Parton models of 3D fractional topological insulators

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JM, X. L. Qi, A. Karch, and S. C. Zhang, arXiv:1004.3628 [PRL 105, 246809 (2010)]

Topological states of matter

	"Thouless" topological order	"Wen" topological order
T-breaking	IQHE	FQHE
T-invariant	QSHE ($\mathbf{Z}_2$ 2D TI)	FQSHE (Bernevig-Zhang, Levin-Stern)
	QME ($\mathbf{Z}_2$ 3D TI)	?

A "fractional" QME?

- IQHE: $\sigma_{xy} = ne^2/h$, FQHE: $\sigma_{xy} = (p/q)e^2/h$
- QME: $\theta = \pi$, "FQME": $\theta = (p/q)\pi$?
- We know materials that have fractional θ, e.g. multiferroics (cf. Joel Moore's talk: Cr_2O_3 has $\theta \approx \pi/24$)… but they break TRS
- Indeed, we know that $\theta = 0$ or π is required by TRS:

$$(-1)^{\theta/\pi} = \text{Fu-Kane-Mele } \mathbf{Z}_2 \text{ invariant}$$

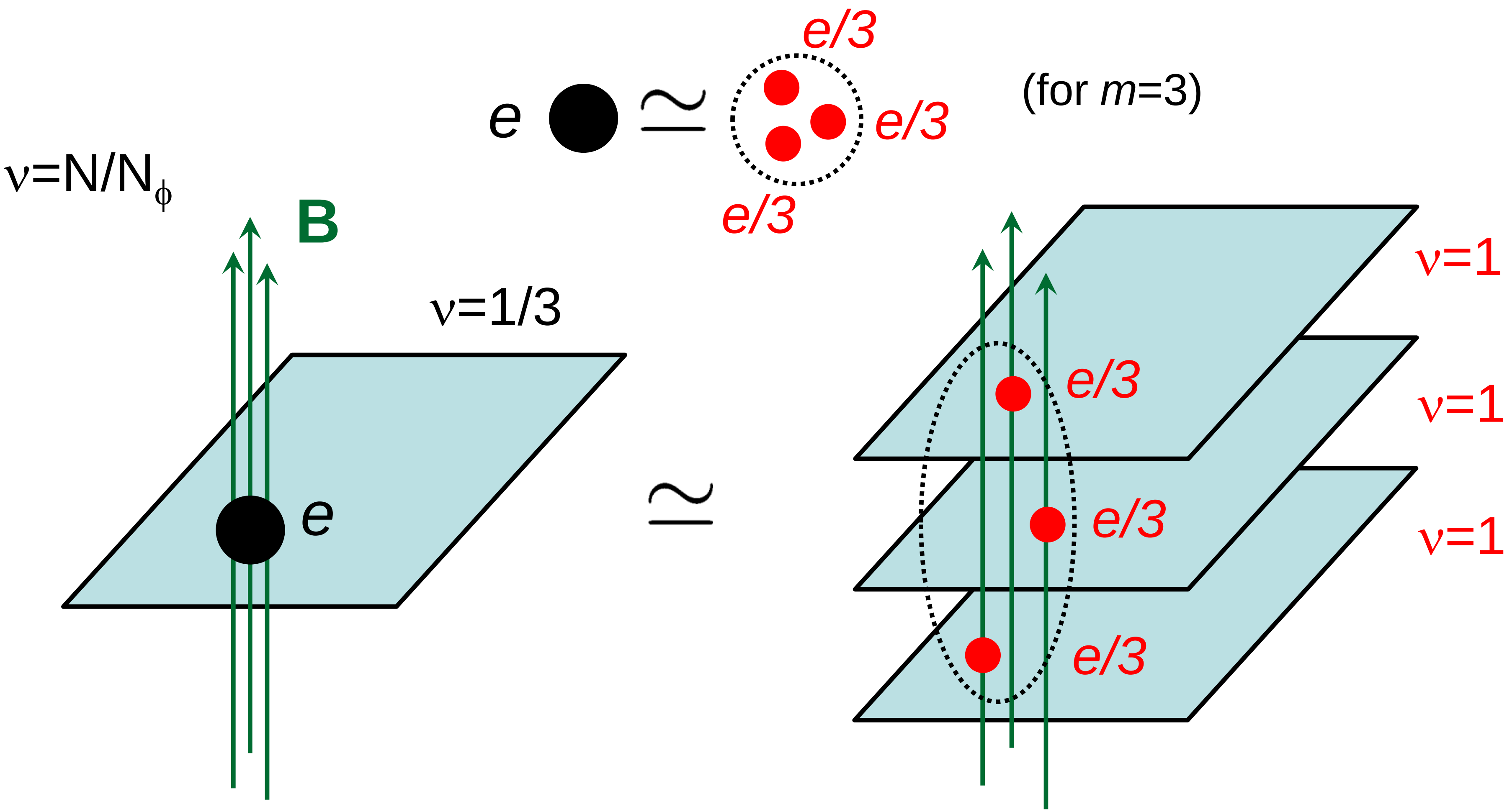
- Even if θ is fractional, why should it be quantized?

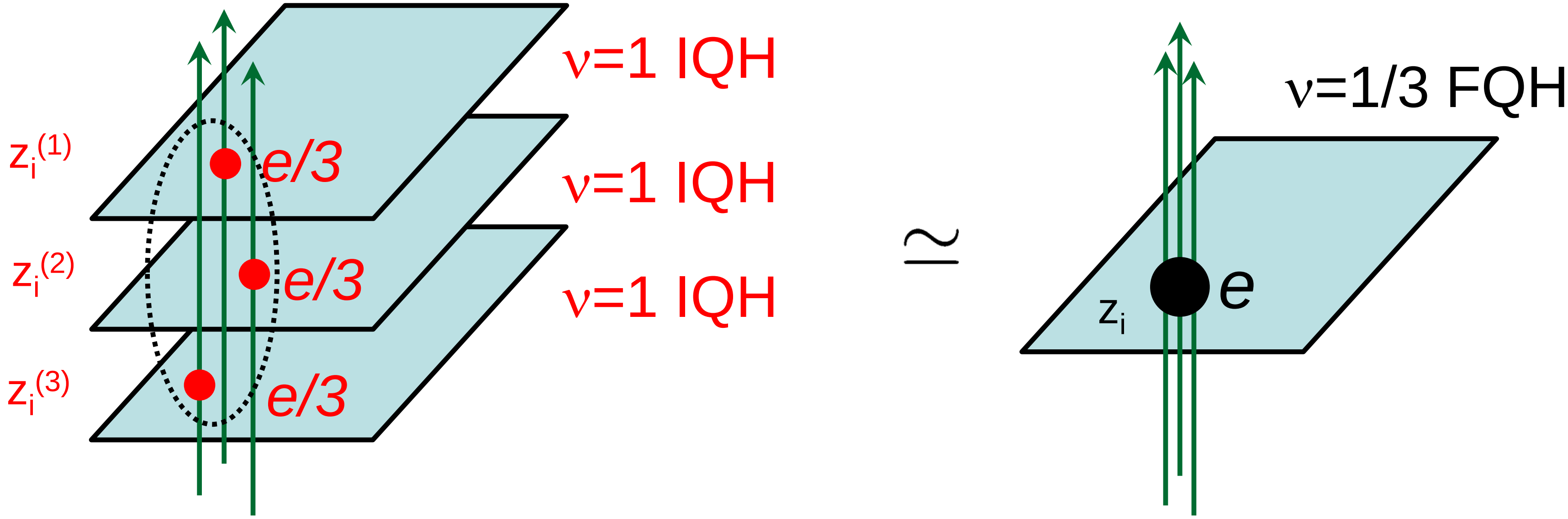
Outline

1. Parton construction of FQH states
2. Parton construction of fractional 3D TI
3. Quantized ME effect and TRS
4. Possible effective gauge theories
5. A microscopic model?

Parton construction of FQH states

- A simple way to understand the FQHE from the IQHE (Jain, 1989; Wen, 1991, 1992, 1999; Barkeshli, Wen, 2010)
- Break electron into *m* partons; each parton forms a $\nu=1$ IQH state





$$\Psi_{\nu=1}(\{z_i^{(1)}\})\Psi_{\nu=1}(\{z_i^{(2)}\})\Psi_{\nu=1}(\{z_i^{(3)}\})$$

$$\sim \prod_{i<j}(z_i^{(1)} - z_j^{(1)}) \prod_{i<j}(z_i^{(2)} - z_j^{(2)}) \prod_{i<j}(z_i^{(3)} - z_j^{(3)})$$

projection

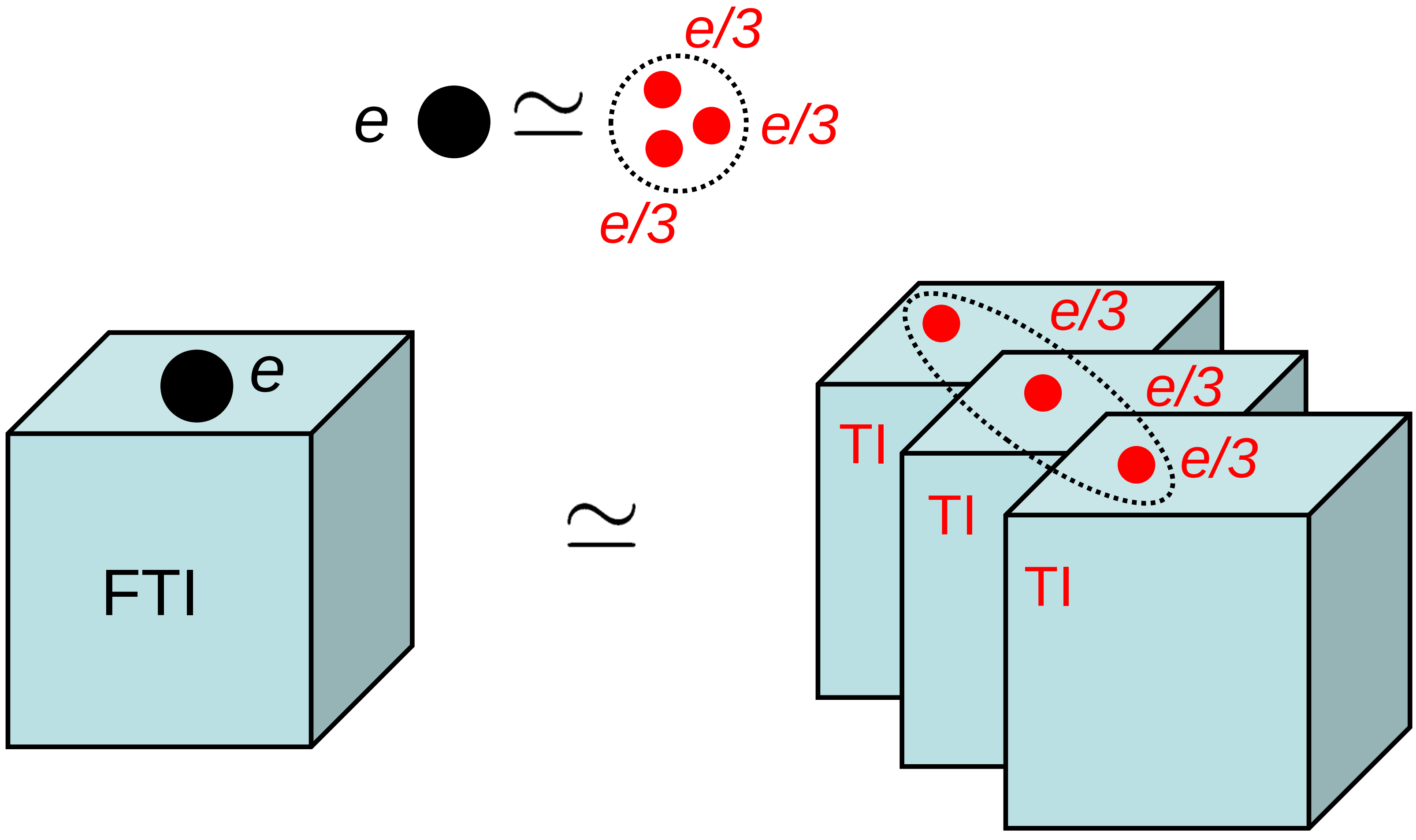
$$z_i^{(1)} = z_i^{(2)} = z_i^{(3)} = z_i$$

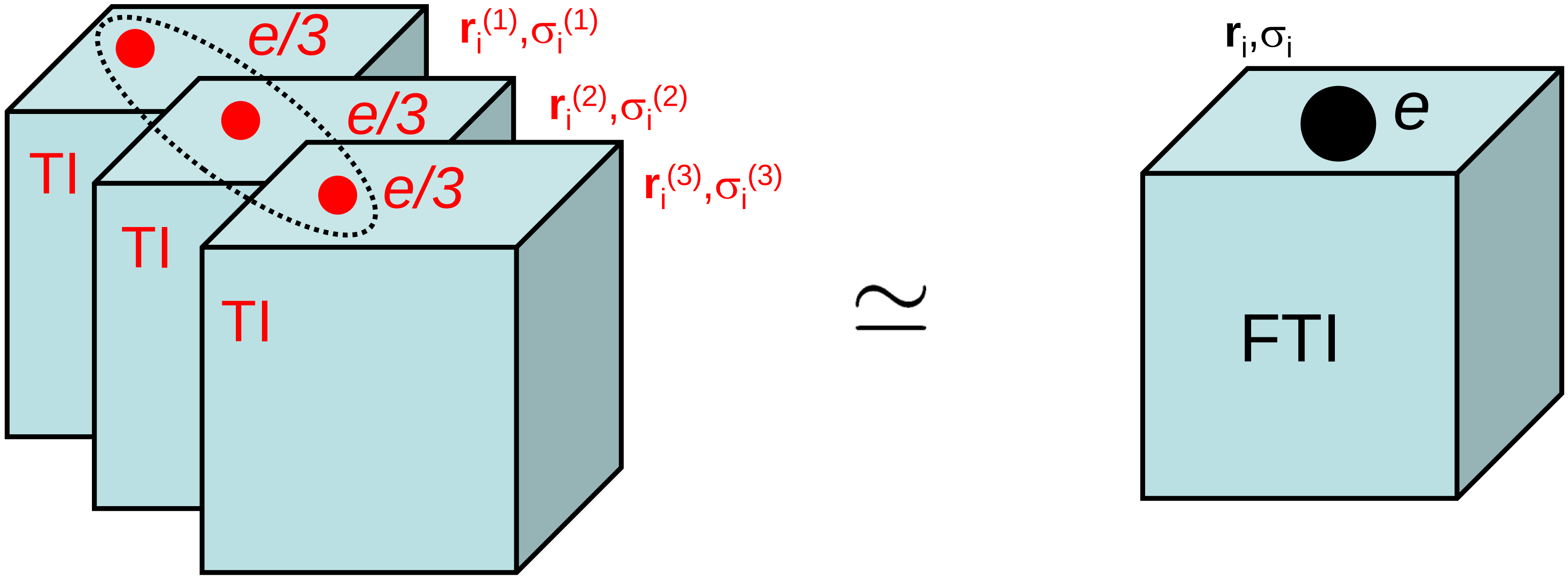
$$\longrightarrow \prod_{i<j}(z_i - z_j)^3 \sim \Psi_{\nu=1/3}(\{z_i\})$$

ν=1/3 Laughlin state

Parton construction of fractional 3D TI

- Use the same construction to define a 3D FTI state based on the topological band insulator state
- Break electron into *m* partons; each parton forms a $\mathbf{Z}_2$ TI





$$\Psi_{\mathbb{Z}_2}(\{\mathbf{r}_i^{(1)},\sigma_i^{(1)}\})\Psi_{\mathbb{Z}_2}(\{\mathbf{r}_i^{(2)},\sigma_i^{(2)}\})\Psi_{\mathbb{Z}_2}(\{\mathbf{r}_i^{(3)},\sigma_i^{(3)}\})$$

projection →

$$(\mathbf{r}_i^{(1)},\sigma_i^{(1)}) = (\mathbf{r}_i^{(2)},\sigma_i^{(2)}) = (\mathbf{r}_i^{(3)},\sigma_i^{(3)}) = (\mathbf{r}_i,\sigma_i)$$

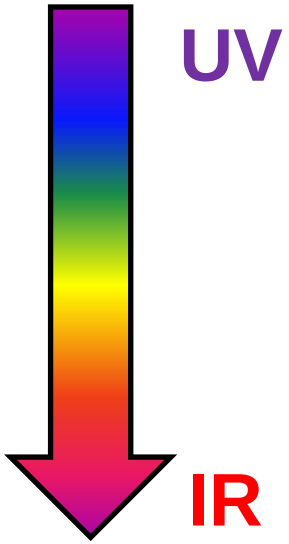
$$[\Psi_{\mathbb{Z}_2}(\{\mathbf{r}_i,\sigma_i\})]^3$$
$$\equiv \Psi_{\mathrm{FTI}}(\{\mathbf{r}_i,\sigma_i\})$$

- This is just one wave function… is it representative of a stable state of matter?
- If yes, what are the physically measurable properties of this state?

Effective field theory

- Beyond writing down a wave function, parton construction can be used to construct an effective field theory (Wen, 1991, 1992, 1999; Barkeshli, Wen, 2010)
- If the effective field theory describes a stable fixed point of the RG, we have a stable phase of matter
- From the effective field theory, we can extract physical properties

- Strategy:
  - Lattice model of interacting electrons
  - Low-energy effective field theory with gapped partons
  - Integrate out gapped partons; topological field theory?



- Imagine adding interactions to a lattice model of $\mathbf{Z}_2$ TI:

$$H = \sum_{ij} h_{ij}^{\alpha\beta} c_{i\alpha}^{\dagger} c_{j\beta} + V(c^{\dagger}, c)$$

single-particle TI Hamiltonian (Dirac model, flat band model…)

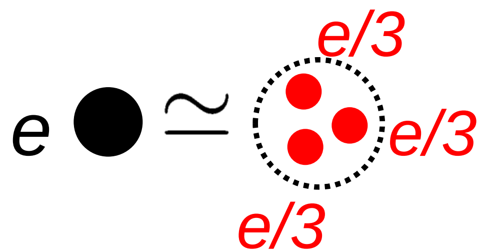
site index

orbital+spin index

strong e-e interactions

- Write the electron operator in terms of fermionic parton (fractionalized) variables:

$$c_{i\alpha} = \psi_{i\alpha}^{(1)} \psi_{i\alpha}^{(2)} \psi_{i\alpha}^{(3)} = \frac{1}{3!} \epsilon_{abc} \psi_{i\alpha}^{(a)} \psi_{i\alpha}^{(b)} \psi_{i\alpha}^{(c)}$$



- We have a complicated model of interacting partons:

$$H = \frac{1}{(3!)^2} \sum_{ij} h_{ij}^{\alpha\beta} \epsilon_{abc} \epsilon_{a'b'c'} \psi_{i\alpha}^{(c)\dagger} \psi_{i\alpha}^{(b)\dagger} \psi_{i\alpha}^{(a)\dagger} \psi_{j\beta}^{(a')} \psi_{j\beta}^{(b')} \psi_{j\beta}^{(c')} + V(\psi^\dagger, \psi)$$

- Do a mean-field approximation:

$$H \simeq \sum_{ij} h_{ij}^{\alpha\beta} U_{ij,aa'}^{\alpha\beta} \psi_{i\alpha}^{(a)\dagger} \psi_{j\beta}^{(a')} + \ldots$$

we have a 3x3 matrix variational parameter:

$$U_{ij,aa'}^{\alpha\beta} = \frac{1}{(3!)^2} \epsilon_{abc} \epsilon_{a'b'c'} \langle \psi_{i\alpha}^{(c)\dagger} \psi_{i\alpha}^{(b)\dagger} \psi_{j\beta}^{(b')} \psi_{j\beta}^{(c')} \rangle$$

- Depending on detail of interactions, we can have various saddle points. Consider a simple saddle point:

$$U^{\alpha\beta}_{ij,aa'} = \eta \delta_{aa'}$$

$$H \simeq \eta \sum_{ij} h^{\alpha\beta}_{ij} \psi^{(a)\dagger}_{i\alpha} \psi^{(a)}_{j\beta} + \dots$$

- The MF Hamiltonian has a global U(3) symmetry

$$\psi^{(a)} \to W_{ab} \psi^{(b)}, \qquad W \in U(3)$$

- Consider another saddle point which is less symmetric:

$$U_{ij,aa'}^{\alpha\beta} = \begin{pmatrix} \eta_1 & & \\ & \eta_2 & \\ & & \eta_3 \end{pmatrix}$$

$$H \simeq \sum_{ij} h_{ij}^{\alpha\beta} \eta_a \psi_{i\alpha}^{(a)\dagger} \psi_{j\beta}^{(a)} + \dots$$

- Now, the MF Hamiltonian only has a global U(1)^3 symmetry

$$\psi^{(a)} \to e^{i\phi_a} \psi^{(a)}, \qquad a = 1, 2, 3$$

$$\Leftrightarrow \psi^{(a)} \to W_{ab} \psi^{(b)}, \qquad W \in U(1)^3$$

- Finally, consider a generic saddle point with no symmetries:

$$U_{ij,aa'}^{\alpha\beta} = \eta_{aa'}$$

$$H \simeq \sum_{ij} h_{ij}^{\alpha\beta} \eta_{aa'} \psi_{i\alpha}^{(a)\dagger} \psi_{j\beta}^{(a')} + \ldots$$

- Now, the MF Hamiltonian only has a global U(1) symmetry

$$\psi^{(a)} \to e^{i\phi} \psi^{(a)}, \qquad a = 1, 2, 3$$

$$\Leftrightarrow \psi^{(a)} \to W_{ab} \psi^{(b)}, \qquad W \in U(1)$$

Fluctuations and emergent gauge structure

- Problem: Hilbert space of partons is bigger than "physical" Hilbert space of original electrons! But dimension of Hilbert space cannot change because of interaction effects…

- Should only keep those unitary transformations which leave the physical electron operator invariant:

$$\psi^{(a)} \to W_{ab}\psi^{(b)}$$

$$c = \frac{1}{3!}\epsilon_{abc}\psi^{(a)}\psi^{(b)}\psi^{(c)} \to \det W \cdot c$$

hence we need **det W = 1**

- Finally, including the fluctuations above MF turns the global symmetry into a (local) gauge structure

- Gauge structure depends on symmetry of saddle point solution:

$$U^{\alpha\beta}_{ij,aa'} = \eta\delta_{aa'} :$$ global U(3) becomes gauge SU(3)

$$U^{\alpha\beta}_{ij,aa'} = \begin{pmatrix} \eta_1 & & \\ & \eta_2 & \\ & & \eta_3 \end{pmatrix} :$$ global $U(1)^3$ becomes gauge U(1)xU(1)

$$U^{\alpha\beta}_{ij,aa'} = \eta_{aa'} :$$ global U(1) becomes gauge $\mathbf{Z}_3$

$$1 = \det W = \det(e^{i\phi}) = e^{3i\phi}$$
$$\Rightarrow W = e^{i\phi} = \sqrt[3]{1} = e^{2\pi i k/3} \in \mathbb{Z}_3, k = 0, 1, 2$$

Low-energy field theory of partons

- Including fluctuations, low-energy effective theory contains **partons in a topological bandstructure** interacting with a **dynamical gauge field**

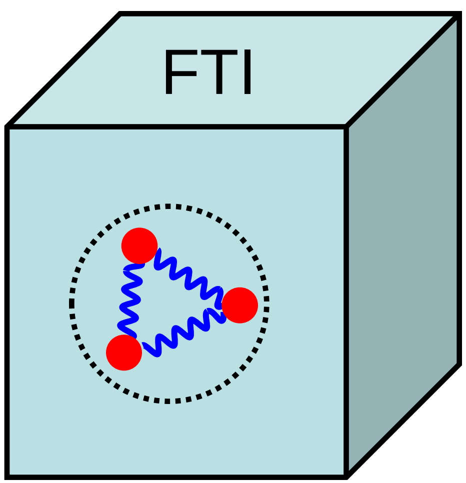
$$\mathcal{L} = \psi^\dagger (iD_0 - H_\theta[-i\mathbf{D}])\psi + \mathcal{L}_{\text{int}}(\psi^\dagger, \psi)$$

θ=π

$$D_\mu = (D_0, -\mathbf{D}) = \partial_\mu + i\frac{e}{\mathcal{N}_c}A_\mu + iga_\mu$$

put back EM field

SU(3), U(1)xU(1), Z_3…



- Does this describe a stable phase of matter?
- If yes, what is its EM response?

Low-energy field theory of partons

- Since the partons are gapped, this is essentially a question about the stable phases of pure gauge theory with gauge group G=SU(3),U(1)xU(1),Z_3…
- SU(3): presumably confining at low energy (although deconfined phases of SU(N) gauge theory are possible if one adds enough gapless, EM neutral matter)
- U(1)xU(1): admits a deconfined, gapless Coulomb phase (Banks, Myerson, Kogut, 1977)
- $\mathbf{Z}_3$: admits a deconfined, gapped phase (Ukawa, Windey, Guth, 1980)
- In the deconfined phase, we can integrate out the partons

Topological field theory of 3D FTI

- Integrate out gapped partons: because of TI bandstructure, get **topological θ-terms (E.B terms)** for the EM and "internal" gauge field a_μ:

over parton degrees of freedom

"internal" field strength

$$\mathcal{L}_{\rm eff} = \frac{\theta}{32\pi^2}\epsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\left(\frac{e}{3}F_{\mu\nu} + g f_{\mu\nu}\right)\left(\frac{e}{3}F_{\lambda\rho} + g f_{\lambda\rho}\right)$$

$$= \frac{\theta_{\rm eff}\, e^2}{32\pi^2}\epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho} + \frac{\theta g^2}{32\pi^2}\epsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\, f_{\mu\nu}f_{\lambda\rho}$$

$$3\times\left(\frac{1}{3}\right)^2 = \frac{1}{3}$$

$$\theta_{\rm eff} = \frac{\theta}{3} = \frac{\pi}{3}$$

DOES THIS PRESERVE TRS?

- Actually a general result: effective θ determined only by the ABJ anomaly coefficient, to all orders in g:

$$C = \sum_i \left(q_i^{\rm em}\right)^2 = \frac{1}{3}$$

Periodicity of θ angle

- Review ordinary TI: why is θ periodic with period 2π?

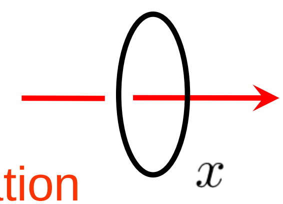
- 3D TI with periodic b.c. on the **electron** wave function: $\mathcal{M} = T^3 \times [t_1, t_2]$
- Choose $\mathbf{E} = E_x \hat{\mathbf{x}}, \mathbf{B} = B_x \hat{\mathbf{x}}$ (Qi, Hughes, Zhang, 2008; Vazifeh, Franz, 2010)

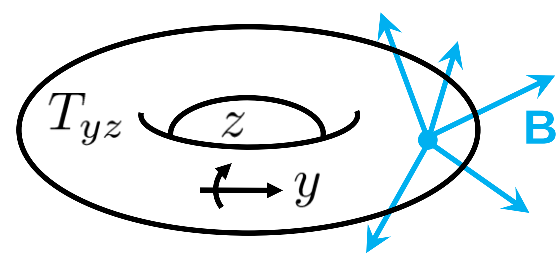
$$\int_{\mathcal{M}} d^4x \, \frac{\theta}{2\pi} \frac{\alpha}{2\pi} \mathbf{E} \cdot \mathbf{B} = \theta \left(\frac{e}{2\pi} \right)^2 \int_{t_1}^{t_2} dt \oint dx \, E_x \int_{T_{yz}} dy \, dz \, B_x \in (2\pi/e)\mathbf{Z}$$

Dirac quantization

$$\oint dx \, A_x(t_1) - \oint dx \, A_x(t_2)$$
$$= (n_1 - n_2) \frac{2\pi}{e} \in (2\pi/e)\mathbf{Z}$$

flux quantization





$$\Rightarrow \int_{\mathcal{M}} d^4x \, \frac{\theta}{2\pi} \frac{\alpha}{2\pi} \mathbf{E} \cdot \mathbf{B} \in \theta \mathbb{Z} \Rightarrow Z(\theta) = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \, e^{iS} = Z(\theta + 2\pi)$$

- What is the period for θ in the **fractional** TI?

Periodicity of θ angle

- Generalized Dirac quantization: $\mathbf{e}_a \cdot \mathbf{m}_b \in 2\pi\mathbb{Z}\delta_{ab}$
  with $\mathbf{e}_a$ an "electric flux vector" (vector in weight lattice), $\mathbf{m}_b$ a "magnetic flux vector" (vector in dual weight lattice); *a*,*b* run over the gauge group generators (Englert, Windey, 1976; Goddard, Nuyts, Olive, 1977)
- Take SU(3) for example: full gauge group is $U(1)_{em}$ x SU(3)
- Cartan generators:

$$H_1 = \frac{g}{\sqrt{2}}\lambda_3 = \frac{g}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, H_2 = \frac{g}{\sqrt{2}}\lambda_8 = \frac{g}{\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, H_3 = \begin{pmatrix} \frac{e}{3} & 0 & 0 \\ 0 & \frac{e}{3} & 0 \\ 0 & 0 & \frac{e}{3} \end{pmatrix},$$

- Fundamental weights:

$$\mathrm{e}_1 = \left(\frac{g}{\sqrt{2}}, \frac{g}{\sqrt{6}}, \frac{e}{3}\right), \mathrm{e}_2 = \left(-\frac{g}{\sqrt{2}}, \frac{g}{\sqrt{6}}, \frac{e}{3}\right), \mathrm{e}_3 = \left(0, -\frac{2g}{\sqrt{6}}, \frac{e}{3}\right)$$

- Solve Dirac quantization condition for the dual fundamental weights:

$$\mathrm{m}_1 = 2\pi\left(\frac{1}{\sqrt{2}g}, \frac{1}{\sqrt{6}g}, \frac{1}{e}\right), \mathrm{m}_2 = 2\pi\left(-\frac{1}{\sqrt{2}g}, \frac{1}{\sqrt{6}g}, \frac{1}{e}\right), \mathrm{m}_3 = 2\pi\left(0, -\frac{2}{\sqrt{6}g}, \frac{1}{e}\right)$$

Periodicity of θ angle

- The allowed magnetic monopoles live in the dual weight lattice:

$$\mathrm{m} = n_1 \mathrm{m}_1 + n_2 \mathrm{m}_2 + n_3 \mathrm{m}_3,\ n_1, n_2, n_3 \in \mathbb{Z}.$$

- Smallest magnetic monopole: only one of n_1,n_2,n_3 is equal to 1 => smallest monopole is "colored" (carries SU(3) magnetic flux)
- Compute action contributed by the θ-term for, say, n_1=1:

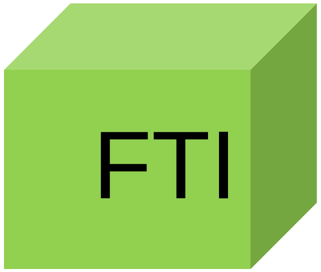
$$\begin{aligned} S_\theta &= \frac{\theta_{\mathrm{eff}} e^2}{32\pi^2} \int_{T^4} d^4x\, \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} + \frac{\theta g^2}{32\pi^2} \int_{T^4} d^4x\, \epsilon^{\mu\nu\lambda\rho} f^a_{\mu\nu} f^a_{\lambda\rho} \\ &= \frac{\theta}{3} \frac{e^2}{4\pi^2} \left(\frac{2\pi}{e}\right)^2 + \frac{\theta g^2}{4\pi^2} \left[\left(\frac{2\pi}{\sqrt{2}g}\right)^2 + \left(\frac{2\pi}{\sqrt{6}g}\right)^2 \right] \\ &= \frac{\theta}{3} + \frac{\theta}{2} + \frac{\theta}{6} = \theta \end{aligned}$$

- Hence θ has period 2π, meaning θ_{eff} has period 2π/3
- Therefore θ_{eff}= π/3 is T-invariant!
- Fractional θ in a T-invariant system: **fractional QME**

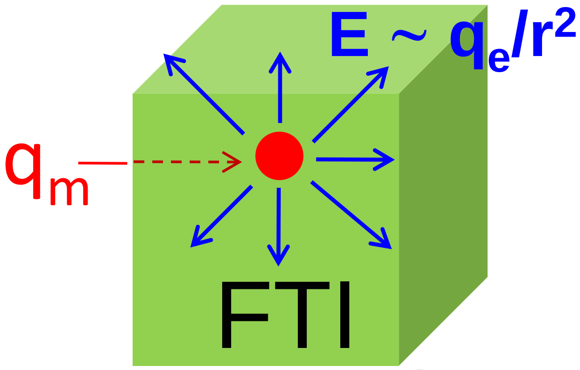
Physical properties of the FTI

- "Halved" FQHE on the surface

$$\sigma_{H,s} = (1/3)\, e^2/2h$$



- Witten effect in the bulk



$$q_e = \frac{\theta_{\text{eff}} e^2}{8\pi^2} q_m$$

Conclusion

- T-invariant $\mathbf{Z}_2$ topological insulators in 2D and 3D exhibit robust gapless surface states that have been experimentally observed
- By analogy with the FQHE, fractional counterparts of these states can be constructed as a field theory of partons
- These FTI states have a fractional θ angle $\theta=\pi/N$, N odd, but do not break TRS
- Physical observables: halved FQHE on the surface, Witten effect on magnetic monopole in the bulk
- Also, nontrivial ground state degeneracy for discrete gauge groups ($\mathbf{Z}_N$)

Open questions

- Relation to Joel Moore’s BF theory?
- Interesting exactly soluble model (M. Levin et al., arXiv:1108.4954) obtains a state with Z_N topological order in the bulk… possibly related
- One needs to construct more realistic models and understand the relationship between different topological field theories “on the market”
- Related work by B. Swingle et al., arXiv:1005.1076, PRB 2011

Thank you!