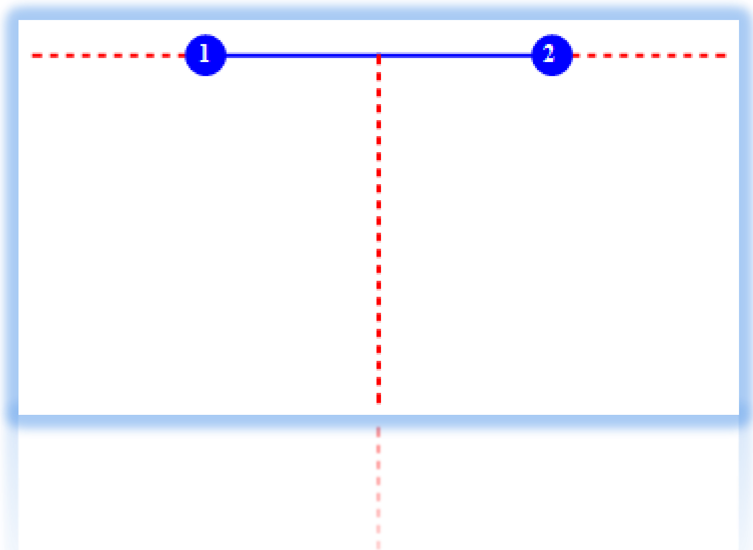
Majorana Fermions in Quantum wires

Yuval Oreg



Santa Barbara November 2011

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Collaborators

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Outline

- Introduction:
  - p Wave SC, and Majorana fermions
  - Realization in FQHE
- Realization in 3D TI, 2D semi conductors with and without FM
- Majoranas in 1D wires
  - Five phases: N,V,H,S,T tuned by μ and TDOS
  - Josephson “transistor”
  - Topological numbers
  - Examples for wave functions
- Exchange and non Abelian physics in 1D wires (embedded in 3D)
  - Relation to effective Spinors, calculation of the Berry phase.

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2D Spin less *px+ipy* SC

$$H = \int \frac{d^2k}{(2\pi)^2}\left[(\varepsilon_k - \mu)\psi_k^+\psi_k + \left(\Delta_k\psi_k\psi_{-k} + h.c.\right)\right]$$

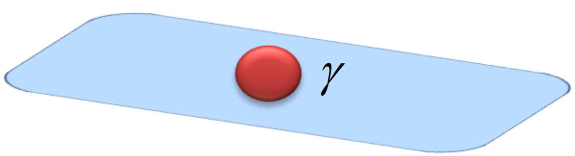
$$\Delta_k \propto k_x + ik_y$$

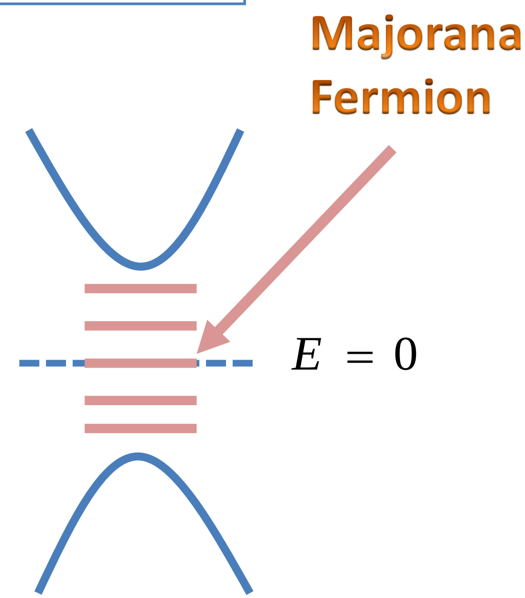
Uniform SC state

is **gapped** in the bulk

Kopnin and Salomaa (1991)

$$E_k = \sqrt{\left|\Delta_k\right|^2 + \left(\varepsilon_k - \mu\right)^2}$$





Spin less *px+ipy* SC vs. S wave SC

Spinless P-wave	S-wave
$\Delta_k \psi_k \psi_{-k} + h.c.$	$\Delta \psi_{k\uparrow} \psi_{-k\downarrow} + h.c.$
$\Delta_k \propto k_x + i k_y$	$\Delta = Const$
$u \psi(r) \partial_r \psi(r) + h.c.$	$\Delta \psi_\uparrow(r) \psi_\downarrow(r) + h.c.$
	$\Delta = Const$
$\lvert\Delta\rvert e^{i\phi} \psi_i \psi_{i+1} + h.c.$	$\Delta \psi_{i\uparrow} \psi_{i\downarrow} + h.c.$
$\Delta = Const$	$\Delta = Const$
$\gamma_k = u_k c_k^\dagger + v_k c_{-k}$	$\gamma_{k0} = u_k c_{k\uparrow}^\dagger + v_k c_{-k\downarrow}$ $\gamma_{k1} = u_k c_{k\downarrow}^\dagger + v_k c_{-k\uparrow}$

Fermion Doubling

Properties of Majorana fermions

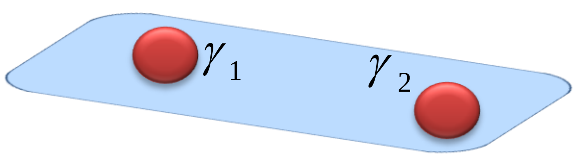
- Majorana Fermion is **its own antiparticle**

$$\gamma = \gamma^{+}$$

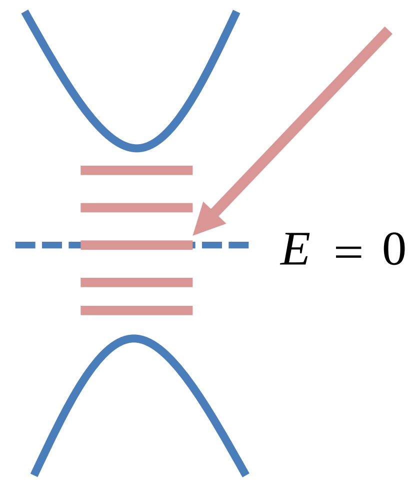
- Existence is **topologically protected**

- One Majorana = **"half" a usual (non local) Dirac fermion**

$$f = \gamma_1 + i\gamma_2$$



Majorana Fermion

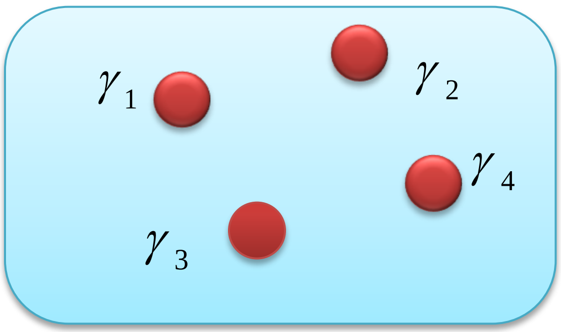


- $2n$ *Majoranas* → ***2^n degenerate ground states***

- Exhibit **non-Abelian braiding statistics**

Majoranas & non Abelian Physics

Four Majoranas



Two non local Dirac Fermions

$$f_A = \gamma_1 + i\gamma_2$$
$$f_B = \gamma_3 + i\gamma_4$$

4 degenerate states

$$|0_A, 0_B\rangle, |0_A, 1_B\rangle$$
$$|1_A, 0_B\rangle, |1_A, 1_B\rangle$$

What happens when we **braid** γ_2 around γ_3 ?

$$|0_A, 0_B\rangle \rightarrow \tfrac{1}{\sqrt{2}}\left(|0_A, 0_B\rangle - i|1_A, 1_B\rangle\right)$$
$$|1_A, 0_B\rangle \rightarrow \tfrac{1}{\sqrt{2}}\left(|1_A, 0_B\rangle - i|0_A, 1_B\rangle\right)$$

Read & Green 2000; Ivanov 2001,
Stern, von Oppen & Mariani 2004

Quantum state changes

Braiding implements unitary rotation within degenerate manifold

Urgently wanted for topological quantum computation

Nayak, Simon, Stern, Freedman, & Das Sarma, RMP 80, 1083 (2008)

How to experimentally realize Px+iPy SC

- Not so easy: - We live in 3D
  - Fermions come usually in pairs (e.g. spin)
  - P+ i P are rare (currently St_2RuO_4)
- One elegant solution v=5/2

Very challenging

1.) Only numerical evidences to Moore Read state

2.) If exist its fragile

3.) Requires strong magnetic field and low temp

ν = 5/2

Composite Fermi sea is unstable towards ***p+ip* pairing!**

$$\Psi_{\mathrm{Pf}} = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^2$$

How to experimentally realize Px+iPy SC

- Not so easy: - We live in 3D
  - Fermions come usually in pairs (e.g. spin)
  - P+ i P are rare (currently St_2RuO_4)
- One elegant solution v=5/2
- Additional promising settings recently proposed
  - Topological insulators
  - Semiconductor heterostructures

See Marcel Franz, Physics **3, 24 (2010)**

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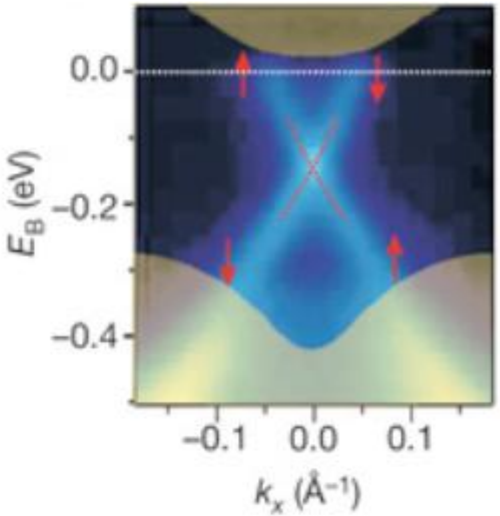
Majoranas in 3D topological insulator

3D topological insulators: inert bulk but **odd # of Dirac cones on the surface**



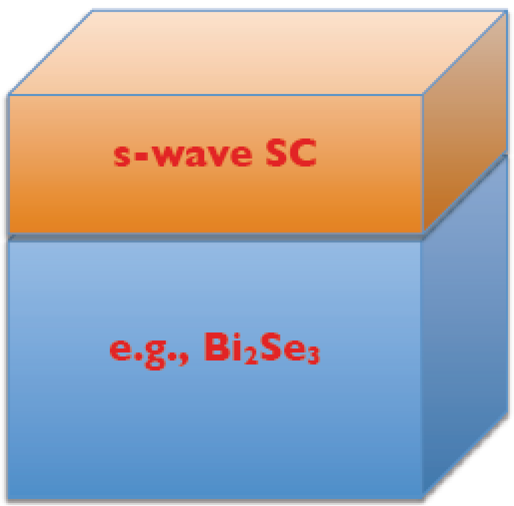
D. Hsieh et al., Nature **460**, 1101 (2009)

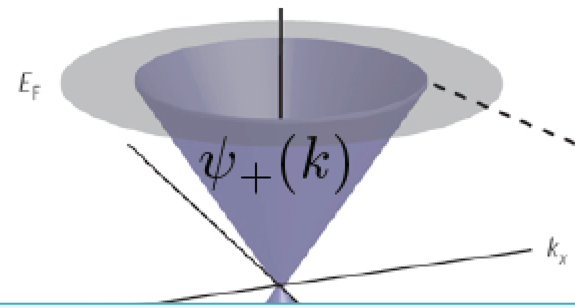
$$H = \int d^2\mathbf{r}\psi^\dagger(-iv\vec{\sigma}\cdot\nabla - \mu)\psi$$

(Fu, Kane, & Mele 2006; Moore & Balents 2006; Roy 2006; Fu & Kane 2008)

"Fermion doubling" solved, but need to generate topological superconductivity...

Majoranas in 3D topological insulator





Promising

1. Topological phase could be very robust (large gap & immunity to disorder)

$$H = \int d^2\mathbf{k} \Big\{ [\epsilon_+(k)\psi_+^\dagger \psi_+ + \epsilon_-(k)\psi_-^\dagger \psi_-]$$
$$+\Delta\left[\left(\frac{k_x - ik_y}{2k}\right)[\psi_-(k)\psi_-(-k) - \psi_+(k)\psi_+(-k)] + h.c.\right]\Big\}$$

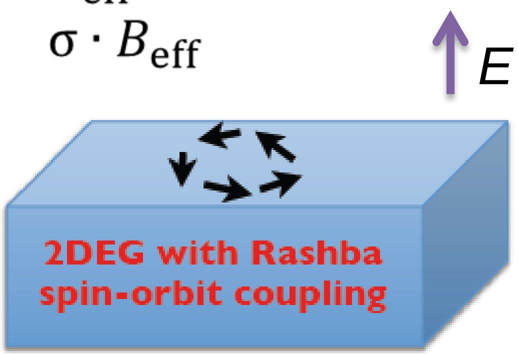
Pairing is *p*+*ip* in this basis!

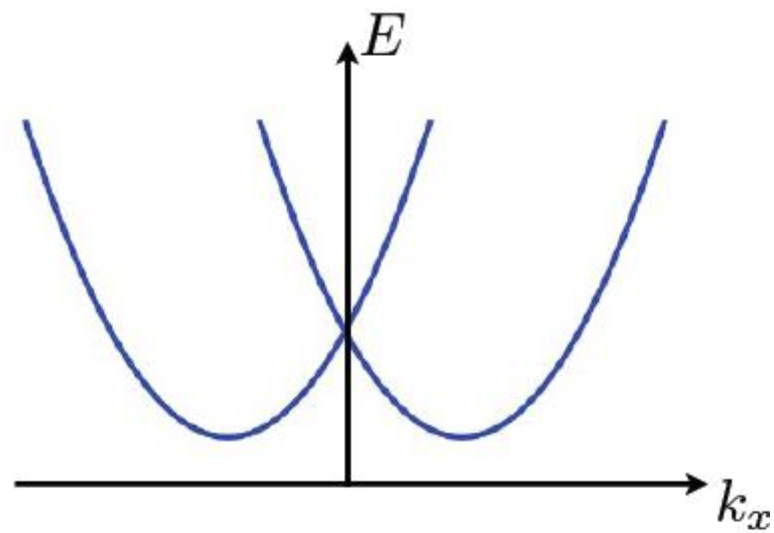
Majoranas in Semi Conductors

Spin Orbit Coupling

$B_{\text{eff}} = \alpha v \times \vec{E}$

$\sigma \cdot B_{\text{eff}}$

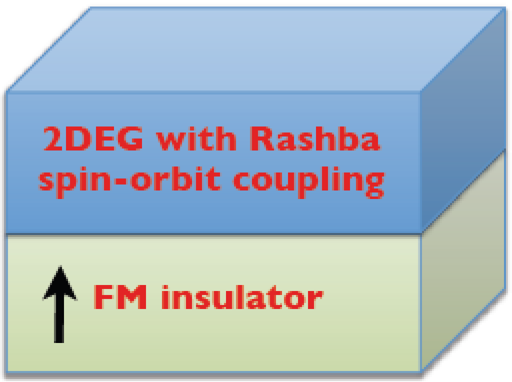


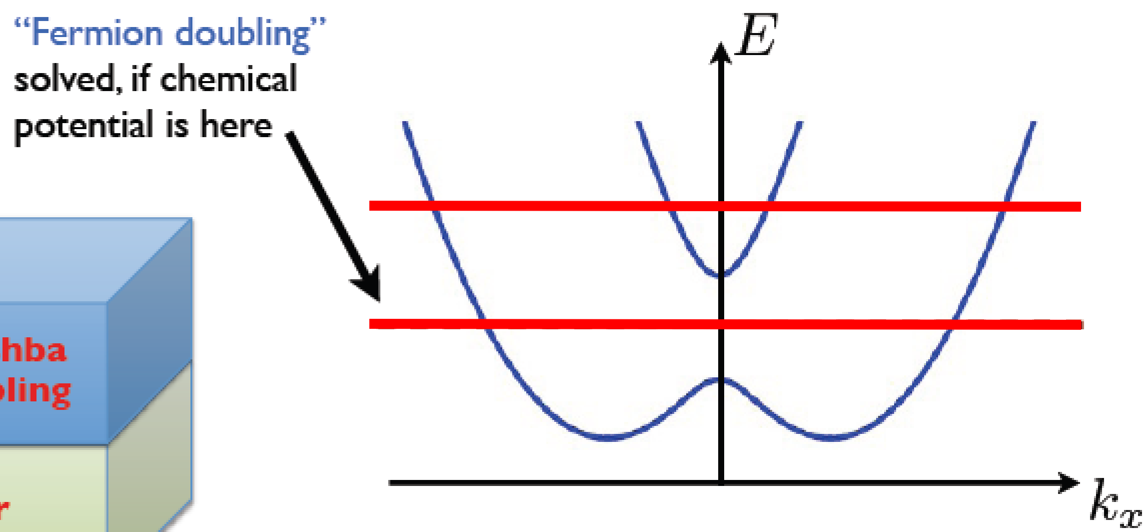


$$H = \int d^2\mathbf{r}\,\psi^\dagger \left[-\frac{\nabla^2}{2m} - \mu - i\alpha(\sigma^x \partial_y - \sigma^y \partial_x) \right] \psi$$

(Sau, Lutchyn, Tewari, & Das Sarma 2009)

Majoranas in Semi Conductors

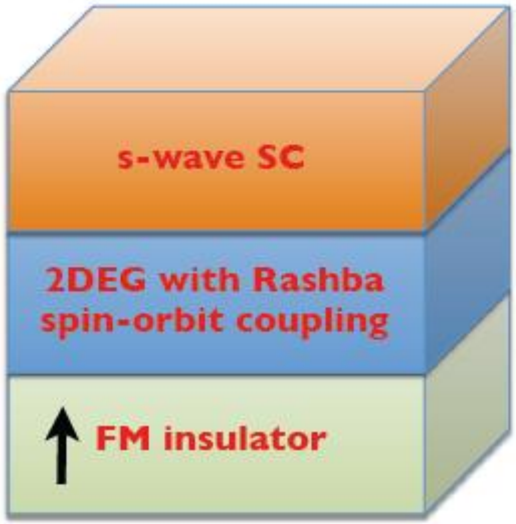


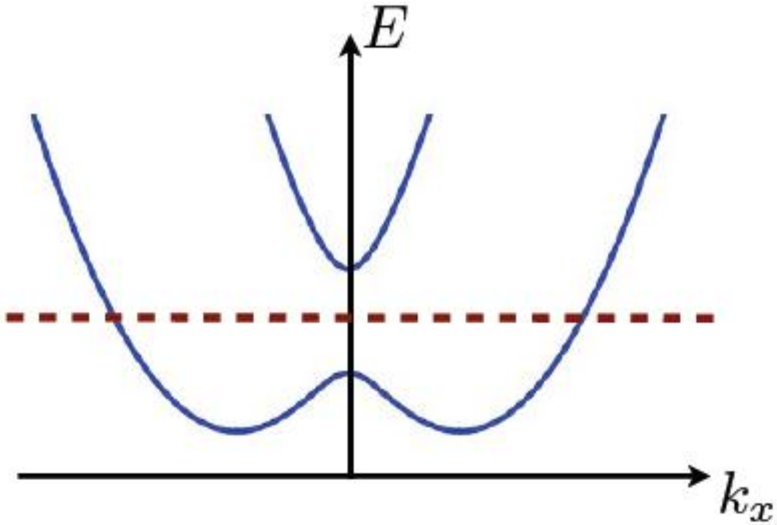


$$H = \int d^2\mathbf{r}\,\psi^\dagger\left[-\frac{\nabla^2}{2m} - \mu - i\alpha(\sigma^x\partial_y - \sigma^y\partial_x) + V_z\sigma^z\right]\psi$$

(Sau, Lutchyn, Tewari, & Das Sarma 2009)

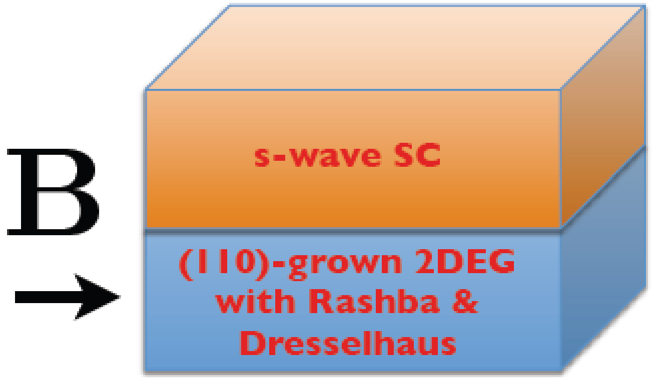
Majoranas in Semi Conductors





$$H = \int d^2\mathbf{r}\psi^\dagger\left[-\frac{\nabla^2}{2m} - \mu - i\alpha(\sigma^x\partial_y - \sigma^y\partial_x) + V_z\sigma^z\right]\psi$$
$$+ \int d^2\mathbf{r}(\Delta\psi_\uparrow\psi_\downarrow + h.c.)$$

Majoranas in Semi Conductors without the FM



Proximity effect generates a topological SC supporting Majorana fermions!

In-plane field plays the role of the FM insulator!

$$H = \int d^2\mathbf{r}\,\psi^\dagger\left[-\frac{\nabla^2}{2m} - \mu - i\alpha(\sigma^x\partial_y - \sigma^y\partial_x) - \boxed{i\beta\sigma^z\partial_x} + V_y\sigma^y\right]\psi$$
$$+ \int d^2\mathbf{r}(\Delta\psi_\uparrow\psi_\downarrow + h.c.)$$

Dresselhaus: tends to align spins **normal** to the 2DEG

Alicea 2010

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1D spin-less P wave SC: Kitaev's Model

$$H = \mu \sum_{x=1}^{N} c_x^\dagger c_x - \sum_{x=1}^{N-1} \left(t c_x^\dagger c_{x+1} + |\Delta| e^{i\phi} c_x c_{x+1} + h.c.\right)$$

$$\mu = 0$$
$$t = |\Delta|$$

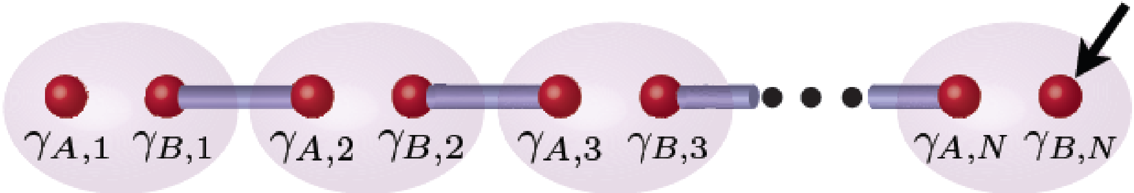
$$c_x = \frac{1}{2} e^{-i\frac{\phi}{2}} (\gamma_{B,x} + i\gamma_{A,x})$$

$$\gamma_{B,x} = c_x e^{i\frac{\phi}{2}} + c_x^\dagger e^{-i\frac{\phi}{2}}$$

$$\gamma_{A,x} = -i c_x e^{i\frac{\phi}{2}} + i c_x^\dagger e^{-i\frac{\phi}{2}}$$

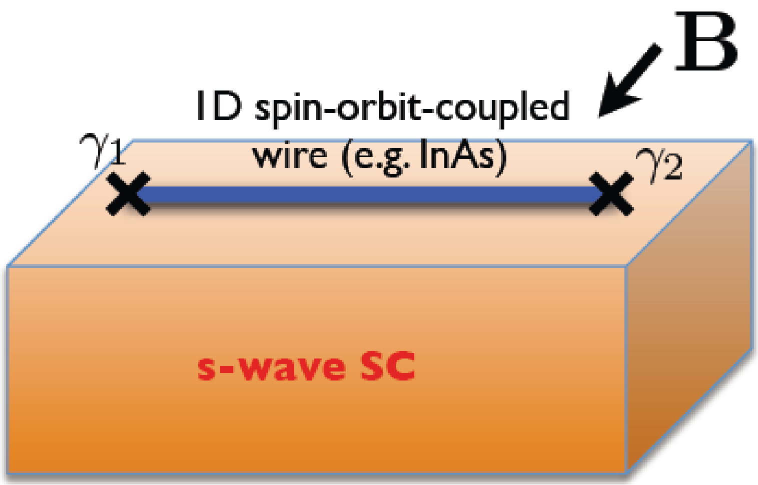
$$\Rightarrow \quad H = -it \sum_{x=1}^{N-1} \gamma_{B,x} \gamma_{A,x+1}$$

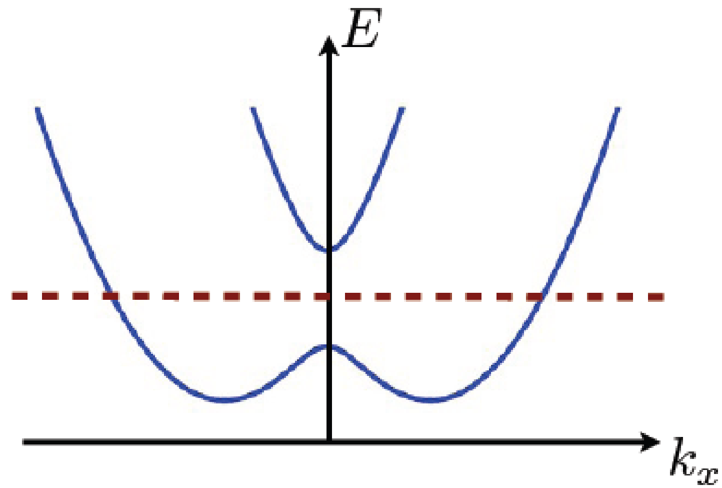
Unpaired end Majorana fermions!

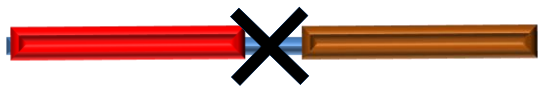


(Kitaev 2001)

Semi conducting wires







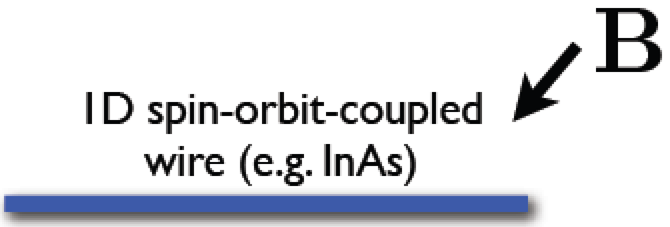
(Lutchyn, Sau, Das Sarma 2010;
YO, Refael, von Oppen 2010)

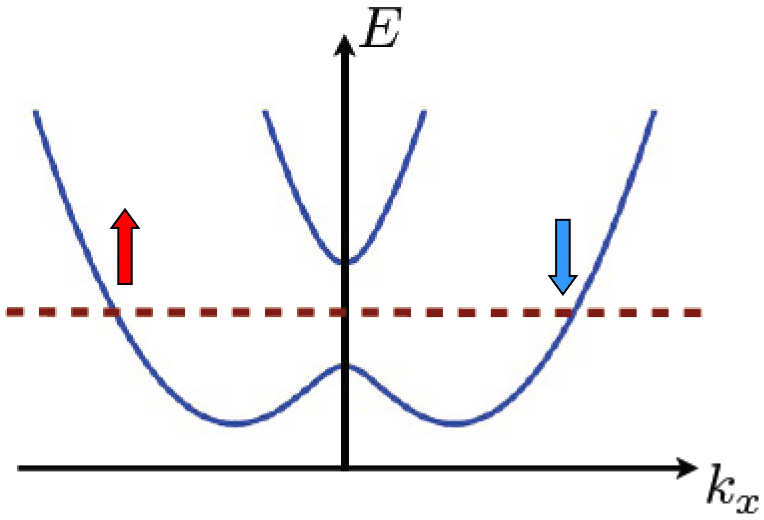
Generates a **topological 1D superconducting state!**

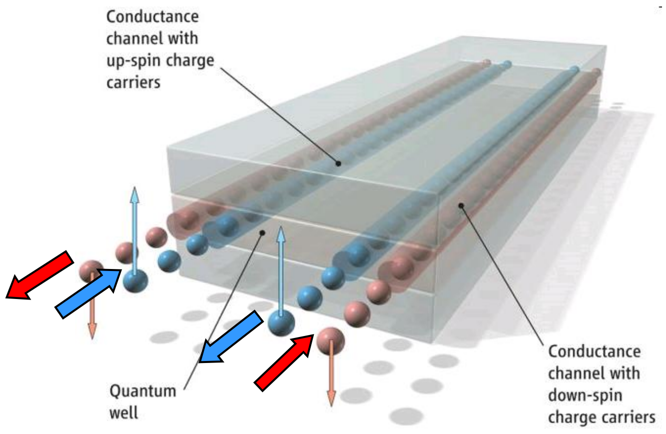
$$H = \int \Psi^\dagger(x)\mathcal{H}(x)dx \qquad \Psi^\dagger = \left(\psi_\uparrow^\dagger, \psi_\downarrow^\dagger, \psi_\downarrow, -\psi_\uparrow\right)$$

$$\mathcal{H} = \left[\frac{p^2}{2m} - \mu(x)\right]\tau_z + up\sigma_y\tau_z + B(x)\sigma_z + \Delta_1(x)\tau_x + \Delta_2(x)\tau_y$$

Equivalence to edge states of 2DTI







Konig et al 2007

Quantum Spin Hall Effect and Topological Phase Transition in HgTe Quantum Wells

B. Andrei Bernevig,[1,2] Taylor L. Hughes,[1] Shou-Cheng Zhang[1*]

Recent Developments

Potter and Lee

sufficient to have odd number of channels in a wire.

Duckheim &Brouwer, Chang & Zhang et al.

Half metal in proxy to superconductor with SOI

Akhmerov, Beenakker, Hassler et al.

Measurement schemes, effects of disorder, Coulomb Island…

Lutchyn and Bonderson

Transfer to a standard qubit

Clarke, Sau & Tewary (Das Sarma)

General properties of exchanging Majoranas on a network

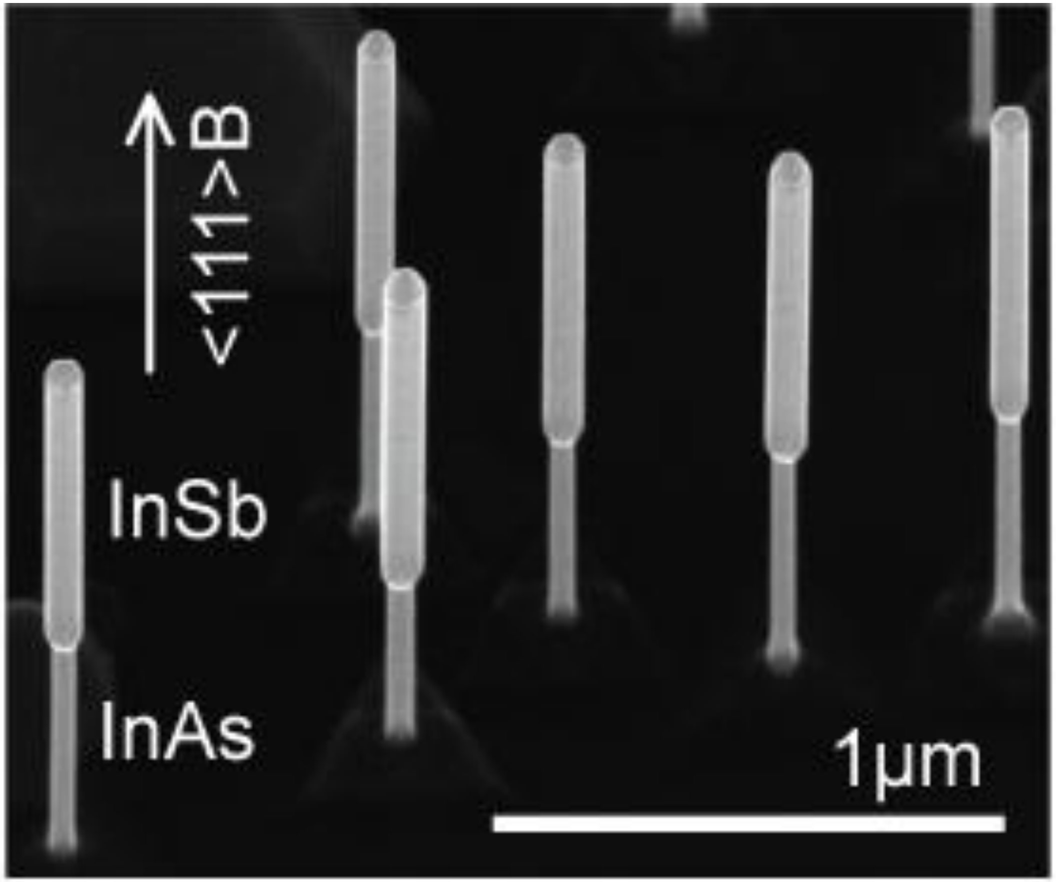
Gangadharaiah, Loss et al.

Interaction effects, helical liquid in CNT…..

Cook and Marcel Franz,

TI wires

InAs/InSb nanowires by MOVPE (Lund)



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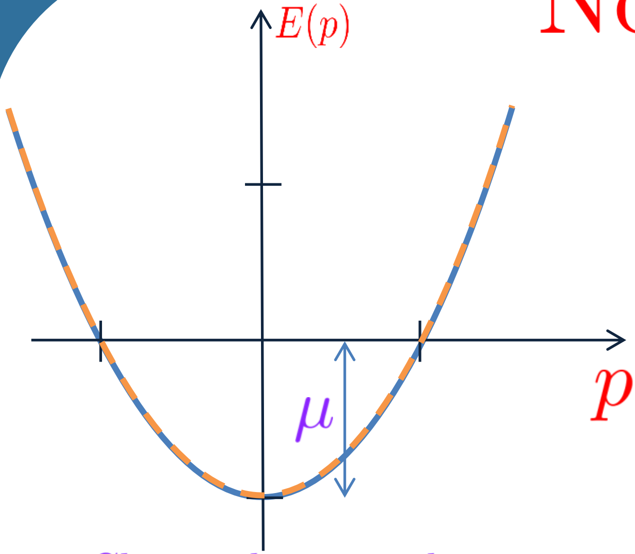
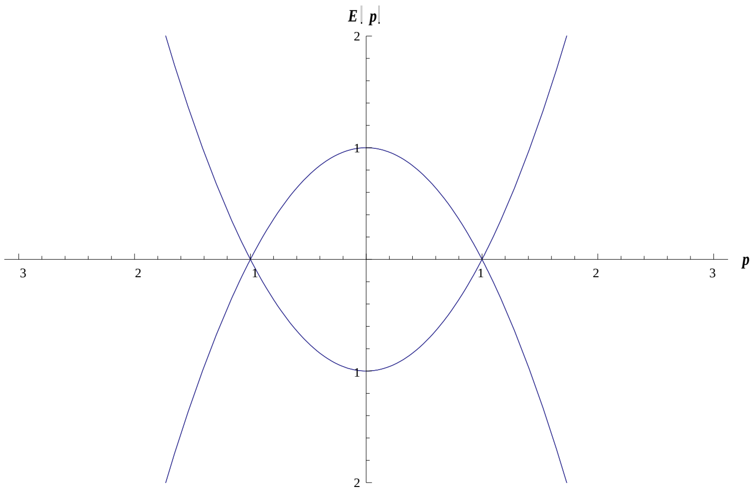
Five Phases

Normal Phase



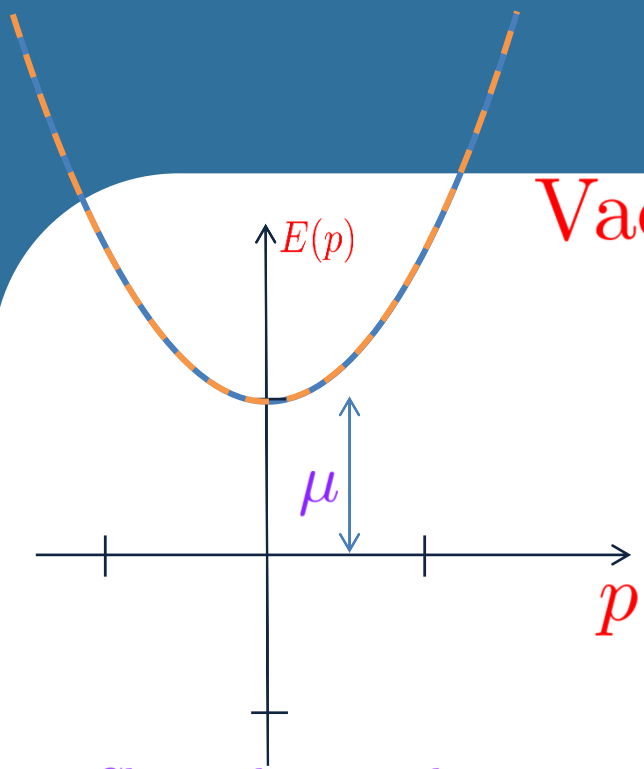
Chemical potential	$\mu = 1$
Spin orbit coupling	$u = 0$
Magnetic Field	$B = 0$
Supercondcutor gap	$\Delta = 0$



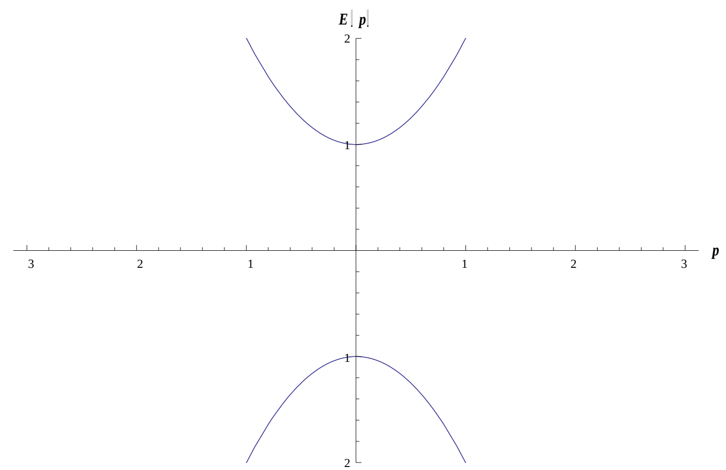
$$G_{2\mathrm{Terminal}} = \frac{e^2}{\hbar} \times 2$$

Five Phases

Vacuum Phase



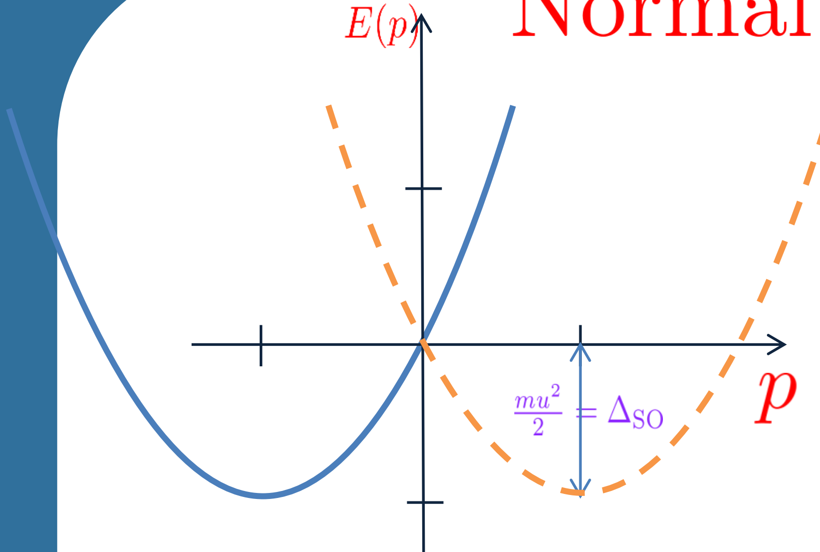
Chemical potential	$\mu = -1$
Spin orbit coupling	$u = 0$
Magnetic Field	$B = 0$
Supercondcutor gap	$\Delta = 0$



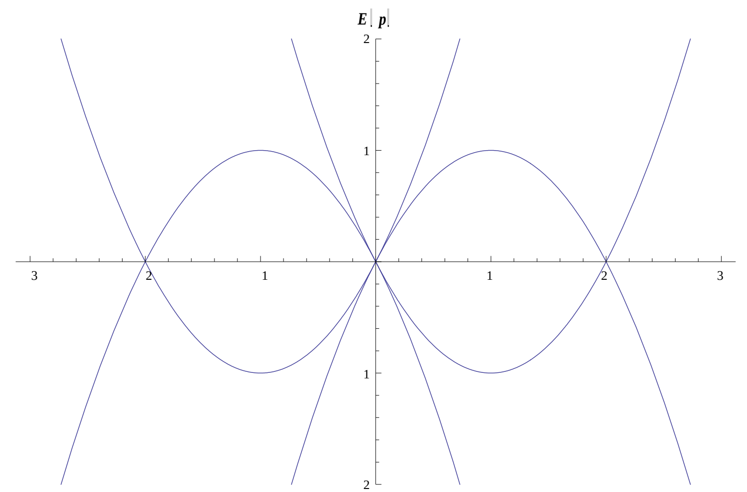
$$G_{2\text{Terminal}} = \frac{e^2}{h} \times 0$$

Five Phases

Normal Phase with SO



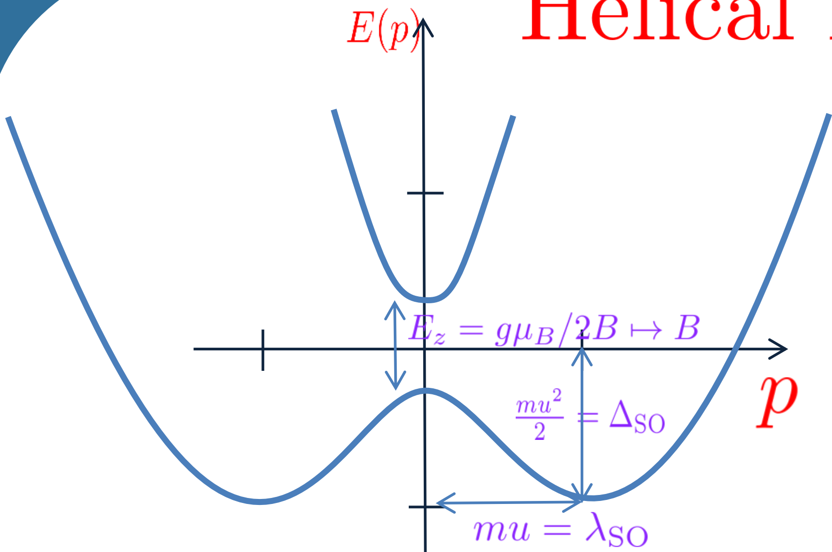
Chemical potential	$\mu = 0$
Spin orbit coupling	$u = 2, (m = 1/2)$
Magnetic Field	$B = 0$
Supercondcutor gap	$\Delta = 0$



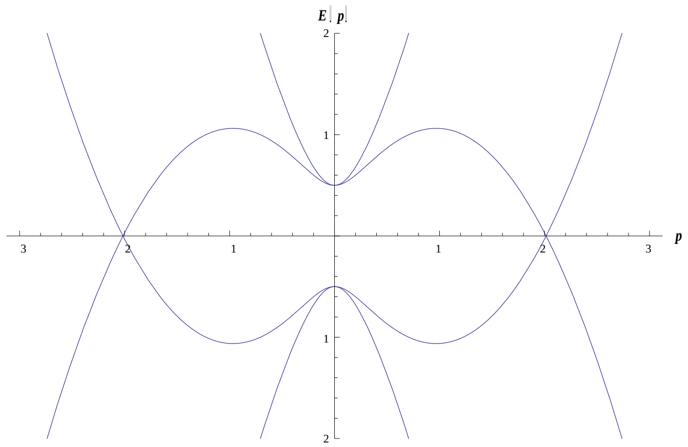
$$G_{2\text{Terminal}} = \frac{e^2}{h} \times 2$$

Five Phases

Helical Phase



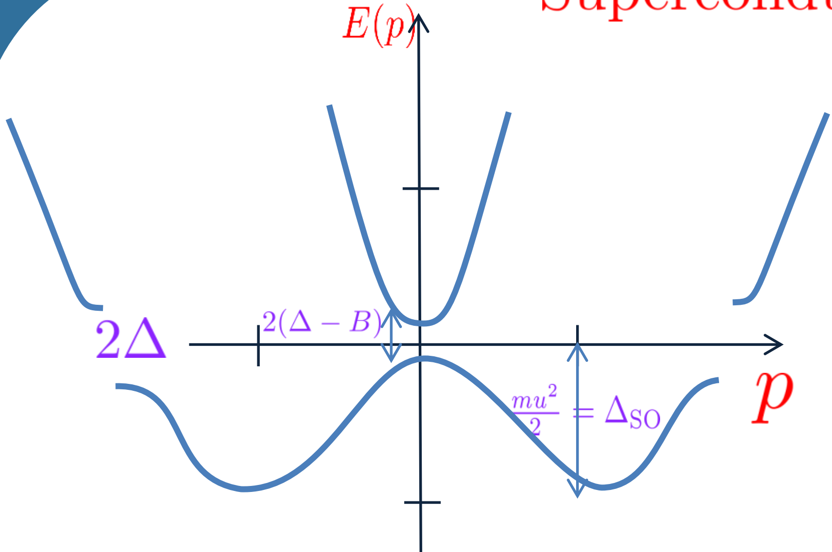
Chemical potential	$\mu = 0$
Spin orbit coupling	$u = 2, (m = 1/2)$
Magnetic Field	$B = 1/2$
Supercondcutor gap	$\Delta = 0$



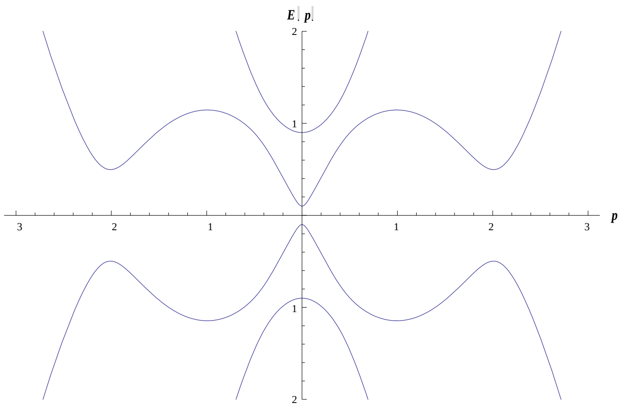
$$G_{2\mathrm{Terminal}} = \frac{e^2}{h} \times 1$$

Five Phases

Superconductor Phase

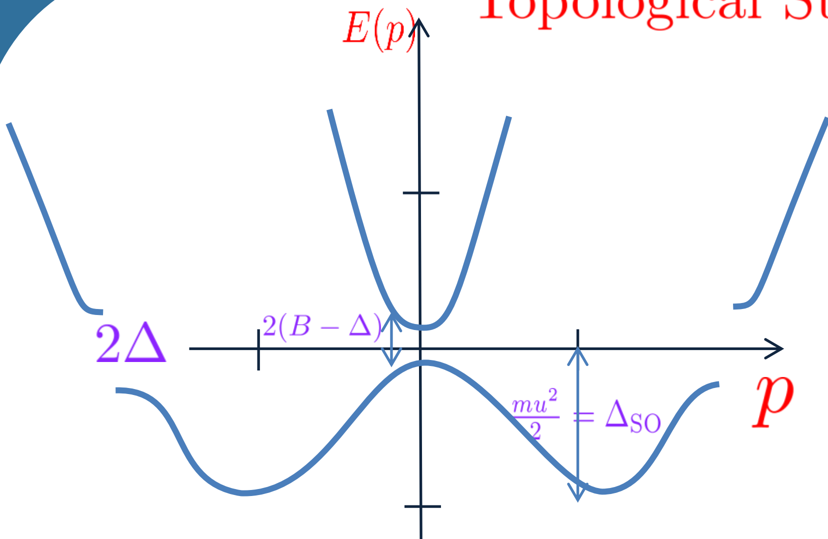


Chemical potential	$\mu = 0$
Spin orbit coupling	$u = 2, (m = 1/2)$
Magnetic Field	$B = .4$
Supercondcutor gap	$\Delta = 0.5$

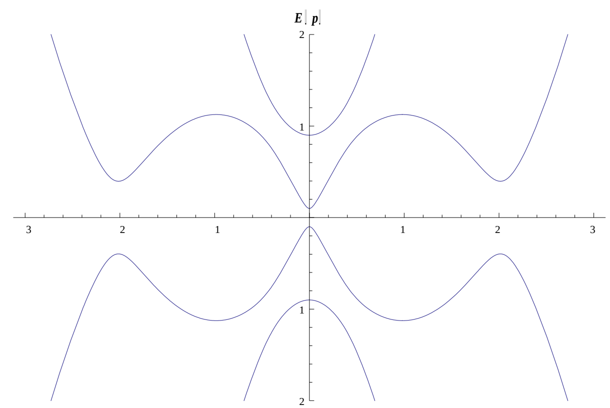


Five Phases

Topological Superconductor Phase



Chemical potential	$\mu = 0$
Spin orbit coupling	$u = 2, (m = 1/2)$
Magnetic Field	$B = 1/2$
Supercondcutor gap	$\Delta = 0.4$



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Tunneling DOS

$$\mathcal{H} = \begin{pmatrix} \frac{p^2}{2m} - \mu + up & B & \Delta & 0 \\ B & \frac{p^2}{2m} - \mu - up & 0 & \Delta \\ \Delta & 0 & -\frac{p^2}{2m} + \mu - up & B \\ 0 & \Delta & B & -\frac{p^2}{2m} + \mu + up \end{pmatrix}. \qquad (1)$$

Green's function:

$$G(\epsilon) = \int_{-\infty}^{\infty} dp(\epsilon - \mathcal{H})^{-1}. \qquad (2)$$

In the Green's function matrix, the element for spin-up electrons is G_{11}. Using $\mathcal{H}|\phi^{(n)}\rangle = E^{(n)}|\phi^{(n)}\rangle$:

$$\begin{aligned} G_{11}(\epsilon) &= \int_{-\infty}^{\infty} dp\langle 1|(\epsilon - \mathcal{H})^{-1}|1\rangle \\ &= \int_{-\infty}^{\infty} dp\langle 1|\sum_n |\phi^{(n)}\rangle\langle\phi^{(n)}|(\epsilon - \mathcal{H})^{-1}|1\rangle \\ &= \int_{-\infty}^{\infty} dp \sum \frac{\left|\langle 1|\phi^{(n)}\rangle\right|^2}{\epsilon - E^{(n)}}, \end{aligned}$$

Most & YO in preparation

Tunneling DOS

The spin up density of states is:

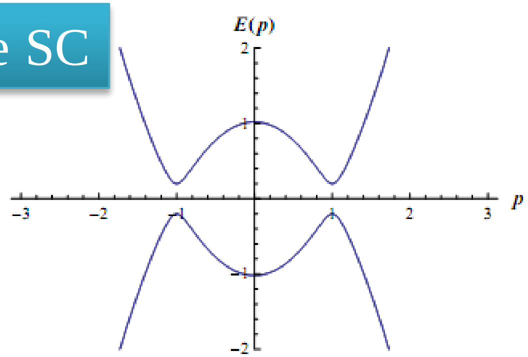
$$D_{\uparrow}(\epsilon) = \frac{1}{\pi} \operatorname{Im} \lim_{\delta \to 0^{+}} G_{11}(\epsilon - i\delta) = \sum_{\{p|E(p)=\epsilon\}} \frac{|\phi_1(p)|^2}{|E'(p)|} \tag{3}$$

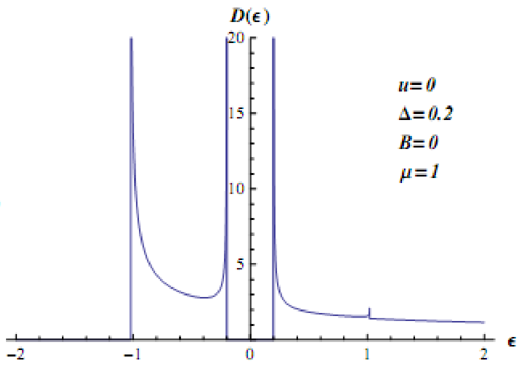
The spin down density of states is:

$$D_{\downarrow}(\epsilon) = \frac{1}{\pi} \operatorname{Im} \lim_{\delta \to 0^{+}} G_{22}(\epsilon - i\delta) = \sum_{\{p|E(p)=\epsilon\}} \frac{|\phi_2(p)|^2}{|E'(p)|} \tag{4}$$

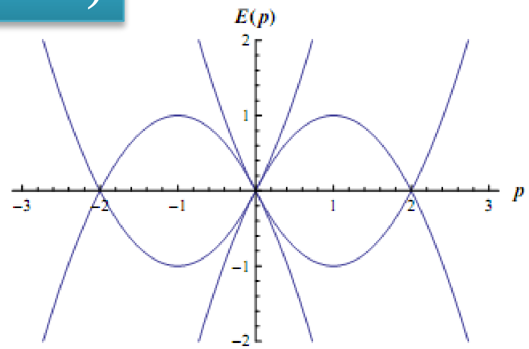
Tunneling DOS

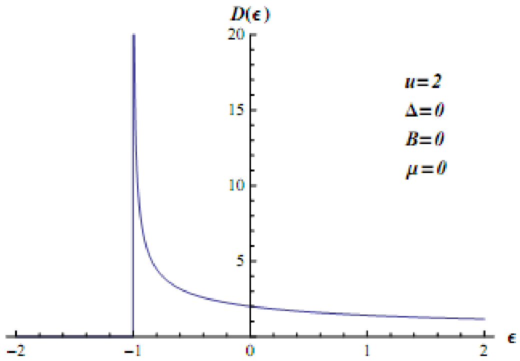
Standard S wave SC



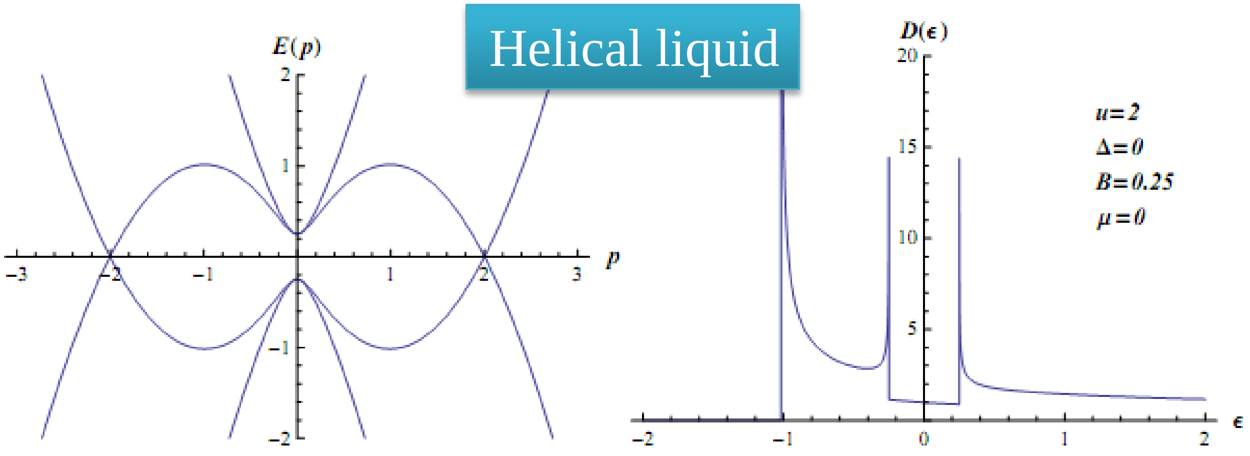


Normal (With SO)

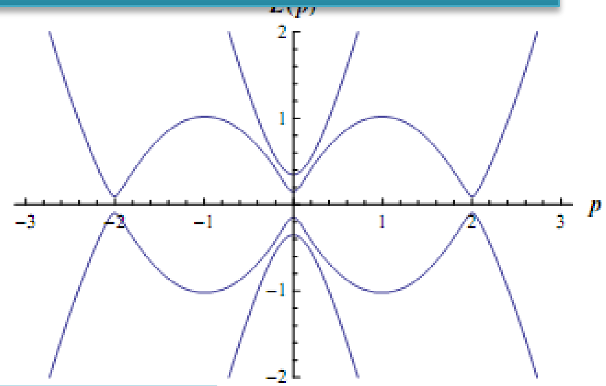


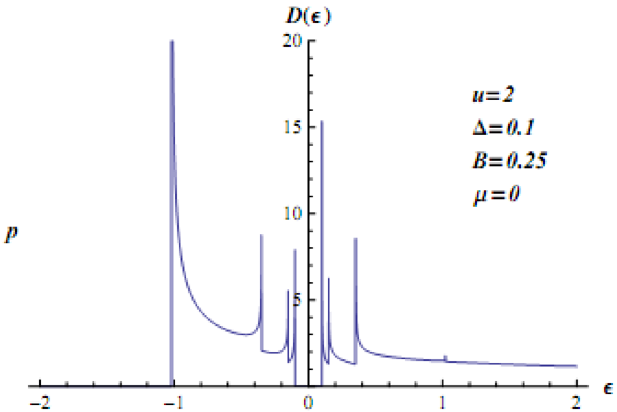


Tunneling DOS

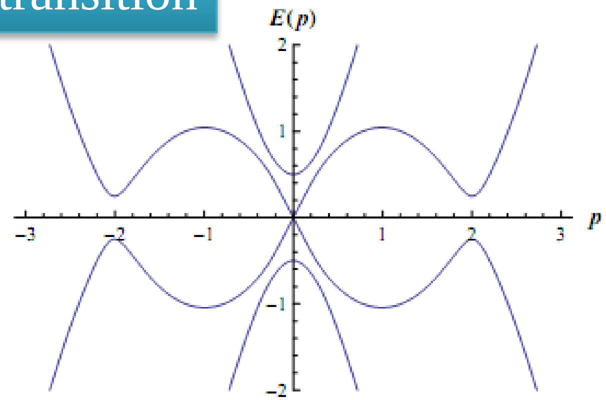


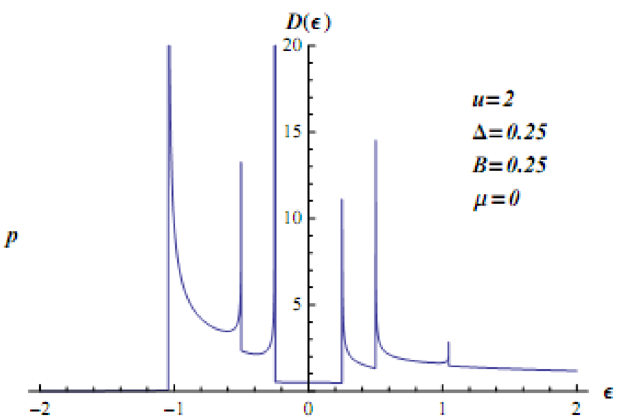
Topological Superconductor





S-T transition





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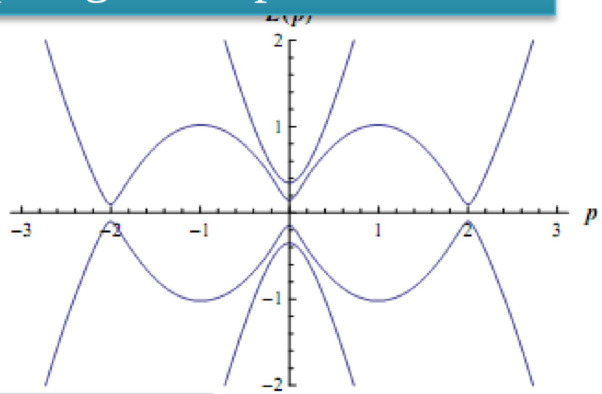
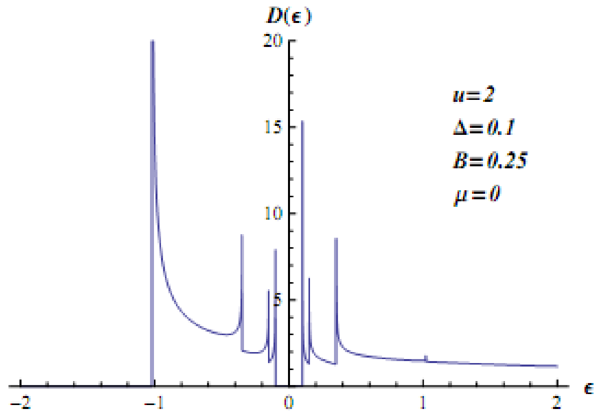
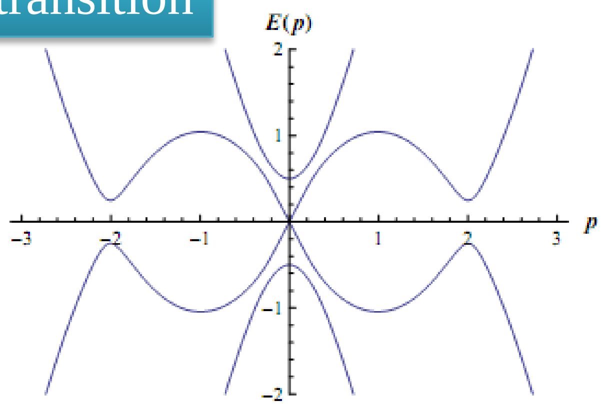
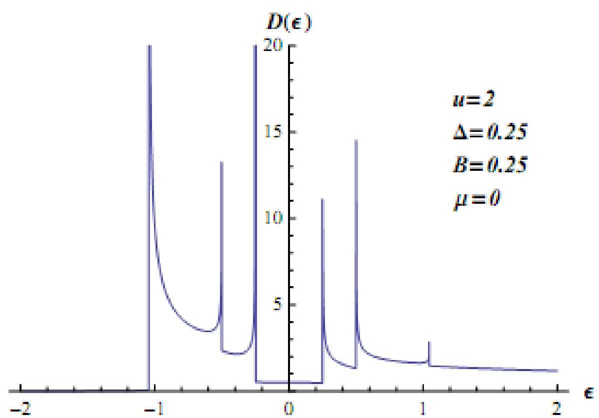


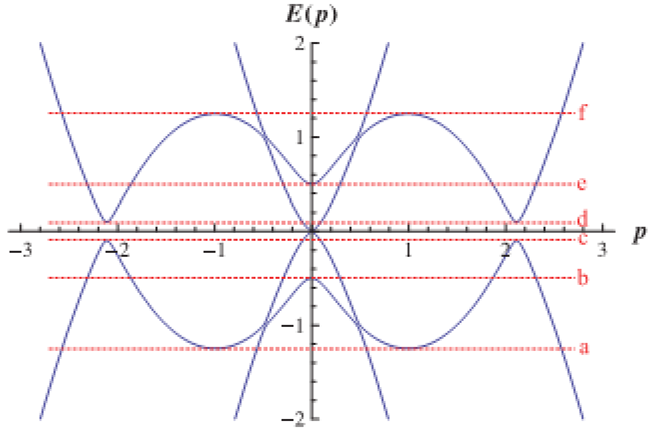


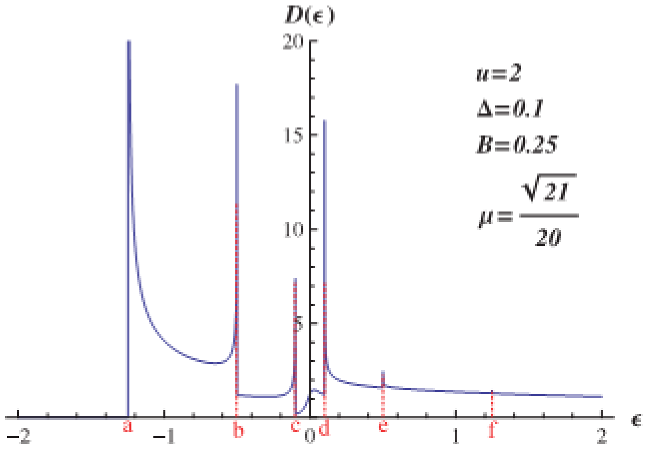
S-T transition





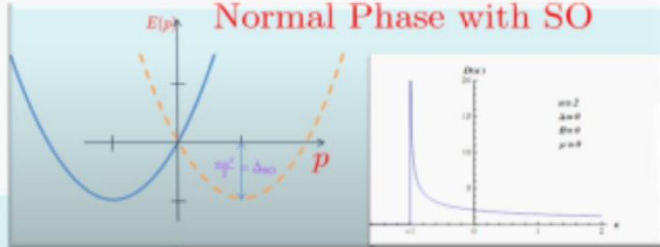
S-T transition

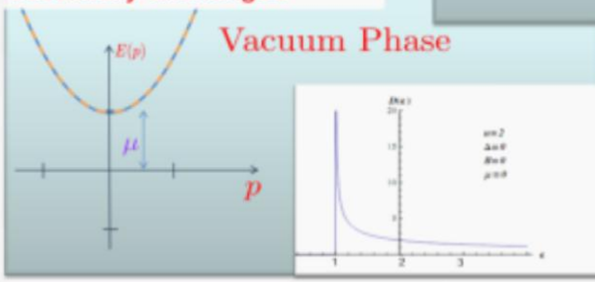




Wire phases:

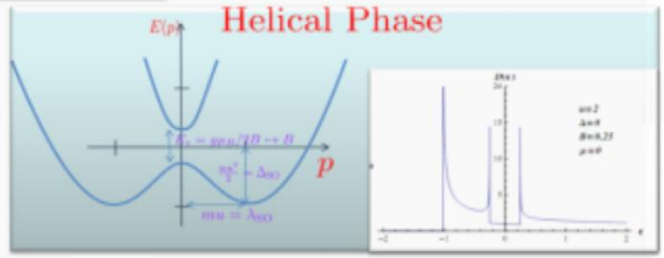
Normal phase (N): *SO coupling yields spin-dependent shifts of the dispersion (left panel), but leaves TDOS (right panel) essentially unchanged*

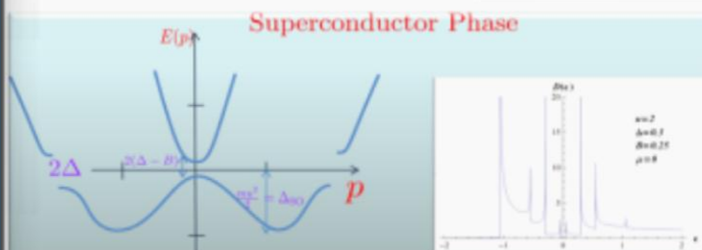




Vacuum phase (V): *occurs when the wire is empty*

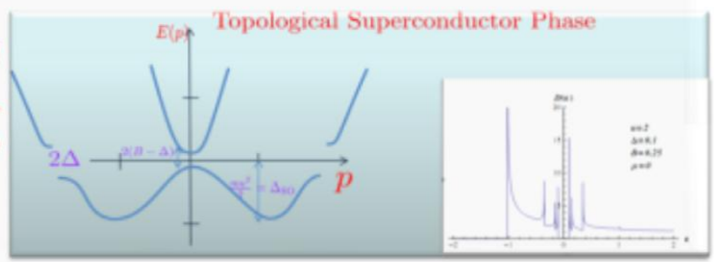
Helical phase (H): *occurs with magnetic field and spin orbit coupling. An additional singularity appears in the TDOS*

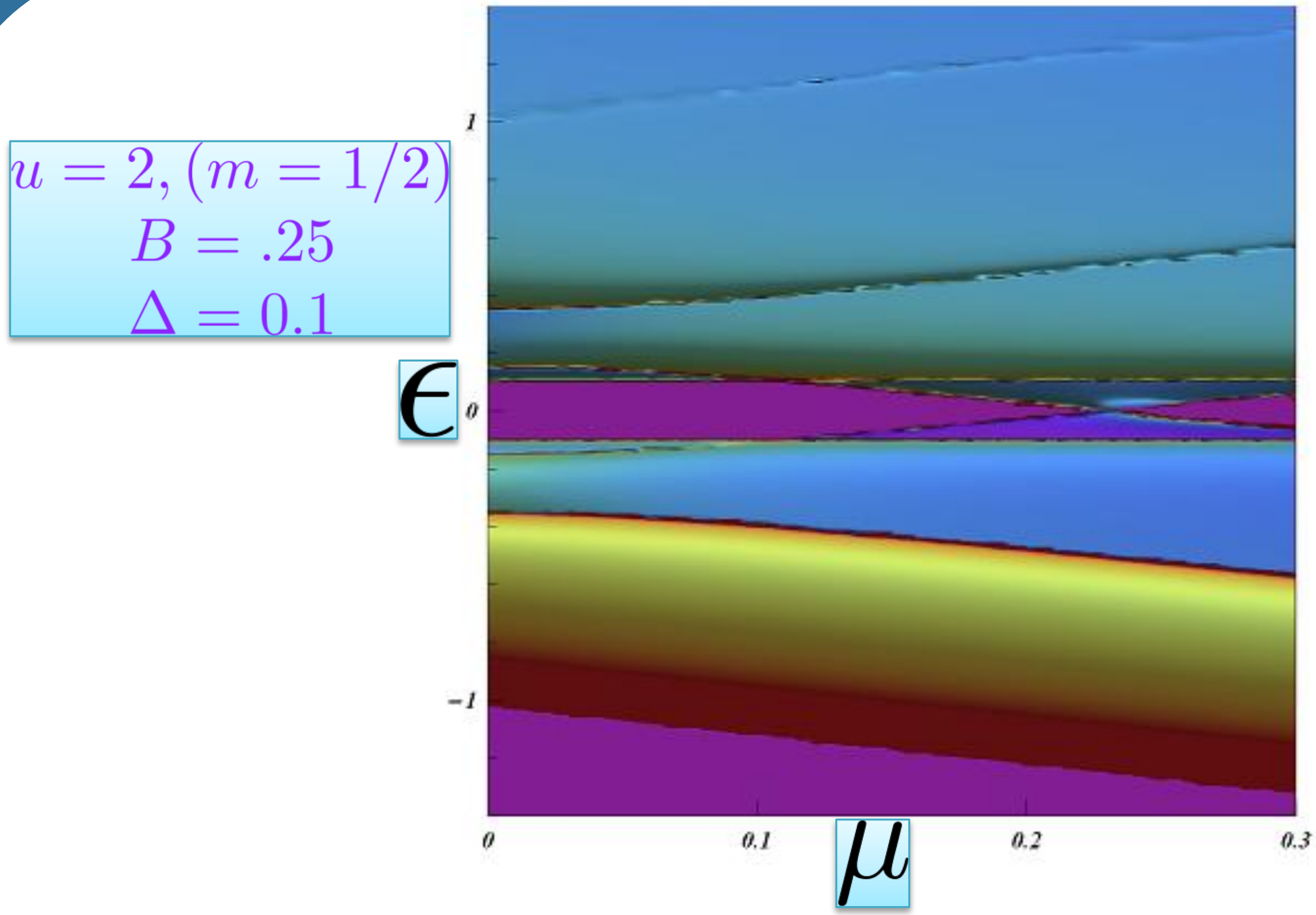




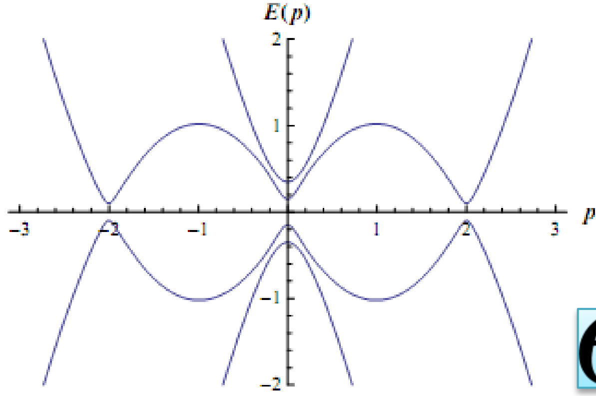
Superconducting phase (S): *occurs when the superconducting gap is larger than the gap due to the magnetic field*

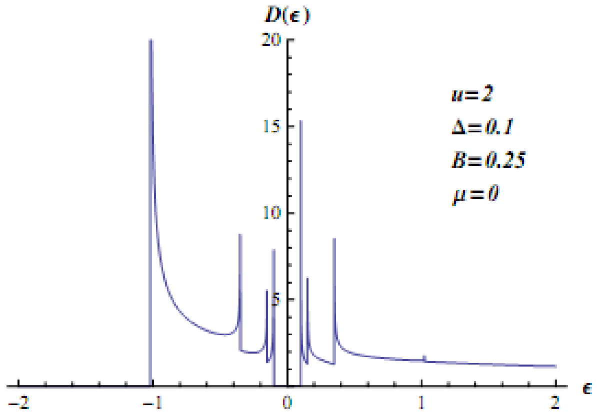
Topological superconducting phase (T): *occurs when the superconducting gap is smaller than the gap due to the magnetic field*

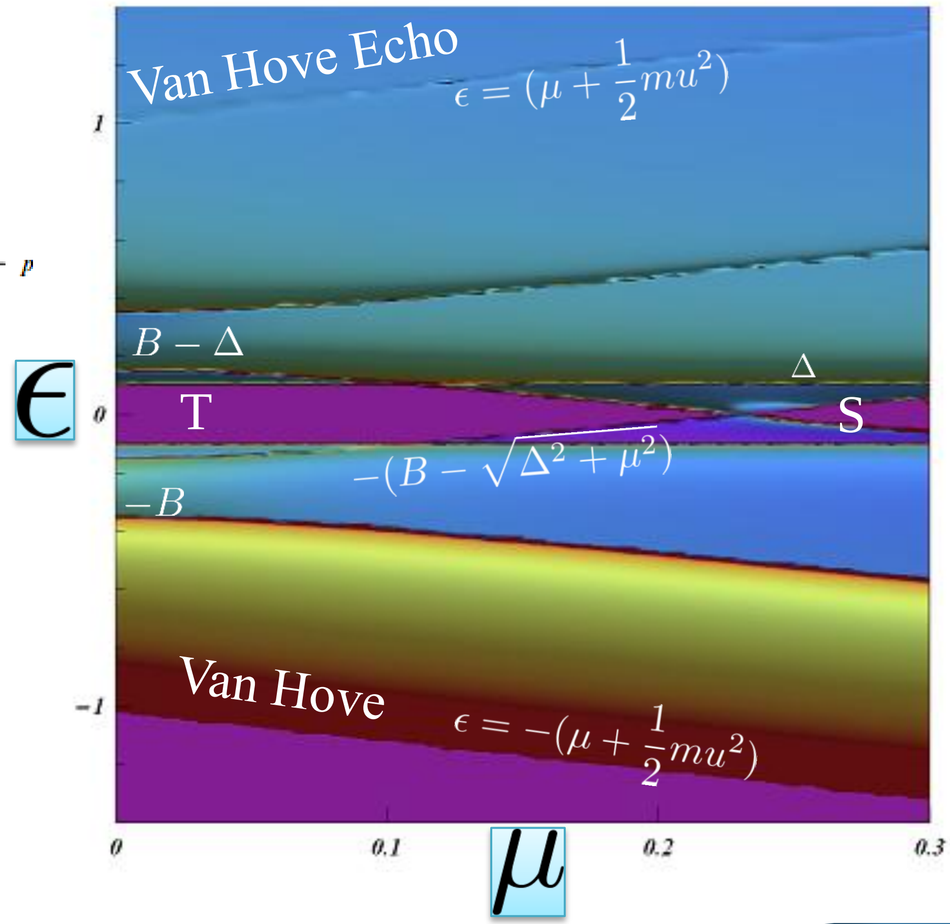



$u = 2, (m = 1/2)$
$B = .25$
$\Delta = 0.1$
ϵ
1
0
-1
0
0.1
0.2
0.3
μ

$$u = 2, (m = 1/2)$$
$$B = 0.25 \mapsto 0.25 \times \Delta_{\mathrm{SO}} = 0.25 \times 3K \approx .75K \to .2T(\text{ for } g = 8)$$
$$\Delta = 0.1 \mapsto 0.1 \times \Delta_{\mathrm{SO}} = 0.1 \times 3K \approx .3K$$







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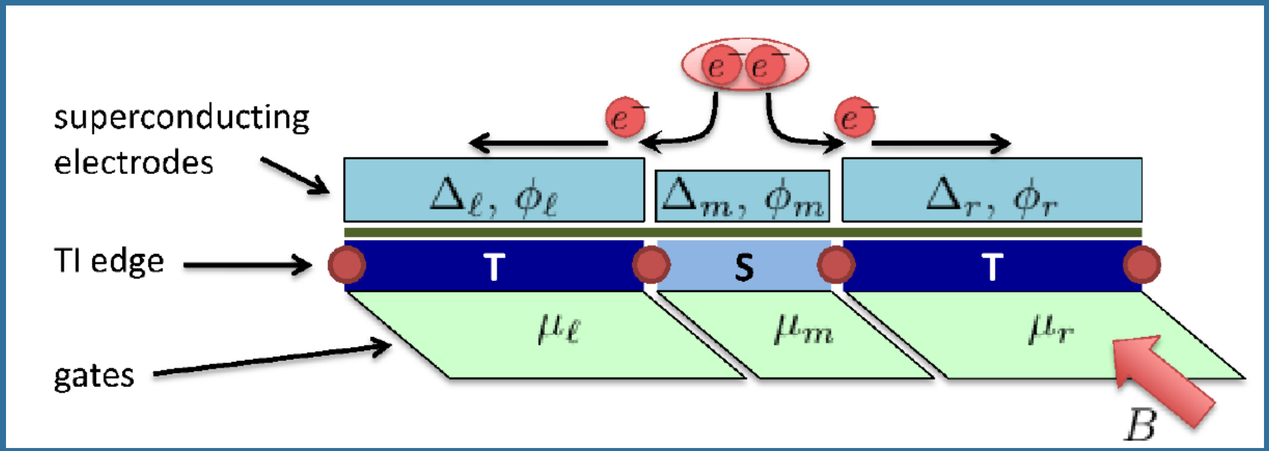
Unconventional Josephson signatures

arXiv:1107.4102
PRL 2011



$$\delta H = -t_m(c^\dagger_{\ell,N} c_{r,1} + h.c.) - \Delta_m(e^{i\phi_m} c_{\ell,N} c_{r,1} + h.c.)$$

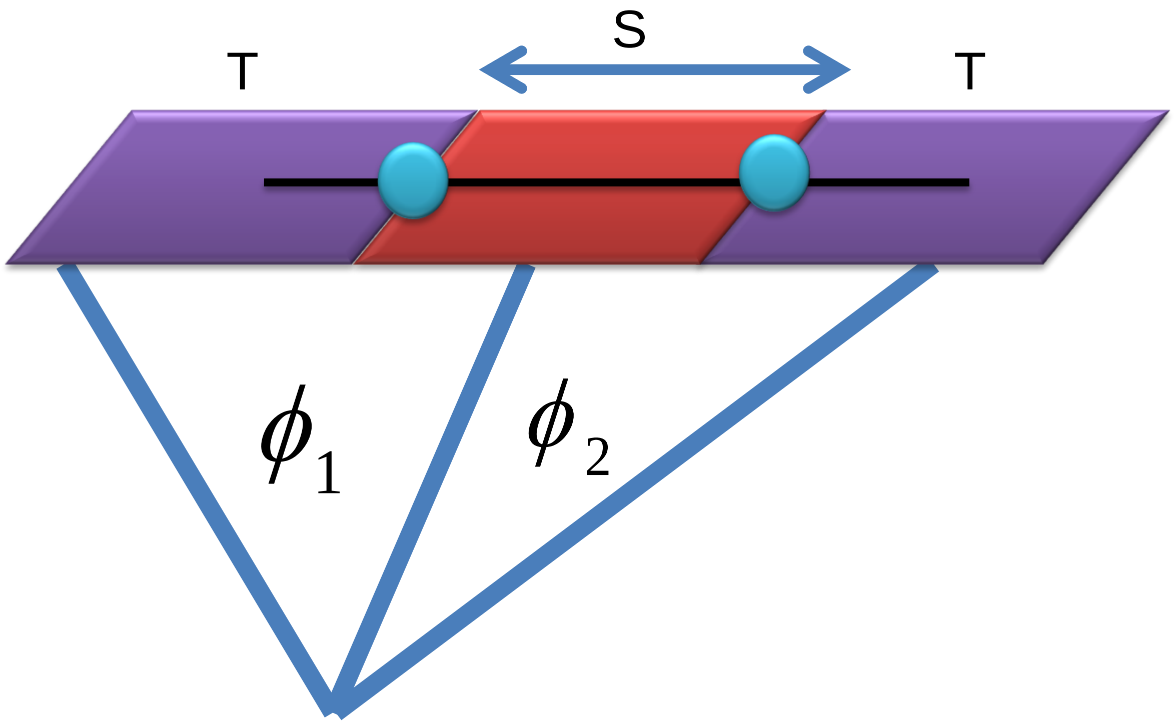
$$\begin{aligned} \delta H &\rightarrow (2f^\dagger f - 1)\{J_M \cos[(\phi_\ell - \phi_r)/2] \\ &+ J_Z \cos[(\phi_\ell + \phi_r)/2 - \phi_m]\}. \end{aligned}$$

$$\boxed{I' = \frac{e}{\hbar} J_Z \sin\left(\frac{\phi_\ell + \phi_r}{2} - \phi_m\right)}$$

Fu & Kane

Josephson „transistor“

Experimental realization



Wire Hamiltonian

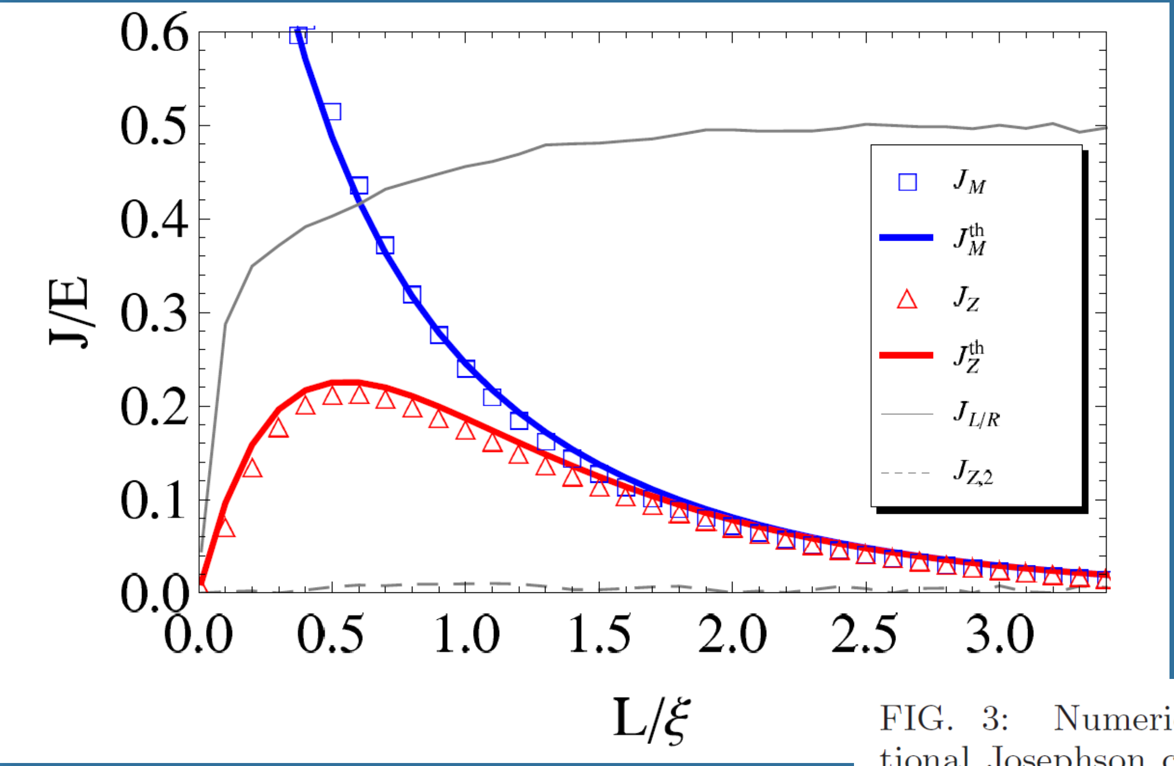


FIG. 3: Numerically determined coefficients of conventional Josephson couplings ($J_{L/R}$), Majorana-induced terms ($J_{M/Z}$), and second harmonic of the J_Z term ($J_{Z,2}$). Our analytical estimates of J_M^{th} and J_Z^{th} agree well with numerics. The energy unit is E and the length unit is $\xi = v/E$. The parameters are $\mu_{l,r} = E, \mu_m = 0$, $\Delta_{l,r} = \sqrt{8}E$, $\Delta_m = E$, and $B_{l,r} = B_m = 2E$. The characteristic lengths are $\lambda_{m+} = \xi/3$ and $\lambda_{m-} = \xi$. For $E = 0.1\text{meV}$ and $v = 10^4\text{m/s}$, the length unit is $\xi = 66\text{nm}$ and the maximum current is $I_Z = \frac{e}{\hbar} J_Z \approx 5.3\text{nA}$.

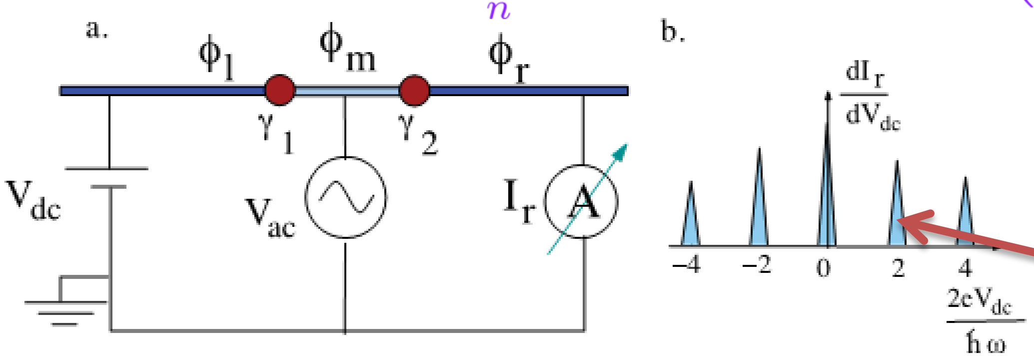
Shapiro steps

PRL Jiang Pekker, Refael, von Open, YO, Alicea

$$V = V_{\rm ac}\cos(\omega t) + V_{dc}$$

$$\Delta\phi = \int V dt = \phi_0 + \frac{2eV_{\rm dc}t}{\hbar} + \frac{2eV_{\rm ac}}{\omega}\sin\omega t$$

$$I(t) = I_c \sin\Delta\phi = I_c \sum_n (-1)^n J_n\left(\frac{2eV_{\rm ac}}{\hbar\omega}\right)\sin\left(\phi_0 + \frac{2eV_{\rm dc}}{\hbar}t - n\omega t\right)$$



$$2eV_{\rm dc} = n\hbar\omega$$

$$\frac{2eV_{\rm dc}}{\hbar\omega} = 2 \Rightarrow eV_{\rm dc} = \hbar\omega$$

Half of the peaks disappear + Nonlocal effect

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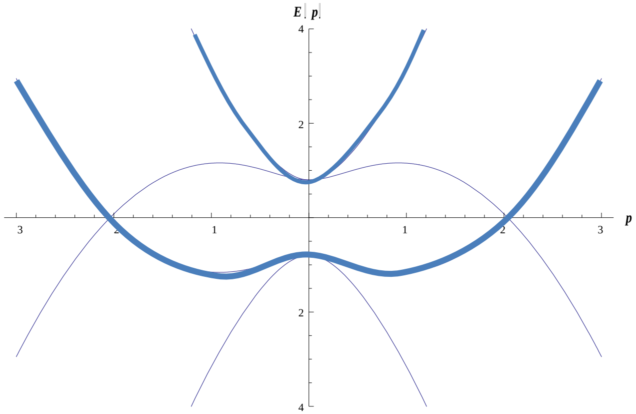
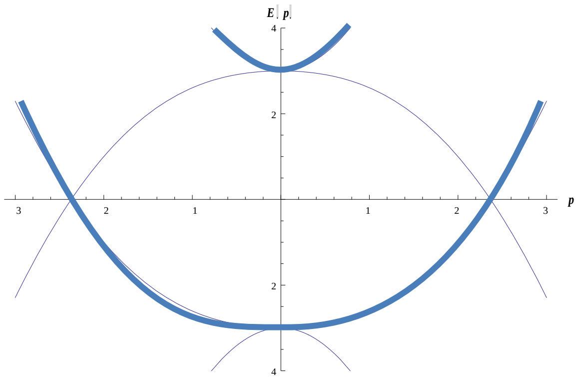
Topological Quantum Numbers & Phase Transitions

Strong Magnetic Field – Maping to a P-wave



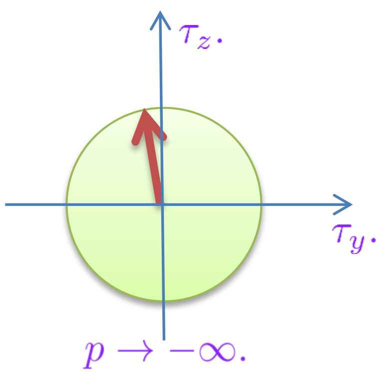
Chemical potential	$\mu = 0$
Spin orbit coupling	$u = 2, (m = 1/2)$
Magnetic Field	$B = 1$
Supercondcutor gap	$\Delta = 0$

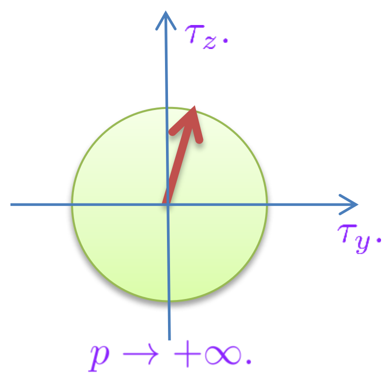


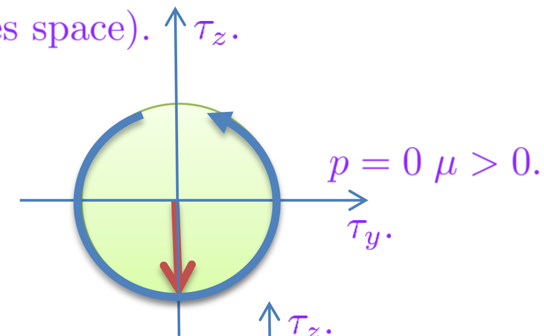
Chemical potential	$\mu = 0$
Spin orbit coupling	$u = 2, (m = 1/2)$
Magnetic Field	$B = 3$
Supercondcutor gap	$\Delta = 0$

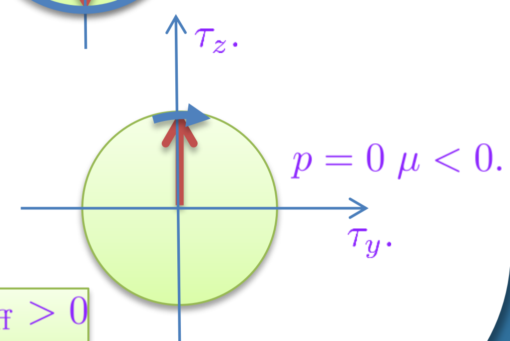
$$h_{p-} = \begin{pmatrix} \frac{1}{2m_{\text{eff}}}p^2 - \mu_{\text{eff}} & iv_{\text{eff}}pe^{-i\phi} \\ -iv_{\text{eff}}pe^{i\phi} & -\frac{1}{2m_{\text{eff}}}p^2 + \mu_{\text{eff}} \end{pmatrix} \mapsto (p^2 - \mu)\tau_z + 2p\tau_y$$

$\mu_{\text{eff}} = \mu + B$; $v_{\text{eff}} = u\Delta/B$; $1/m^* = 1/m(1 - mu^2/B)$.
The 2×2 is in the particle-hole (describes by pauli τ -matrices space).









$$\boxed{2\times \text{ Berry Phase} = \int_{-\infty}^{+\infty} dp \, \langle \psi_G(p) | \, \partial_p \, | \psi_G(p) \rangle = \int d\theta = \begin{cases} 2\pi \text{ for } \mu_{\text{eff}} > 0 \\ 0 \text{ for } \mu_{\text{eff}} < 0 \end{cases}}$$

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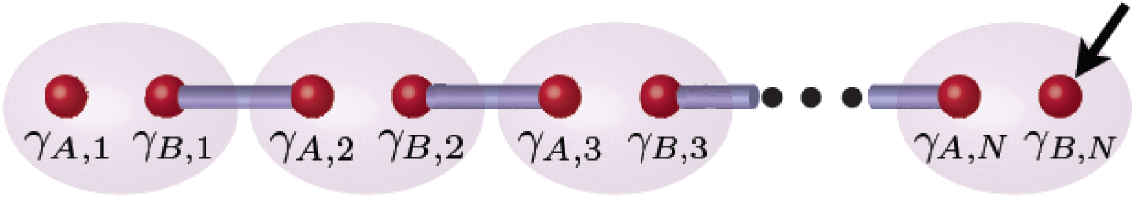
1D spin-less P wave SC

$$H = \mu \sum_{x=1}^{N} c_x^\dagger c_x - \sum_{x=1}^{N-1} \left(t c_x^\dagger c_{x+1} + |\Delta| e^{i\phi} c_x c_{x+1} + h.c. \right)$$

$$\boxed{\begin{array}{ll} \mu = 0 & \\ t = |\Delta| & \qquad c_x = \frac{1}{2} e^{-i\frac{\phi}{2}} \left(\gamma_{B,x} + i\gamma_{A,x} \right) \end{array}}$$

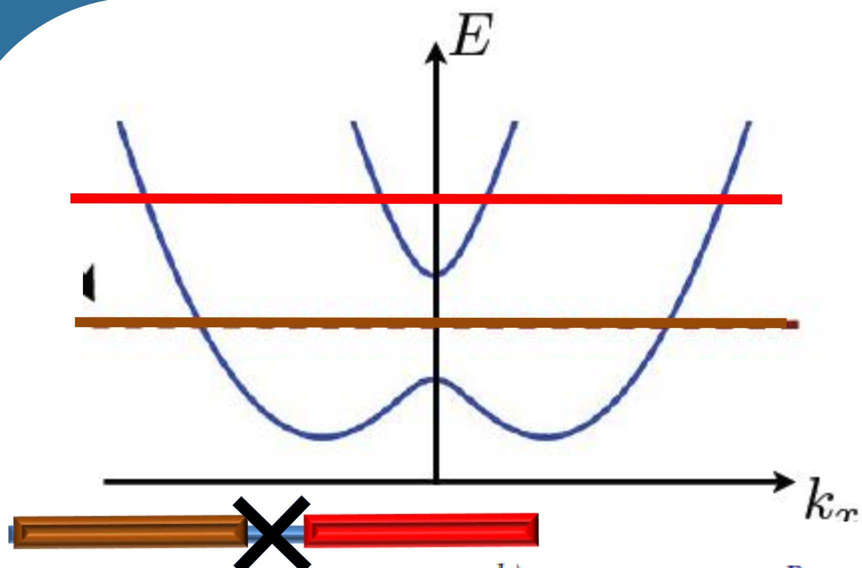
$$\Rightarrow \quad H = -it \sum_{x=1}^{N-1} \gamma_{B,x} \gamma_{A,x+1}$$

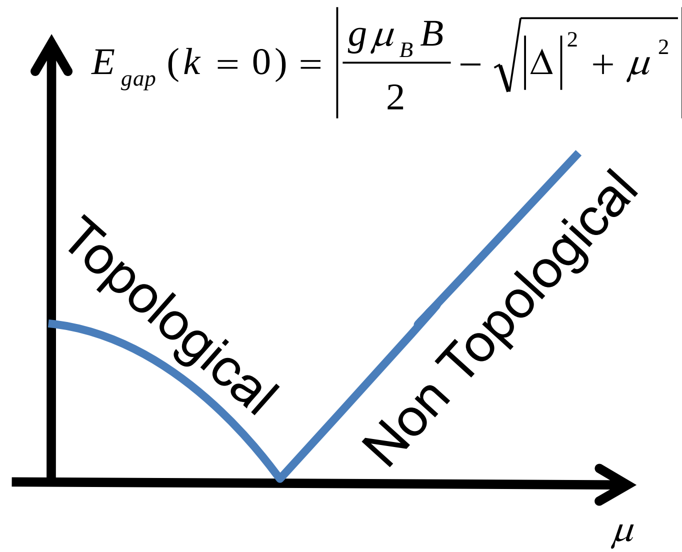
Unpaired end Majorana fermions!

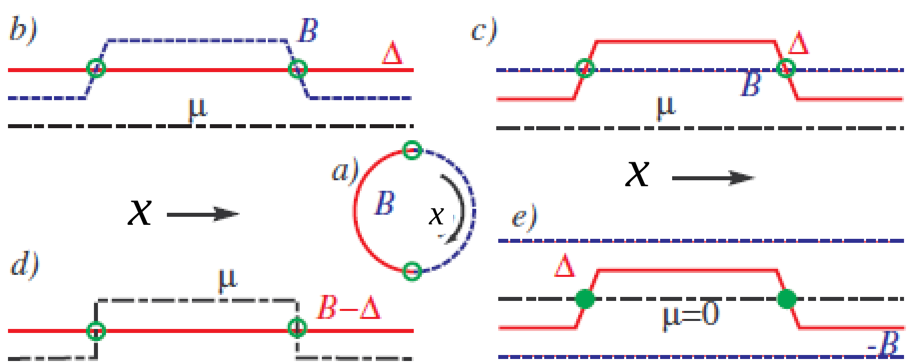


(Kitaev 2001)

Majoranas





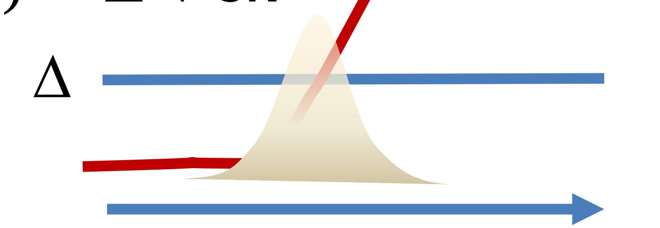


(Lutchyn, Sau, Das Sarma 2010;
YO, Refael, von Oppen 2010)

Shape of GS wave function

$$H = up\,\sigma_z\tau_z - \mu(x)\tau_z + B(x)\sigma_x + \Delta(x)\tau_x$$

$$B(x) = \Delta + bx$$



$$E_0 = 0;\, E_1 = \sqrt{2ub}$$

$$\varphi_0(x) = \left(\frac{b}{u\pi}\right)^{1/4} e^{-\frac{bx^2}{2u}}$$

$$\gamma = \gamma^+ = \frac{1}{2}\left(\psi_\uparrow - i\psi_\downarrow + i\psi_\downarrow^+ + \psi_\uparrow^+\right)$$

$$h_{p-} = \begin{pmatrix} \frac{1}{2m_{\rm eff}}p^2 - \mu_{\rm eff} & iv_{\rm eff}pe^{-i\phi} \\ -iv_{\rm eff}pe^{i\phi} & -\frac{1}{2m_{\rm eff}}p^2 + \mu_{\rm eff} \end{pmatrix} \mapsto (p^2-\mu)\tau_z + 2p\tau_y$$

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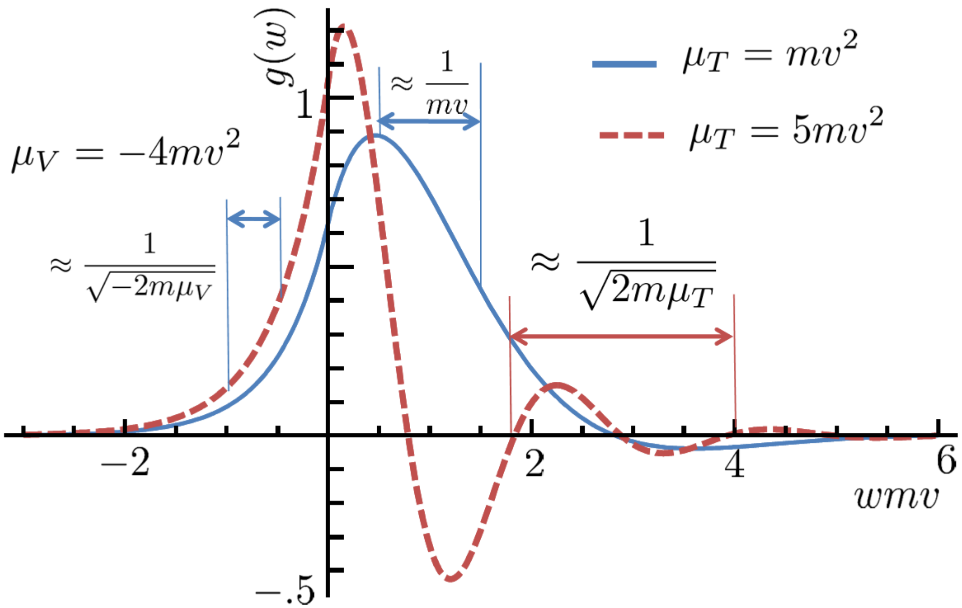

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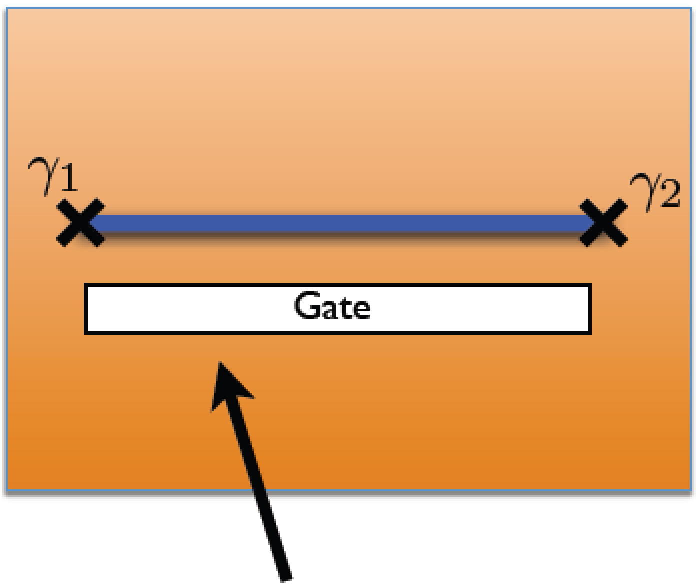
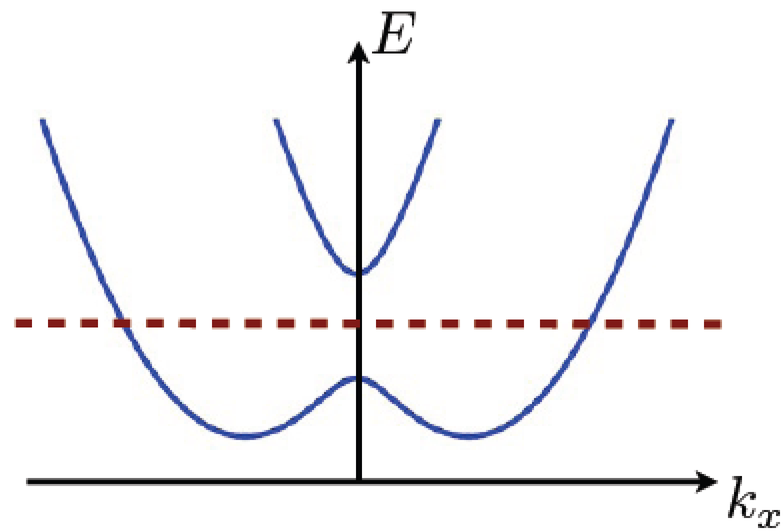
Quantum computing with 1D wires??

- At a minimum, we'd need the ability to
  - Adiabatically transport Majoranas
  - Create pairs of Majoranas out of the vacuum
  - Fuse Majoranas back into the vacuum
  - Braid Majoranas
  - Realize non-Abelian statistics

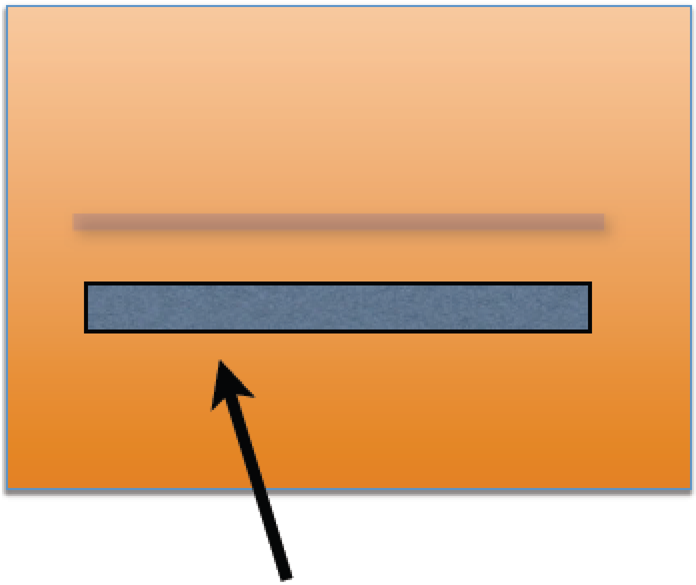
Manipulating Majoranas in 1D wires

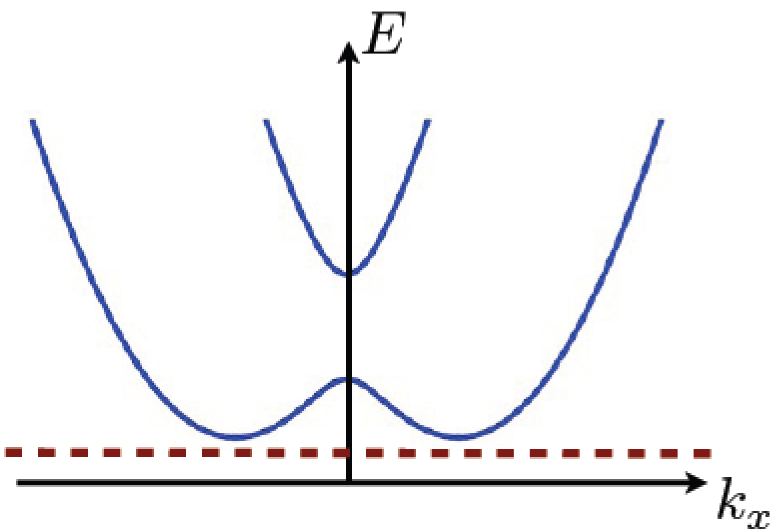




Manipulate Majoranas by changing chemical potential via gate voltage

Manipulating Majoranas in 1D wires

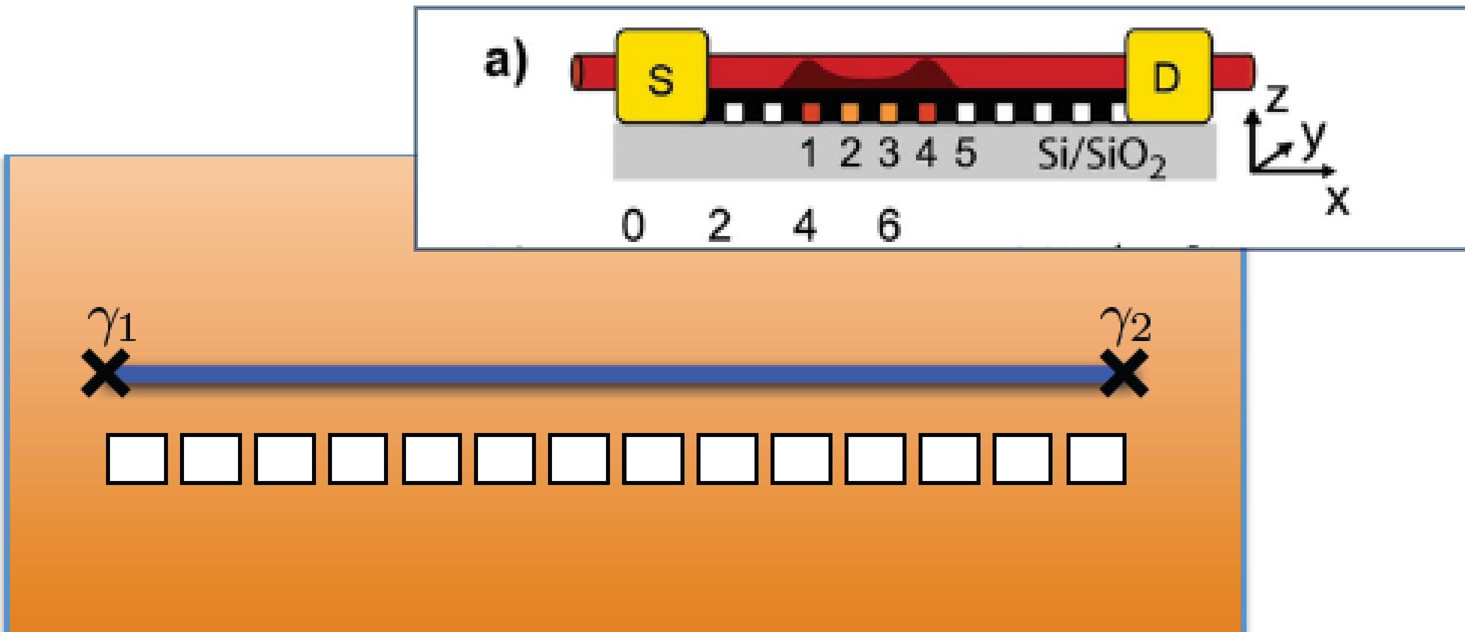




Manipulate Majoranas by changing chemical potential via gate voltage

Manipulating Majoranas in 1D wires

Better: use a `**keyboard' of gates!**



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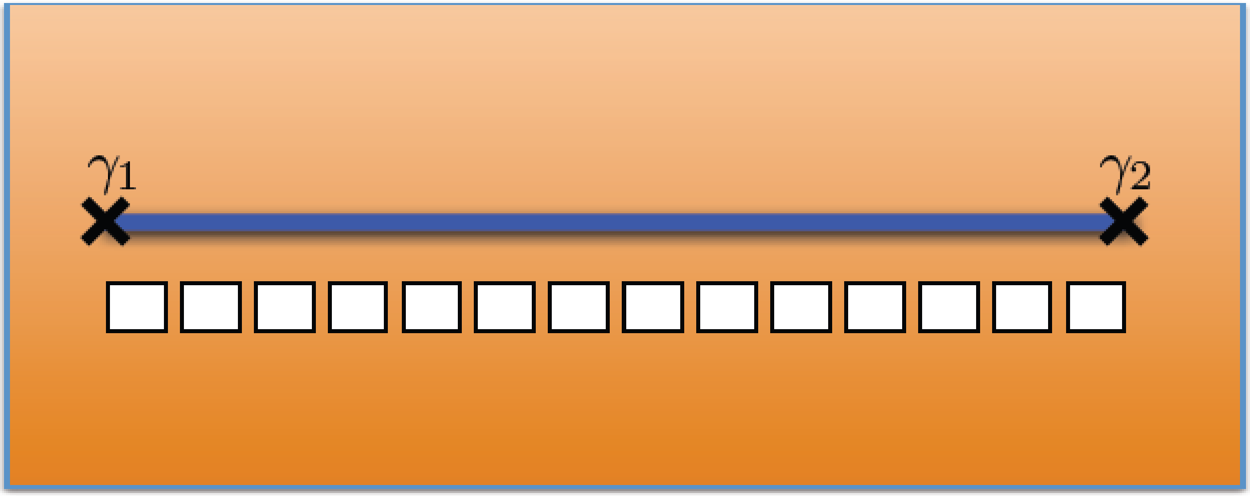

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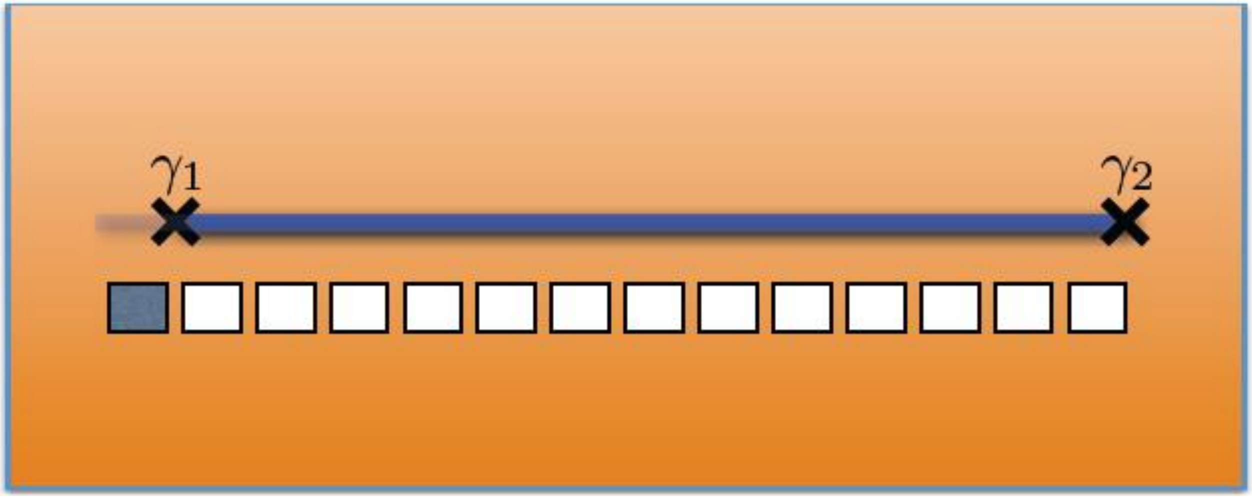
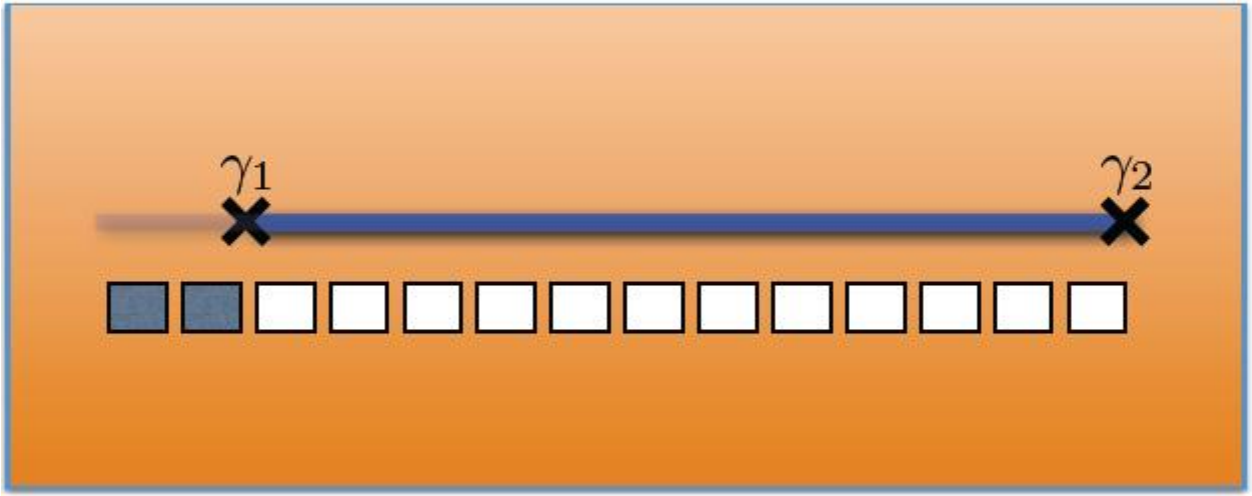


Manipulating Majoranas in 1D wires

Better: use a **`keyboard' of gates!**



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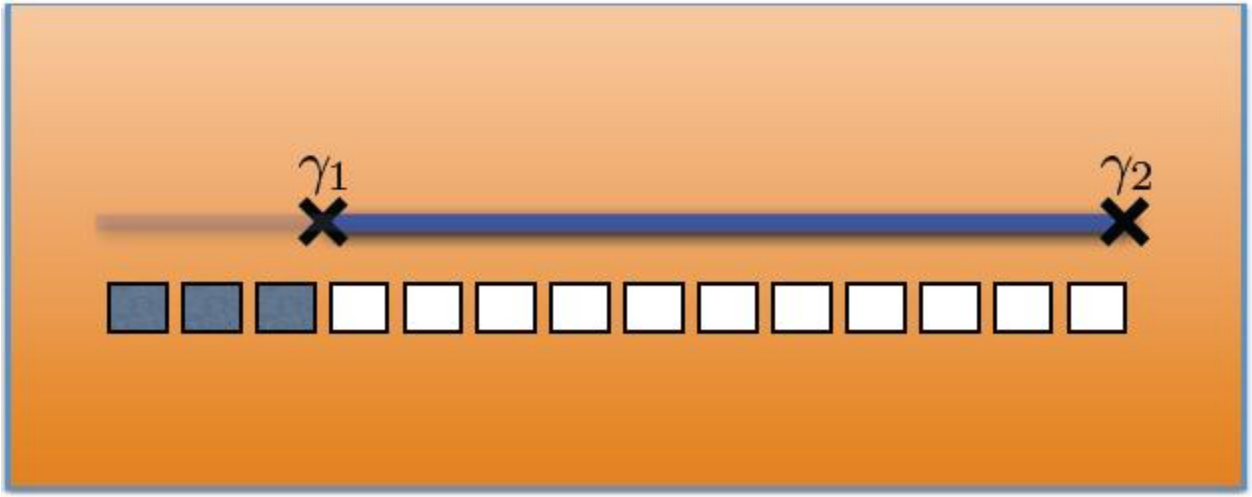

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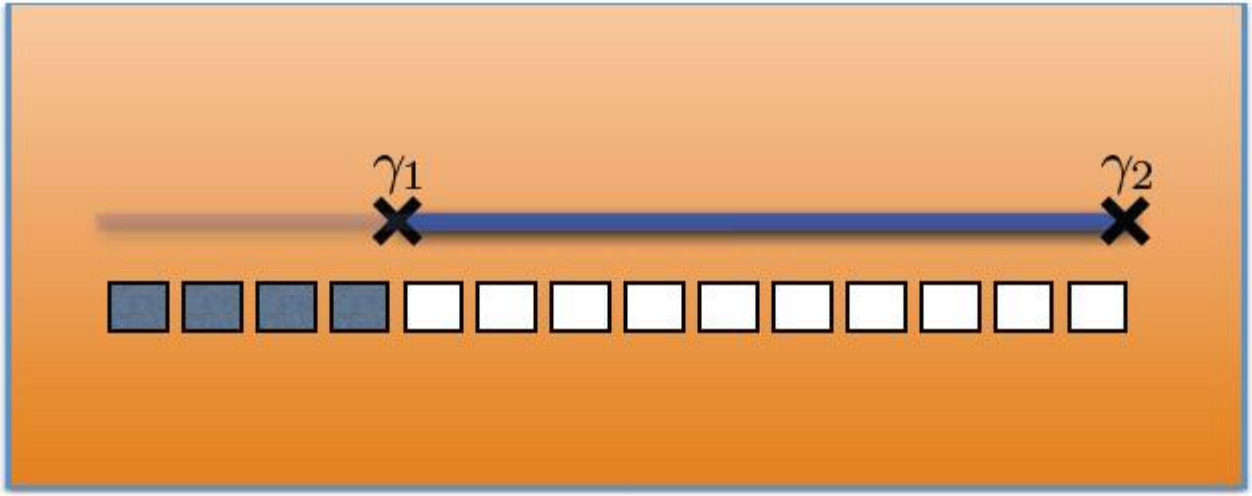

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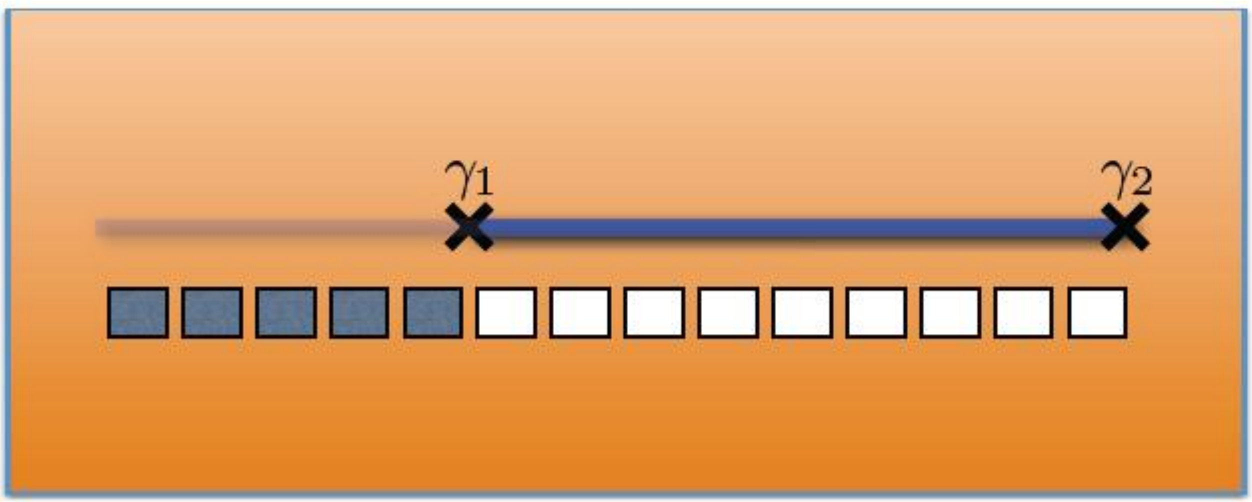

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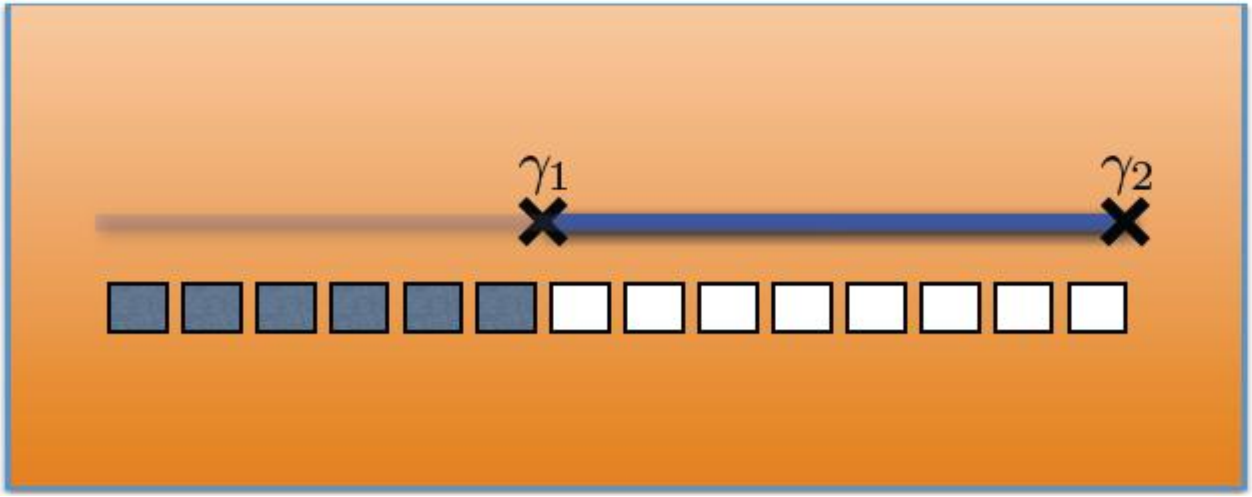

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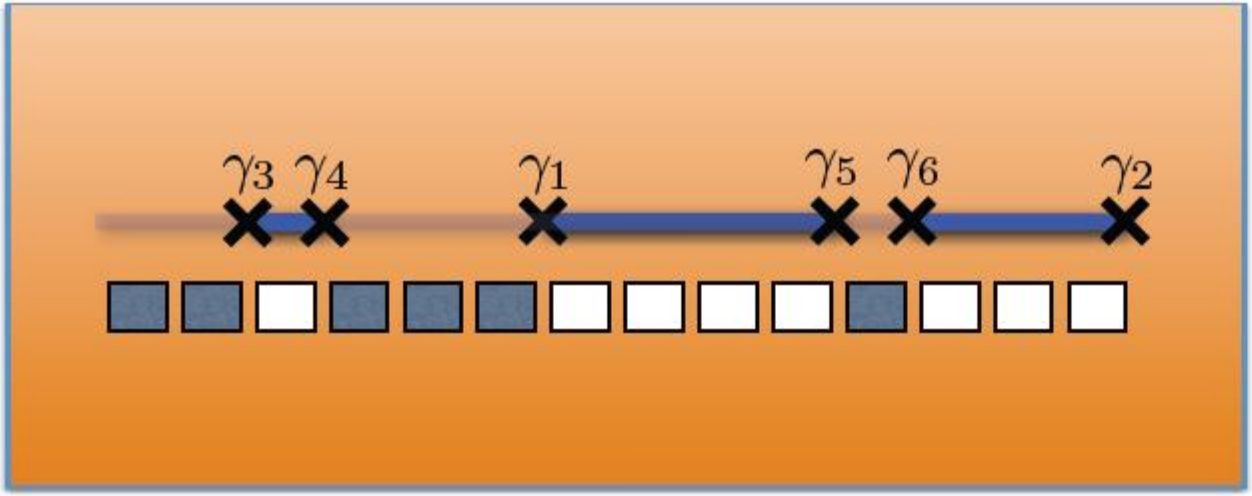

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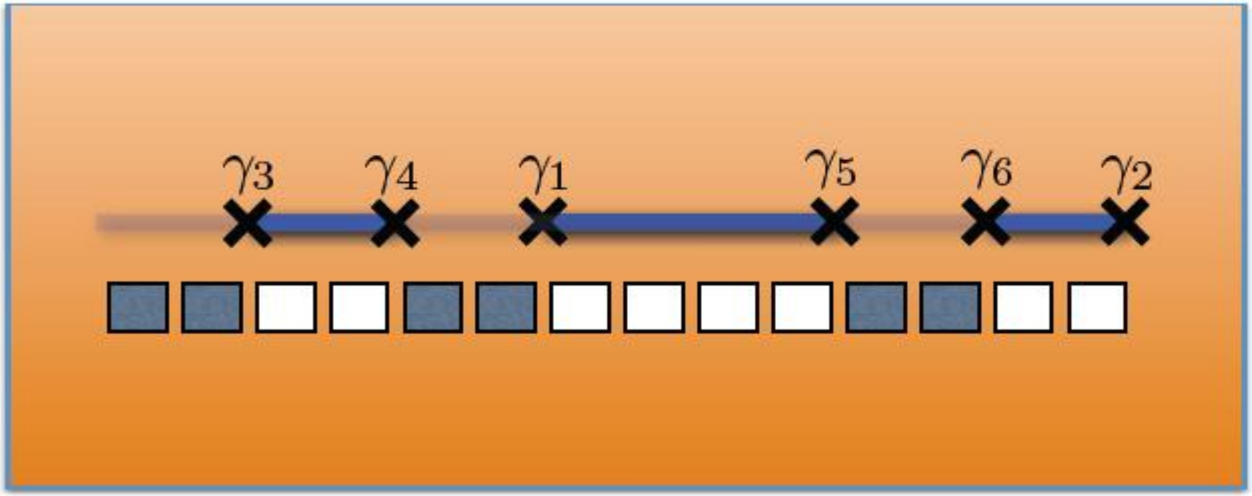

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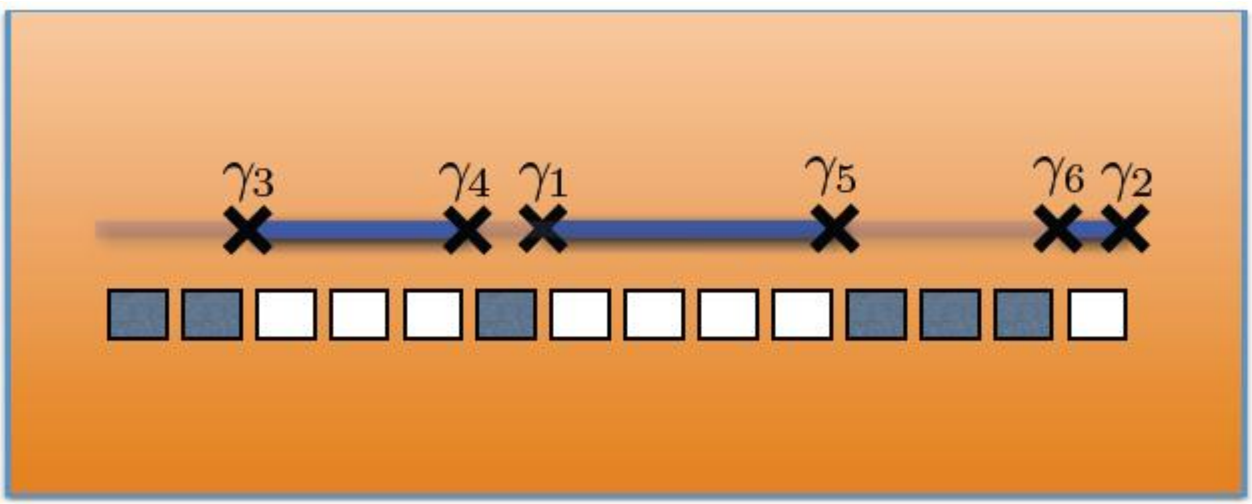
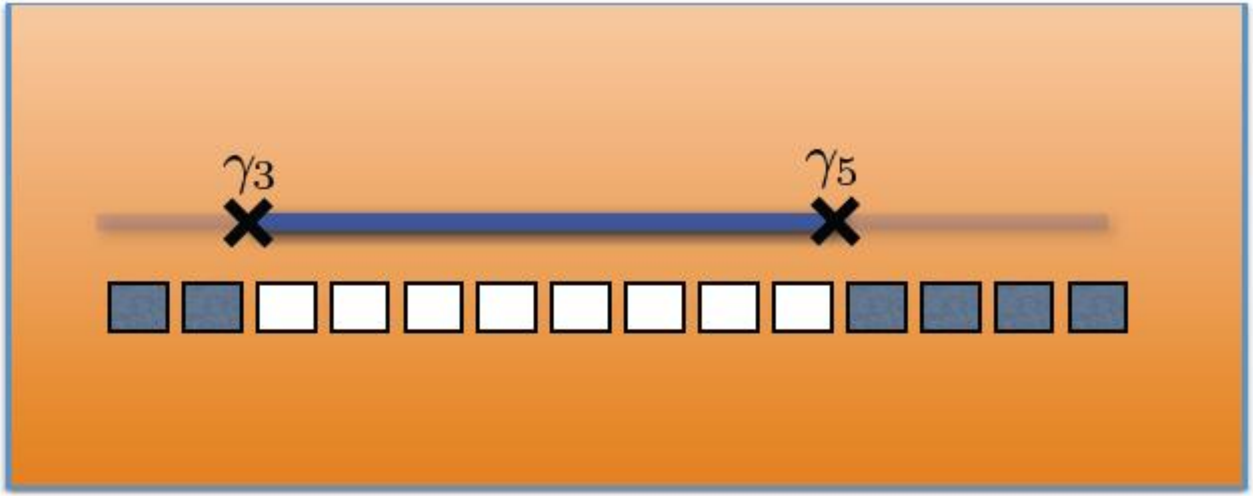


Manipulating Majoranas in 1D wires

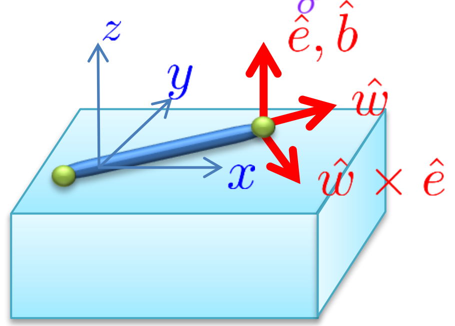
Better: use a `**keyboard' of gates!**

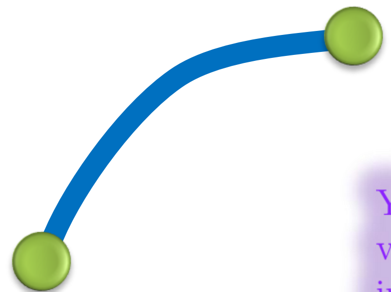
Can manipulate shift and fuse, what about braiding?



Exchange in 3D : weak B (4 components)

$$\gamma_{\vec{\Omega}} = \sum_{\sigma} \int d^3\vec{r} \left[f_{\sigma}^{\vec{\Omega}}(\vec{r})\psi_{\sigma}(\vec{r}) + \text{h.c.} \right] .$$

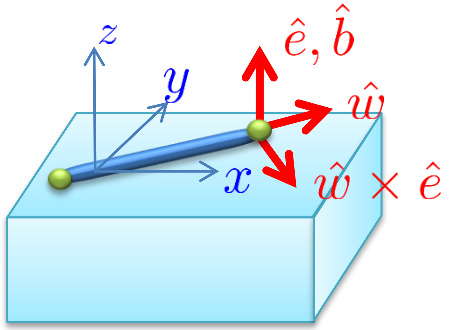




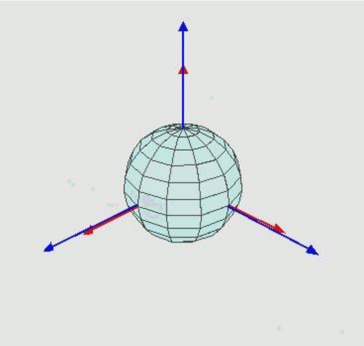
YO, Stern, Halperin
von Oppen, Refael, Alicea,
in preparation

$$\vec{\Omega} = \begin{array}{ll} \text{Superconducting gap phase (and amplitude)} & \theta \\ \text{Direction of the wire end} & \hat{w} \\ \text{Direction (and amplitude) of the electric field} & \hat{e} \\ \text{Direction (and amplitude) of the magnetic field} & \hat{b} \end{array}$$

Exchange in 3D :



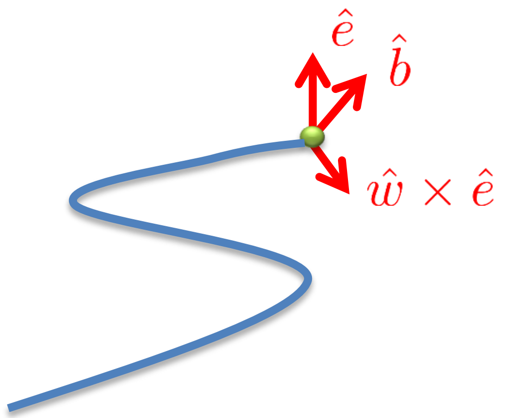
$$|MF\rangle = \frac{1}{2}\begin{pmatrix} f_\uparrow \\ f_\downarrow \\ f_\downarrow^\dagger \\ -f_\uparrow^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_p \\ \hat{T}\psi_p \end{pmatrix}.$$



Manipulation of the wire $\Rightarrow$ Rotation of the 2 comp spinor.

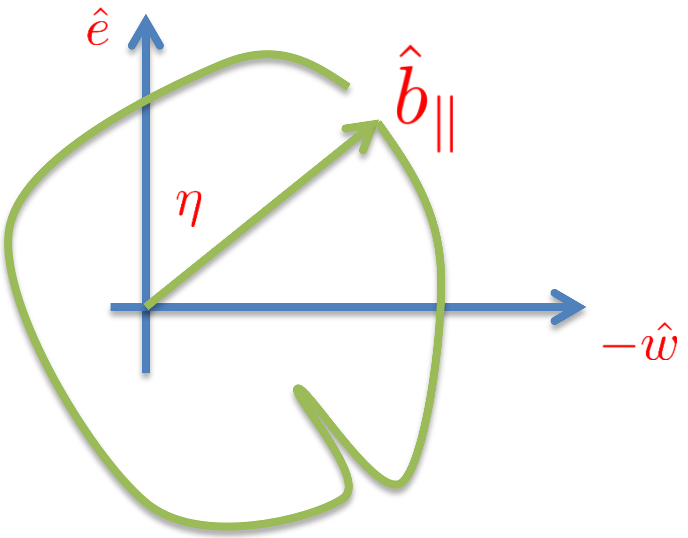
- Rotation by an angle α around $\hat{e}||\hat{b} \perp w \Rightarrow \psi_p \to e^{-i\sigma_z\alpha/2}\psi_p$
- Rotation around $\hat{w}$...
- Rotaion of the tripod $\hat{e}, \hat{w}, \hat{e} \times \hat{w}$ by 360° causes a multiplication of the Majoranas by -1
- Rotaion around e by 180 causes only one of the Majoranas to be multiplied by -1

Exchange in 3D



The gap closes for:

$0 \neq \hat{b} \times (\hat{e} \times \hat{w}) = \hat{e}(\hat{b} \cdot \hat{w}) - \hat{w}(\hat{b} \cdot \hat{e})$



If $\hat{b}$ is in the plane defined by $\hat{e}$ and $\hat{w}$ $e^{i\eta} = \hat{b} \cdot (\hat{e} + i\hat{w})$

Strong Magnetic field

2 components vector

$$|gs(\phi)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\phi/2} \\ e^{i\phi/2} \end{pmatrix}$$

Single-valued vs. Multi-valued

Reminder: Berry phase: Toy problem: spin in magnetic field

$$H = B\sigma_x \cos(\phi) + \sigma_y \sin(\phi)$$
$$= B \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}$$

$$\vec{B} = B \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$$

adiabatic ground state

$$|gs(\phi)\rangle = e^{i\chi(\phi)} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\phi} \\ 1 \end{pmatrix}$$

w/ arbitrary

$$\chi(\phi)$$

Geometric phase

Adiabatic evolution of ϕ (from ϕ_0 to $\phi(t)$):

1.) Dynamical phase $$e^{iBt}$$

2.) Berry phase $$e^{-i\theta_B}$$

$$\theta_B[\phi(t)] = \mathrm{Im} \int_{\phi_0}^{\phi(t)} d\phi \, \langle gs(\phi)| \, \partial_\phi \, |gs(\phi)\rangle$$

3.) Explicit monodromy of $\chi(\phi)$

geometric phase = Berry phase + monodromy

Berry vs monodromy

Berry phase

$$\theta_B[\phi_0 + 2\pi] = -\pi + [\chi(\phi_0 + 2\pi) - \chi(\phi_0)]$$

$$e^{-i\theta_B} = e^{-i\theta_B[\phi_0+2\pi] = -\pi + \underbrace{[\chi(\phi_0+2\pi)-\chi(\phi_0)]}_{\text{cancelled by monodromy}}}$$

Full rotation of $\vec{B}$ Geometric phase of $e^{i\pi}$

Berry vs monodromy

1.) Single valued and continous

$$\chi(\phi) = 0$$

vanishing monodromy
geometric phase = Berry phase

2.) Multi- valued and continous

$$\chi(\phi) = \phi/2$$

$$|gs(\phi)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\phi/2} \\ e^{i\phi/2} \end{pmatrix}$$

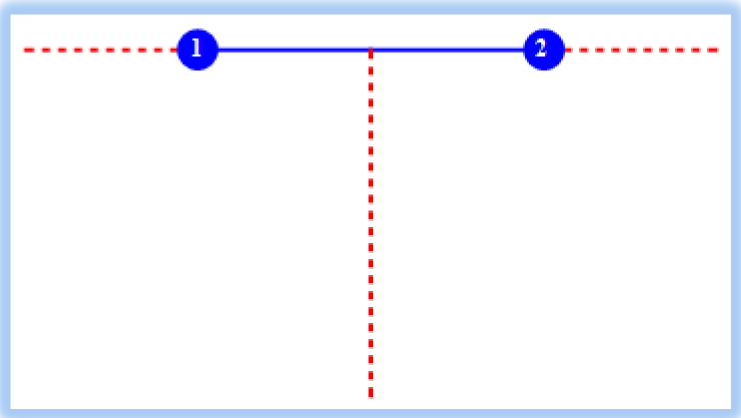
vanishing Berry phase
geometric phase = monodromy

3.) Singled valued Majorana representation

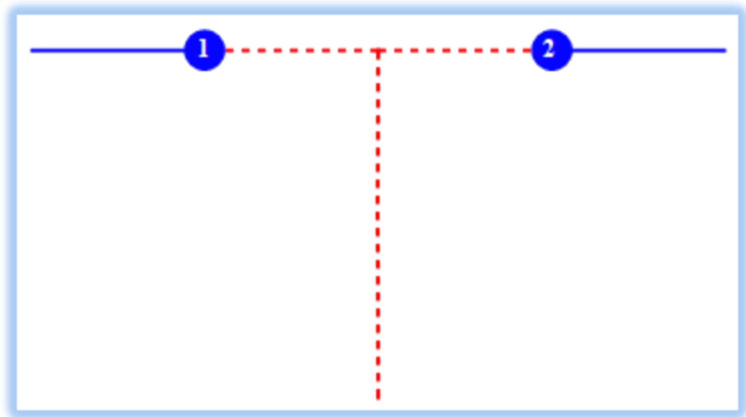
$$\chi(\phi) = (\phi \bmod 2\pi)/2)$$

vanishing monodromy; stepwise accumulation of Berry

Exchanging Majoranas in 1D Wires



Use a T (or Cross) Junction



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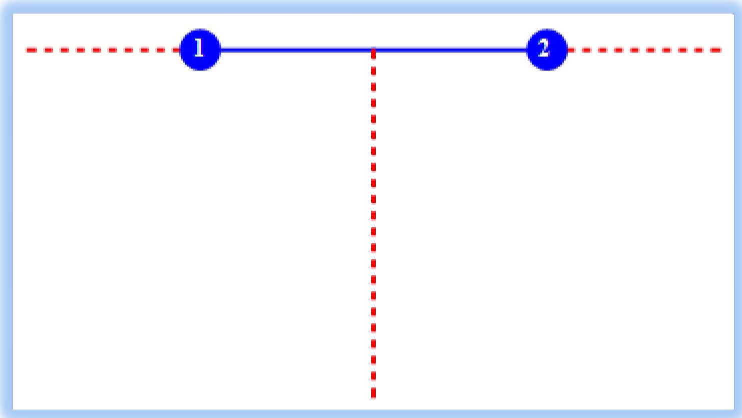

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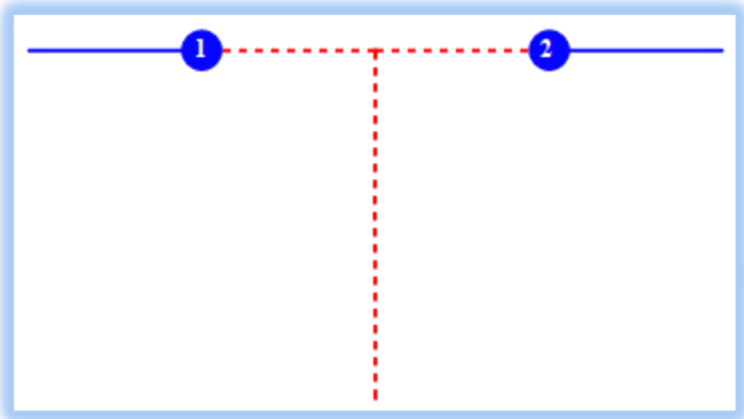

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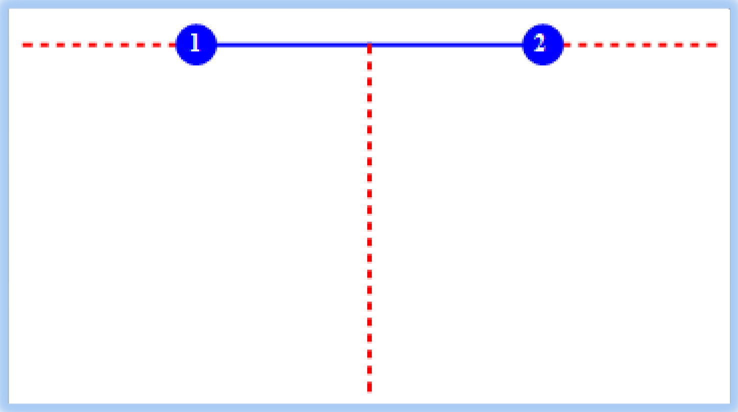

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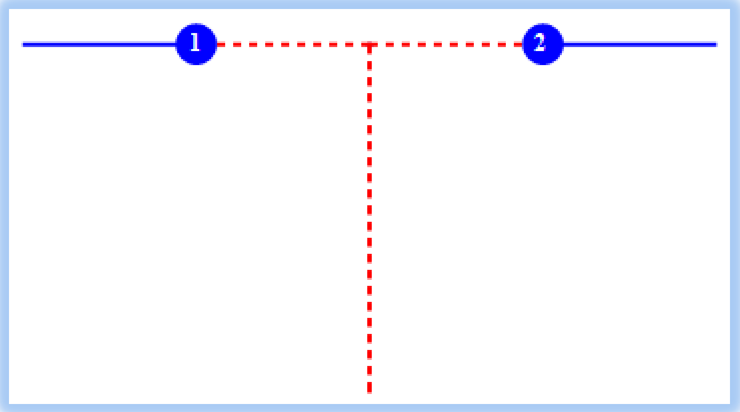
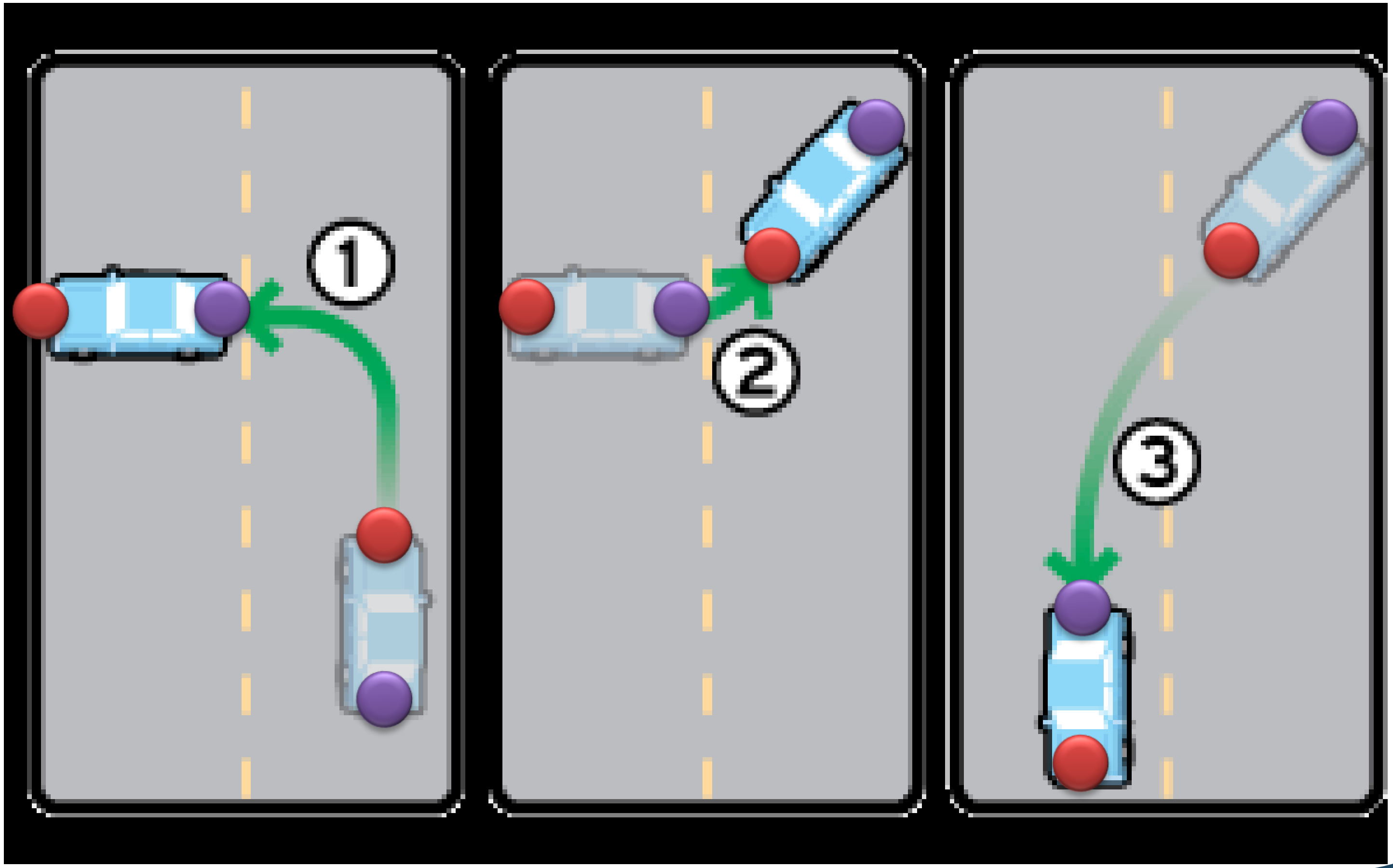


There Point Turn



Exchanging Majoranas in 1D Wires

Recall from 2D p+ip

$$\gamma_1 \longrightarrow \gamma_2$$
$$\gamma_2 \longrightarrow -\gamma_1$$

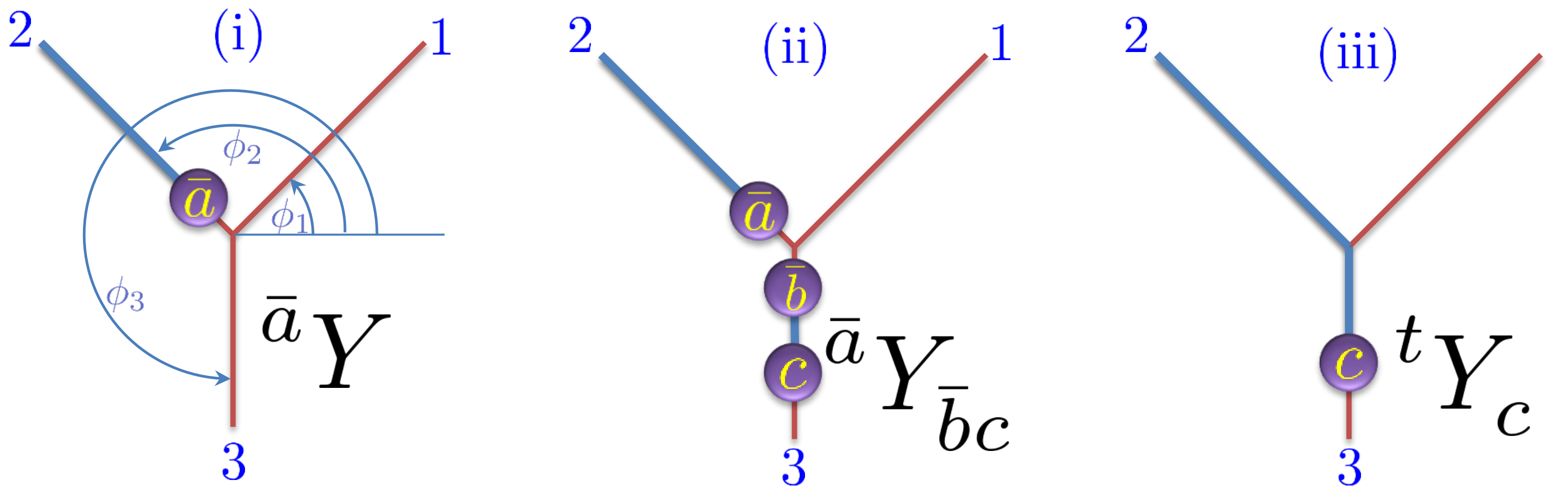
Vortices are crucial here!

Read & Green, Ivanov, Sterrn, von Oppen & Mariani

Statistics encoded in Berry phase...

$$\chi_n \equiv \mathrm{Im} \int dt \langle n|\partial_t|n\rangle$$

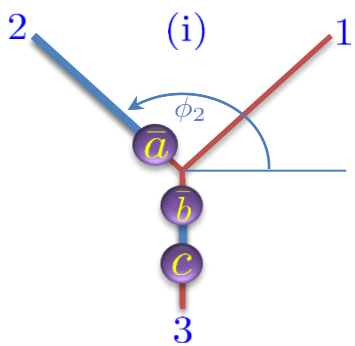
Y-junction



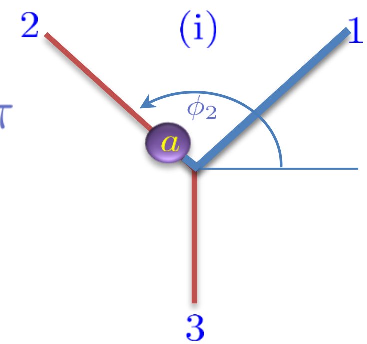
See also recent paper by B. van Heck, A. R. Akhmerov, F. Hassler, M. Burrello, and C. W. J. Beenakker arXiv:1111.6001v1 [cond-mat.mes-ha] 25 Nov 2011

At strong magnetic field

$$\mathcal{H} = i\gamma_a\gamma_b g_{ab} \sin\left(\frac{\alpha_a - \alpha_b}{2}\right) + i\gamma_b\gamma_c g_{bc} \sin\left(\frac{\alpha_c - \alpha_c}{2}\right) + i\gamma_a\gamma_c g_{ac} \sin\left(\frac{\alpha_a - \alpha_c}{2}\right).$$



$$\alpha_a = \phi_2 + \pi$$



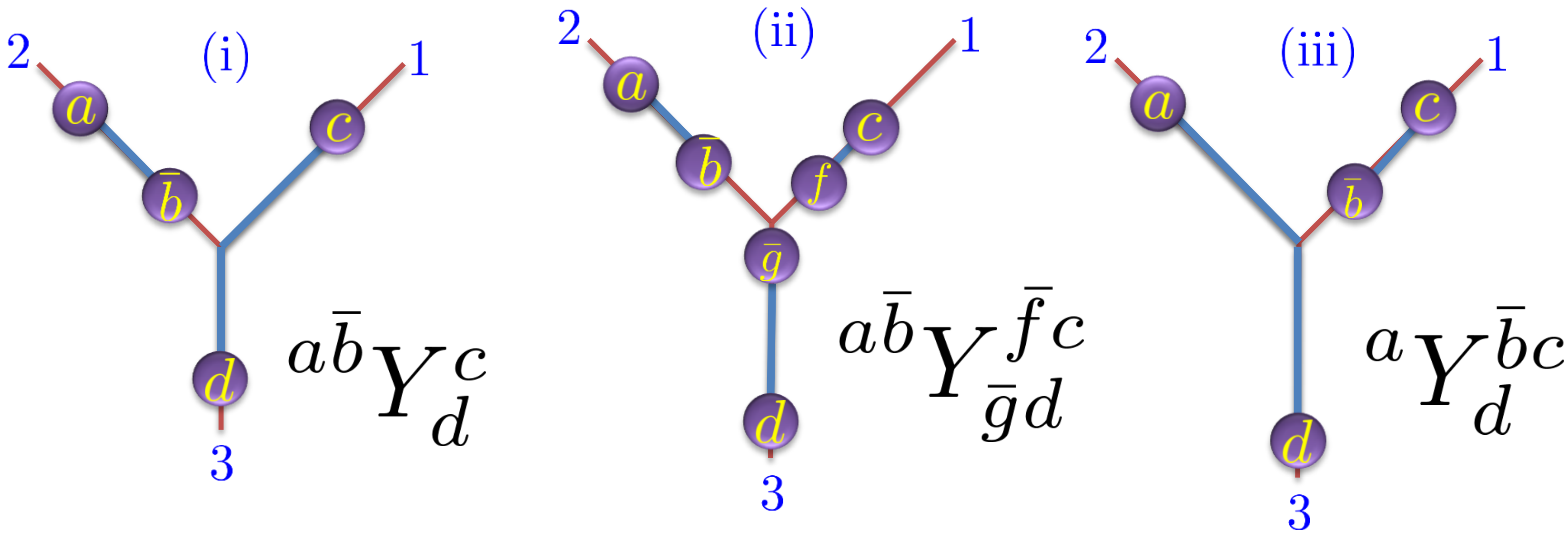
$$\alpha_a = \phi_2$$

$$\gamma[t] \sim g_{bc}(t) \sin\left[\frac{\alpha_c - \alpha_b}{2}\right]\gamma_a - g_{ac}(t) \sin\left[\frac{\alpha_a - \alpha_c}{2}\right]\gamma_b + g_{ab}(t) \sin\left[\frac{\alpha_b - \alpha_a}{2}\right]\gamma_c$$

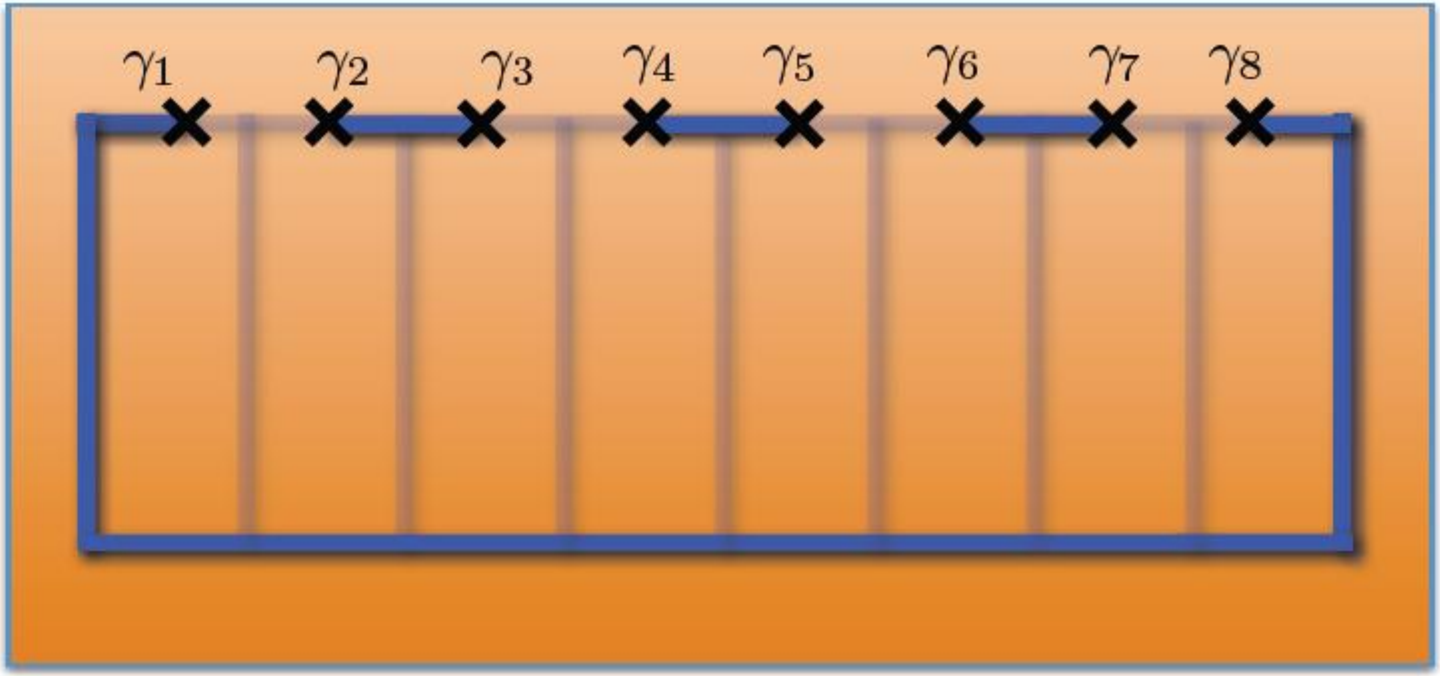
$$\mathcal{H} = i\gamma_a\gamma_b g_{ab}\sin\left(\frac{\alpha_a-\alpha_b}{2}\right)+i\gamma_b\gamma_c g_{bc}\sin\left(\frac{\alpha_c-\alpha_c}{2}\right)+i\gamma_a\gamma_c g_{ac}\sin\left(\frac{\alpha_a-\alpha_c}{2}\right).$$

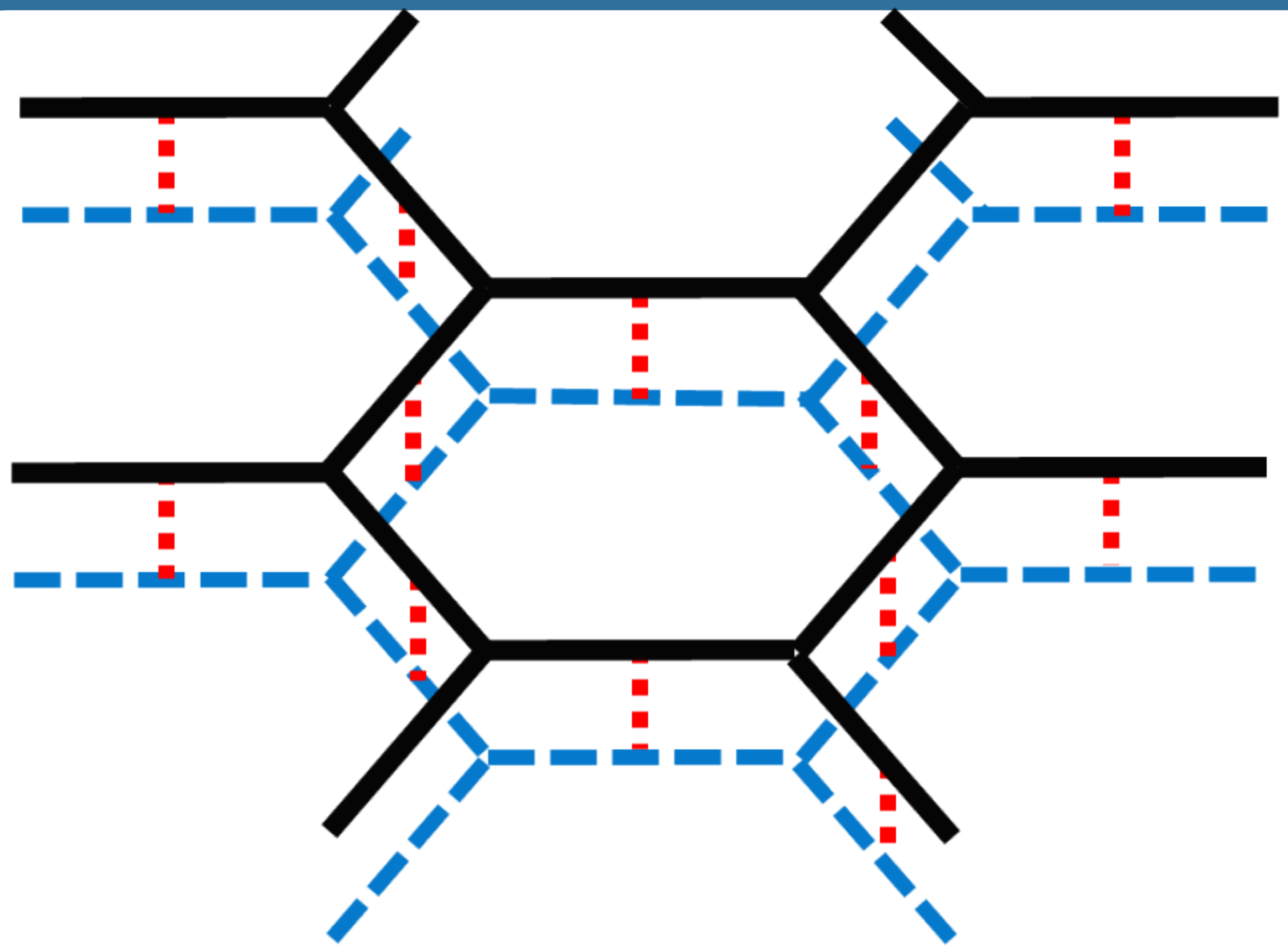
$$\begin{pmatrix} 0 & ig_{ab} & ig_{ac} \\ -ig_{ab} & 0 & ig_{bc} \\ -ig_{ac} & -ig_{bc} & 0 \end{pmatrix} \begin{pmatrix} g_{bc} \\ -g_{ac} \\ g_{ab} \end{pmatrix} = 0$$

$$\gamma[t] \sim g_{bc}(t)\sin\left[\frac{\alpha_c-\alpha_b}{2}\right]\gamma_a - g_{ac}(t)\sin\left[\frac{\alpha_a-\alpha_c}{2}\right]\gamma_b + g_{ab}(t)\sin\left[\frac{\alpha_b-\alpha_a}{2}\right]\gamma_c$$

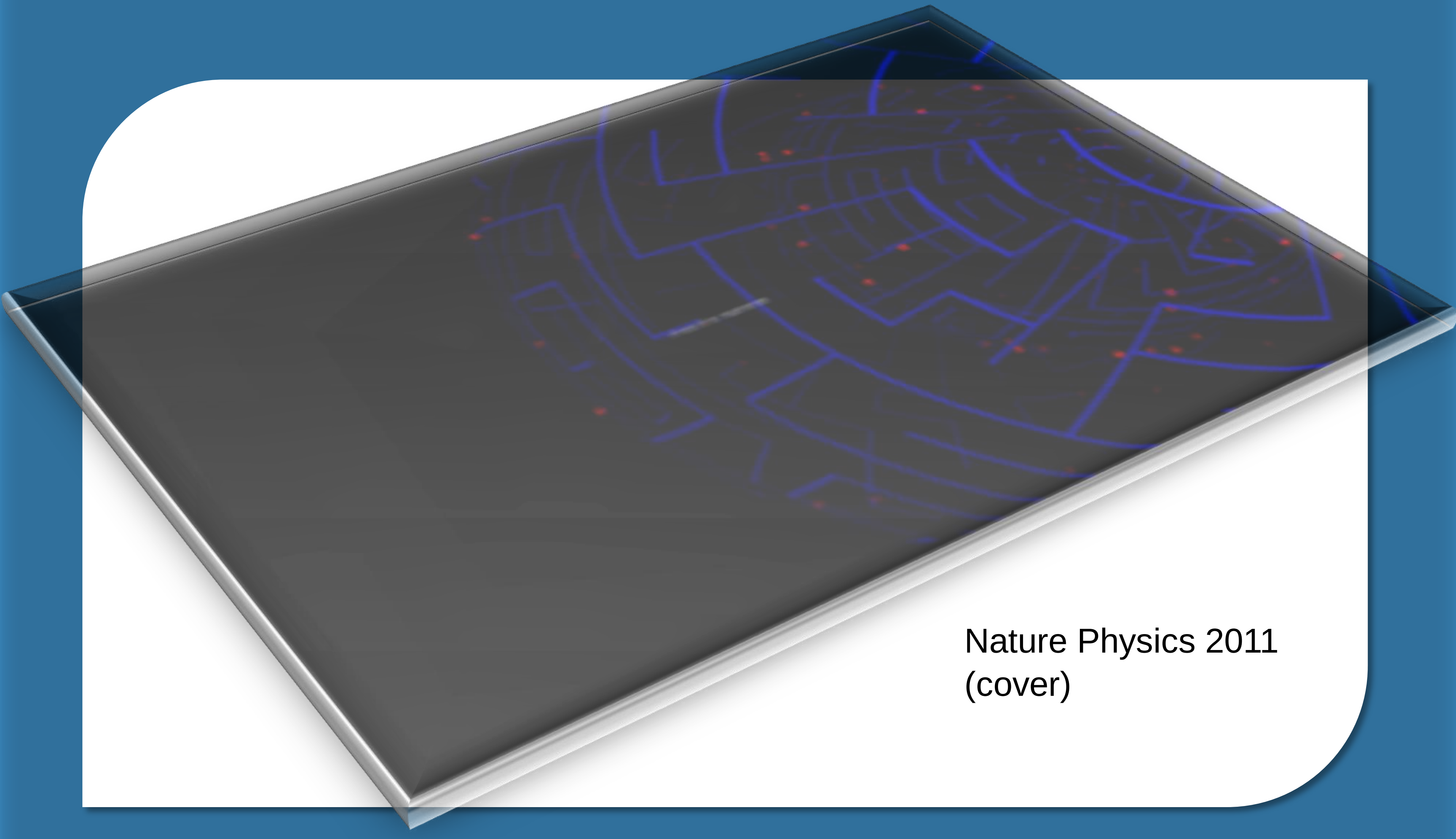

(i)
2
1
a
$\bar{b}$
c
d
3
${}^{a\bar{b}}Y_d^{c}$
(ii)
2
1
a
$\bar{b}$
c
f
$\bar{g}$
d
3
${}^{a\bar{b}}Y_{\bar{g}d}^{\bar{f}c}$
(iii)
2
1
a
$\bar{b}$
c
d
3
${}^{a}Y_d^{\bar{b}c}$

A net work?





In 3D?



Nature Physics 2011
(cover)

Summary

- Spin orbit with magnetic field in proxy to superconductors can host Majoranas & non-Abelian statistics in 1D+
- All that without vortexes
- Topological insulator & semiconductor heterostructures
- Many open questions! (Which materials to use, universal quantum computation, better measurement schemes, connection to 2D p+ip, etc.)
- Manipulations of Majoranas at the ends of 3D wires.