Topological nematic states and non-Abelian lattice dislocations

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Outline

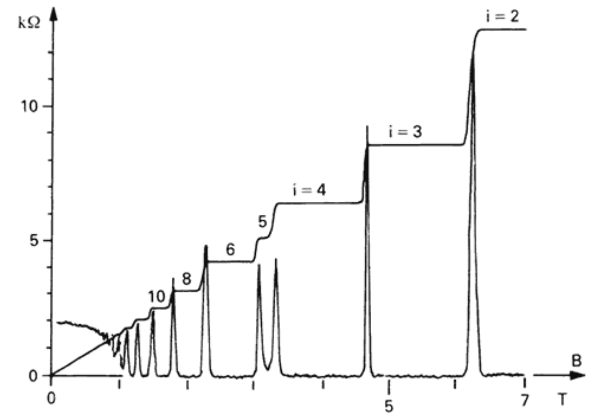
- Fractional quantum anomalous Hall (FQAH) states---fractional quantum Hall states in translation invariant lattice models
- 1D Wannier state description of FQAH states
- FQAH states with higher Chern number and the topological nematic states
- Topological degeneracy induced by lattice dislocations
- Edge state picture and topological field theory description

Ref: XLQ, Phys Rev Lett. **107**, 126803 (2011)

Maissam Barkeshli & XLQ, in preparation

Integer quantum Hall (IQH) state

$$\sigma_{xy} = ne^2/h$$ (K von Klitzing 1980)

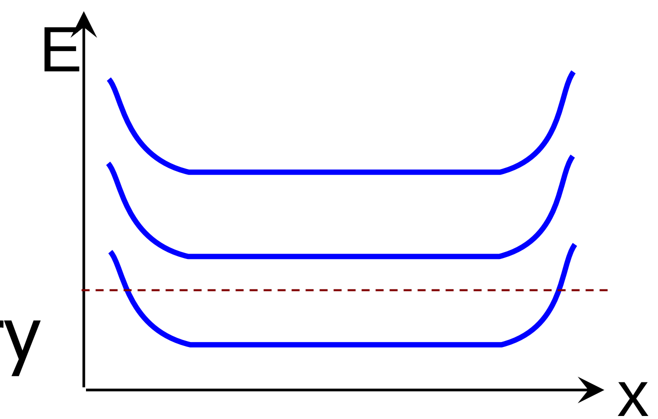


- Topological origin of the quantized Hall conductance:
- Bulk gap (Landau level gap)
- The first Chern number (TKNN number)

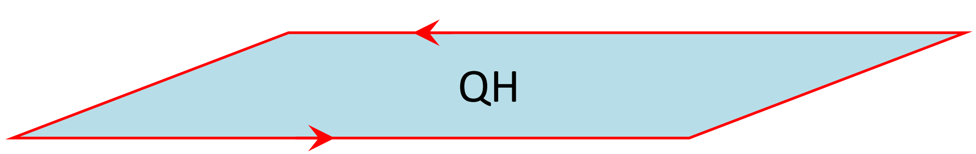
(Laughlin PRB 1981, Thouless, et al, PRL 1982)

$$n = \frac{1}{2\pi}\int d^2k\left(\partial_x a_y - \partial_y a_x\right)$$

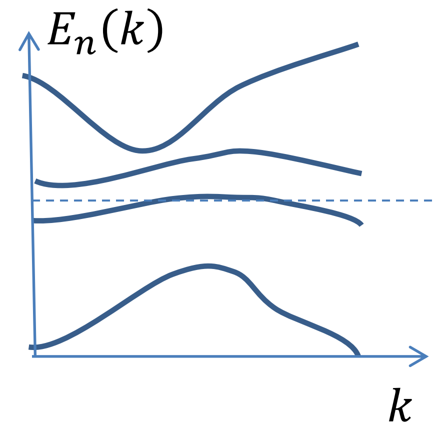
$$a_i(\boldsymbol{k}) = -i\langle \boldsymbol{k}|\partial_i|\boldsymbol{k}\rangle$$



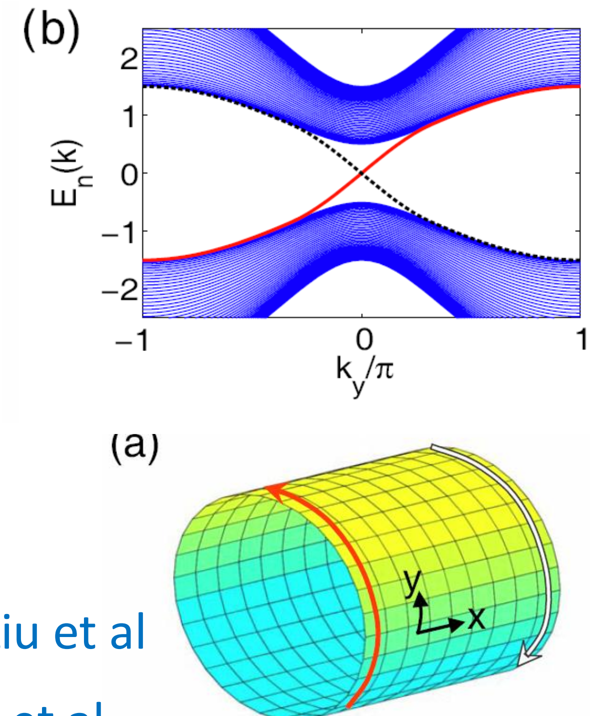
- Chiral edge states on the boundary



(Integer) Quantum Anomalous Hall States



- A lattice model with nonzero Chern number in the occupied band
- General lattice Hamiltonian with translation symmetry $H = \sum_k c_k^+ h(k) c_k$
- There are n bands $|n, \boldsymbol{k}\rangle$
  Chern number $C_1 = \frac{1}{2\pi} \int d^2 k \nabla \times \vec{a}_n$ defined for each band, $\vec{a}_n(\boldsymbol{k}) = -i\langle n\boldsymbol{k}|\partial_i|n\boldsymbol{k}\rangle$



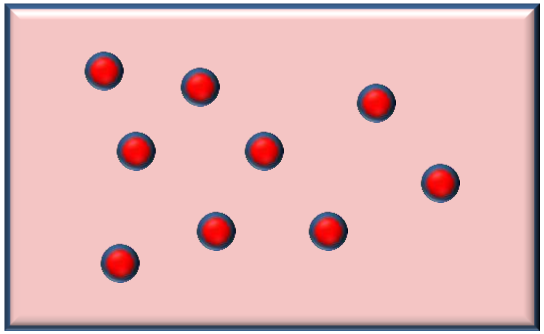
- Example: two-band models $H = \sum_a d_a(\boldsymbol{k})\sigma^a$ (Haldane 1988, Qi Wu Zhang 2005)
- Mateiral proposals: Hg(Mn)Te/CdTe (Liu et al PRL 2008), Cr or Fe doped Bi2Se3 film (Yu et al Science 2010)

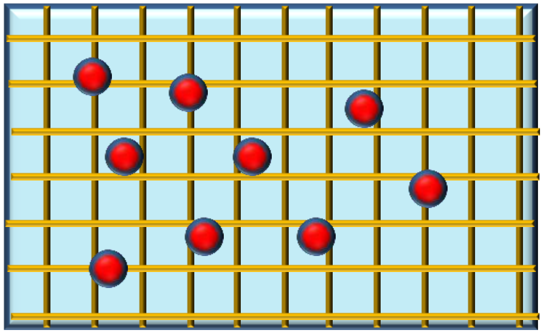
Fractional Quantum Hall (FQH) States

- In partially filled Landau levels, electron interaction can lead to FQH states with nontrivial topological order and fractionalized quasiparticles (Tsui et al 1982)
- FQH states can be described by many-body wavefunctions such as the Laughlin wavefunction (Laughlin 1983)
- $\Psi_{\frac{1}{m}}(\{z_i\}) = \prod_{i<j}(z_i - z_j)^m \exp(-\sum_i |z_i|^2/2l_B^2)$
- Moore-Read wavefunction for a non-Abelian state
- $\Psi_{MR}(\{z_i\}) = \mathrm{Pf}(\frac{1}{z_i - z_j}) \prod_{i<j}(z_i - z_j)^q \exp(-\sum_i |z_i|^2/2l_B^2)$
- Wavefunctions can be constructed systematically to describe many FQH states (e.g., Bernevig&Haldane2008, Wen&Wang 2008)

Can the QAH state be generalized to fractional QH states?

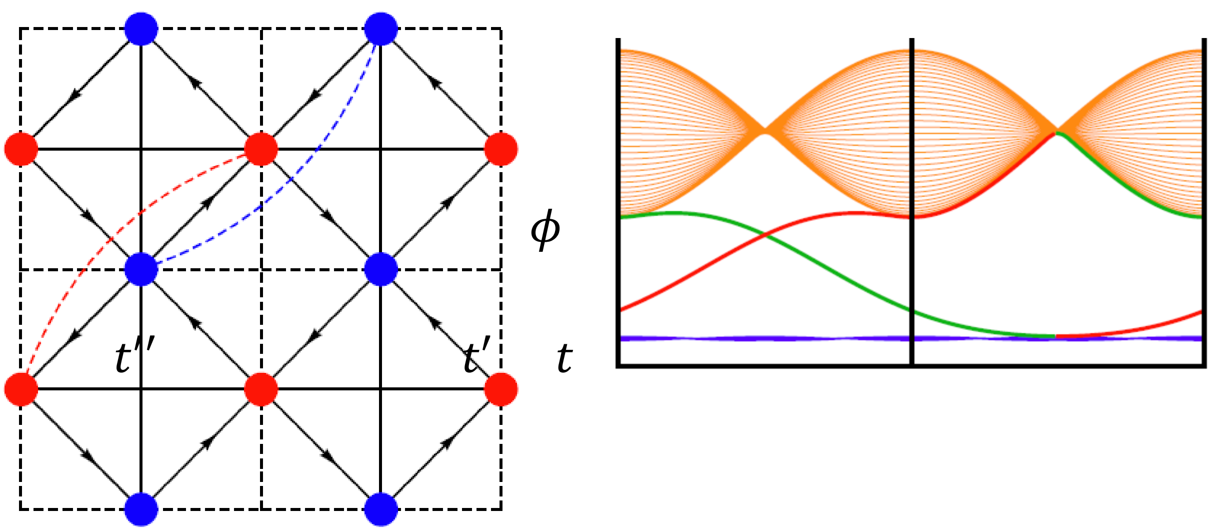
	With magnetic field	No magnetic field Nontrivial band structure
Integer filling non-interacting	Integer Quantum Hall	Quantum anomalous Hall
Fractional filling interacting	Fractional Quantum Hall	Fractional quantum anomalous Hall





Fractional quantum anomalous Hall (FQAH) states

- FQAH can be realized in a topologically nontrivial flat band (Sun *et al*, Neupert, *et al*, Tang *et al*, PRL 2011, Sheng *et al* Nat. Comm. 2011)

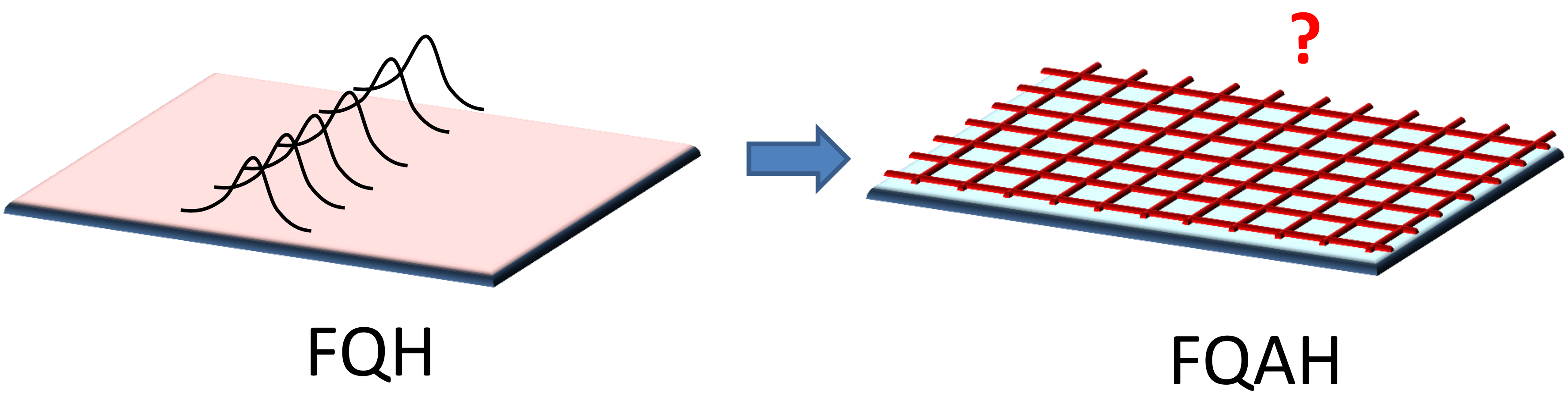


Flat band for $t = 1, t' = \frac{1}{2+\sqrt{2}}, t'' = \frac{1}{2+2\sqrt{2}}, \phi = \frac{\pi}{4}$
(Sun *et al 2011)*

- Numerical evidence of FQH states have been found (Neupert, *et al*, Tang *et al*, PRL 2011, Sheng *et al* Nat. Comm. 2011, Regnault&Bernevig 1105.4867, Wu, Bernevig&Regnault arXiv:1111.1172)

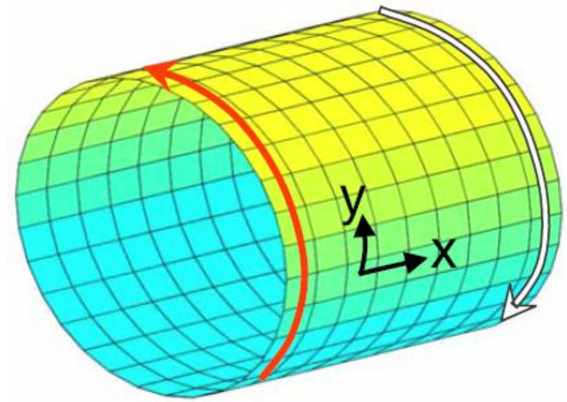
Wave-function description of FQAH states

- What are the many-body wavefunctions describing FQAH states?
- Related to many other questions about FQAH, e.g., what states can be realized on the lattice?
- Idea: Finding the single-particle basis corresponding to the Landau level wavefunctions in the ordinary QH states.

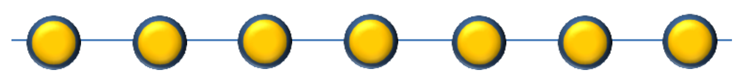


Wave-function description of FQAH states: 1D Wannier functions

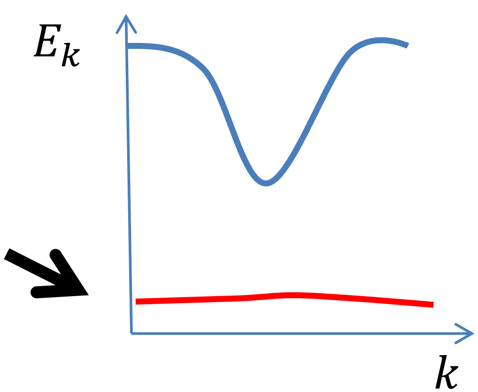
- The proper basis can be found by using 1D Wannier functions
- Consider FQAH state on a cylinder
- The states for each fixed k_y forms a 1D chain.
- 1D Wannier functions:
  a local basis for the 1D system.
  Fourier transform of Bloch states
- $|W_{nk_y}\rangle = \frac{1}{\sqrt{L_x}} \sum_{k_x} e^{ik_x n} e^{i\varphi(\boldsymbol{k})} |k_x, k_y\rangle$



k_y eigenstates

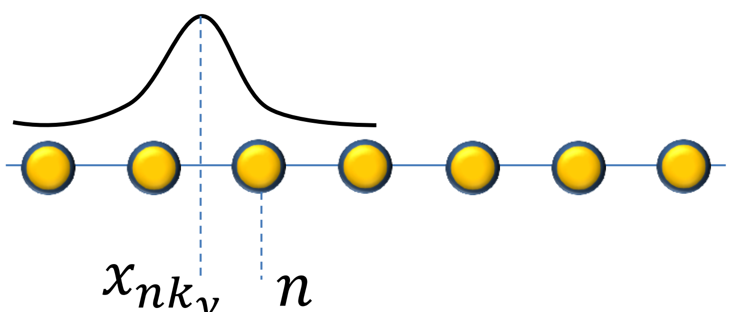


$h_{nm}(k_y)$



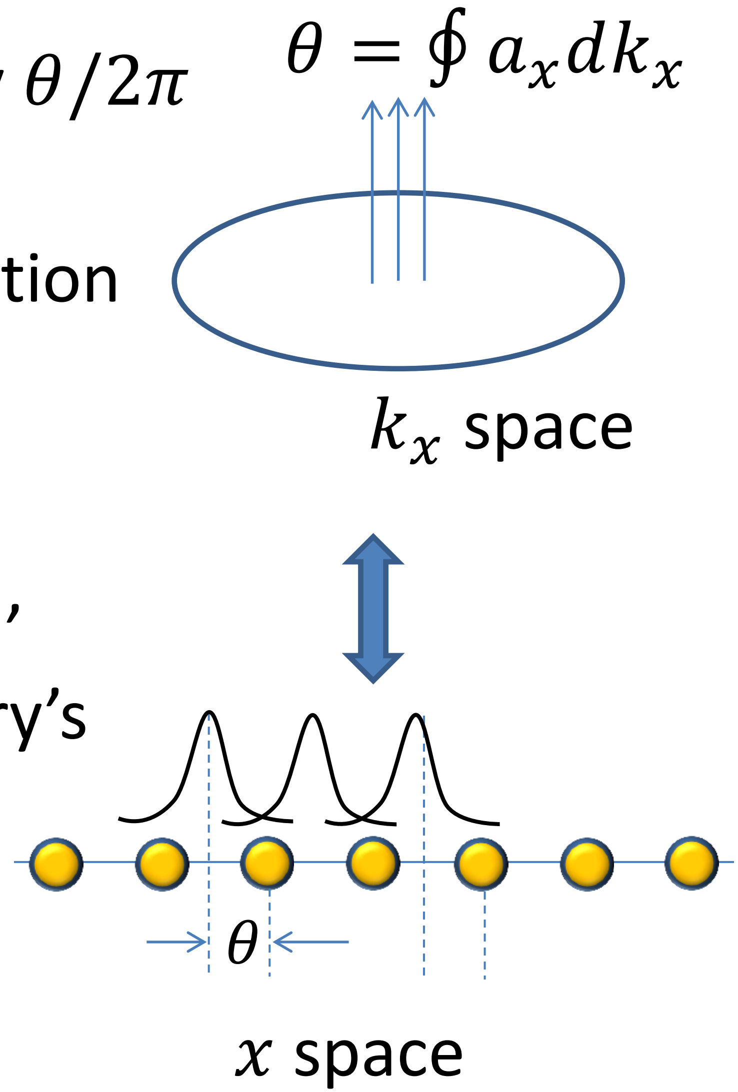
1D Wannier functions

- $|W_{nk_y}\rangle = \frac{1}{\sqrt{L_x}} \sum_{k_x} e^{ik_x n} e^{i\varphi(\boldsymbol{k})} |k_x, k_y\rangle$
- The ambiguity of $\varphi(\boldsymbol{k})$ can be fixed by requiring the Wannier functions to be maximally localized, i.e., by minimizing
  $(\Delta x)^2 = \langle W_{nk_y}|x^2|W_{nk_y}\rangle - \langle W_{nk_y}|x|W_{nk_y}\rangle^2$
- In 1D, the maximally localized Wannier function (MLWF) can be obtained by diagonalizing the projected x operator (Kivelson 1982):
- $\hat{x} = P\_xP\_$ with $P\_ = \sum_{\boldsymbol{k}} |\boldsymbol{k}\rangle\langle\boldsymbol{k}|$ projection to the occupied band.
- $\hat{x}|W_{nk_y}\rangle = x_{nk_y}|W_{nk_y}\rangle$
- $x_{nk_y} = n - \theta(k_y)/2\pi$



1D Wannier functions

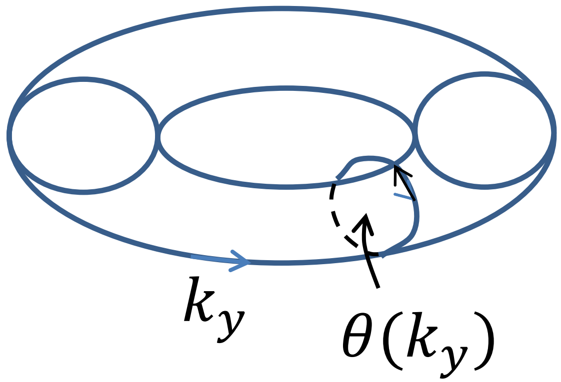
- Wannier functions are shifted by $\theta/2\pi$ with respect to the lattice sites
- Correspondingly, charge polarization $P = -\theta/2\pi$.
- Since $x = i\partial/\partial k_x$, the projected position operator $\hat{x} = i\frac{\partial}{\partial k_x} - a_x$, $a_x(\boldsymbol{k}) = -i\langle \boldsymbol{k} | \partial_{k_x} | \boldsymbol{k} \rangle$ is the Berry's phase gauge field
- The shift of eigenvalues of $\hat{x}$ is determined by the flux of a_x



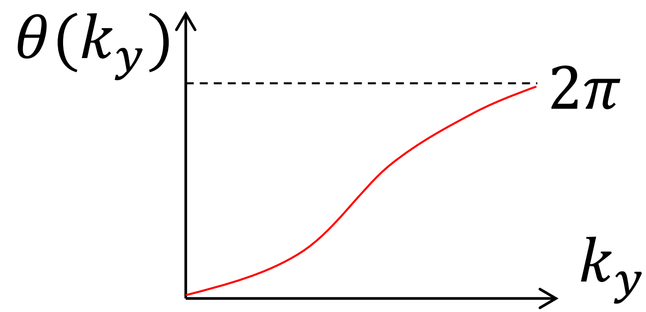
Coh & Vanderbilt, 2009 PRL

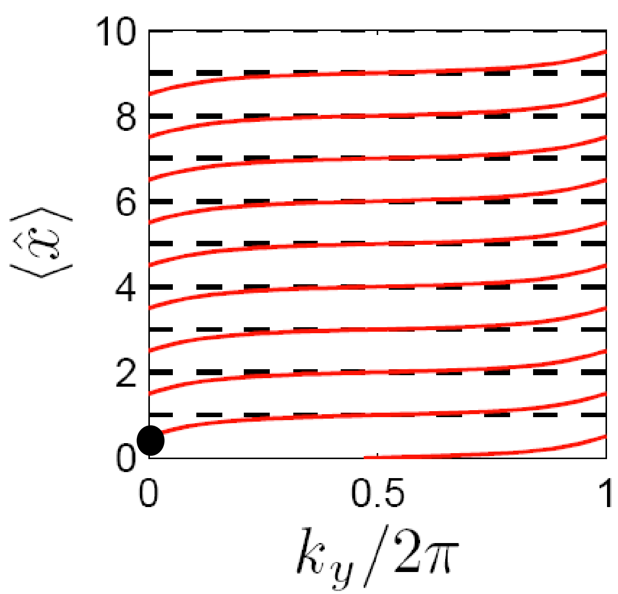
1D Wannier functions and the Chern number

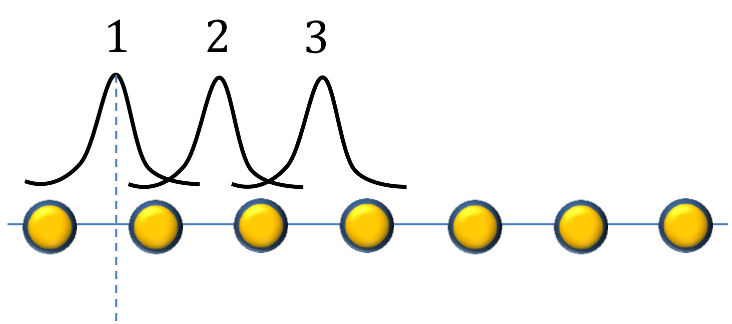
- Chern number on the Brillouin zone torus is the winding number of the flux $\theta(k_y)$



$$\frac{\sigma_H}{h/e^2} = C_1 = -\frac{1}{2\pi}\int_0^{2\pi} \partial_{k_y}\theta(k_y)\, dk_y$$

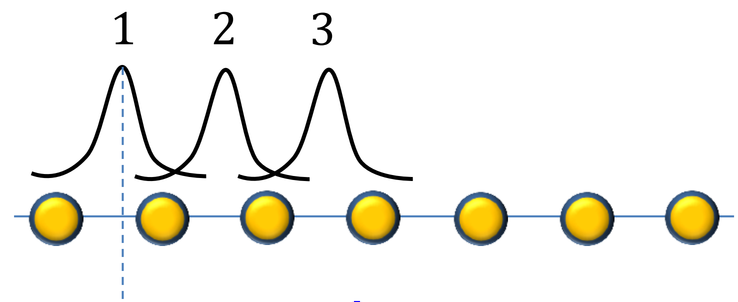




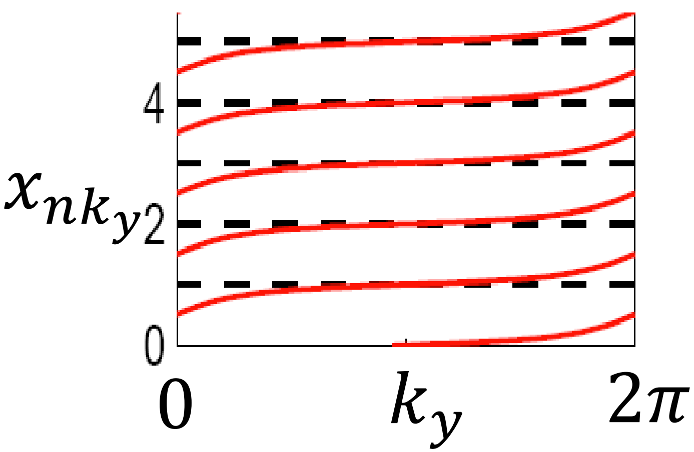


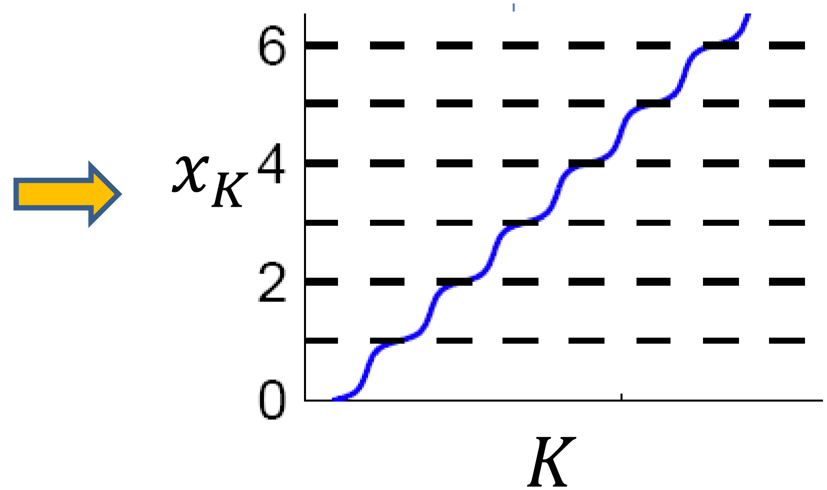
1D Wannier functions in QAH states

- "Twisted" boundary condition for Wannier functions
- $k_y \to k_y + 2\pi, |W_{nk_y}\rangle \to |W_{n+1,k_y}\rangle$
- A extended momentum K can be defined, only if the Chern number is nontrivial



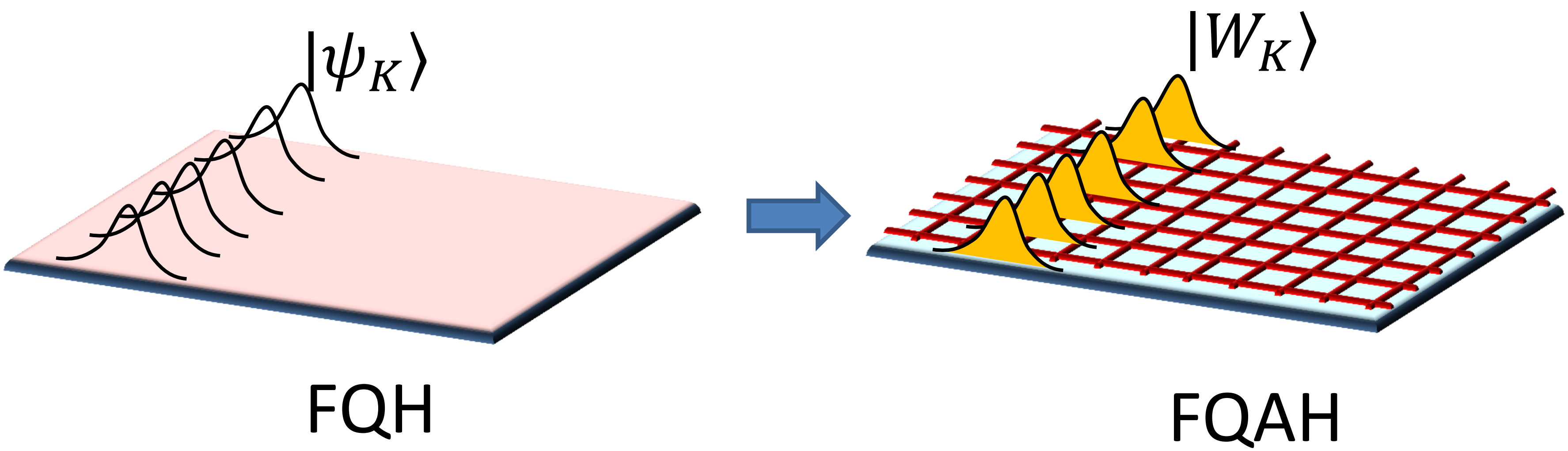
- $|W_{nk_y}\rangle = |W_K\rangle$





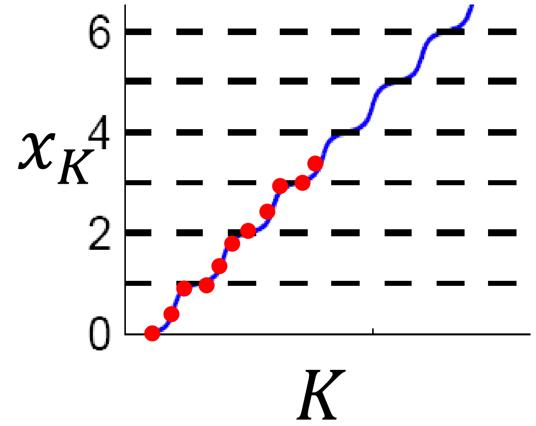
Using 1D Wannier functions to describe FQAH states

- After the redefinition, Wannier functions $|W_K\rangle$ are analog of Landau level wavefunctions
- $\psi_K(x,y) = e^{iky} e^{-\left(x-Kl_B^2\right)^2/2l_B^2}$



Using 1D Wannier functions to describe FQAH states

- Using this mapping of basis, every FQH wavefunction is mapped to the lattice FQAH states



- FQH:

$$|\Psi\rangle = \sum_{\{n_K\}} \Phi(\{n_K\}) \prod_{n_K=1} |\psi_K\rangle$$

$$n_K = 0,1$$

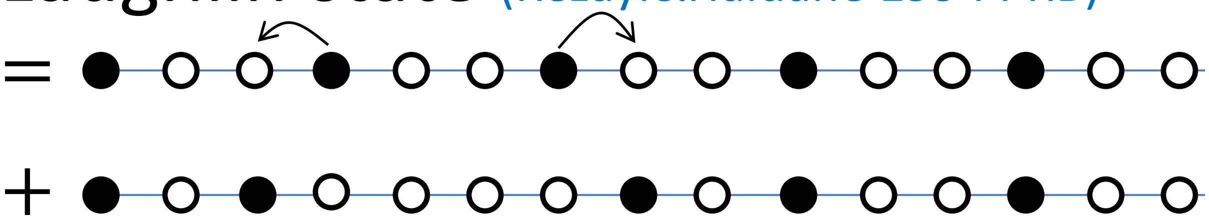
- ➔FQAH:

$$|\Psi\rangle = \sum_{\{n_K\}} \Phi(\{n_K\}) \prod_{n_K=1} |W_K\rangle$$

Using 1D Wannier functions to describe FQAH states

- The occupation number wavefunction $\Phi(\{n_K\})$ is known for many FQH states
- For Laughlin state (Rezayi&Haldane 1994 PRB)

$|\Psi_{1/3}\rangle =$

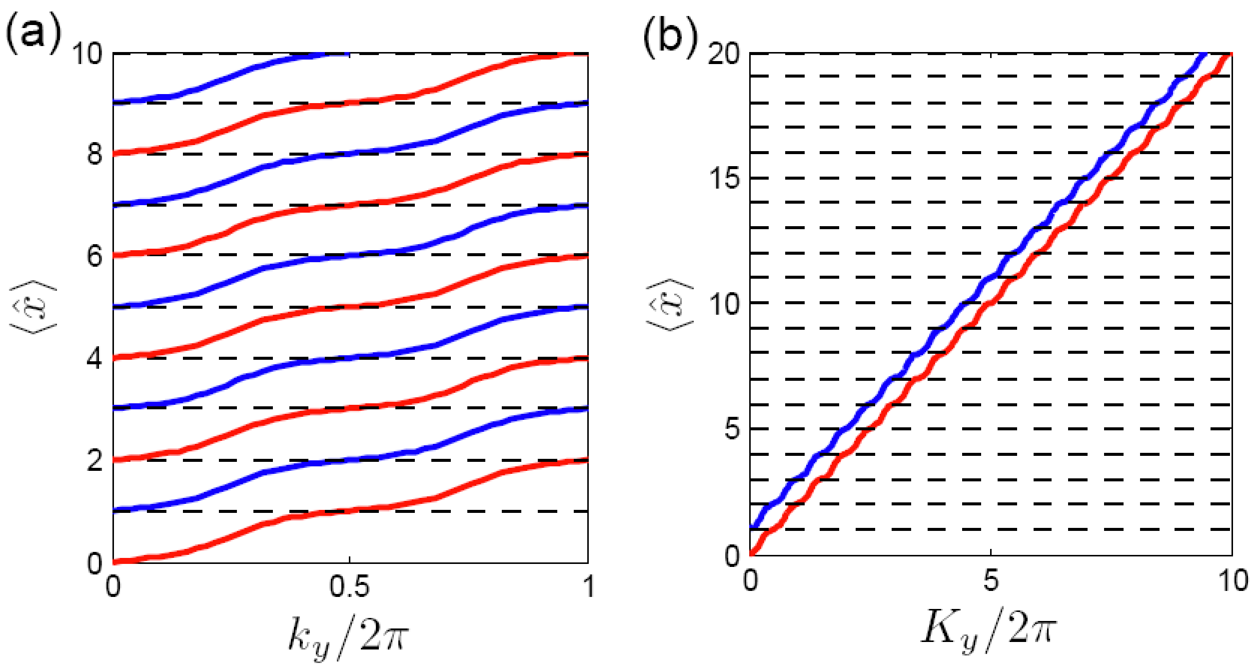


$+ \ldots\ldots$

- A generic construction by Jack polynomials (Bernevig&Haldane 2008 PRL)
- All knowledge on FQH wavefunctions can now be used to construct lattice wavefunctions with the same topological properties.
- Pseudo-potential Hamiltonians can be constructed on the lattice

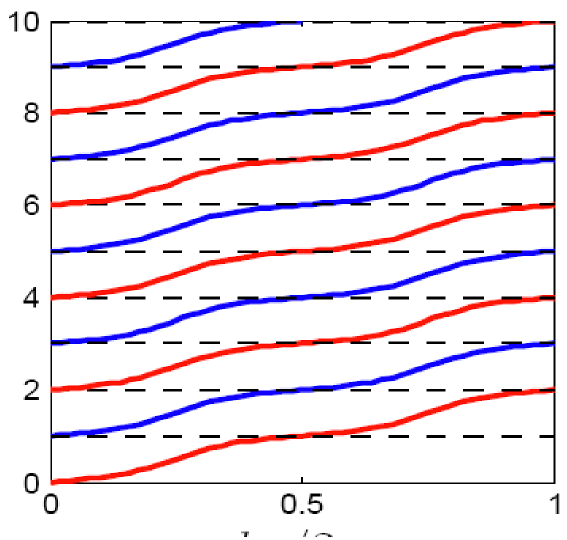
FQAH state with higher Chern number

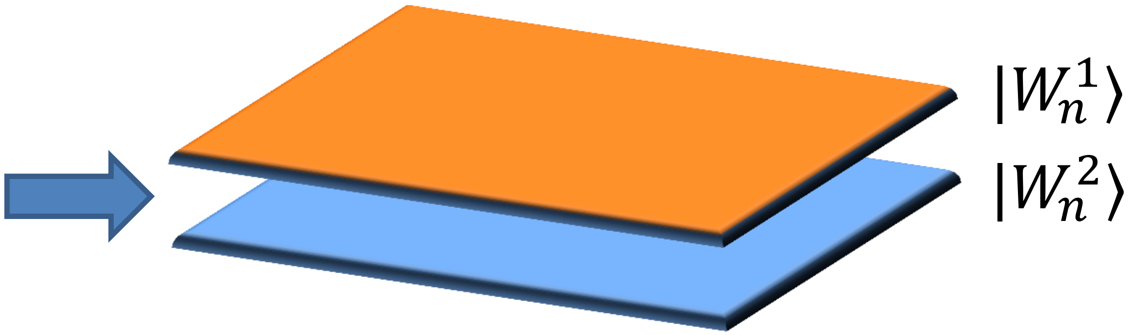
- Are there new states in the FQAH system that are absent in the ordinary FQH?
- Similar approach can be generalized to bands with Chern number >1
- Higher winding number of the Wannier state position



Realizing multi-layer FQH states in one band

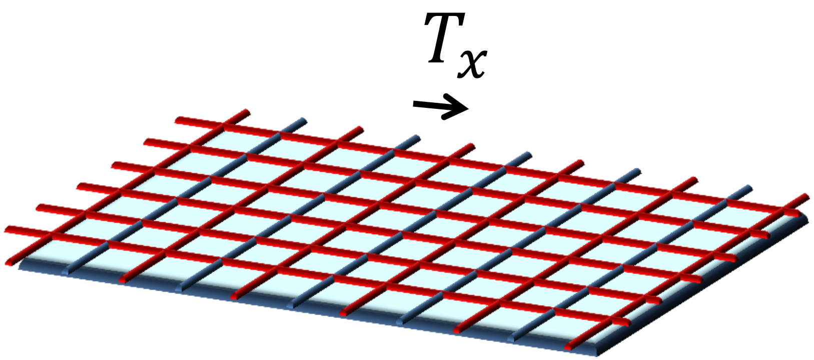
- For Chern number $C_1 = 2$, the Wannier states form two groups $|W_n^1\rangle, |W_n^2\rangle$, with each group equivalent to a Landau level
- ➔Double-layer FQH states can be realized in a single band

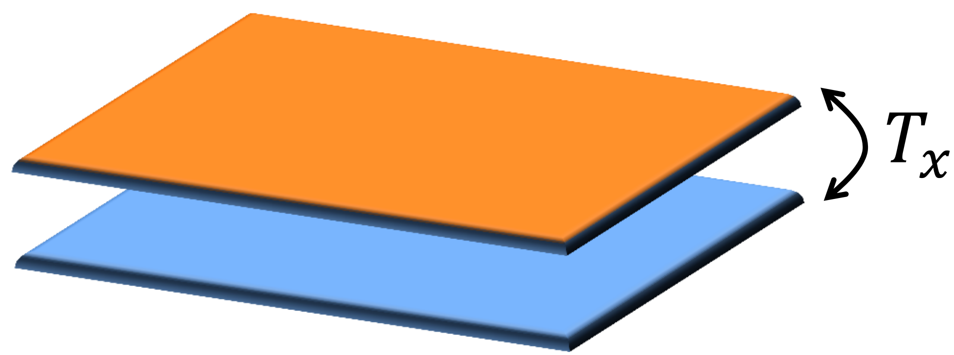




Nontrivial representation of lattice translation symmetry

- Lattice translations T_x, T_y acts differently on this basis
- $T_x|W_n^1\rangle = |W_n^2\rangle,\ T_x|W_n^1\rangle = |W_n^1\rangle$
- $T_y\left|W_n^{1,2}\right\rangle = e^{in2\pi/L_y}|W_n^{1,2}\rangle$



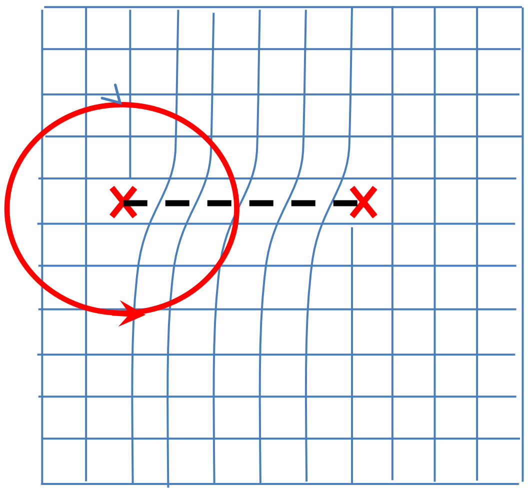


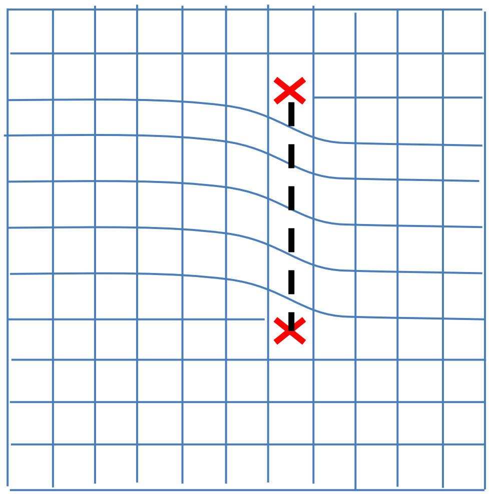
Topological nematic states

- Consider the Halperin (mnl) states (Halperin '83)
- $\Phi(z_i, w_j) = \prod_{i<j}(z_i - z_j)^m (w_i - w_j)^n \prod_{i,j}(z_i - w_j)^l$
- Lattice translation T_x exchanges the two "layers".
- For $m = n$ the state is translation invariant. However, the 4-fold lattice rotation symmetry (for a square lattice) is broken.
- We name such a state as a topological nematic state
- Lattice dislocations in a topological nematic state carry nontrivial topological degeneracy

Dislocations in topological nematic states

- Dislocations are described by the Burgers vector $\vec{b} = (b_x, b_y)$



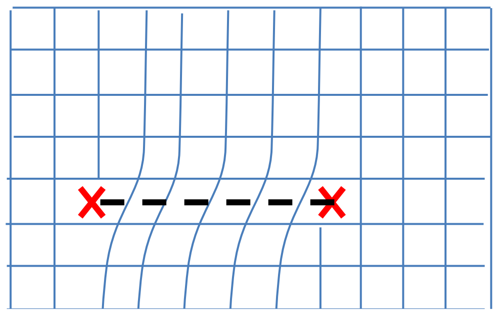


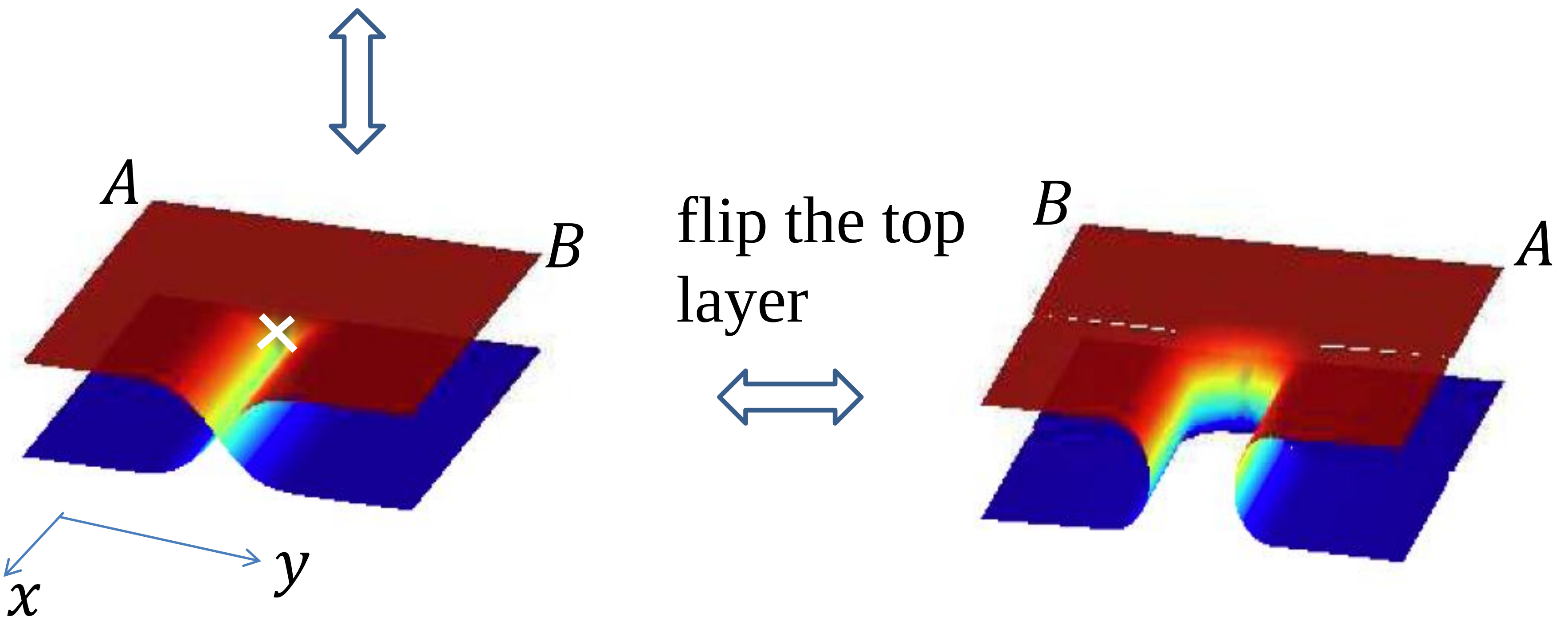
x-dislocation $\vec{b} = \hat{x}$ $\qquad$ y-dislocation $\vec{b} = \hat{y}$

- Across the "branch-cut" of the x-dislocation, the two layers are exchanged!

Dislocations in topological nematic states

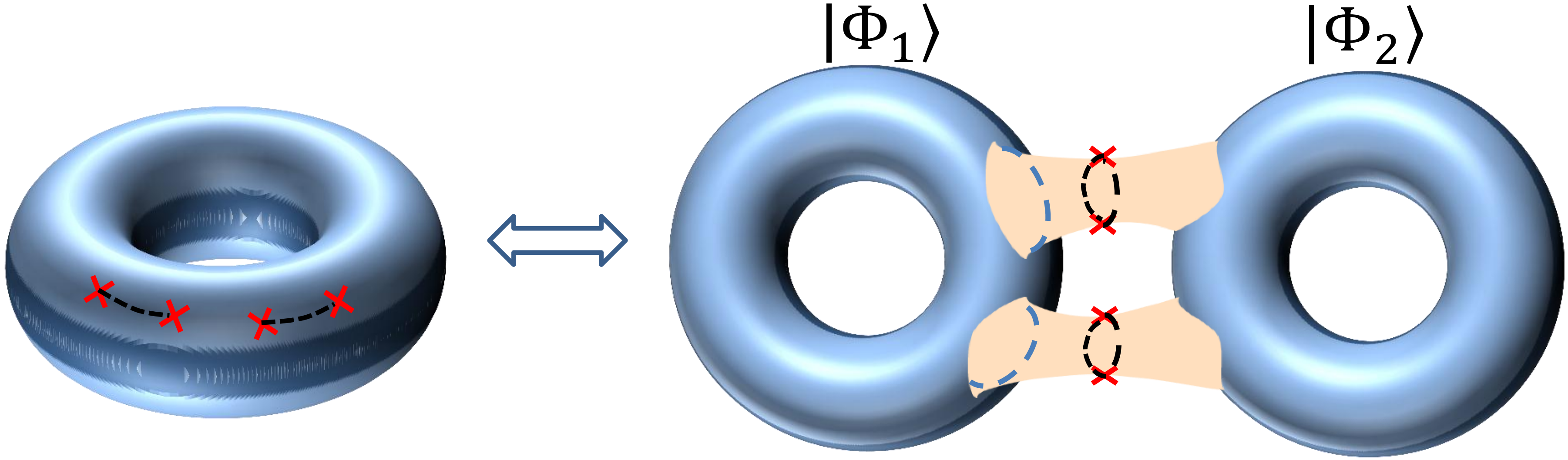
- A pairs of x-dislocations is equivalent to a “worm-hole”





Dislocations in topological nematic states

- Consider a simple case of $(mm0)$ state, which is a direct product of two Laughlin states
- $|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle$
- Consider two pairs of dislocations on the torus
- Ground state degeneracy $N = m^3$
- Degeneracy $d = \sqrt{m}$ per dislocation



Dislocations in topological nematic states

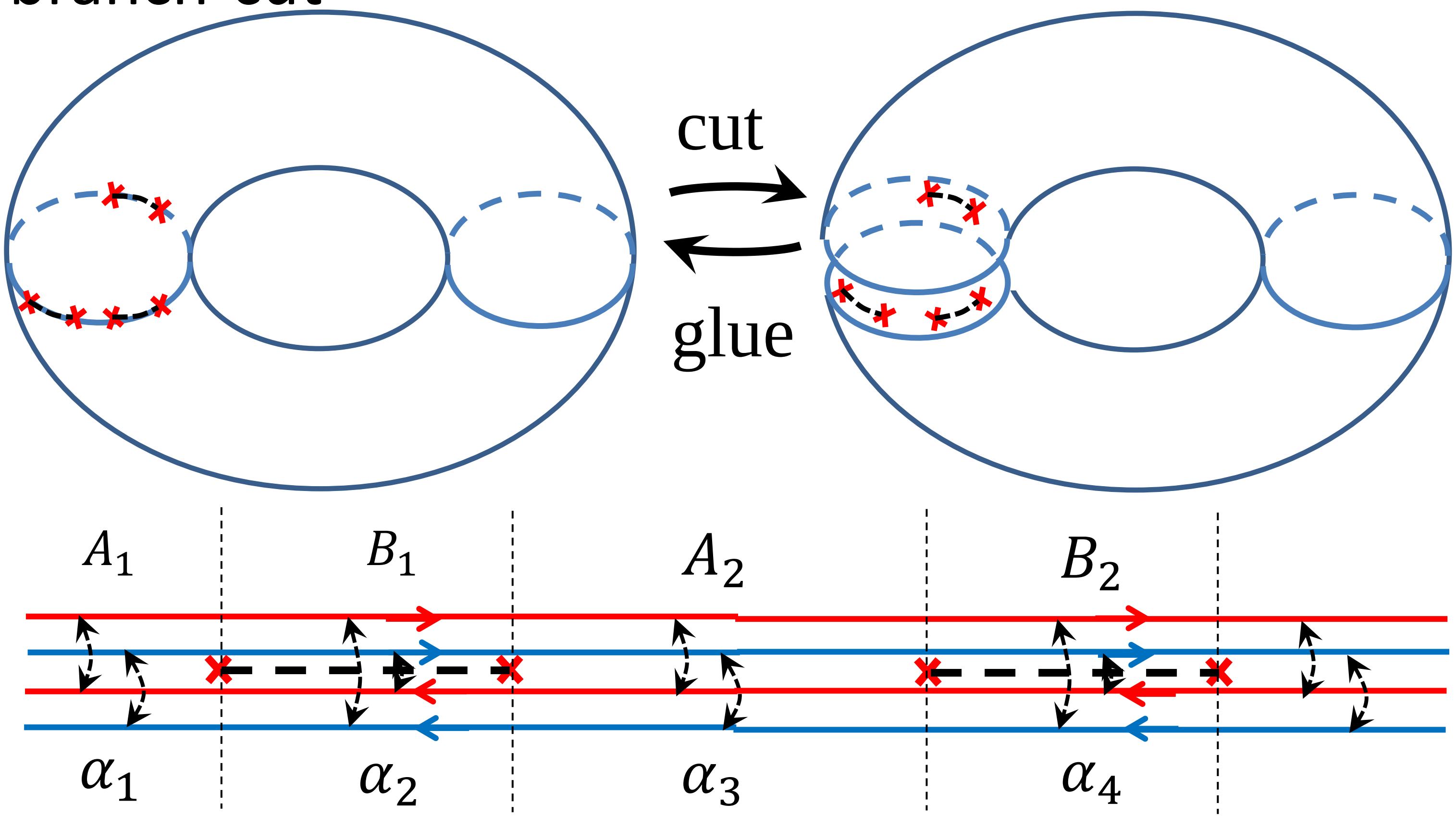
- For more general (mml) states, the topological degeneracy with n pairs of dislocation is
- $N = |m^2 - l^2| \cdot |m - l|^{n-1}$
- The degeneracy per dislocation, i.e., quantum dimension is $d = \sqrt{|m - l|}$
- This degeneracy can be obtained by studying the Chern-Simons theory with branch-cuts

$$\mathcal{L} = \frac{1}{4\pi} a_\mu^I K_{IJ} \partial_\nu a_\tau^J$$ (Barkeshli&Wen 2010)

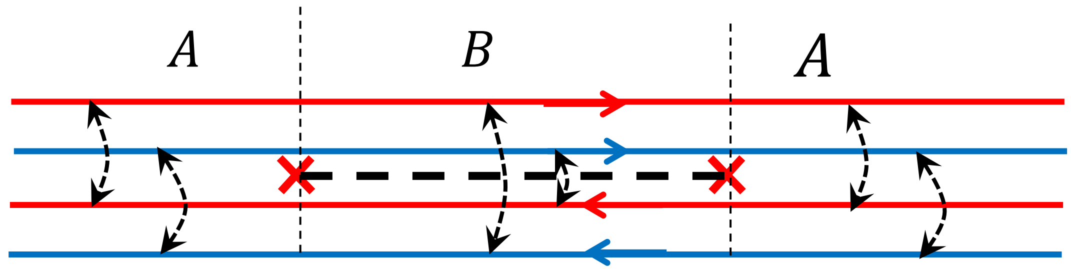
- Alternatively, it can be understood from an edge state picture

Edge state picture of dislocation-induced degeneracy

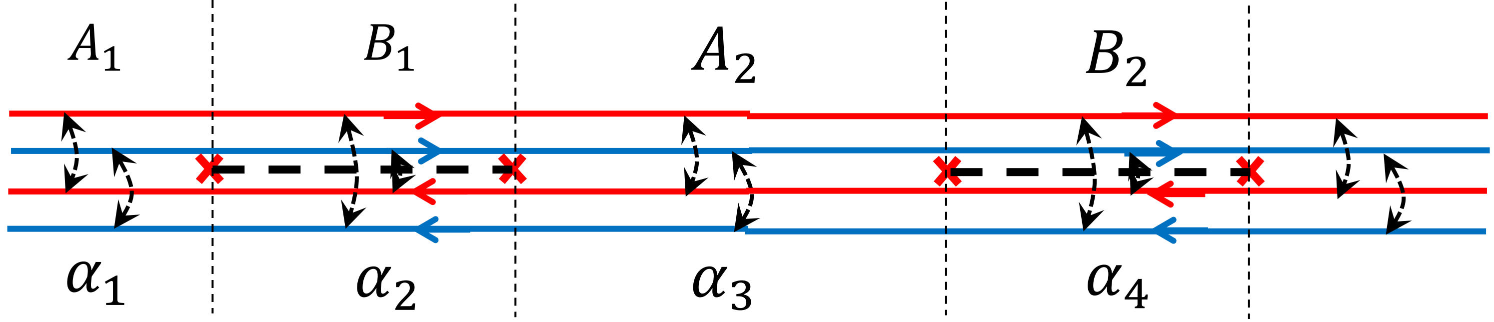
- Consider the torus as a cylinder glued along the edge
- Inter-edge tunneling exchanges the two layers across the branch-cut



Edge state picture of dislocation-induced degeneracy

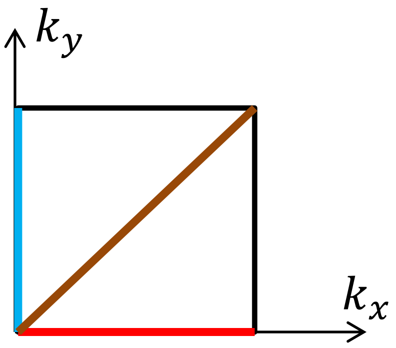


- The edge states are described by the chiral Luttinger liquid theory
- $\mathcal{L} = \frac{1}{2\pi}\left(\partial_t \phi_L^I K_{IJ} \partial_x \phi_L^J - \partial_x \phi_L^I V_{IJ} \partial_x \phi_L^J\right) + \frac{1}{2\pi}\left(-\partial_t \phi_R^I K_{IJ} \partial_x \phi_R^J - \partial_x \phi_R^I V_{IJ} \partial_x \phi_R^J\right) + \mathcal{L}_{int}$
- Electron operators $c_{L,R}^I = e^{iK_{IJ}\phi_{L,R}^J}$
- Interedge tunneling
- $\mathcal{L}_{int} = \begin{cases} g(c_L^{1+} c_R^1 + c_L^{2+} c_R^2 + h.c.), & \text{A region} \\ g(c_L^{1+} c_R^2 + c_L^{2+} c_R^1 + h.c.), & \text{B region} \end{cases}$

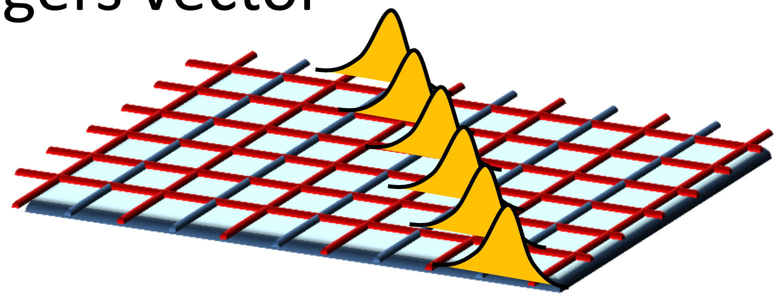


- Without dislocation, the inter-edge coupling potential $g\sum_I \cos K_{IJ}(\phi_L^J - \phi_R^J)$ has $\det|K|$ number of minima in the "Brillouin zone" $\phi_L^{1,2} - \phi_R^{1,2} \in [0,2\pi)$, leading to the degeneracy of $\det|K| = m^2 - l^2$ of the torus
- With n pairs of dislocations, each A region contributes $m^2 - l^2$, each B region contributes $m + l$, because the two operators $\phi_L^1 - \phi_R^1$ in A and $\phi_L^1 - \phi_R^2$ in B do not commute
- Each dislocation has a constraint $Q_i = \int_{\alpha_i}^{\alpha_{i+1}} \partial_x(\phi_1+\phi_2)dx = 0$
- ➔Degeneracy $N = \frac{(m^2-l^2)^n (m+l)^n}{(m+l)^{2n-1}} = (m^2 - l^2)(m + l)^{n-1}$

Classification of topological nematic states



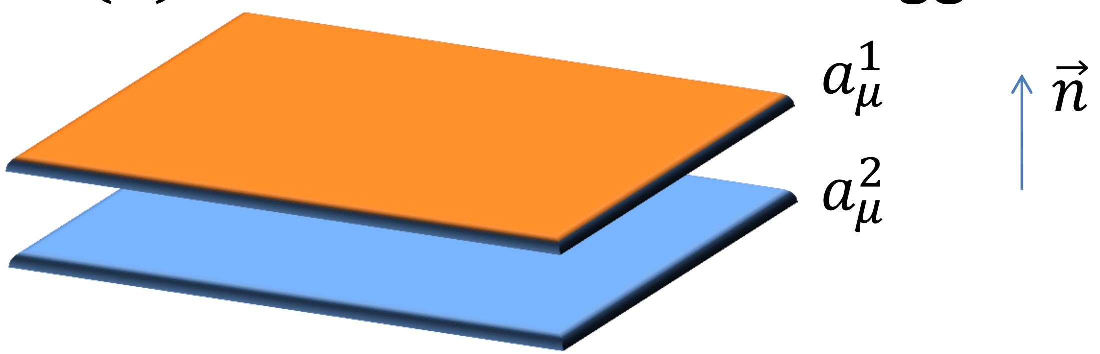
- The topological nematic states discussed above are sensitive to x-dislocations but not y-dislocations.
- Apparently, different topological nematic states can be obtained.
- Generically, 1D Wannier functions can be defined along any reciprocal lattice direction $\vec{K} = 2\pi(n_x, n_y)$
- The corresponding topological nematic states is sensitive to dislocations with burgers vector $\vec{b} \cdot \frac{\vec{K}}{2\pi}$ odd.



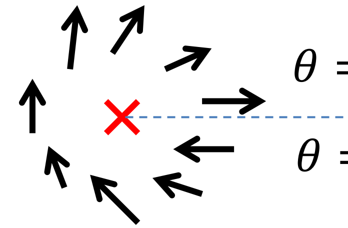
- ➔ 3 types of topological nematic states $(0,1), (1,0), (1,1)$
- The ordinary Halperin state can be viewed as a trivial class $(0,0)$

A topological field theory description of topological nematic states

- Without dislocations, the effective theory is an Abelian $U(1) \times U(1)$ Chern-Simons theory $\mathcal{L} = \frac{1}{4\pi} a_\mu^I K_{IJ} \partial_\nu a_\tau^J$
- Around a dislocation, a_μ^1 and a_μ^2 are exchanged
- To describe this effect we introduce a $U(2)$ gauge field A_μ and a Higgs field $H = \sigma \cdot \vec{n} e^{i\theta}$ which breaks $U(2) \to U(1) \times U(1)$. The manifold of the Higgs field is $S^2 \times U(1)/Z_2$



A topological field theory description of topological nematic states



- Consider the Chern-Simons-Higgs theory
- $\mathcal{L} = \frac{m-l}{4\pi} \epsilon^{\mu\nu\tau} tr\left[A_\mu \partial_\nu A_\tau + \frac{2}{3} A_\mu A_\nu A_\tau\right] + \frac{l}{4\pi} \epsilon^{\mu\nu\tau} tr[A_\mu] \partial_\nu tr[A_\tau] + J tr[D_\mu H^\dagger D_\mu H]$
- A constant H breaks $U(2)$ to diagonal $U(1) \times U(1)$, leading to the Abelian Chern-Simons theory.
- A dislocation corresponds to a half vortex of $H = \sigma \cdot \vec{n} e^{i\theta}$.
- The two $U(1)$ are exchanged around the dislocation
- Generically, $\theta = \pi \vec{u} \cdot \vec{N}$ with $\vec{u}$ displacement field, and $\vec{N} = (n_x, n_y)$ the type of the topological nematic state.

Summary

- 1D Wannier functions provide the proper basis for characterizing fractional topological states in lattice systems with nontrivial band structure.
- FQH states can be mapped to lattice models
- A band with higher Chern number is mapped to a multi-layer FQH state
- New states named as topological nematic states can be realized, with non-Abelian dislocations even if the state itself is Abelian.
- Provide the possibility of experimentally observe the topological degeneracy, which otherwise requires a high genus surface.