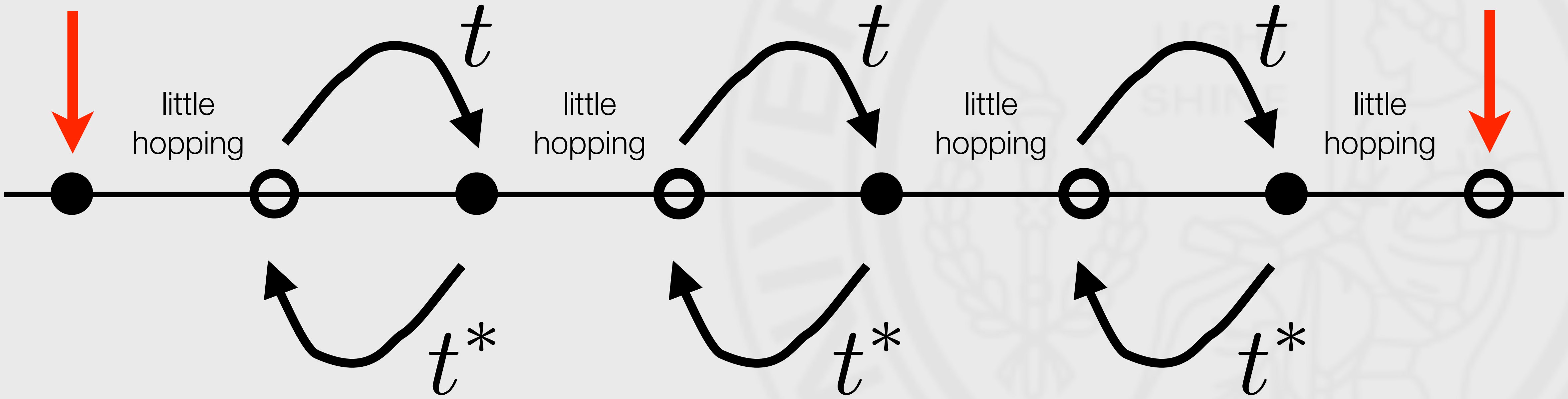

Top invariant

$$N_1 = \mathrm{tr} \int \frac{dp}{4\pi i} \Sigma G^{-1} \partial_p G \Big|_{\omega=0} = 1$$

Yet large single particle gap ***U***

Where are the edge states?

Zeros at the edge

$$G_{ij}(0)\psi_j = 0$$

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1D interacting model

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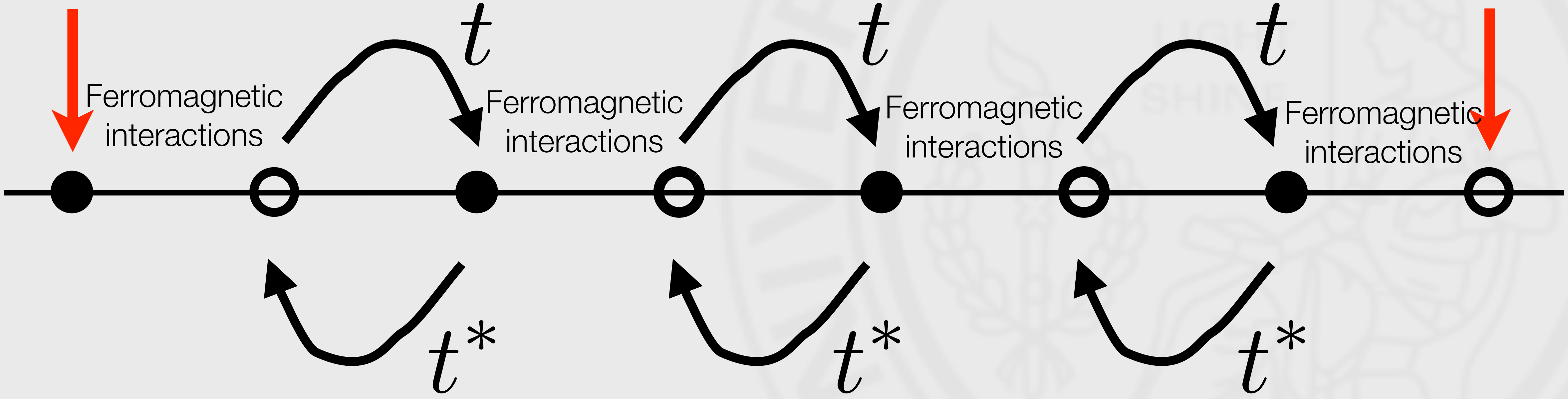

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Yet large single particle gap ***U***

Where are the edge states?

Haldane chain

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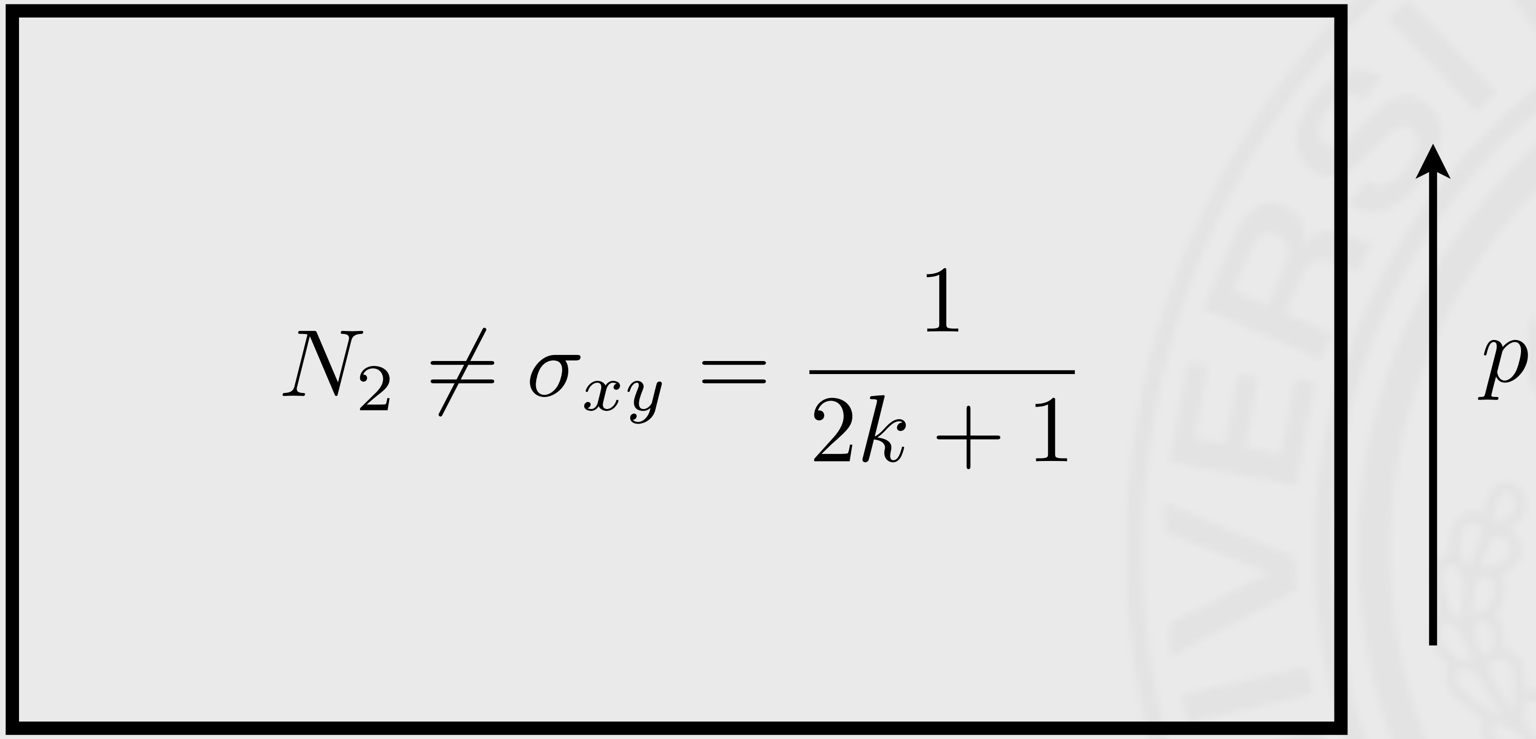
Application 4: FQHE at simple fractions



$$N_2 \neq \sigma_{xy} = \frac{1}{2k+1}$$

Edge Green's function

$$G = \frac{(i\omega + vp)^{2k}}{i\omega - vp}$$

X.G. Wen, 1989

$$N_0(p) = \int \frac{d\omega}{2\pi i} K \partial_\omega G = \frac{1}{2} \sum_n \left[\underbrace{\operatorname{sign} \epsilon_n(p)}_{\text{poles}} - \underbrace{\operatorname{sign} r_n(p)}_{\text{zeros}} \right]$$

$$N_0(\Lambda) - N_0(-\Lambda) = 2k + 1 = N_2$$

From Thouless' to Wen's topological order?

Application 5: topological insulators (class AII)

$\mathbb{Z}_2$ structure has to be studied by dimensional reduction

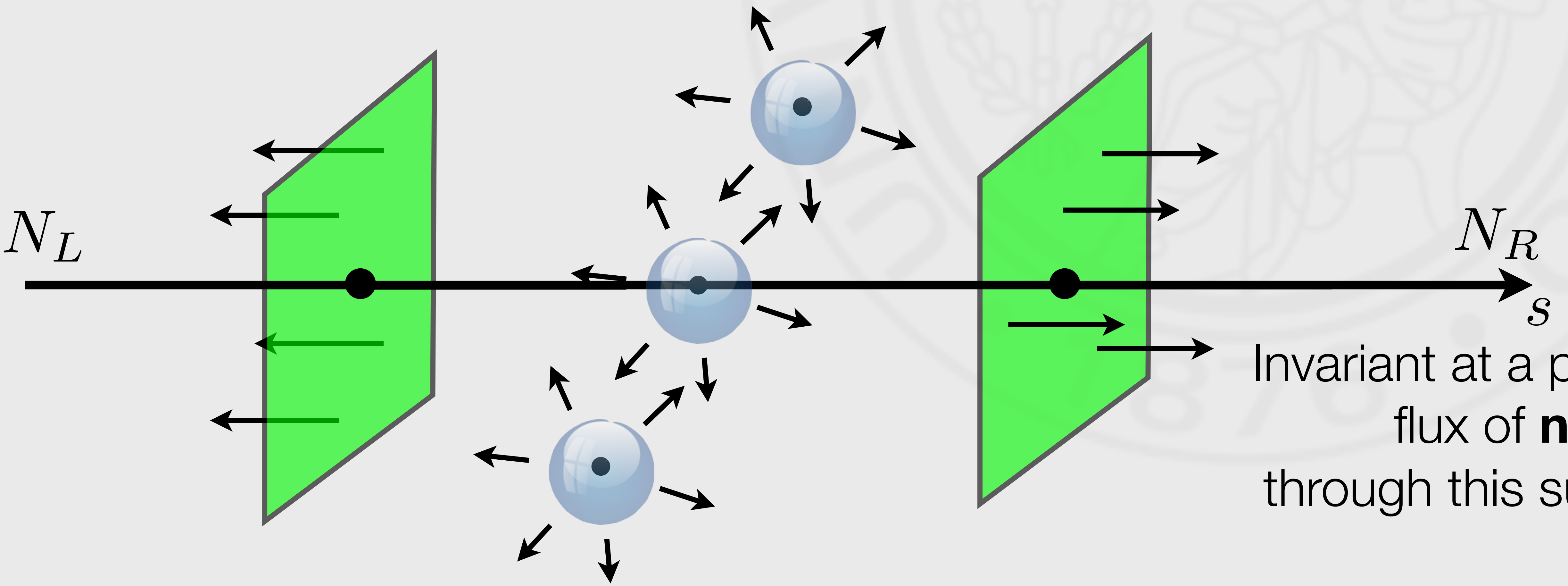
$$G(\omega, p_1, p_2) \to G(\omega, p_1, p_2, q_1, q_2)$$

unphysical momenta

$$G(\omega, \mathbf{p}, \mathbf{q}) = \sigma_y G^T(\omega, -\mathbf{p}, -\mathbf{q})\sigma_y$$

TR invariance

$$N_4 = \text{odd}$$



Invariant at a point s:
flux of **n**
through this surface

Odd N_4 = Odd # edge modes = 1 mode at ***q***=0 =
physical edge mode

Application 6: Fractional topological insulators

Edge Green's functions

1. Two FQHE with opposite chirality

$$G(\omega,p)=\begin{pmatrix}\frac{(i\omega+vp)^{2k}}{i\omega-vp} & 0\\ 0 & \frac{(i\omega-vp)^{2k}}{i\omega+vp}\end{pmatrix}$$

2. Add ***q***-dependence

$$G=\begin{pmatrix}\frac{i\omega+p}{\omega^2+p^2+q^2}\left(\frac{(i\omega+p)^2}{\omega^2+\Lambda^2}\right)^n & \frac{q_x+iq_y}{\omega^2+p^2+q^2}\\ \frac{q_x-iq_y}{\omega^2+p^2+q^2} & \frac{i\omega-p}{\omega^2+p^2+q^2}\left(\frac{(i\omega-p)^2}{\omega^2+\Lambda^2}\right)^n\end{pmatrix}$$

3. Calculate $N_4 = N_2|_{p=\Lambda} - N_2|_{p=-\Lambda} = 2k+1$

4. Topologically protected since $N_4 = \mathrm{odd}$

Conclusions

Bulk-edge correspondence is an interesting tool.

The end