Electron Transport in Topological Insulators: Disorder, Interaction and Quantum Criticality

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PRL 98, 256801 (2007); PRL 105, 036803 (2010)

in "50 Years of Anderson Localization", ed. by E. Abrahams
(World Scientific, 2010); reprinted in Int J Mod Phys B 24, 1577 (2010)

and, assuming the time permits, also:

Multifractality and interaction:
Enhancement of superconductivity by Anderson localization

in collaboration with

I. Burmistrov, Landau Institute, Chernogolovka

I. Gornyi, Karlsruhe Institute of Technology & Ioffe Inst., St.Petersburg

arXiv:1102.3323, to be published in PRL

Anderson localization

Philip W. Anderson

1958 "Absence of diffusion
in certain random lattices"

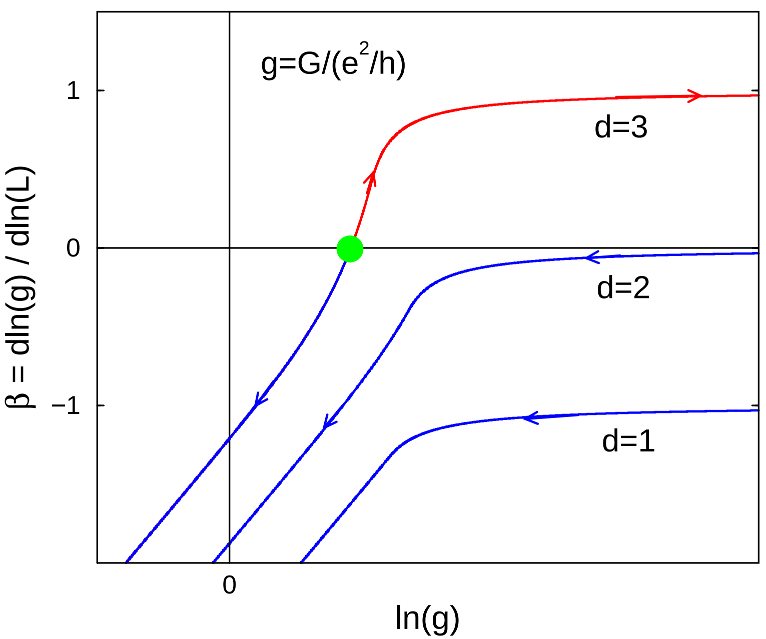
sufficiently strong disorder ⟶ quantum localization

⟶ eigenstates exponentially localized, no diffusion

⟶ Anderson insulator

Nobel Prize 1977

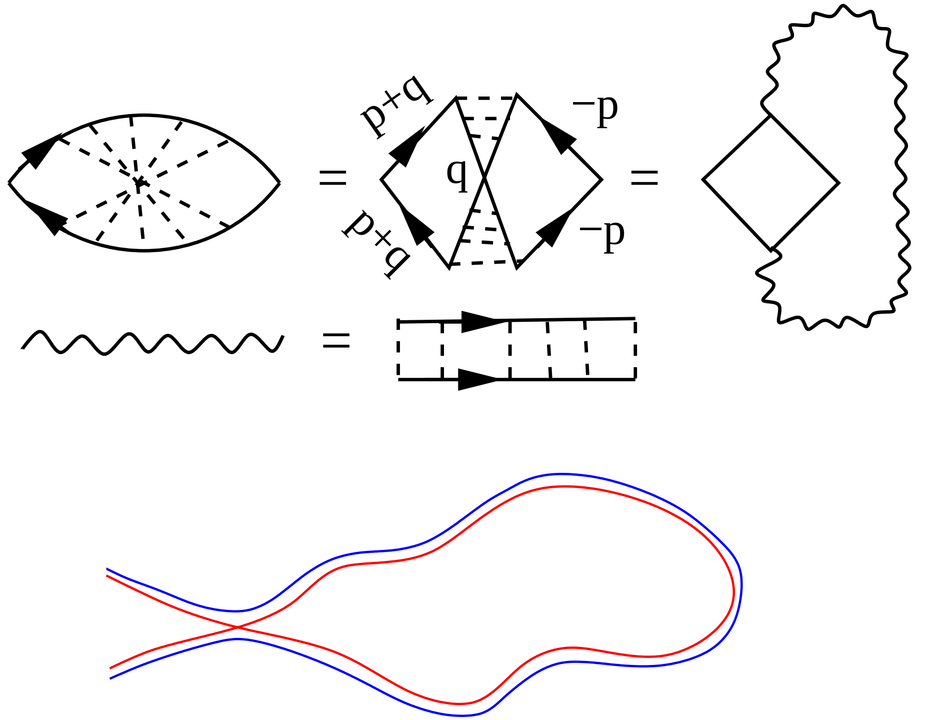
Anderson Insulators & Metals



Scaling theory of localization:
Abrahams, Anderson, Licciardello, Ramakrishnan '79

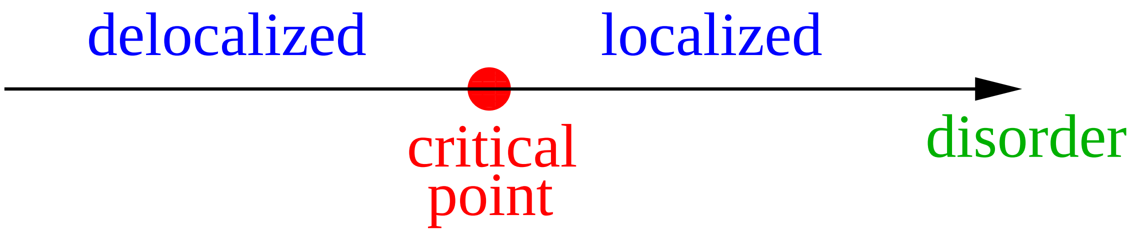
Modern approach:
RG for field theory (σ-model)

large g: weak localization



quasi-1D, 2D : all states are localized

$d > 2$: Anderson metal-insulator transition



review: Evers, ADM, Rev. Mod. Phys. 80, 1355 (2008)

Field theory: non-linear σ-model

$$S[Q] = \frac{\pi\nu}{4}\int d^d r \, \mathrm{Str}\, [-D(\nabla Q)^2 - 2i\omega\Lambda Q], \qquad Q^2(\mathrm{r}) = 1$$

Wegner'79 (replicas); Efetov'83 (supersymmetry)

σ-model manifold:

e.g., unitary class (broken time-reversal symmetry):

- fermionic replicas: $\mathrm{U}(2n)/\mathrm{U}(n) \times \mathrm{U}(n)$, $n \to 0$
- bosonic replicas: $\mathrm{U}(n,n)/\mathrm{U}(n) \times \mathrm{U}(n)$, $n \to 0$
- supersymmetry: $\mathrm{U}(1,1|2)/\mathrm{U}(1|1) \times \mathrm{U}(1|1)$

fermionic replicas: "sphere"

bosonic replicas: "hyperboloid"

SUSY: {"sphere" × "hyperboloid"} "dressed" by anticommuting variables

with Coulomb interaction: Finkelstein'83

Disordered electronic systems: Symmetry classification

Altland, Zirnbauer '97

Conventional (Wigner-Dyson) classes

	T	spin rot.	symbol
GOE	$+$	$+$	AI
GUE	$-$	$+/-$	A
GSE	$+$	$-$	AII

Chiral classes

	T	spin rot.	symbol
ChOE	$+$	$+$	BDI
ChUE	$-$	$+/-$	AIII
ChSE	$+$	$-$	CII

$$\boldsymbol{H} = \begin{pmatrix} \mathbf{0} & \mathrm{t} \\ \mathrm{t}^{\dagger} & \mathbf{0} \end{pmatrix}$$

Bogoliubov-de Gennes classes

	T	spin rot.	symbol
	$+$	$+$	CI
	$-$	$+$	C
	$+$	$-$	DIII
	$-$	$-$	D

$$\boldsymbol{H} = \begin{pmatrix} \mathrm{h} & \boldsymbol{\Delta} \\ -\boldsymbol{\Delta}^{*} & -\mathrm{h}^{T} \end{pmatrix}$$

Disordered electronic systems: Symmetry classification

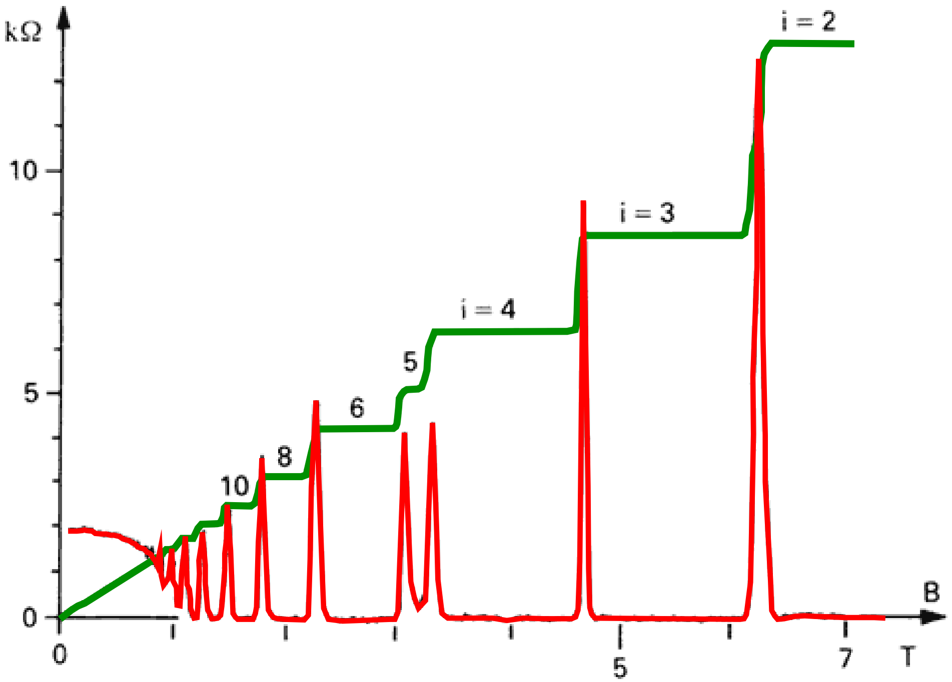
Ham. class	RMT	T	S	compact symmetric space	non-compact symmetric space	σ-model B|F	σ-model compact sector $\mathcal{M}_F$
Wigner-Dyson classes							
A	GUE	$-$	$\pm$	$U(N)$	$GL(N,\mathbb{C})/U(N)$	AIII|AIII	$U(2n)/U(n)\times U(n)$
AI	GOE	$+$	$+$	$U(N)/O(N)$	$GL(N,\mathbb{R})/O(N)$	BDI|CII	$Sp(4n)/Sp(2n)\times Sp(2n)$
AII	GSE	$+$	$-$	$U(2N)/Sp(2N)$	$U^*(2N)/Sp(2N)$	CII|BDI	$O(2n)/O(n)\times O(n)$
chiral classes							
AIII	chGUE	$-$	$\pm$	$U(p+q)/U(p)\times U(q)$	$U(p,q)/U(p)\times U(q)$	A|A	$U(n)$
BDI	chGOE	$+$	$+$	$SO(p+q)/SO(p)\times SO(q)$	$SO(p,q)/SO(p)\times SO(q)$	AI|AII	$U(2n)/Sp(2n)$
CII	chGSE	$+$	$-$	$Sp(2p+2q)/Sp(2p)\times Sp(2q)$	$Sp(2p,2q)/Sp(2p)\times Sp(2q)$	AII|AI	$U(n)/O(n)$
Bogoliubov - de Gennes classes							
C		$-$	$+$	$Sp(2N)$	$Sp(2N,\mathbb{C})/Sp(2N)$	DIII|CI	$Sp(2n)/U(n)$
CI		$+$	$+$	$Sp(2N)/U(N)$	$Sp(2N,\mathbb{R})/U(N)$	D|C	$Sp(2n)$
BD		$-$	$-$	$SO(N)$	$SO(N,\mathbb{C})/SO(N)$	CI|DIII	$O(2n)/U(n)$
DIII		$+$	$-$	$SO(2N)/U(N)$	$SO^*(2N)/U(N)$	C|D	$O(n)$

Symmetry alone is not always sufficient to characterize the system.

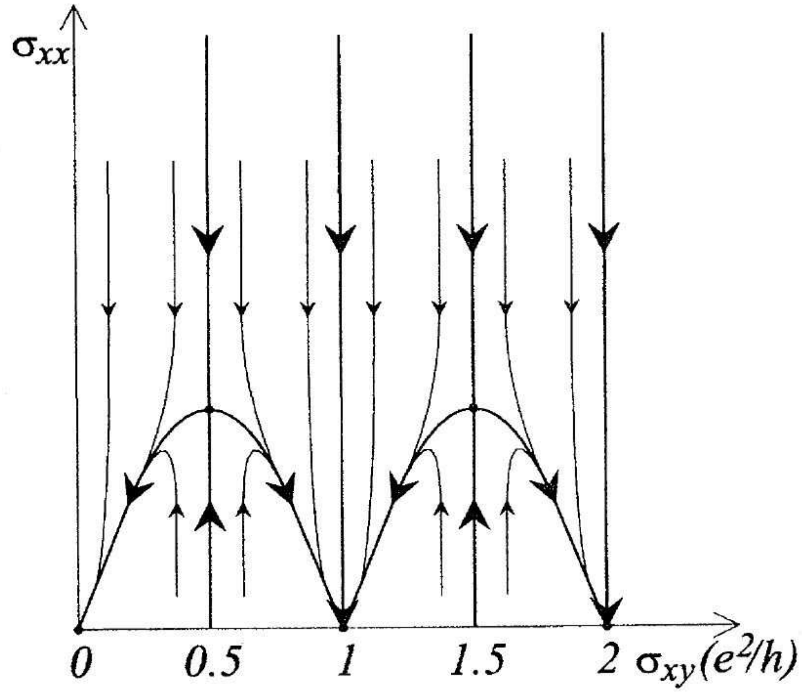
There may be also a non-trivial topology.

It may protect the system from localization.

IQHE: $\mathbb{Z}$ topological insulator

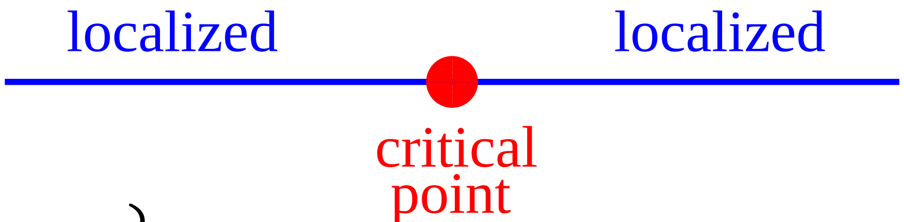


von Klitzing '80 ; Nobel Prize '85



IQHE flow diagram

Khmelnitskii' 83, Pruisken' 84



Field theory (Pruisken):

σ-model with topological term

$$S = \int d^2 r \left\{ -\frac{\sigma_{xx}}{8} \mathrm{Tr} (\partial_\mu Q)^2 + \frac{\sigma_{xy}}{8} \mathrm{Tr} \epsilon_{\mu\nu} Q \partial_\mu Q \partial_\nu Q \right\}$$

QH insulators $\longrightarrow$ $n = \ldots, -2, -1, 0, 1, 2, \ldots$ **protected edge states**

$\longrightarrow$ $\mathbb{Z}$ **topological insulator**

$\mathbb{Z}_2$ topological protection from localization: 1D

Many-channel 1D systems: 1D σ-model

Zirnbauer '92 ; ADM, Müller-Groeling, Zirnbauer '94

Exact calculation of $\langle g \rangle (L/\xi)$ and $\langle g^2 \rangle (L/\xi)$

for all Wigner-Dyson classes (A, AI, AII)

class AII: $\langle g \rangle \to 1/2, \quad \langle g^2 \rangle \to 1/2 \quad$ for $\; L/\xi \to \infty$

One channel remains delocalized!

This result included both σ-model with $\theta = 0$

and with $\theta = \pi$ topological term

$\theta = 0$ even number of channels, topologically trivial, $g \to 0$

$\theta = \pi$ odd number of channels, $g \to 1$

$\longleftrightarrow$ edge of 2D topological insulator (QSH system)

$\mathbb{Z}_2$ topological protection from localization: 2D

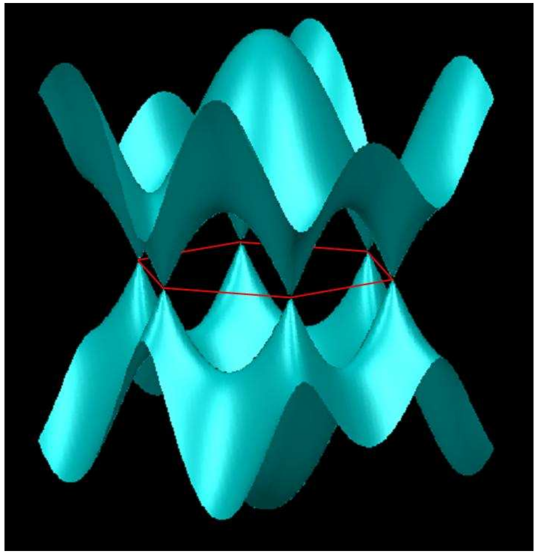
Fendley, **"Critical points in two-dimensional replica sigma models"**

arXiv:cond-mat/0006360, lecture at NATO ASI

Abstract: I survey the kinds of critical behavior believed to be exhibited in two-dimensional disordered systems. I review the different replica sigma models used to describe the low-energy physics, and discuss how critical points appear because of WZW and theta terms.

RMT	replica sigma model	possible 2D critical behavior
GUE	$U(2N)/U(N)\times U(N)$	Pruisken phase
C	$Sp(2N)/U(N)$	Pruisken phase
D	$O(2N)/U(N)$	Pruisken phase, metallic phase
CII	$U(N)/O(N)$	$\theta=\pi\rightarrow U(N)_1$; Gade phase
GSE	$O(2N)/O(N)\times O(N)$	$\theta=\pi\rightarrow O(2N)_1$; metallic phase
AIII	$U(N)\times U(N)/U(N)$	WZW term; Gade phase
CI	$Sp(2N)\times Sp(2N)/Sp(2N)$	WZW term
DIII	$O(N)\times O(N)/O(N)$	WZW term; metallic phase
BDI	$U(2N)/Sp(2N)$	Gade phase?
GOE	$Sp(4N)/Sp(2N)\times Sp(2N)$	none!

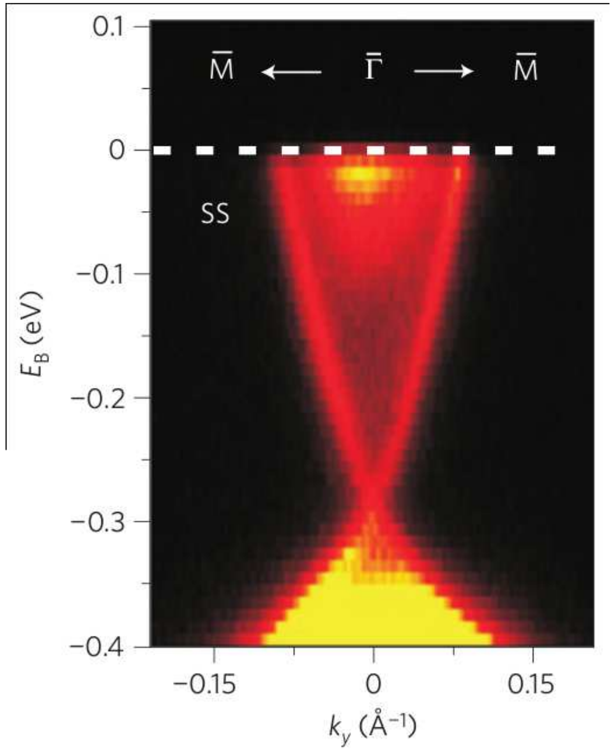
2D massless Dirac fermions



Graphene

Geim, Novoselov'04

Nobel Prize'10



Surface of 3D topological insulators

BiSb, BiSe, BiTe Hasan group '08

σ-model field theory for disordered 2D Dirac fermions

Ostrovsky, Gornyi, ADM '07

- Graphene: long-range disorder (no valley mixing)
- Surface states of 3D TI: no restriction on disorder range

2D Dirac fermions: σ-models with topological term

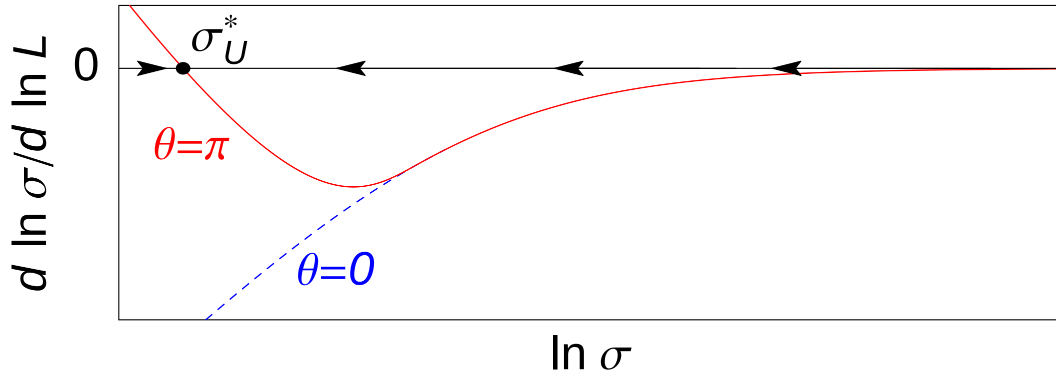
- **Generic disorder (broken TRS)** $\Longrightarrow$ **class A (unitary)**

$$S[Q] = \frac{1}{8}\,\mathrm{Str}\left[-\sigma_{xx}(\nabla Q)^2 + Q\nabla_x Q \nabla_y Q\right] \quad = \quad -\frac{\sigma_{xx}}{8}\,\mathrm{Str}(\nabla Q)^2 + i\pi N[Q]$$

topol. invariant $N[Q] \in \pi_2(\mathcal{M}) = \mathbb{Z}$

$\Longrightarrow$ Quantum Hall critical point

$$\sigma \;=\; 4\sigma_U^* \;\simeq\; 4 \times (0.5 \div 0.6)\frac{e^2}{h}$$



- **Random potential (preserved TRS)** $\Longrightarrow$ **class AII (symplectic)**

$$S[Q] = -\frac{\sigma_{xx}}{16}\,\mathrm{Str}(\nabla Q)^2 + i\pi N[Q]$$

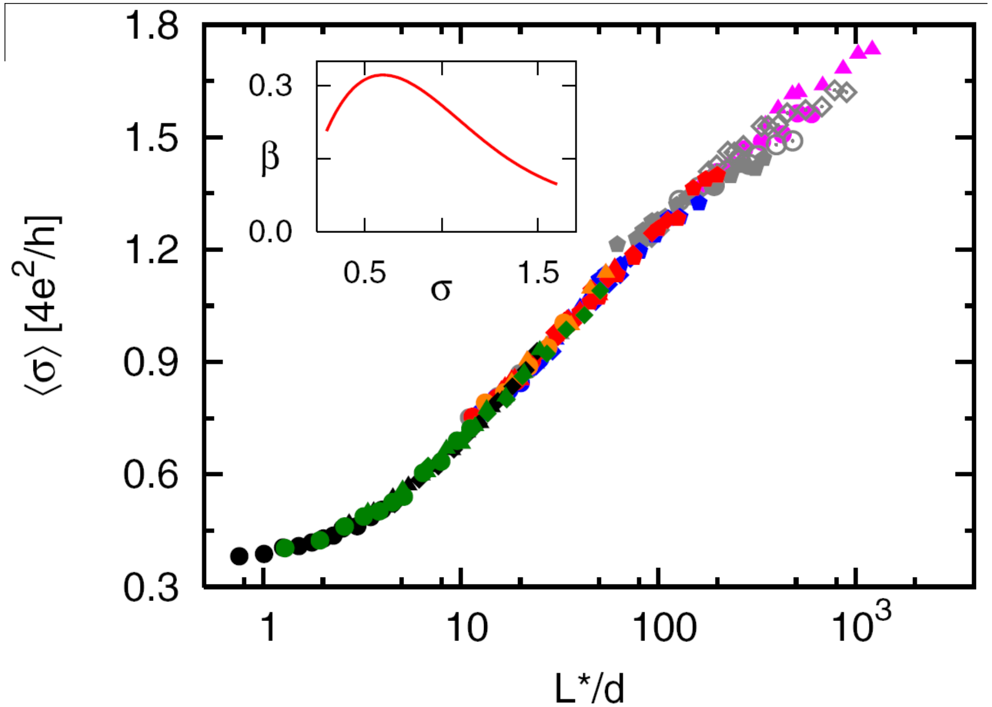
topological invariant: $N[Q] \in \pi_2(\mathcal{M}) = \mathbb{Z}_2 = \{0, 1\}$

Topological protection from localization !

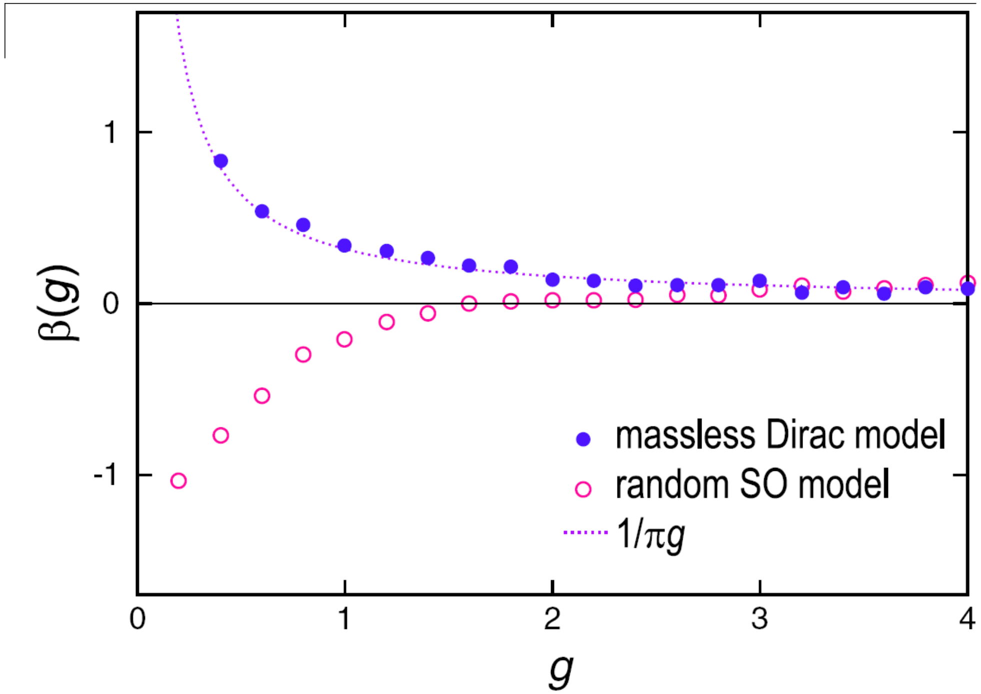
Ostrovsky, Gornyi, ADM, PRL 98, 256801 (2007)

Dirac fermions in random potential: numerics

Bardarson, Tworzydło, Brouwer, Beenakker, PRL '07

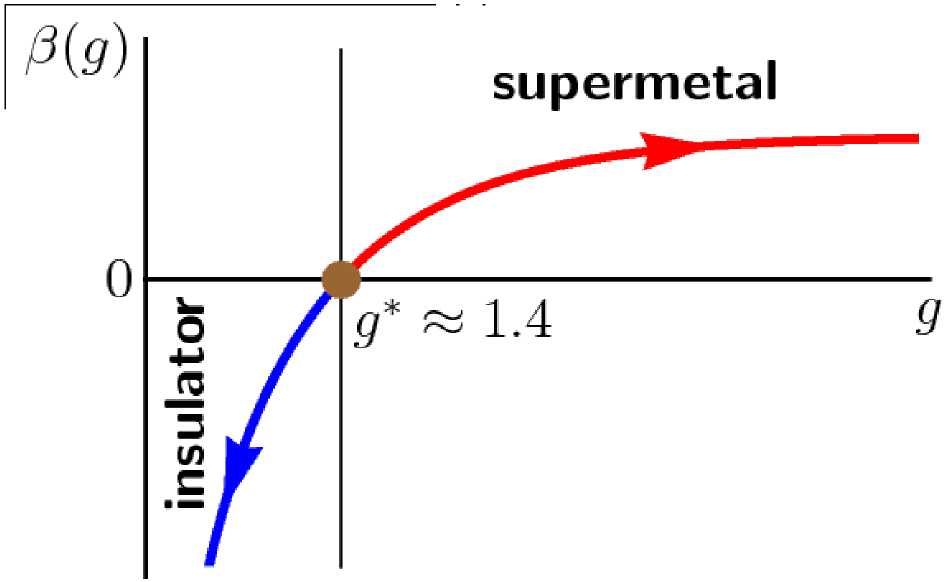


Nomura, Koshino, Ryu, PRL '07

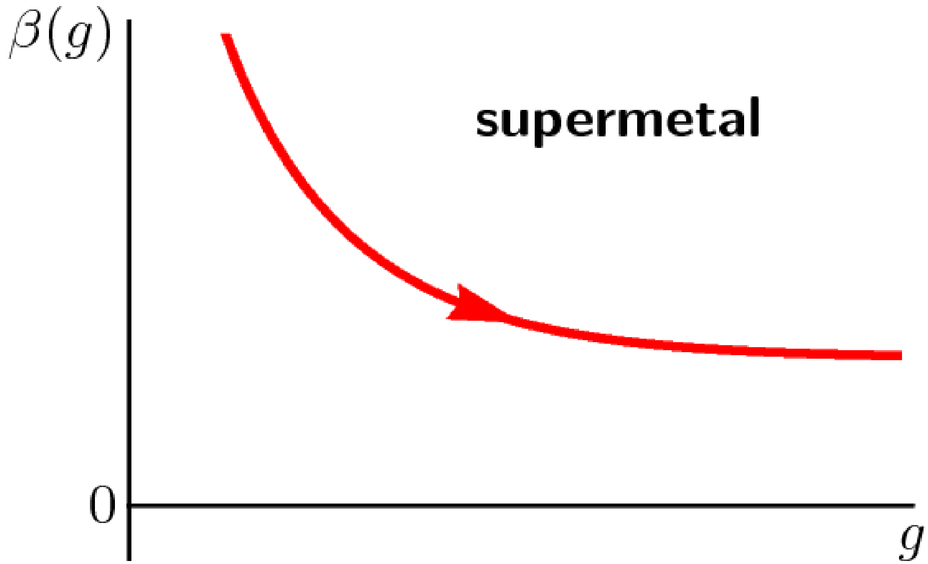


- **absence of localization confirmed**
- **log scaling towards the perfect-metal fixed point** $\boldsymbol{\sigma \to \infty}$

Schematic beta functions for 2D systems of symplectic class AII



Conventional spin-orbit systems



Dirac fermions
(topological protection)

surface of 3D top. insulator
or
graphene without valley mixing

Periodic table of Topological Insulators

	Symmetry classes				Topological insulators			
p	H_p	R_p	S_p	$\pi_0(R_p)$	d=1	d=2	d=3	d=4
0	AI	BDI	CII	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
1	BDI	BD	AII	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
2	BD	DIII	DIII	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
3	DIII	AII	BD	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
4	AII	CII	BDI	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
5	CII	C	AI	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
6	C	CI	CI	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
7	CI	AI	C	0	0	0	$\mathbb{Z}$	0
$0'$	A	AIII	AIII	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$1'$	AIII	A	A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

H_p – symmetry class of Hamiltonians

R_p – sym. class of classifying space (of Hamiltonians with eigenvalues $\to \pm 1$)

S_p – symmetry class of compact sector of σ-model manifold

Kitaev'09; Schnyder, Ryu, Furusaki, Ludwig'09; Ostrovsky, Gornyi, ADM'10

Classification of Topological insulators

Two ways to detect existence of TIs of class p in d dimensions:

(i) by inspecting the topology of classifying spaces R_p:

$$\begin{cases}\text{TI of type } \mathbb{Z} \\ \text{TI of type } \mathbb{Z}_2\end{cases} \Longleftrightarrow \quad \pi_0(R_{p-d}) = \begin{cases}\mathbb{Z} \\ \mathbb{Z}_2\end{cases}$$

(ii) by analyzing homotopy groups of the σ-model manifolds:

$$\begin{cases}\text{TI of type } \mathbb{Z} \Longleftrightarrow \pi_d(S_p) = \mathbb{Z} & \text{Wess-Zumino term} \\ \text{TI of type } \mathbb{Z}_2 \Longleftrightarrow \pi_{d-1}(S_p) = \mathbb{Z}_2 & \theta = \pi \ \ \text{topological term}\end{cases}$$

WZ and $\theta = \pi$ terms make boundary excitations "non-localizable"

TI in d $\Longleftrightarrow$ topological protection from localization in $d - 1$

Bott periodicity: $\pi_d(R_p) = \pi_0(R_{p+d})$, **periodicity 8**

2D Dirac surface states of a 3D TI:
Disorder and interaction

Surface of 3D $\mathbb{Z}_2$ TI:

single 2D massless Dirac mode

With disorder:

Topological protection from localization,

RG flow towards supermetal

What is the effect of Coulomb interaction?

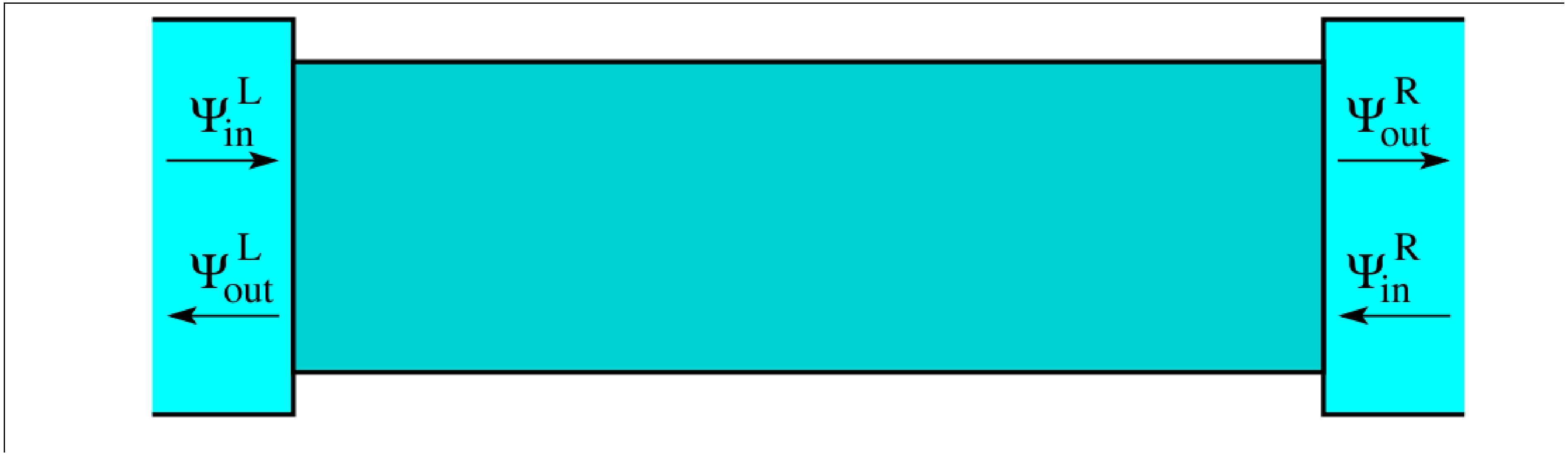
assume not too strong interaction $\quad r_s = \sqrt{2}e^2/\epsilon v_F \lesssim 1$

$\Longrightarrow$ **no instabilities, no symmetry-breaking**

$\Longrightarrow$ **topological protection from localization persists**

But interaction may destroy the supermetal phase!

Absence of localization in a symplectic wire with odd number of channels



Scattering matrix of a symplectic system

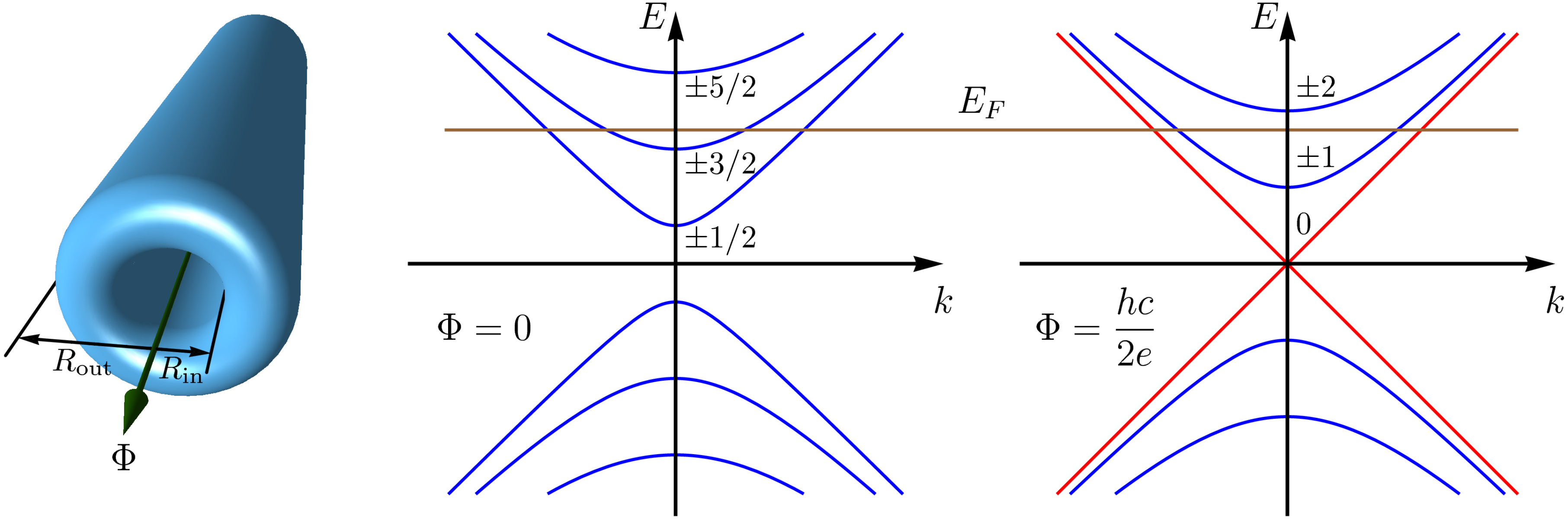
$$\begin{pmatrix}\Psi^L_{out}\\ \Psi^R_{out}\end{pmatrix} = \begin{pmatrix} r & t' \\ t & r'\end{pmatrix}\begin{pmatrix}\Psi^L_{in}\\ \Psi^R_{in}\end{pmatrix}$$

TI symmetry $\Longrightarrow$ $r = -r^T$, $r' = -r'^T$, $t = t'^T$

For N channels:

$\det r = (-1)^N \det r^T$ $\Longrightarrow$ **no localization if N is odd ! ! !**

Topological protection: Reduction to 1D



Hollow cylinder threaded with magnetic flux $\boldsymbol{\Phi}$

Surface states: $$E_n(p) = \pm\sqrt{p^2 + \left(n + \frac{1}{2} - \frac{e\Phi}{hc}\right)^2}$$

Time-reversal symmetry preserved for $e\Phi/hc$ integer or half-integer

Half-integer $e\Phi/hc$ $\Longrightarrow$ odd number of 1D channels

$\Longrightarrow$ no 1D localization $\Longrightarrow$ no 2D localization

Coulomb interaction in symplectic class AII: RG

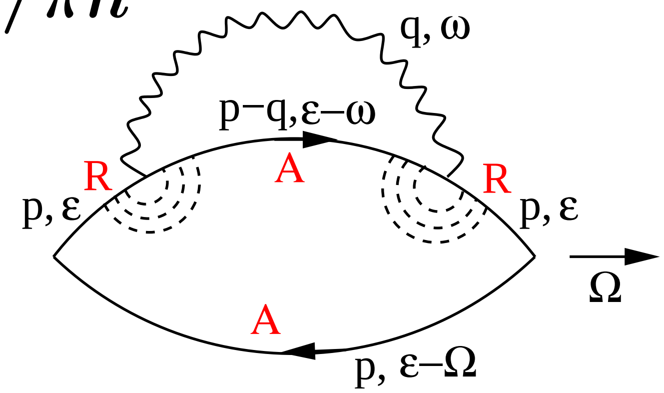
cf. Altshuler, Aronov '79; Finkelstein '83

$$\beta(g) = \frac{dg}{d\ln L} \quad = \quad \frac{N}{2} \quad - \quad 1 \quad + \quad (N^2-1)\mathcal{F}$$

weak antilocalization − ee-singlet + ee-multiplet

g – dimensionless conductance in units $2e^2/\pi h$

N – # of flavors (spin, valleys, etc)



Graphene: $N = 4$ (2 valleys, 2 spins)

⟶ WAL wins ⟶ supermetal survives

Surface of a 3D TI: $N = 1$

⟶ $\beta(g) = -1/2 < 0$ ⟶ ee-interaction wins

⟶ conductance decreases upon RG

⟶ Coulomb repulsion destroys supermetal phase

more about RG: symplectic class, $N = 1$

with Coulomb interaction:

$$\frac{dg}{d\ln L} = -\frac{1}{2} + \gamma_c$$

$$\frac{d\gamma_c}{d\ln L} = \frac{1+\gamma_c}{2g} - 2\gamma_c^2$$

γ_c – Cooper-channel interaction constant

assume $\gamma_c > 0$ (no superconductivity)

under RG $\gamma_c \to (2g)^{-1/2} \ll 1$

$\longrightarrow$ γ_c does not affect scaling towards smaller g

more about RG: symplectic class, $N = 1$

for comparison, with weak short-range interaction:

$$\frac{dg}{d\ln L} = \frac{1}{2} + \frac{\gamma_s}{2} + \gamma_c$$

$$\frac{d\gamma_s}{d\ln L} = -\frac{\gamma_s + 2\gamma_c}{2g}$$

$$\frac{d\gamma_c}{d\ln L} = -\frac{\gamma_s}{2g} - 2\gamma_c^2$$

γ_s – spin-singlet interaction constant

under RG both interaction amplitudes remain small:

$\gamma_c \to 1/2g \ll 1, \quad \gamma_s \to -1/g \ll 1$

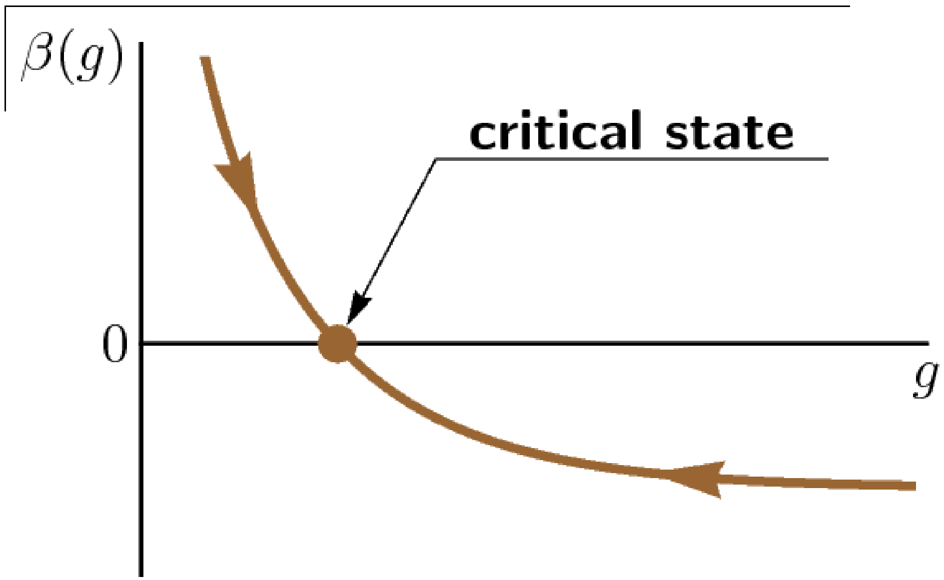
$\longrightarrow$ interaction does not affect scaling $g \to \infty$ ("supermetal")

Interaction-induced quantum criticality in 3D TI

Ostrovsky, Gornyi, ADM, PRL 105, 036803 (2010)

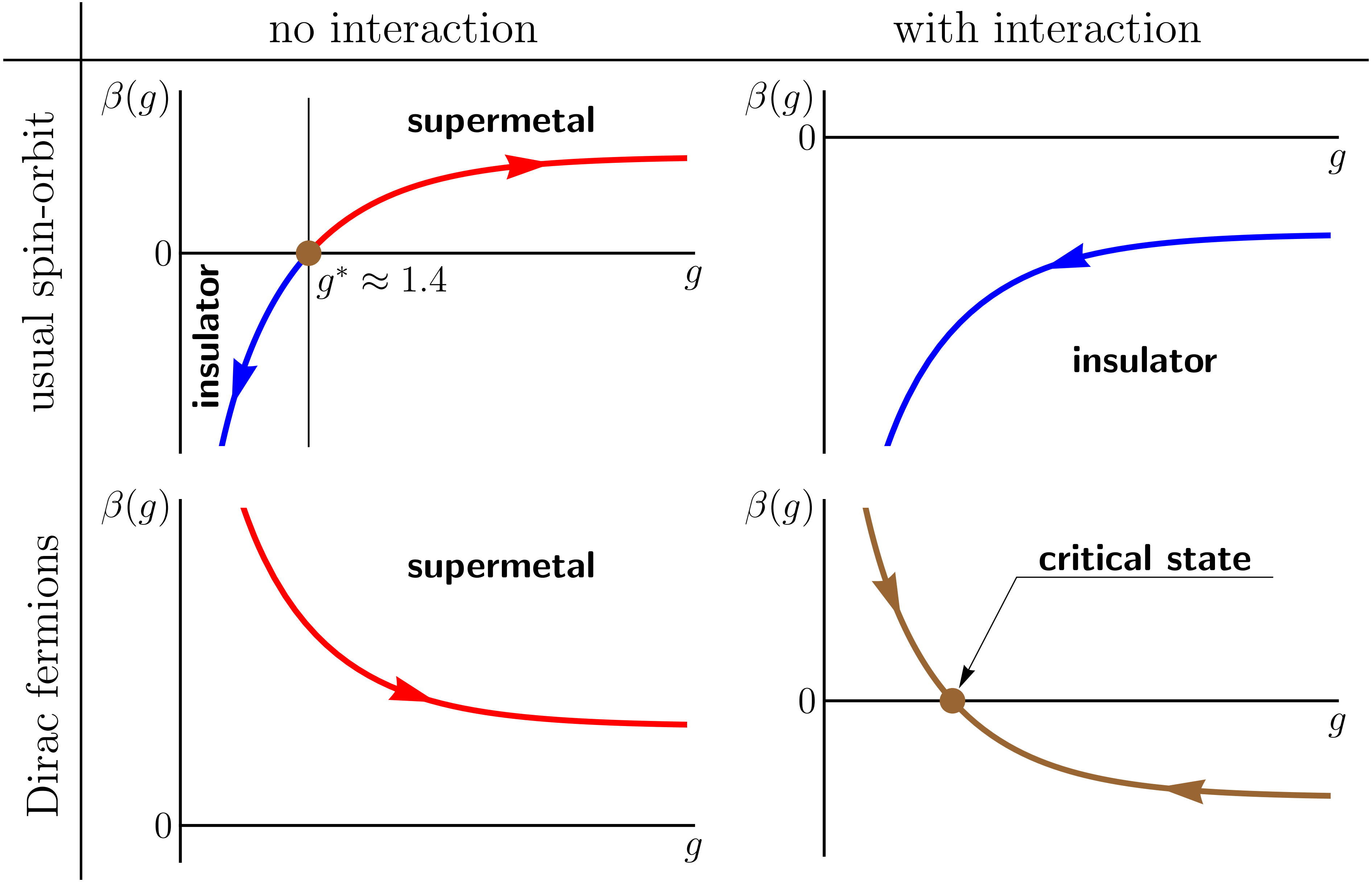
- **Interaction** $\longrightarrow$ tendency to localization at $g \gg 1$
- **Topology** $\longrightarrow$ protection from strong localization (no flow towards $g \ll 1$)

$\longrightarrow$ novel quantum critical point should emerge at $g \sim 1$

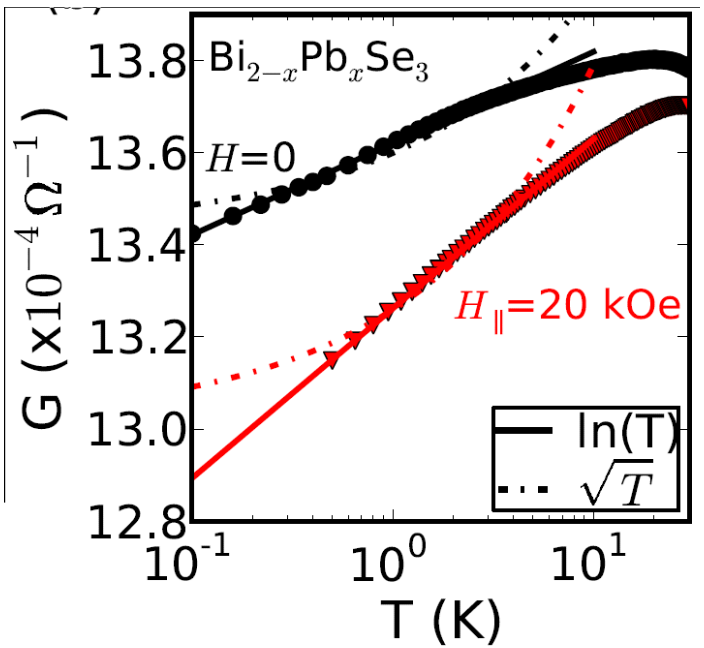


analogous to QH critical point, but here induced by interaction

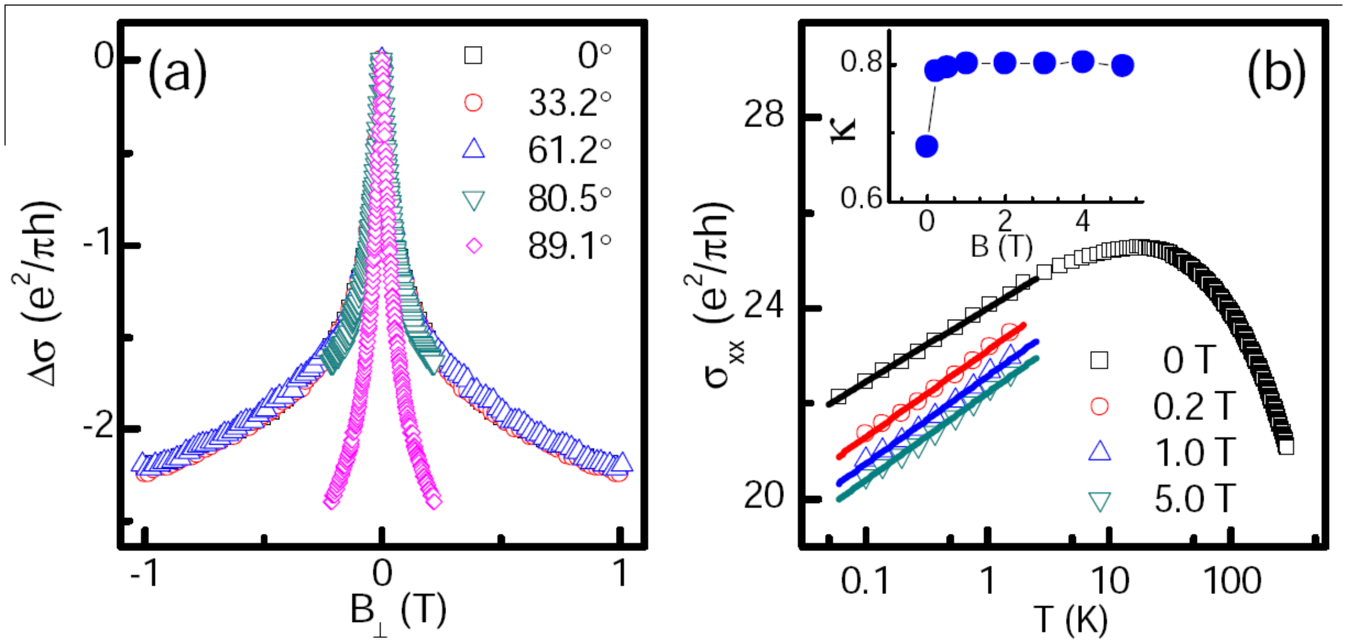
β functions for symplectic class: Interaction and Topology



Experiment: Conductivity via surface of 3D top. insulator



Wang et al, PRB'11



Chen et al, PRB '11

Weak Antilocalization magnetoresistance as expected for class AII

But: **Localizing T dependence** as expected for Coulomb interaction

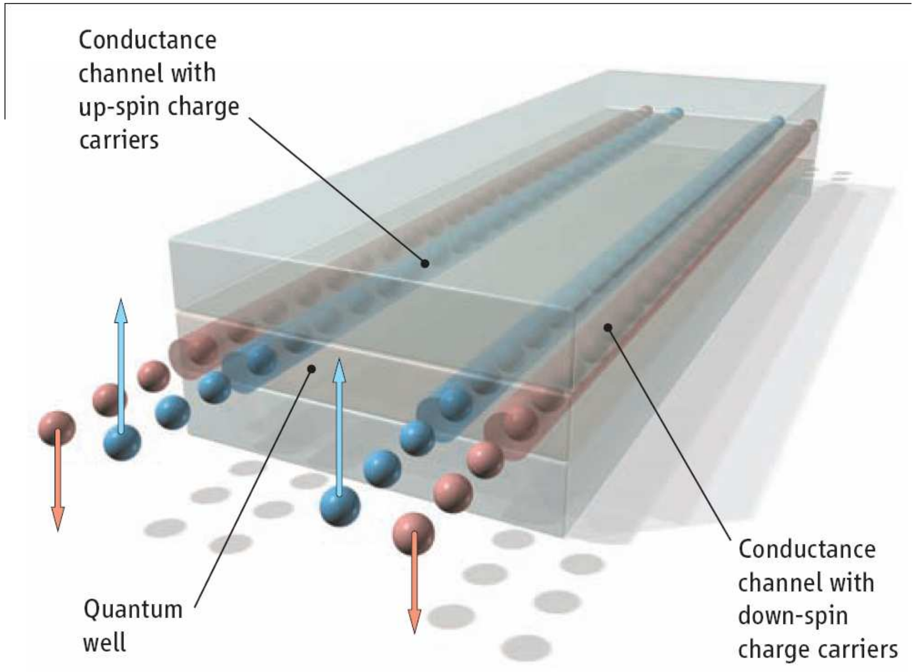
Experimental challenges (to study the predicted critical state):

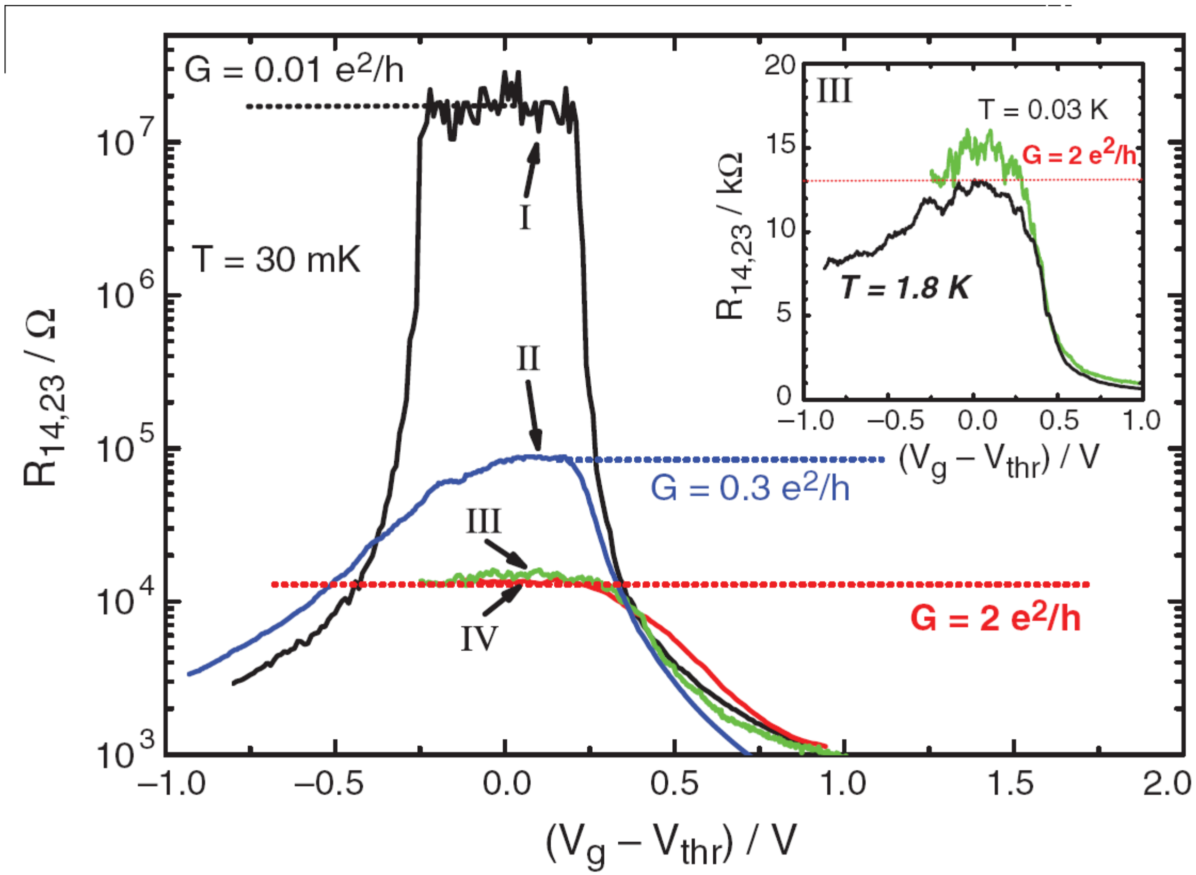
- solely surface conduction
- tuning chemical potential to a vicinity of Dirac point

Quantum spin Hall transition between 2D topological and normal insulators

QSHE in CdTe/HgTe/CdTe quantum wells: Experiment

Molenkamp group '07





I — normal insulator, $d = 5.5$ nm

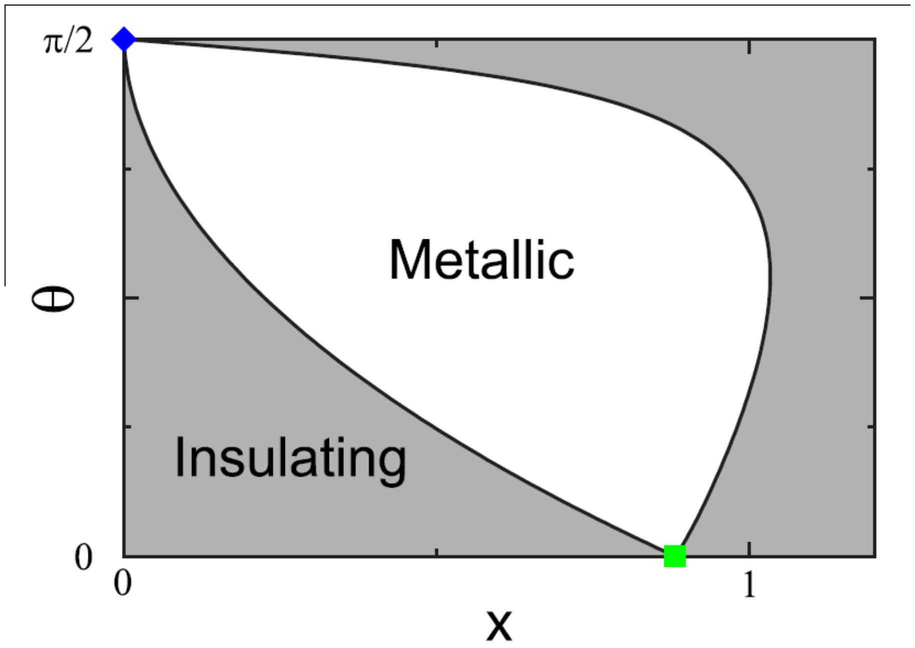
II, III, IV — inverted quantum well structure, $d = 7.3$ nm

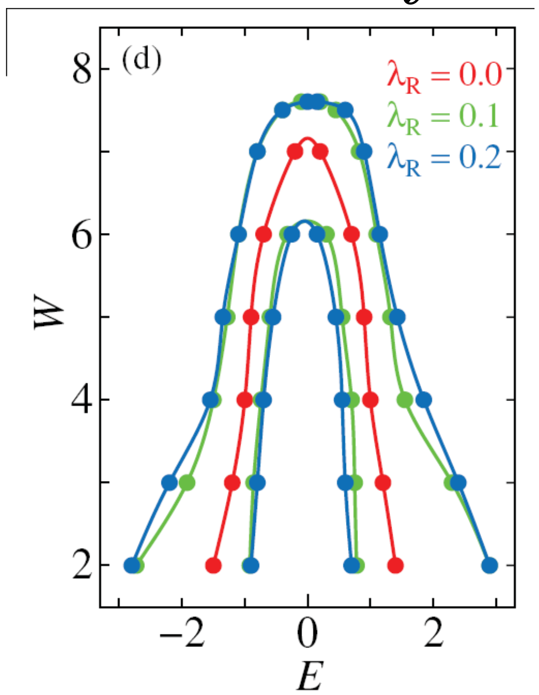
$\longrightarrow$ 2D topological insulator

2D TIs: QSHE phase diagram

In the presence of disorder, TI and normal insulator phases are separated by the supermetal phase

transitions TI–supermetal and supermetal–NI are in the coventional symplectic MIT universality class

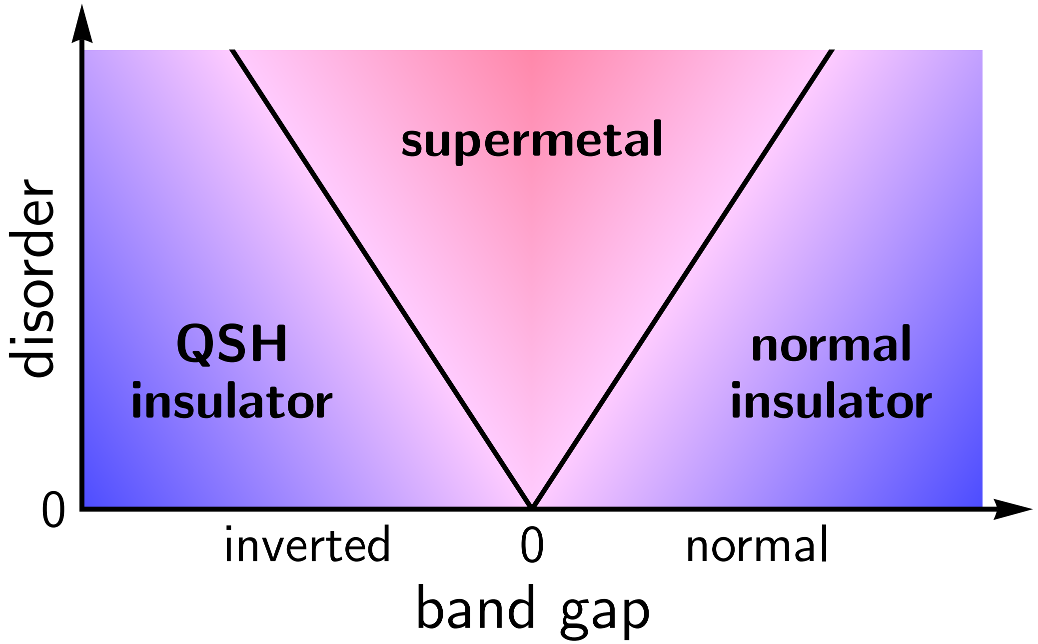


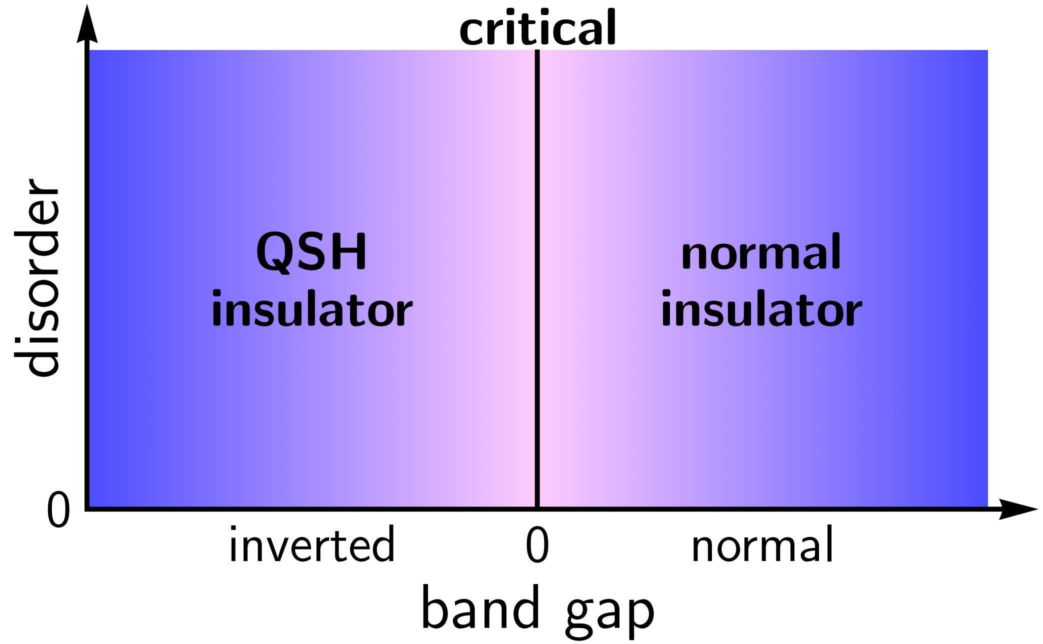


Onoda, Avishai, Nagaosa ’07; Obuse et al ’07

Effect of Coulomb interaction on phase diagram — ?

2D TIs: QSHE phase diagram (cont'd)





Coulomb interaction "kills" the supermetal phase

⟶ **quantum critical point of Quantum Spin Hall transition**
with conductivity $\sim e^2/h$

alternative: critical phase (seems less likely)

$\mathbb{Z}_2$ edge in the presence of Coulomb interaction

Edge of 2D TI: single propagating mode in each direction

Impurity backscattering prohibited (symplectic time reversal invariance)

Coulomb interaction $\longrightarrow$ **Luttinger liquid** , conductance e^2/h

Xu, Moore '06; Wu, Bernevig, Zhang '06:

Umklapp processes (uniform or random)

$\partial\mathcal{D}_2/\partial \ln L = (3-8K)\mathcal{D}_2$ $\qquad$ K – **Luttinger liquid parameter**

Coulomb 1/r interaction: $\qquad K(q) = \left(1 + 2\alpha \ln \frac{q_0}{q}\right)^{-1/2} \qquad \alpha = e^2/\pi^2\epsilon h v_F$

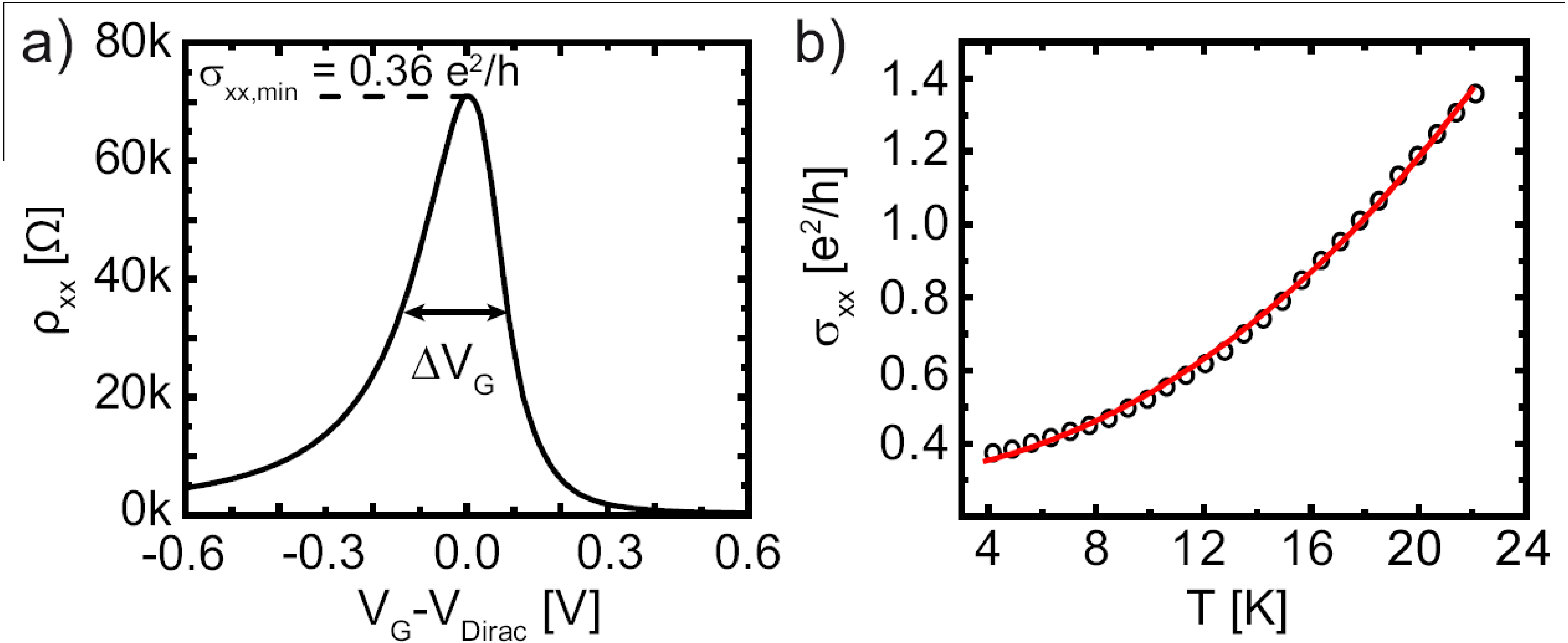
$\longrightarrow$ $\mathcal{D}_2$ processes negligible up to the scale $\qquad L_0 \sim q_0^{-1} \exp \frac{80}{9\alpha}$

What happens with TI beyond this scale is an interesting question
but purely academic for not too strong interaction:

$r_s = 1 \longrightarrow L_0 \sim 10^{60}\,\text{nm} \; > \;$ **size of Universe**

Thus, TI phase persists in the presence of not too strong Coulomb interaction

Experiment: HgTe/CdTe critical well thickness



Büttner, . . . , Molenkamp, arXiv:1009.2248

conductivity saturates with decreasing T at a value $\sim e^2/h$, in agreement with theory

TI–NI transition: other realization

2D TI–NI transition can also be realized

on a surface of a 3D weak topological insulator

Recent works (no e-e interaction):

Ringel, Kraus, Stern, arXiv:1105.4351

Mong, Bardarson, Moore, arXiv:1109.3201

Multifractality and interaction:
Enhancement of superconductivity by Anderson localization

in collaboration with

I. Burmistrov, Landau Institute, Chernogolovka

I. Gornyi, Karlsruhe Institute of Technology & Ioffe Inst., St.Petersburg

arXiv:1102.3323, to be published in PRL

Multifractality at the Anderson transition

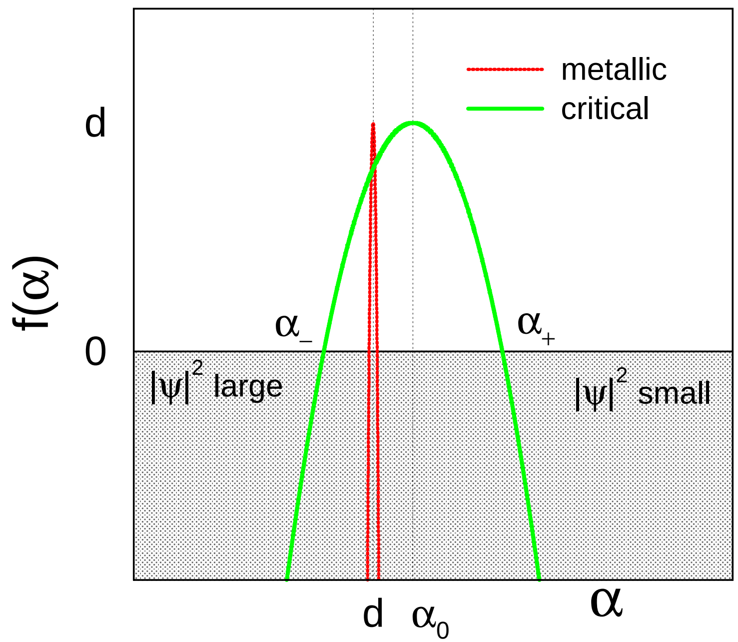
$P_q = \int d^d r |\psi(\mathrm{r})|^{2\mathrm{q}}$ inverse participation ratio

$$\langle P_q \rangle \sim \begin{cases} L^0 & \text{insulator} \\ L^{-\tau_q} & \text{critical} \\ L^{-d(q-1)} & \text{metal} \end{cases}$$

$$\tau_q = \underbrace{d(q-1)}_{\text{normal}} + \underbrace{\Delta_q}_{\text{anomalous}} \equiv D_q(q-1) \quad \text{multifractality}$$

$\tau_q \longrightarrow$ Legendre transformation

$\longrightarrow$ singularity spectrum $f(\alpha)$



wave function statistics:

$$\mathcal{P}(\ln|\psi^2|) \sim L^{-d+f(\ln|\psi^2|/\ln L)}$$

$L^{f(\alpha)}$ – measure of the set of points where $|\psi|^2 \sim L^{-\alpha}$

Multifractality and the field theory

- Δ_q – scaling dimensions of operators $\mathcal{O}^{(q)} \sim (Q\Lambda)^q$

$d = 2 + \epsilon$: $\Delta_q = -q(q-1)\epsilon + O(\epsilon^4)$ Wegner '80

- Infinitely many operators with negative scaling dimensions
- wave function correlations $\longleftrightarrow$ Operator Product Expansion

Wegner 85 ; Duplantier, Ludwig 91

- $\Delta_1 = 0 \longleftrightarrow \langle Q \rangle = \Lambda$ naive order parameter uncritical

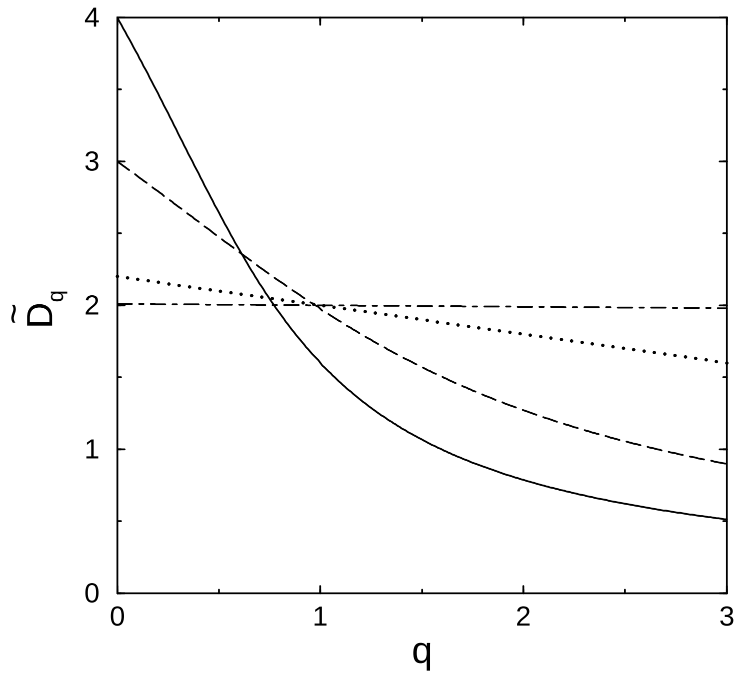
Transition described by an order parameter function $F(Q)$

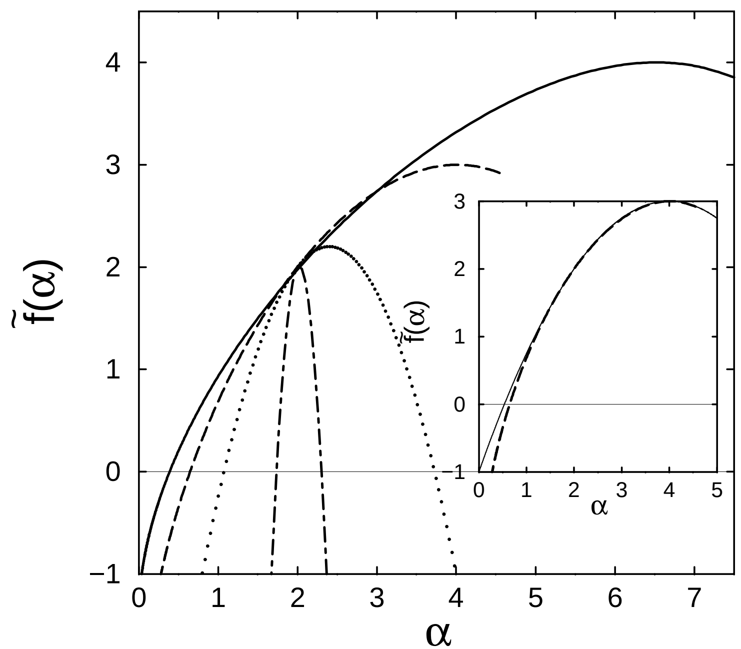
Zirnbauer 86, Efetov 87

$\longleftrightarrow$ distribution of local Green functions and wave function amplitudes

ADM, Fyodorov '91

Dimensionality dependence of multifractality





Analytics (2 + ϵ, one-loop) and numerics

$$\tau_q = (q-1)d - q(q-1)\epsilon + O(\epsilon^4)$$

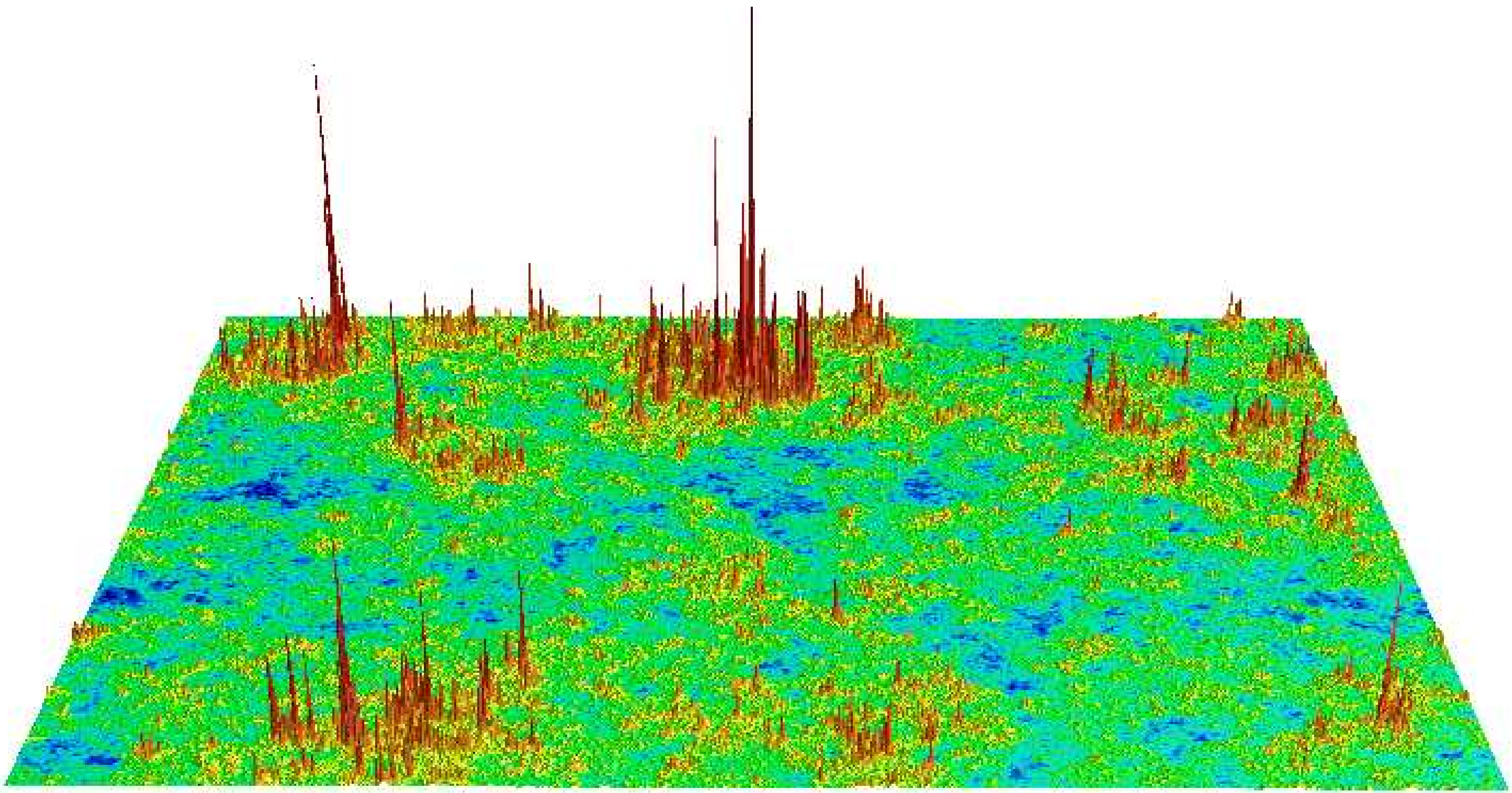
$$f(\alpha) = d - (d+\epsilon-\alpha)^2/4\epsilon + O(\epsilon^4)$$

$d = 4$ (full)
$d = 3$ (dashed)
$d = 2+\epsilon,\ \ \epsilon = 0.2$ (dotted)
$d = 2+\epsilon,\ \ \epsilon = 0.01$ (dot-dashed)

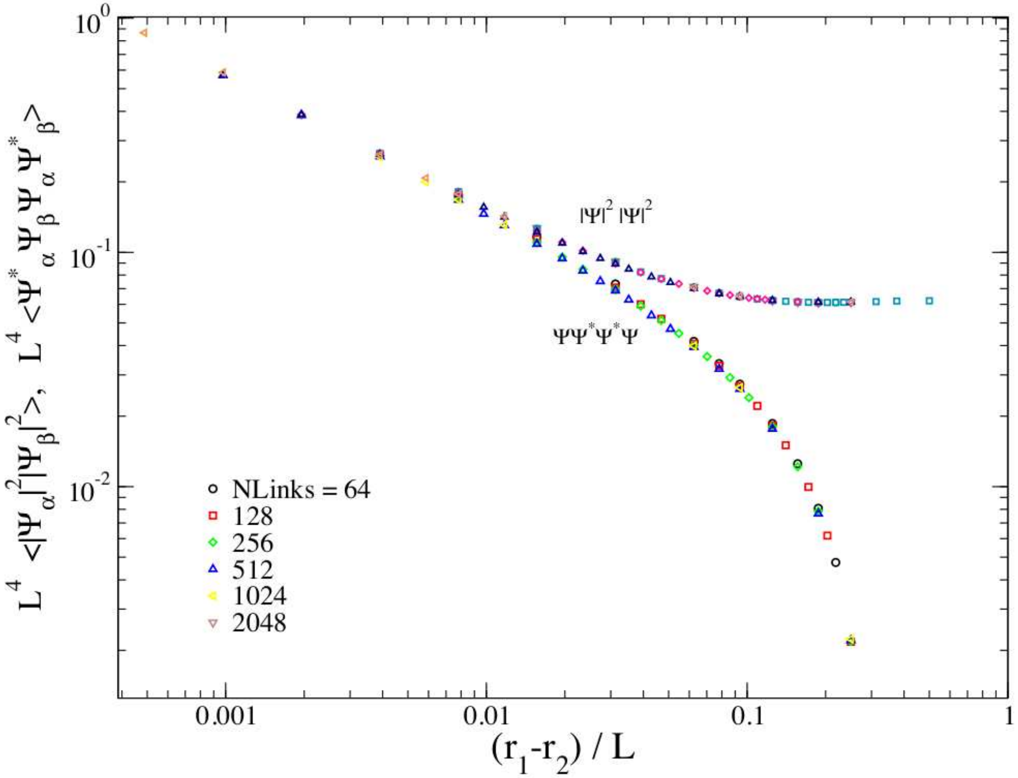
Inset: $d = 3$ (dashed)
vs. $d = 2+\epsilon,\ \ \epsilon = 1$ (full)

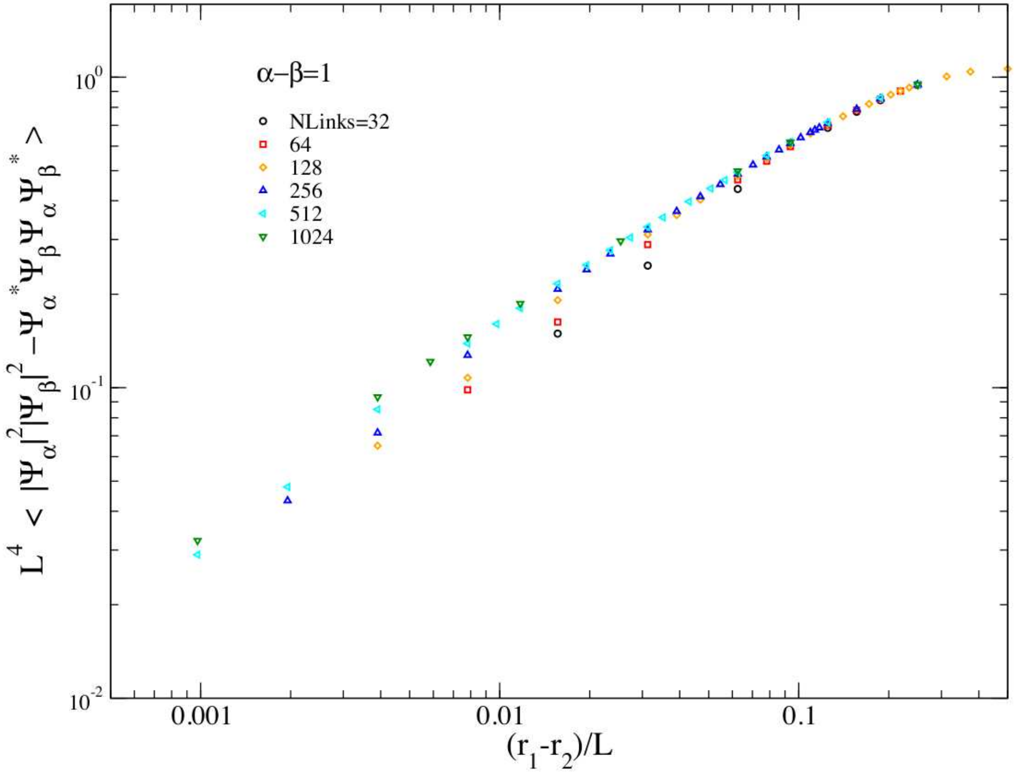
Mildenberger, Evers, ADM '02

Multifractal wave functions at the Quantum Hall transition



Interaction scaling at criticality





Hartree, Fock

enhanced by multifractality

exponent $\Delta_2 \simeq -0.52 < 0$

Hartree − Fock

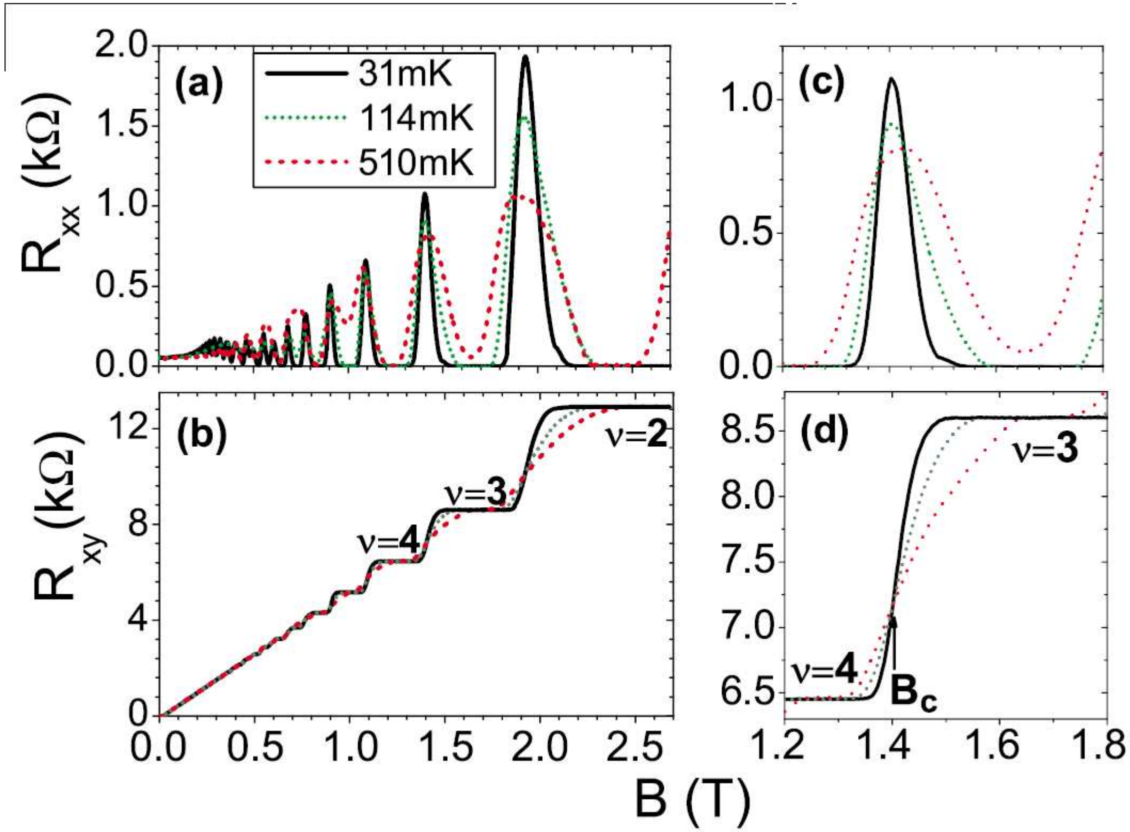
suppressed by multifractality

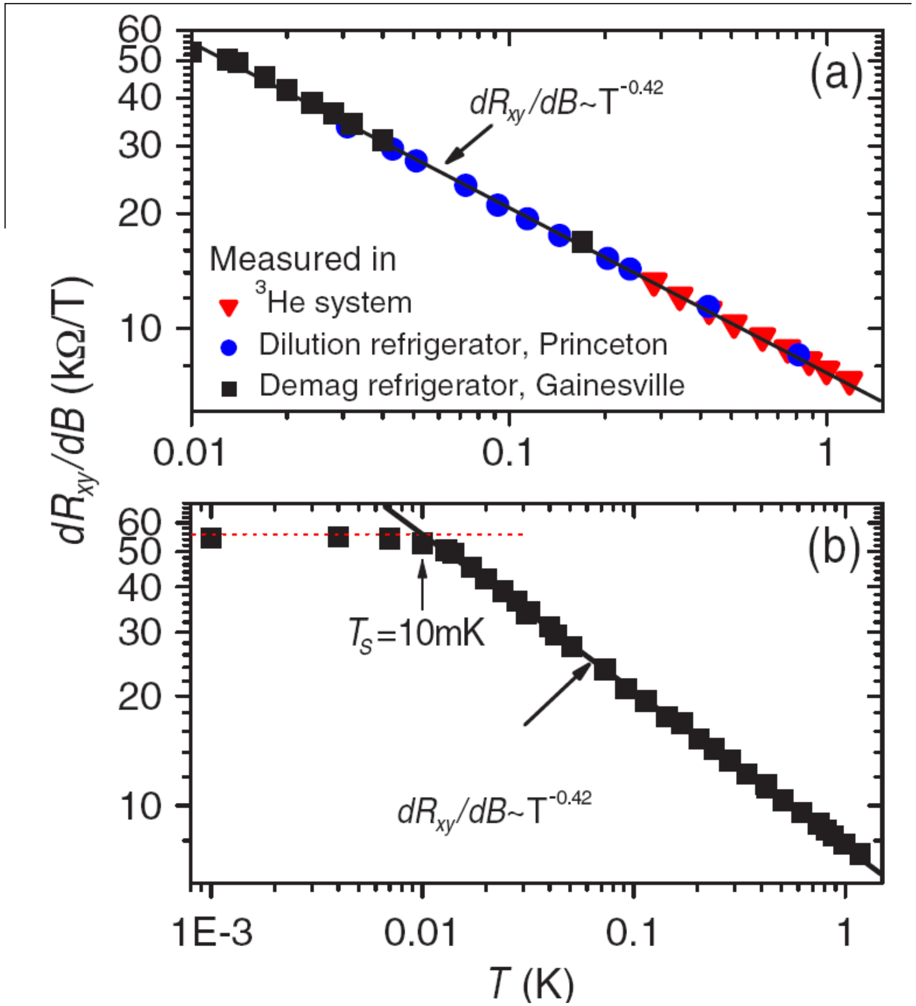
exponent $\mu_2 \simeq 0.62 > 0$

Burmistrov, Bera, Evers, Gornyi, ADM, Annals Phys. 326, 1457 (2011)

$\longrightarrow$ **Dephasing at QH and MI transitions**

Temperature scaling of quantum Hall transition





Transition width exponent $\kappa = 1/\nu z_T = 0.42 \pm 0.01$

Wei, Tsui, Paalanen, Pruisken, PRL'88 ; Li et al., PRL'05, PRL'09

Scaling at QH transition: Theory and experiment

- **Theory (short-range interaction):**

$\longrightarrow$ **dephasing rate** $\tau_\phi^{-1} \propto T^p$ with $p = 1 + 2\mu_2/d$

dephasing length $L_\phi \propto T^{-1/z_T}$ $\qquad z_T = d/p$

Transition width exponent $$\kappa = \frac{1}{z_T \nu} = \frac{1 + 2\mu_2/d}{\nu d}$$

$\mu \simeq 0.62 \quad \longrightarrow \quad p \simeq 1.62 \quad \longrightarrow \quad z_T \simeq 1.23$

$\nu \simeq 2.35$ (Huckestein et al '92, ...) $\longrightarrow$ $\kappa \simeq 0.346$

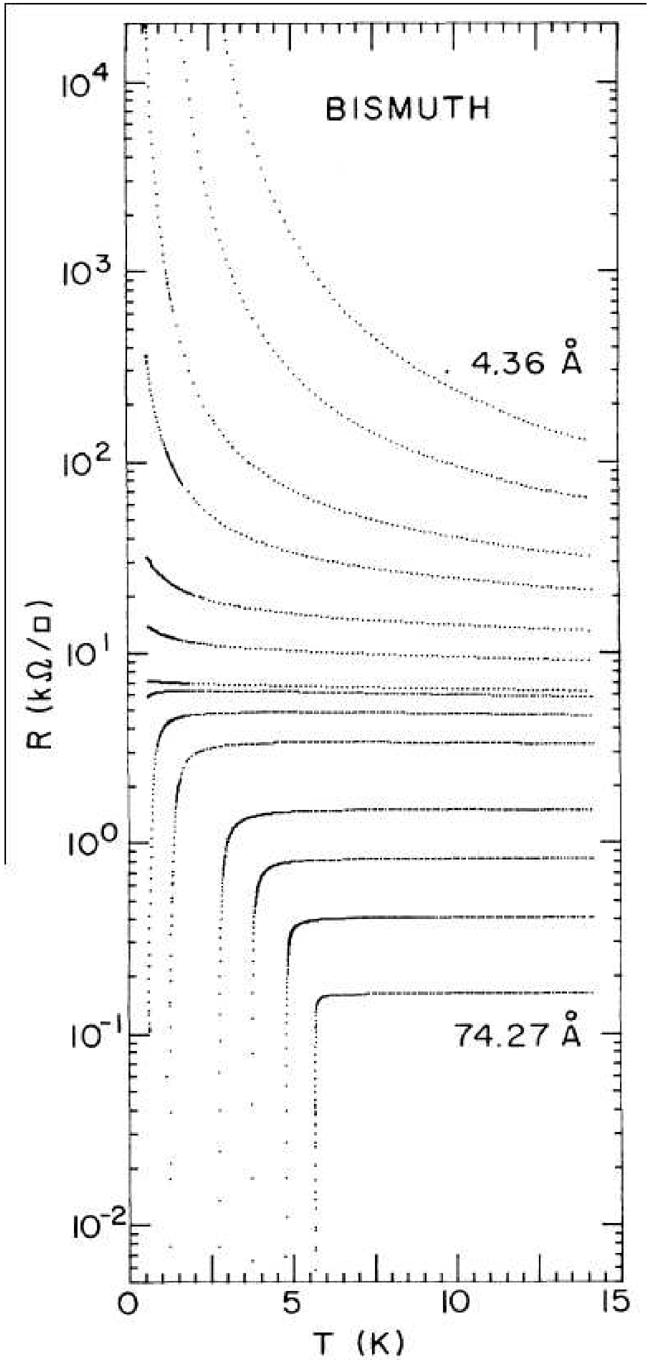
$\nu \simeq 2.59$ (Ohtsuki, Slevin '09) $\longrightarrow$ $\kappa \simeq 0.314$

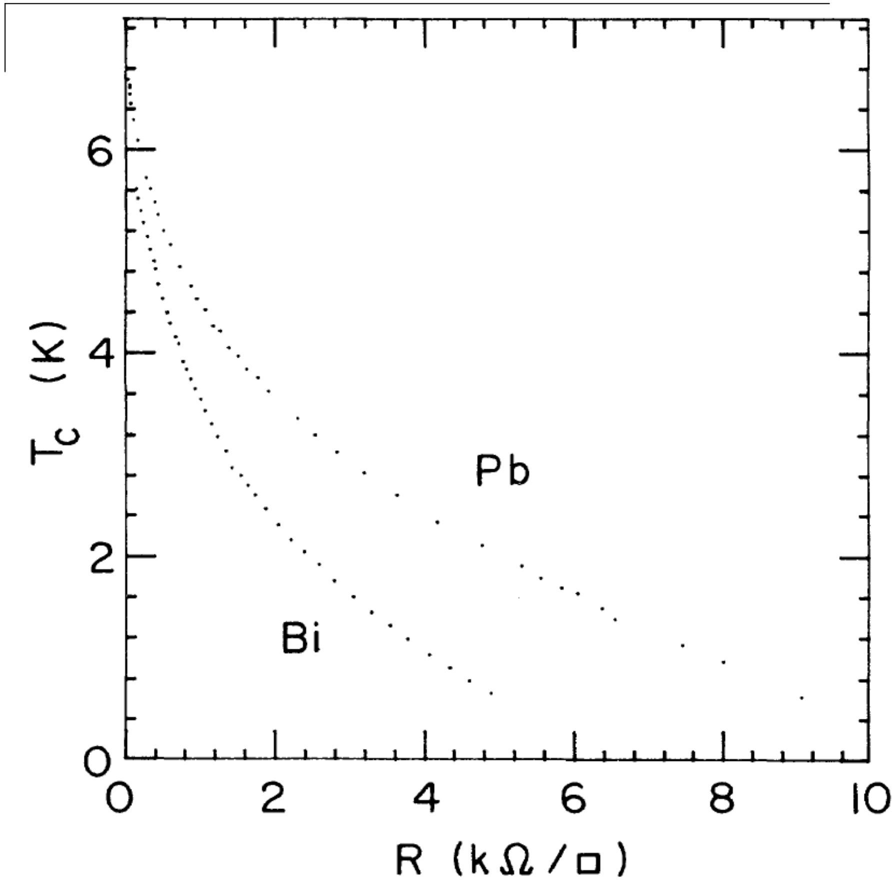
- **Experiment** (long-range $1/r$ Coulomb interaction):

$\kappa = 0.42 \pm 0.01$

Difference in κ fully consistent with short-range and Coulomb ($1/r$) problems being in different universality classes

Superconductor-Insulator Transition





Haviland, Liu, Goldman, PRL'89

Bi and Pb films

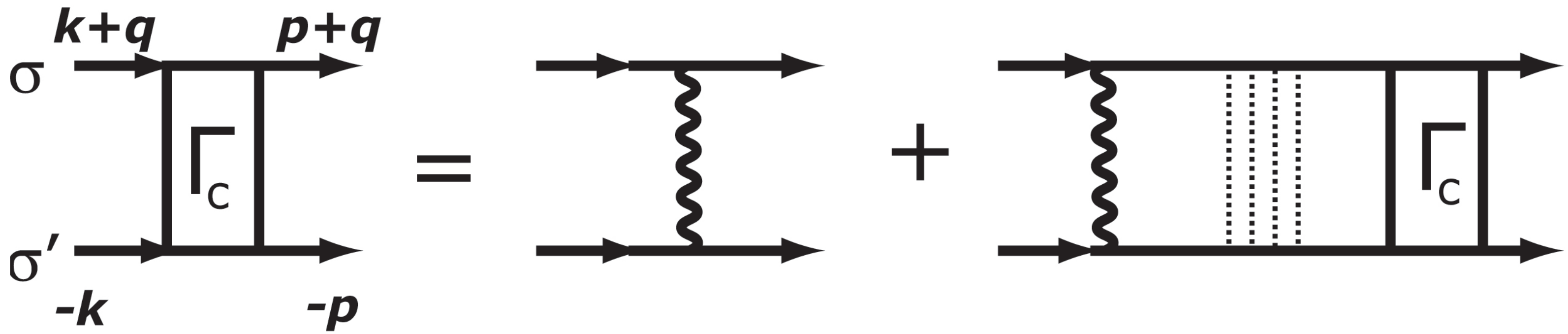
Suppression of T_c by disorder

Anderson theorem

Abrikosov, Gorkov'59 ; Anderson'59

non-magnetic impurities do not affect s-wave superconductivity:

Cooper instability unaffected by diffusive motion



mean free path does not enter the expression for T_c

Anderson Theorem vs Anderson Localization – ?

Suppression of T_c of disordered films due to Coulomb repulsion

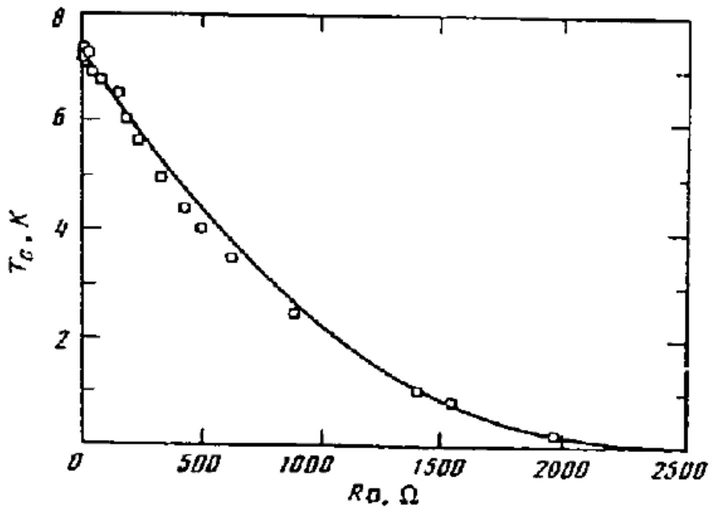
Combined effect of disorder and Coulomb (long-range) interaction

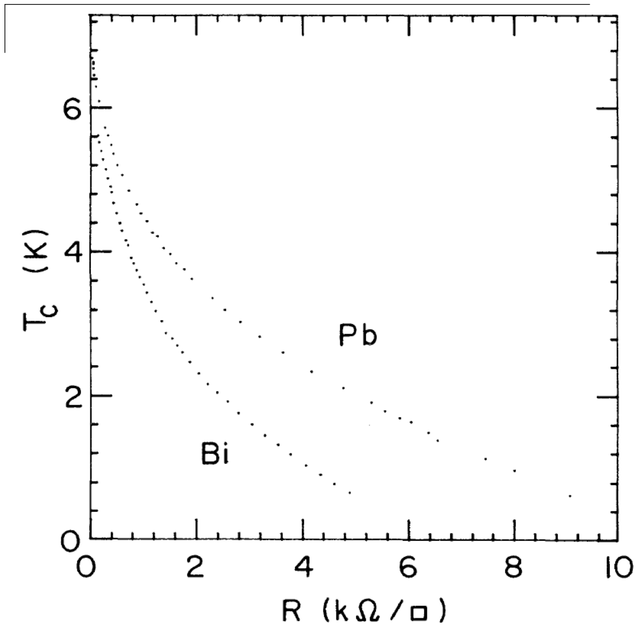
First-order perturbative correction to T_c: Maekawa, Fukuyama'81

RG theory: Finkelstein '87

T_c suppressed; monotonously decays with increasing resistivity

This suppression is observed in many experiments





Mo-Ge films, Graybeal, Besley'84

Bi and Pb films, Haviland, Liu, Goldman'89

Enhancement of superconductivity by multifractality

short-range interaction

Feigelman, Ioffe, Kravtsov, Yuzbashyan, Cuevas, PRL '07, Ann. Phys.'10 :

multifractality of wave functions near MIT in 3D

$\longrightarrow$ enhancement of Cooper-interaction matrix elements

$\longrightarrow$ enhancement of T_c as given by self-consistency equation

Questions:

- **Can suppression of T_c for Coulomb repulsion and enhancement due to multifractality be described in a unified way?**
- **What are predictions of RG ? Does the enhancement hold if the repulsion in particle-hole channels is taken into account ?**
- **Effect of disorder on T_c in 2D systems ?**

SIT in disordered 2D system: Orthogonal symmetry class

σ-model RG with short-range interaction:

$$\frac{dt}{dy} = t^2 - (\frac{\gamma_s}{2} + 3\frac{\gamma_t}{2} + \gamma_c)t^2$$

$$\frac{d\gamma_s}{dy} = -\frac{t}{2}(\gamma_s + 3\gamma_t + 2\gamma_c)$$

$$\frac{d\gamma_t}{dy} = -\frac{t}{2}(\gamma_s - \gamma_t - 2\gamma_c)$$

$$\frac{d\gamma_c}{dy} = -\frac{t}{2}(\gamma_s - 3\gamma_t) - 2\gamma_c^2 \qquad\qquad y \equiv \ln L$$

Interactions: singlet γ_s, triplet γ_t, Cooper γ_c

$\gamma_s \to -1 \quad \longrightarrow \quad$ Finkelstein's RG for Coulomb interaction

Disorder: dimensionless resistivity $t = 1/G$

Assume small bare values: $t_0\,,\ \gamma_{i,0} \ll 1$

SIT in disordered 2D system: Orthogonal class (cont'd)

Weak interaction $\longrightarrow$ discard $\gamma_i t^2$ contributions to $dt/d\ln L$

$$\frac{d}{dy}\begin{pmatrix}\gamma_s\\ \gamma_t\\ \gamma_c\end{pmatrix} = -\frac{t}{2}\begin{pmatrix}1 & 3 & 2\\ 1 & -1 & -2\\ 1 & -3 & 0\end{pmatrix}\begin{pmatrix}\gamma_s\\ \gamma_t\\ \gamma_c\end{pmatrix} - \begin{pmatrix}0\\ 0\\ 2\gamma_c^2\end{pmatrix}\ ; \qquad \frac{dt}{dy} = t^2$$

Eigenvalues and -vectors of linear problem (without BCS term γ_c^2):

$$\lambda = 2t\ :\ \begin{pmatrix}-1\\ 1\\ 1\end{pmatrix}\ ; \qquad \lambda' = -t\ :\ \begin{pmatrix}1\\ 1\\ -1\end{pmatrix}\ \text{and}\ \begin{pmatrix}1\\ -1\\ 2\end{pmatrix}$$

2D system is "weakly critical" (on scales shorter than ξ)

The eigenvalues λ, λ' are exactly multifractal exponents:

$\lambda \equiv -\Delta_2 > 0$ (RG relevant), $\lambda' = -\mu_2 < 0$ (RG irrelevant)

SIT in disordered 2D system: Orthogonal class (cont'd)

Couplings that diagonalize the linear system:

$$\begin{pmatrix} \gamma \\ \gamma' \\ \gamma'' \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \gamma_s \\ \gamma_t \\ \gamma_c \end{pmatrix}$$

Upon RG γ increases, whereas γ', γ'' decrease.

Solution approaches the λ–eigenvector, i.e.. $\gamma_s = -\gamma_t = -\gamma_c$

$\longrightarrow$ neglect γ', γ'' and keep γ only:

$$\frac{d\gamma}{dy} = 2t\gamma - \frac{2}{3}\gamma^2 \qquad\qquad t(y) = \frac{t_0}{1 - t_0 y}$$

Superconductivity may develop if the starting value

$$\gamma_0 = \frac{1}{6}(-\gamma_{s,0} + 3\gamma_{t,0} + 2\gamma_{c,0}) < 0$$

SIT in disordered 2D systems, orthogonal class: Results

$$T_c \sim \exp\{-1/|\gamma_{c,0}|\} \quad \text{(BCS)}\ , \qquad G_0 \gtrsim |\gamma_0|^{-1}$$

$$T_c \sim \exp\{-2G_0\}\ , \qquad |\gamma_0|^{-1/2} \lesssim G_0 \lesssim |\gamma_0|^{-1}$$

$$\text{insulator}\ , \qquad G_0 \lesssim |\gamma_0|^{-1/2}$$

Non-monotonous dependence

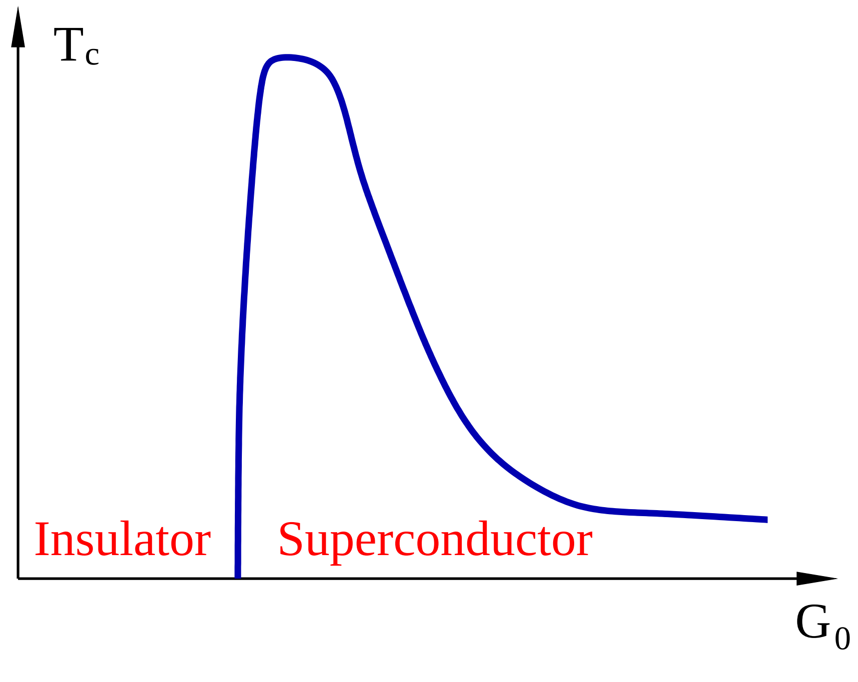
of T_c on disorder (G_0)

Exponentially strong enhancement

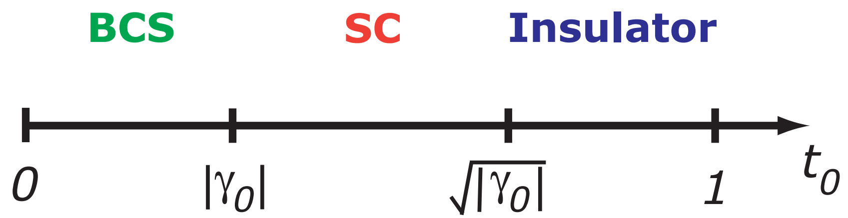
of superconductivity by multifractality

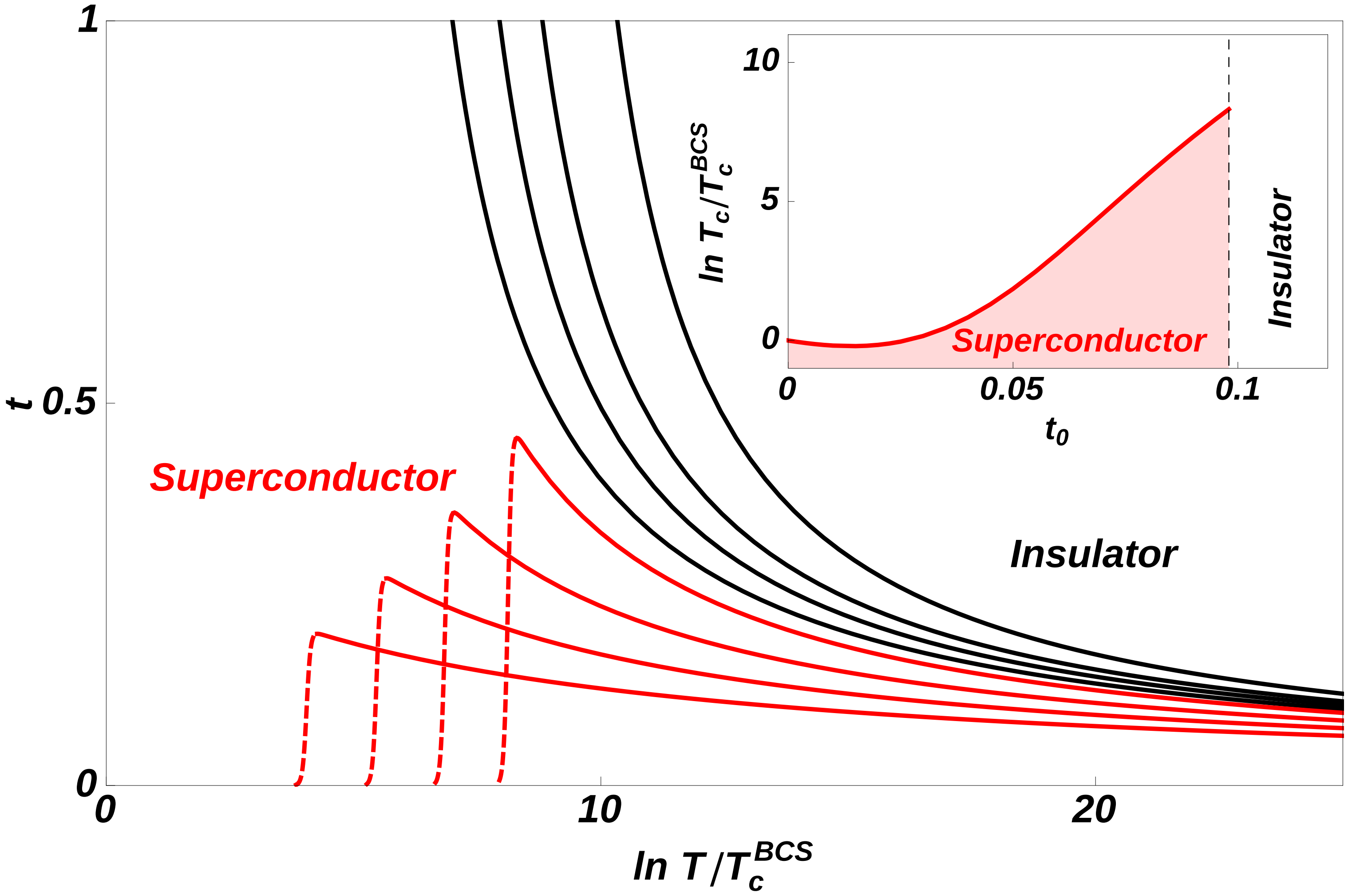
in the intermediate disorder range,

$|\gamma_0|^{-1/2} \lesssim G_0 \lesssim |\gamma_0|^{-1}$



SIT in disordered 2D system, orth. class: Results (cont'd)





Inset: $T_c(t_0)$

$t(T)$ for $\gamma_{c0} = 0.04$, $\gamma_{s0} = -0.005$, $\gamma_{t0} = 0.005$, and $t_0 = 0.065 \div 0.12$

Disordered 2D system: Symplectic symmetry class

Strong spin-orbit interaction $\longrightarrow$ only spin-singlet modes survive

$$\frac{dt}{dy} = -\frac{1}{2}t^2 - (\frac{\gamma_s}{2} + \gamma_c)t^2$$

$$\frac{d\gamma_s}{dy} = -\frac{t}{2}(\gamma_s + 2\gamma_c)$$

$$\frac{d\gamma_c}{dy} = -\frac{t}{2}\gamma_s - 2\gamma_c^2$$

$$t(y) = \frac{t_0}{1 + yt_0/2}$$ antilocalization

$$\frac{d}{dy}\begin{pmatrix}\gamma_s \\ \gamma_c\end{pmatrix} = -\frac{t}{2}\begin{pmatrix}1 & 2 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\gamma_s \\ \gamma_c\end{pmatrix} - \begin{pmatrix}0 \\ 2\gamma_c^2\end{pmatrix}$$

Disordered 2D system: Symplectic class (cont'd)

Eigenvalues and -vectors of linear system:

$$\lambda = \frac{t}{2} \; : \; \begin{pmatrix} -1 \\ 1 \end{pmatrix} \; ; \qquad \lambda' = -t \; : \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Again λ, λ' are multifractal exponents:

$\lambda \equiv -\Delta_2 > 0$ **(RG relevant),** $\quad \lambda' = -\mu_2 < 0$ **(RG irrelevant)**

Couplings that diagonalize the linear system:

$$\begin{pmatrix} \gamma \\ \gamma' \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \gamma_s \\ \gamma_c \end{pmatrix}$$

Upon RG γ increases, whereas γ' decreases.

Solution approaches the λ–eigenvector, i.e.. $\gamma_s = -\gamma_c$

$\longrightarrow$ **neglect γ' and keep γ only:**

$$\frac{d\gamma}{dy} = \frac{t}{2}\gamma - \frac{2}{3}\gamma^2 \qquad\qquad t(y) = \frac{t_0}{1 + t_0 y/2}$$

Disordered 2D system, symplectic class: Results

Superconductivity if $\gamma_0 = \frac{1}{3}(-\gamma_{s,0} + 2\gamma_{c,0}) < 0$

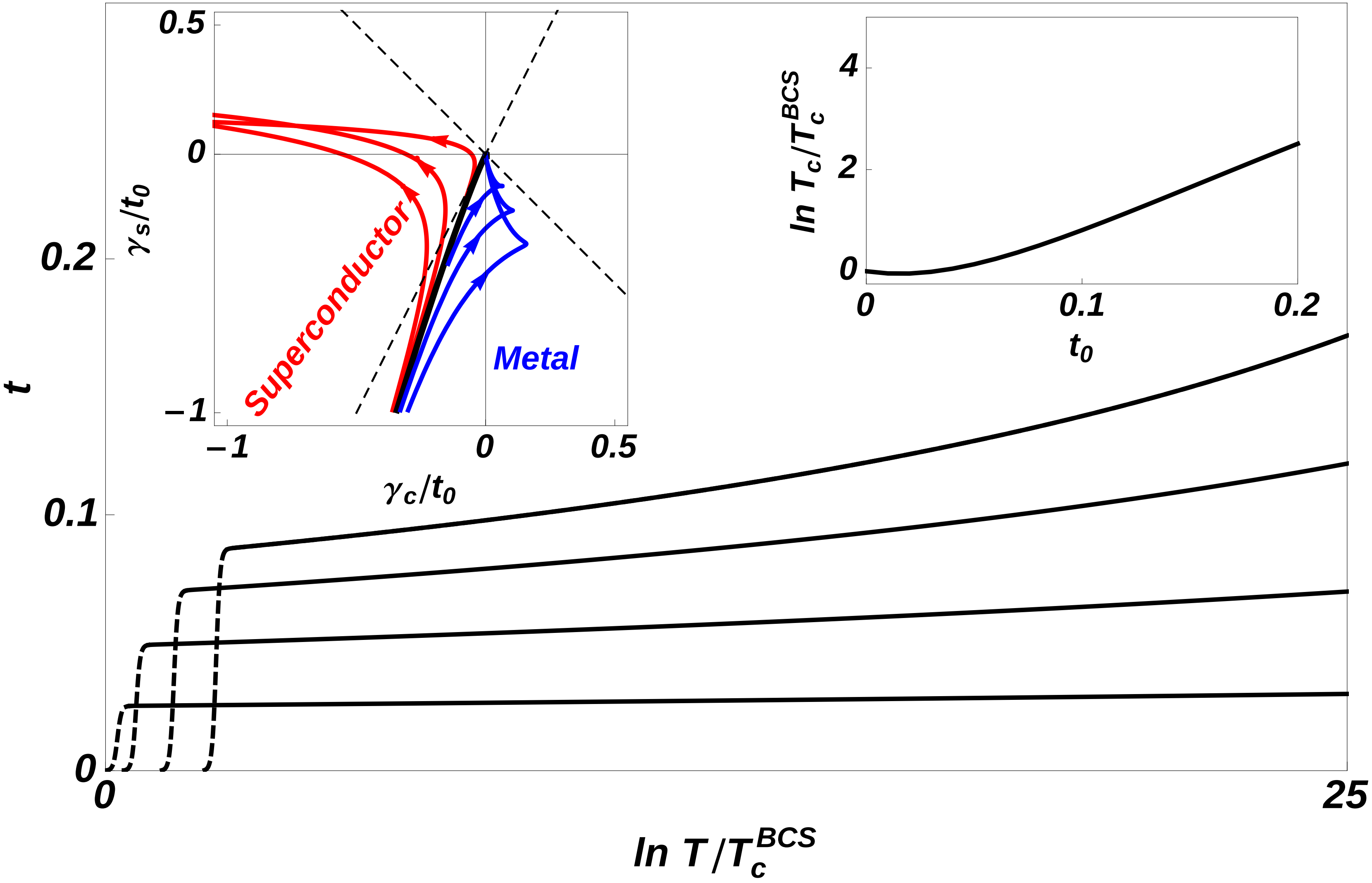
$$T_c \sim \exp\left\{-1/|\gamma_{c,0}|\right\} \quad \text{(BCS)}\ , \qquad G_0 \gtrsim |\gamma_0|^{-1}$$

$$T_c \sim \exp\left\{-c(G_0/|\gamma_0|)^{1/2}\right\}\ , \qquad 1 \lesssim G_0 \lesssim |\gamma_0|^{-1}$$

Exponentially strong enhancement of superconductivity

by multifractality at $1 \lesssim G_0 \lesssim |\gamma_0|^{-1}$

Disordered 2D system, symplectic class: Results (cont'd)



Before the system becomes insulator at $t_0 > t_* \sim 1$:
further enhancement of T_c near Anderson transition point t_*

SIT near Anderson transition

Consider system at Anderson localization transition

in 2D (symplectic symmetry class) or 3D

$$\frac{d\gamma}{dy} = -\Delta_2\gamma - \gamma^2$$

Superconductivity if $\gamma_0 < 0$

$\Delta_2 < 0$ – multifractal exponent at Anderson transition point

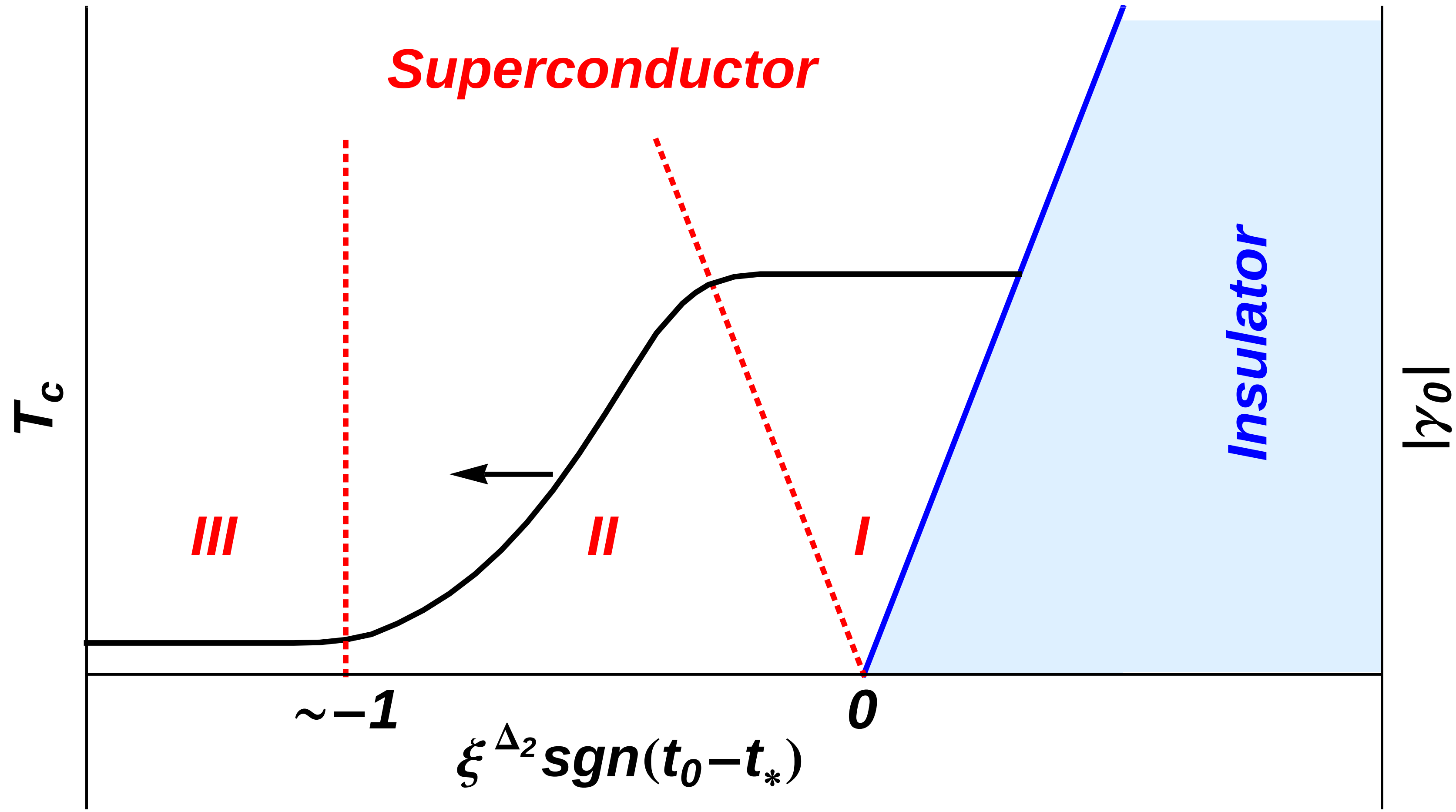
$$T_c \sim |\gamma_0|^{d/|\Delta_2|}$$

Exponentially strong enhancement of superconductivity:

Power-law instead of exponential dependence of T_c on interaction!

Agrees with Feigelman et al.

SIT near Anderson transition: Results



III : BCS

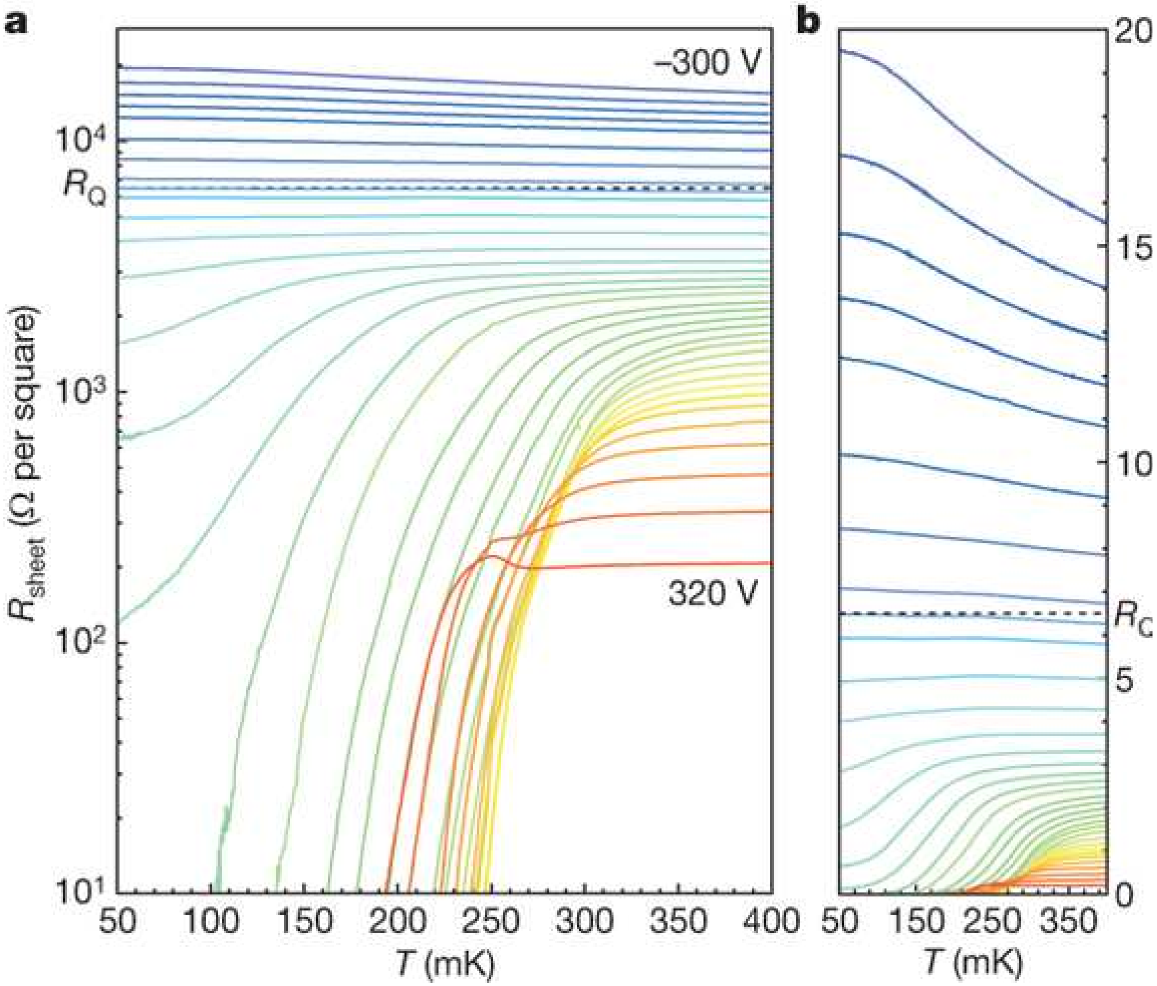
I : $T_c \sim |\gamma_0|^{d/|\Delta_2|}$

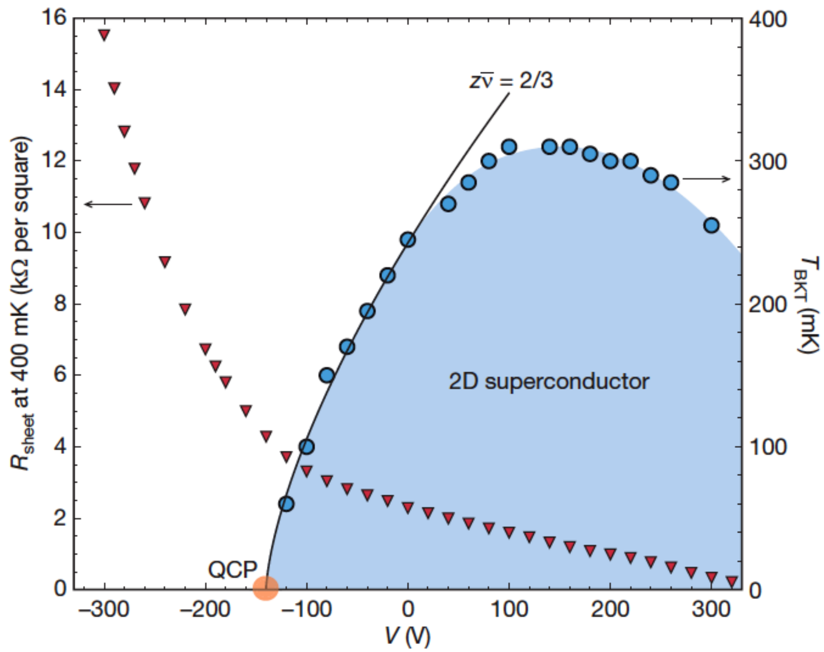
II : crossover: $T_c \sim \xi^{-3} \exp(-c\xi^{\Delta_2}/|\gamma_0|)$ (3D)

Experimental realizations ?

Key assumption: **short-range character of interaction**

⟶ **systems with strongly screened Coulomb interaction**





Caviglia,. . . ,Mannhart,
Triscone, Nature'08

$LaAlO_3/SrTiO_3$ interface

$\epsilon \approx 10^4$

2D: Comments

I. BCS and BKT

Calculating T_c, we treated superconductivity on the BCS level.

Because of phase fluctuations, the actual transition in 2D is of the BKT character.

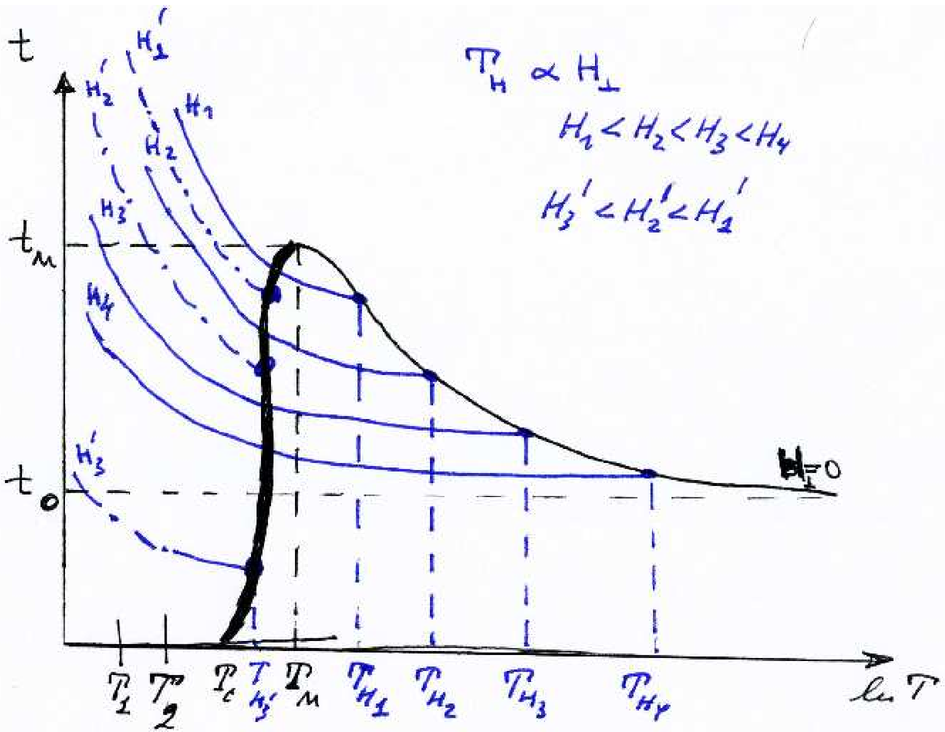
However, the temperatures are close, $T_{\mathrm{BKT}} \simeq T_c$

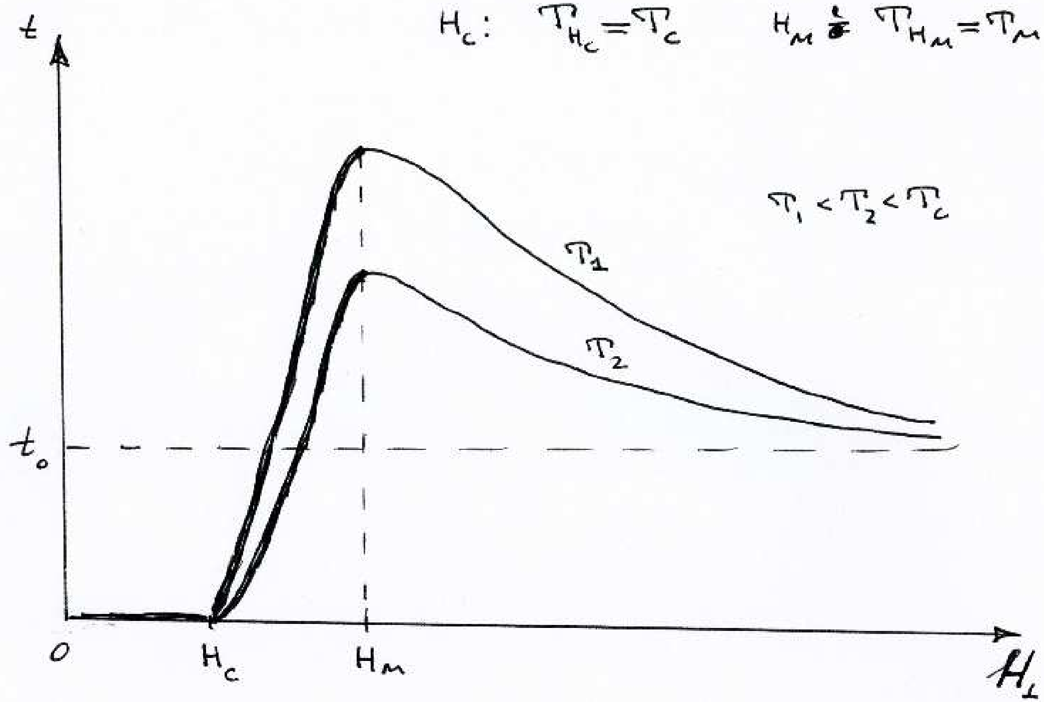
Beasley, Mooij, Orlando, PRL '79
Halperin, Nelson, JLTP'79
Kadin, Epstein, Goldman, PRB'83
Benfatto, Castellani, Giamarchi, PRB'09

Since the obtained enhancement of T_c is exponentially large, it is equally applicable to T_{BKT}

2D: Comments

II. Magnetoresistance in transverse field for $|\gamma_0| < t_0 < |\gamma_0|^{1/2}$





Summary

I. 2D transport in topological insulator systems:

interplay of localization, interaction, and topology

Coulomb inter. $\longrightarrow$ quantum criticality, conductivity $\sim e^2/h$

- surface of a 3D top. insulator
- QSH transition between normal and topol. insulator in 2D

II. 2D disordered superconductors:

Short-range interaction $\longrightarrow$
non-monotonous dependence of T_c on resistivity;

exponential enhancement of superconductivity by multifractality

- in 2D systems at intermediate disorder, $|\gamma_0| < t_0 < |\gamma_0|^{1/2}$
- near Anderson transition

Possible realizations: systems with strongly screened Coulomb interaction (large background ϵ, metallic gate)

THE END