Surface-Edge State
& Half-Quantized Hall Conductance
in Topological Insulators

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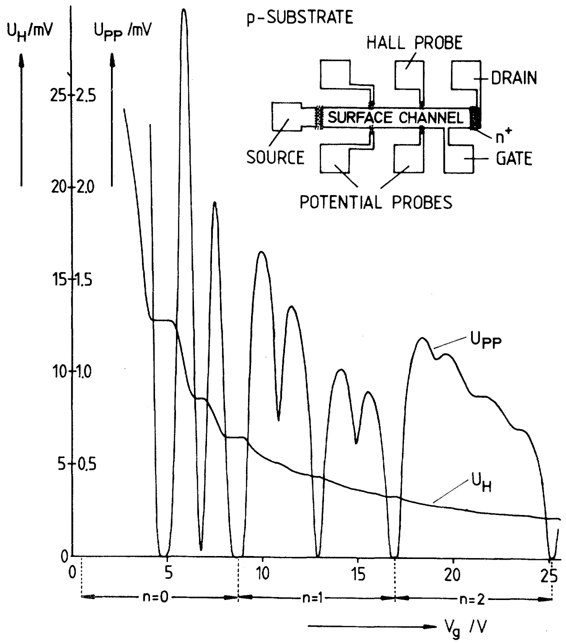
- Reference: R. L. Chu, J. R. Shi and S. Q. Shen, Surface edge state and half-quantized Hall conductance in topological insulators, Phys. Rev. B 84, 085312 (2011)

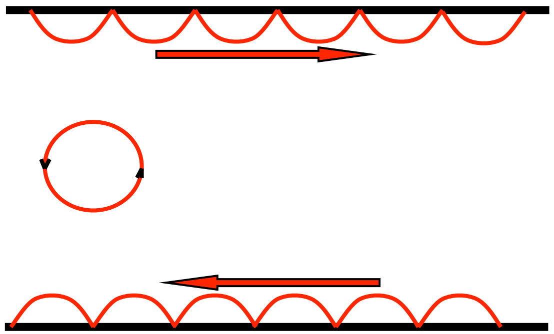
Integer Quantum Hall Effect & Edge State



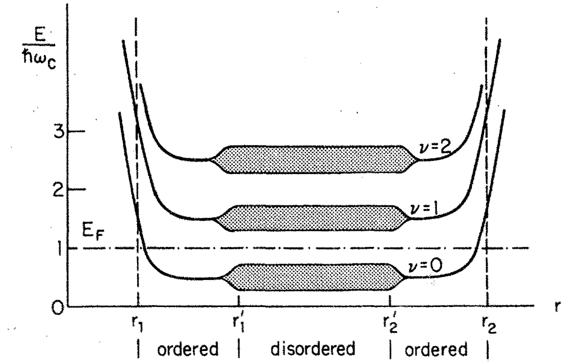
Klitzing, K. von; Dorda, G.; Pepper, M. "New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance". *Phys. Rev. Lett.* **45**, 494 (1980).

$$\rho_{xy} = -\frac{\sigma_{xy}}{\sigma_{xx}^2 + \sigma_{xy}^2} = -\sigma_{xy}^{-1} = -\frac{h}{ne^2}$$

n is an integer, the number of edge states.



$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_C$$



$k_x \to y_0 / l_b^2$

$$i_n = -\frac{e}{L}\sum \langle v_x \rangle_n = -\frac{e}{L}\sum_{k_x} \langle \partial H / \hbar \partial k_x \rangle_n$$
$$= -\frac{e}{2\pi\hbar}\int dk_x \frac{\partial E_n}{\partial k_x} \to -\frac{e}{h}\int dy_0 \frac{\partial E_n}{\partial y_0}$$
$$= -\frac{e}{h}\left[E_n(L/2) - E_n(-L/2)\right]$$
$$= \frac{e^2}{h} V$$

Halperin, 82; MacDonald & Streda, 84

The key feature:
Bulk electrons are localized
while edge electrons are extended.

Half-Quantized Hall Conductance in Graphene?

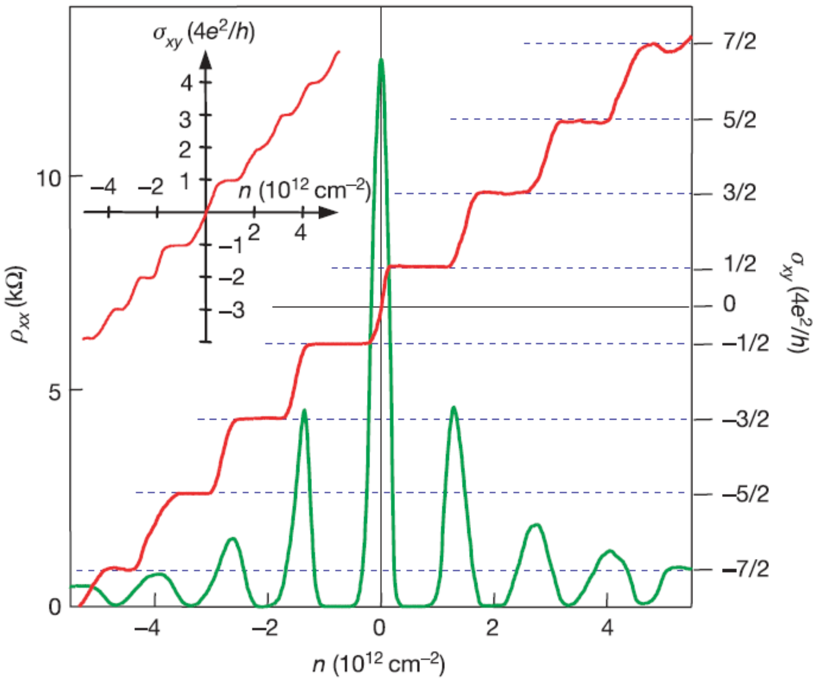


Figure 4 | QHE for massless Dirac fermions. Hall conductivity σ_{xy} and longitudinal resistivity ρ_{xx} of graphene as a function of their concentration at $B = 14$ T and $T = 4$ K. $\sigma_{xy} \equiv (4e^2/h)\nu$ is calculated from the measured dependences of $\rho_{xy}(V_g)$ and $\rho_{xx}(V_g)$ as $\sigma_{xy} = \rho_{xy}/(\rho_{xy}^2 + \rho_{xx}^2)$. The behaviour of $1/\rho_{xy}$ is similar but exhibits a discontinuity at $V_g \approx 0$, which is avoided by plotting σ_{xy}. Inset: σ_{xy} in 'two-layer graphene' where the quantization sequence is normal and occurs at integer ν. The latter shows that the half-integer QHE is exclusive to 'ideal' graphene.

Two valleys +
Double spin degeneracy

$$\sigma_H = (n + 1/2)\frac{4e^2}{\hbar}$$

$$= (4n + 2)\frac{e^2}{\hbar}$$

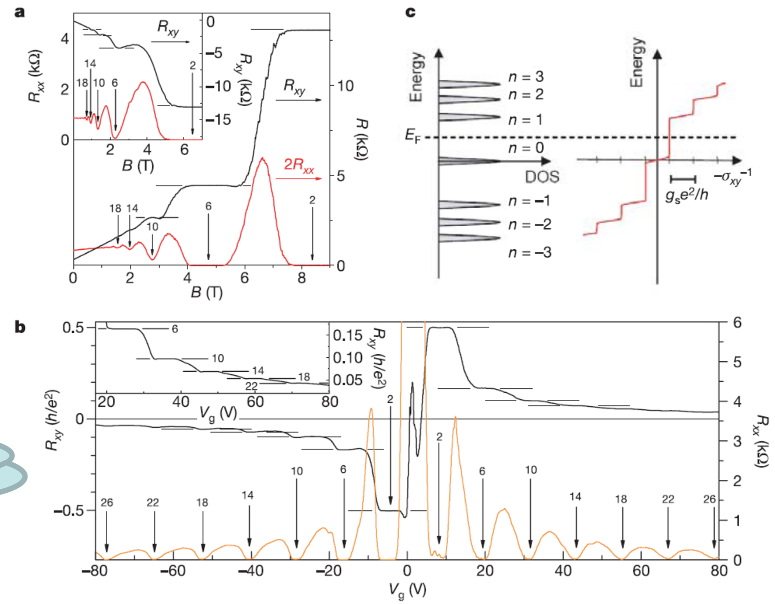
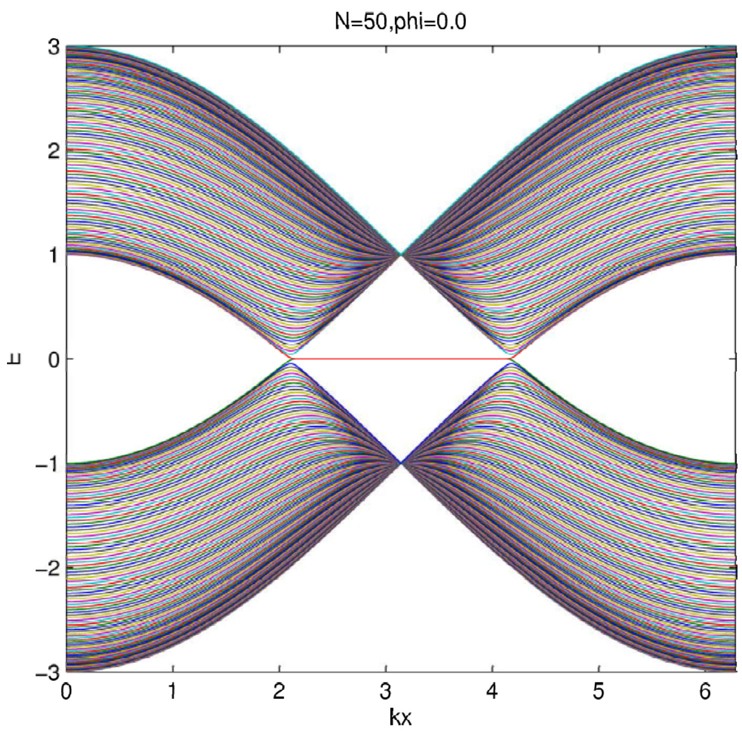
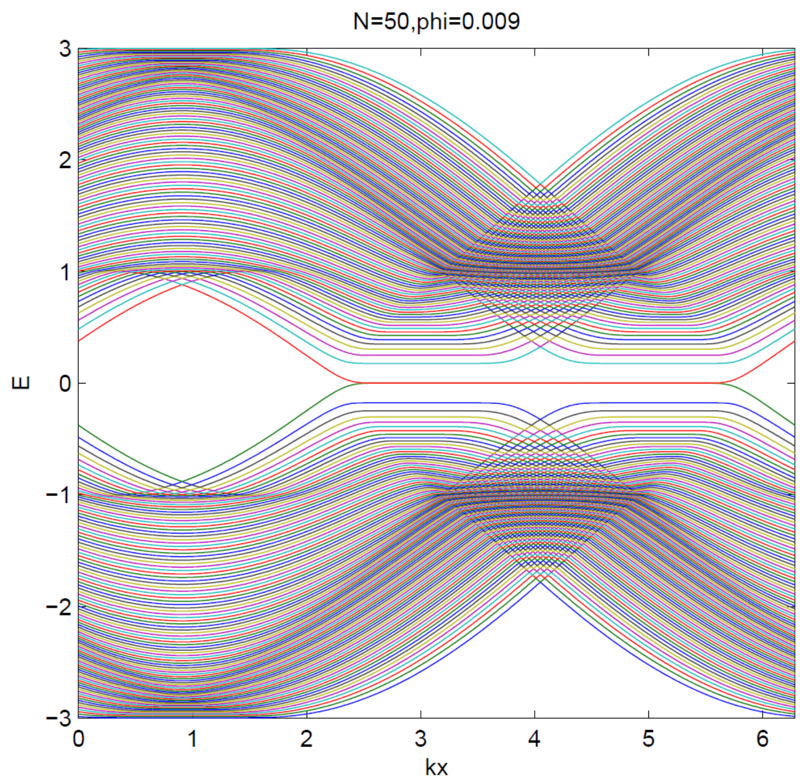


Figure 2 | Quantized magnetoresistance and Hall resistance of a graphene device. a, Hall resistance (black) and magnetoresistance (red) measured in the device in Fig. 1 at $T = 30$ mK and $V_g = 15$ V. The vertical arrows and the numbers on them indicate the values of B and the corresponding filling factor ν of the quantum Hall states. The horizontal lines correspond to $h/e^2\nu$ values. The QHE in the electron gas is shown by at least two quantized plateaux in R_{xy}, with vanishing R_{xx} in the corresponding magnetic field regime. The inset shows the QHE for a hole gas at $V_g = -4$ V, measured at 1.6 K. The quantized plateau for filling factor $\nu = 2$ is well defined, and the second and third plateaux with $\nu = 6$ and $\nu = 10$ are also resolved. **b**, Hall resistance (black) and magnetoresistance (orange) as a function of gate voltage at fixed magnetic field $B = 9$ T, measured at 1.6 K. The same convention as in **a** is used here. The upper inset shows a detailed view of high-filling-factor plateaux measured at 30 mK. **c**, A schematic diagram of the Landau level density of states (DOS) and corresponding quantum Hall conductance (σ_{xy}) as a function of energy. Note that, in the quantum Hall states, $\sigma_{xy} = -R_{xy}^{-1}$. The LL index n is shown next to the DOS peak. In our experiment the Fermi energy E_F can be adjusted by the gate voltage, and R_{xy}^{-1} changes by an amount $g_s e^2/h$ as E_F crosses a LL.

Zhang et al, Nature 438, 201 (2005)

Band Structure of Graphene in B field





The zero mode edge states connects two valleys. So the zero-mode conductance originates from the one edge state connecting two valleys, NOT from two one-halfs of two valleys

The Hall Conductance and the Chern Number

The system Hamiltonian

$$H = \epsilon(p) + \sum_{\alpha=x,y,z} d_\alpha(p)\sigma_\alpha$$

Thouless, Kohmoto*, Nightingale, and den Nijs, **Phys. Rev. Lett. 49, 405(1982)**

The Kubo formula for the conductance: a result of linear response theory

$$\sigma_{ij} = \frac{e^2\hbar}{\Omega} \sum_{p,\mu\neq\mu'} \frac{(f_{p\mu} - f_{p\mu'})\mathrm{Im}(\langle p\mu|v_i|p\mu'\rangle\langle p\mu'|v_j|p\mu\rangle)}{(E_{p\mu} - E_{p\mu'})(E_{p\mu} - E_{p\mu'} + i\delta^+)}$$

$$\sigma_{ij} = -\frac{e^2\hbar}{2\Omega}\sum_p \frac{(f_{p,-} - f_{p,+})}{d^3}\epsilon_{\alpha\beta\gamma}\frac{\partial d_\alpha}{\partial p_i}\frac{\partial d_\beta}{\partial p_j}d_\gamma.$$

$$\sigma_H = \nu\frac{e^2}{h}$$

Hall conductance

The Chern number is an integer if the Brillouin zero is finite.

Half-Quantized Hall Conductance for Two Dimensional Massive Dirac Gas

Redlich, PRD 29, 2366(84); Jackiw, PRD 29, 2375(84)
Qi et al, PRB (2008)

$$H = v_F \hbar k \cdot \sigma + m v_F^2 \sigma_z = d \cdot \sigma$$

$$\sigma_H = -\frac{1}{2} \sum_k \frac{d \cdot (\partial_{k_x} d \times \partial_{k_y} d)}{d^3}$$

$$\sigma_H = -\frac{e^2}{2h} \mathrm{sgn}(m)$$

Integer Quantized Hall Conductance

Lu, Shan, Chu, Niu & Shen, PRB 81, 115407(10)

$$H = v_F \hbar k \cdot \sigma + (m v_F^2 - B k^2)\sigma_z$$

$$\sigma_H = -\frac{e^2}{2h}\left[\mathrm{sgn}(m) + \mathrm{sgn}(B)\right]$$

Half-Quantization in B-field

$$H = v_F\, p \cdot \sigma \rightarrow v_F\,(p - \frac{e}{c} A) \cdot \sigma$$

$$H = v_F \begin{pmatrix} 0 & \Pi_- \\ \Pi_+ & 0 \end{pmatrix} = \sqrt{2}\,\frac{\hbar v_F}{l_B} \begin{pmatrix} 0 & a \\ a^+ & 0 \end{pmatrix}$$

$$E_n = \pm\sqrt{n\hbar eB};\left|n,\pm\right\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \left|n-1\right\rangle \\ \pm\left|n\right\rangle \end{pmatrix}$$

$$E_{n=0} = 0;\left|n=0\right\rangle = \begin{pmatrix} 0 \\ \left|0\right\rangle \end{pmatrix}$$

$$\boxed{\sigma_H = (n+1/2)\frac{e^2}{h}}$$

$$j_{xy} = \frac{1}{2}\frac{e^2}{h}E_y \quad ?$$

Q: Is this formula correct?
If yes, can we measure one half-quantized Hall conductance?

3D Topological Insulator

Under the time reversal

$$\Theta H(k)\Theta^{-1} = H(-k)$$

Time Reversal Invariant Momentum

$$k = -k + G; \quad k = G/2$$

(Reciprocal lattice vector G)

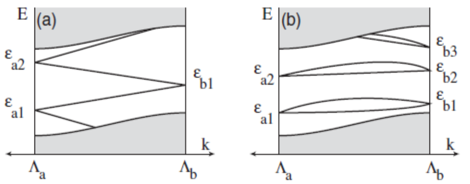
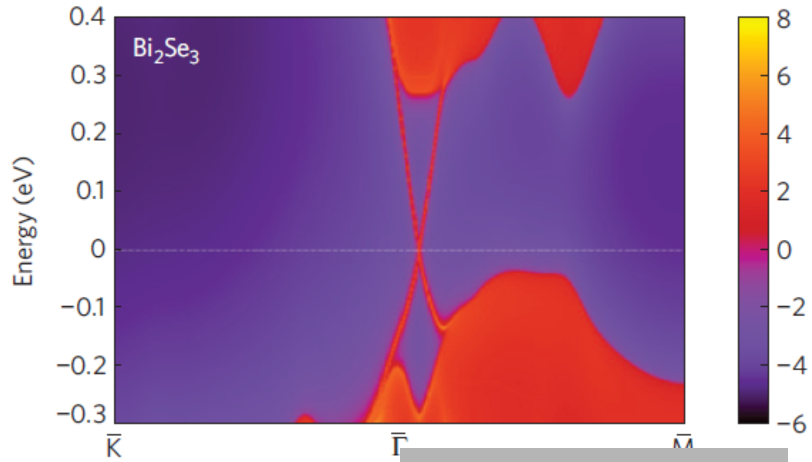


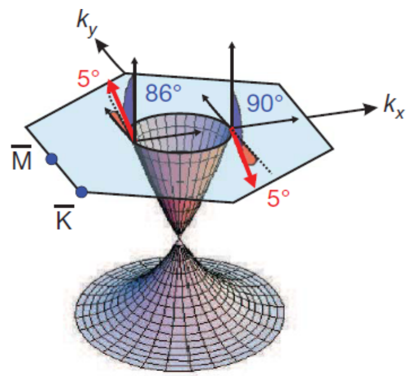
FIG. 1. Schematic surface (or edge) state spectra as a function of momentum along a line connecting Λ_a to Λ_b for (a) $\pi_a\pi_b = -1$ and (b) $\pi_a\pi_b = +1$. The shaded region shows the bulk states. In (a) the TRP changes between Λ_a and Λ_b, while in (b) it does not.

Fu and Kane, PRL 98, 106803 (2007)

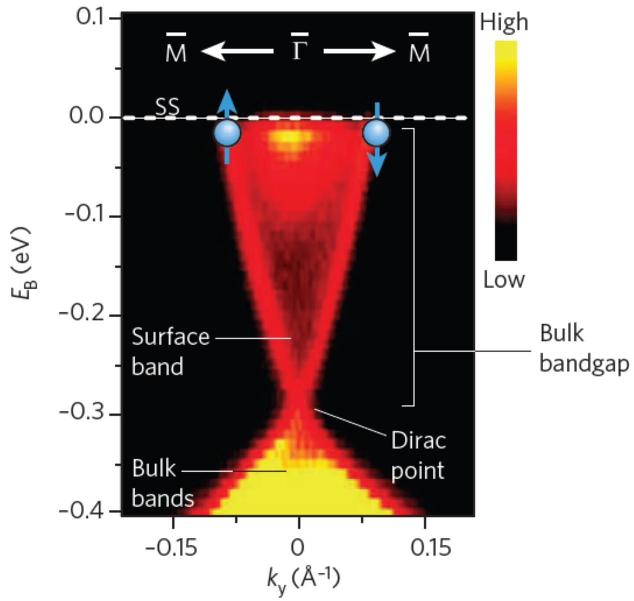
The surface states in the gap:
1. The Fermi surface encloses an odd number of Dirac cones;
2. The Fermi surface has a single spin state at each momentum, the lock-in relation of electron momentum and spin;
3. The Berry phase arround the Fermi surface is π.



Zhang et al, Nat. Phys (2009)



Hsieh et al, Nature (2009)



Xia et al, Nat. Phys, (2009)

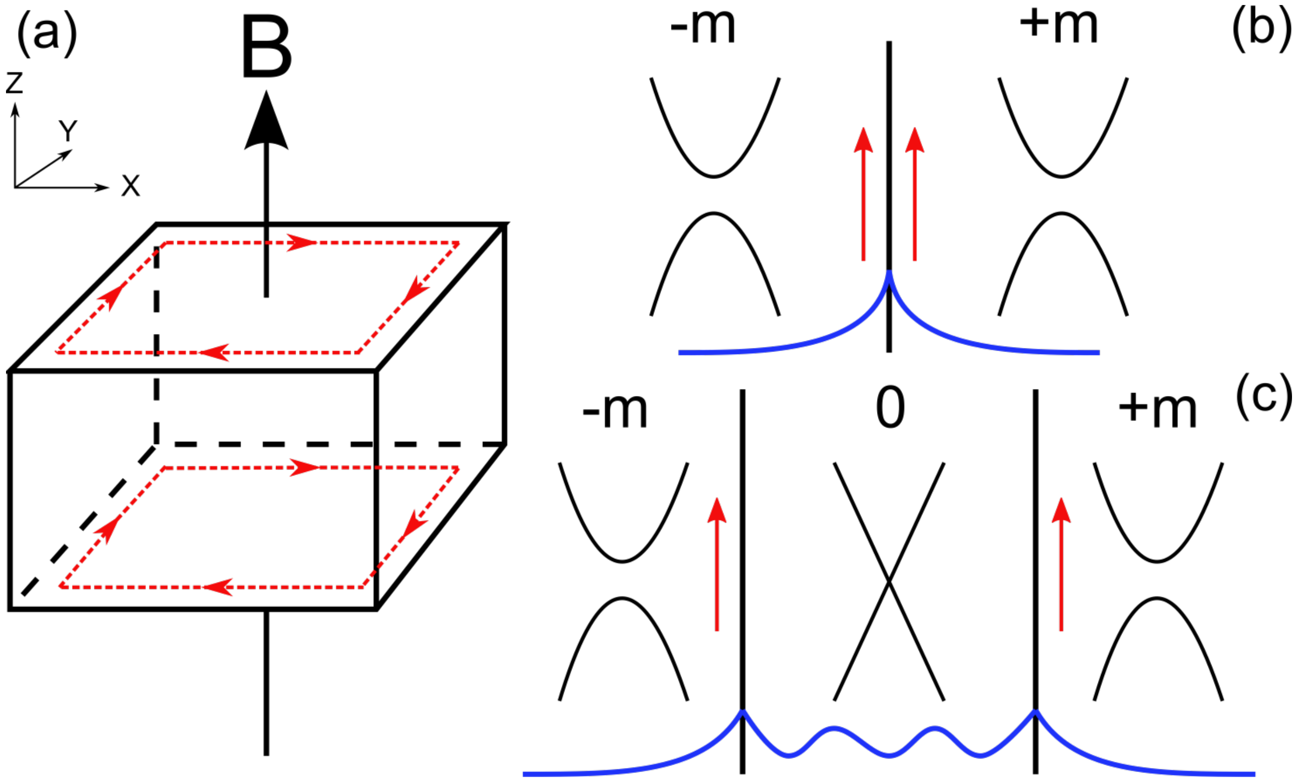
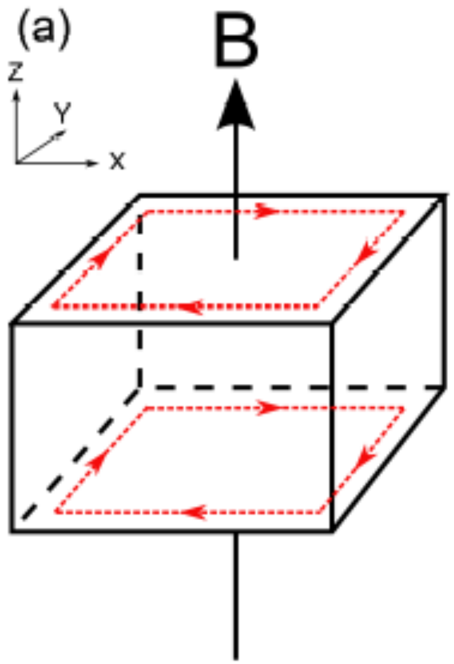
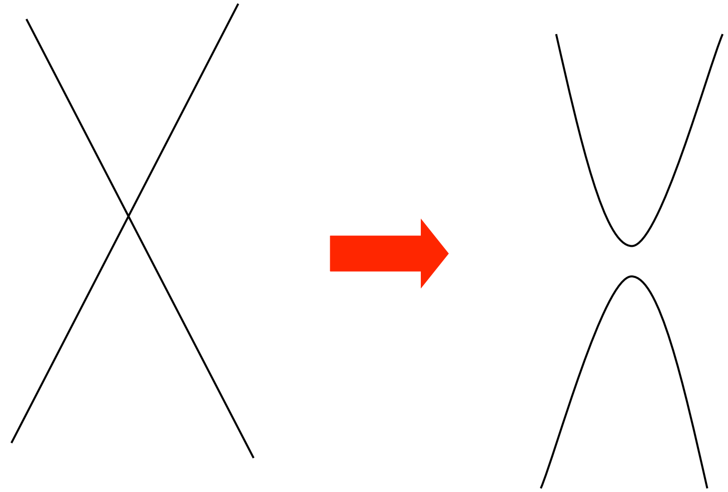


FIG. 1: (Color online) (a) Schematic of a 3D TI in a weak magnetic field, and the formation of chiral current on the top and bottom surface boundaries. (b) A bound state at the interface of 2D Dirac fermions with positive and negative masses, whose wave function is illustrated. The arrow indicates the flow of edge current. (c) Splitting of the bound state separated by massless Dirac fermions in the side surface, which wave function is illustrated.

Effective Model for Surface States with the Zeeman Splitting

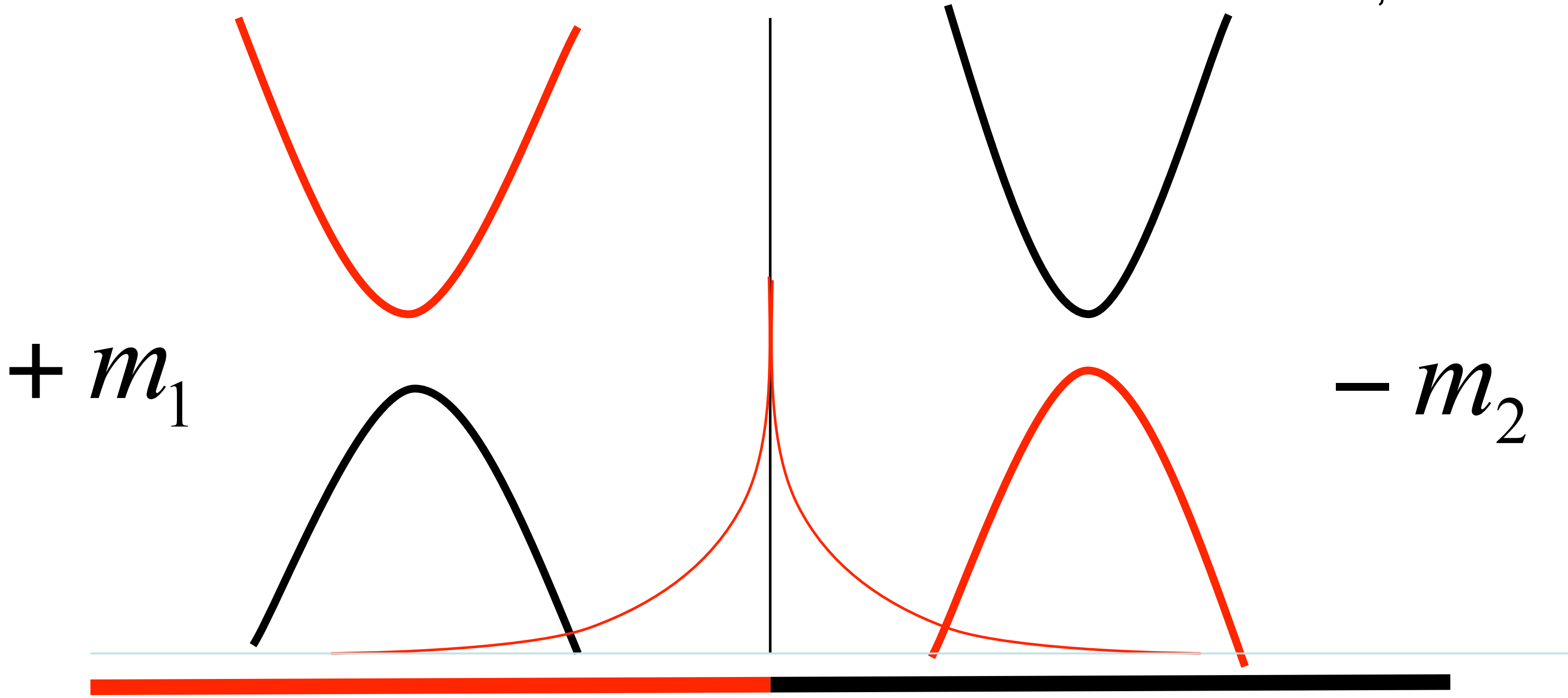




$$H_{\text{eff}}(\boldsymbol{k}) = v\,(\boldsymbol{k} \times \boldsymbol{\sigma}) \cdot \boldsymbol{n} - g_{\parallel}\mu_B h_{\parallel}\sigma_{\parallel} - g_{\perp}\mu_B \boldsymbol{h}_{\perp} \cdot \boldsymbol{\sigma}_{\perp}.$$

1D example:
Zero Energy Bound State:

Jackiw & Rebbi, PRD 13, 3398(76)

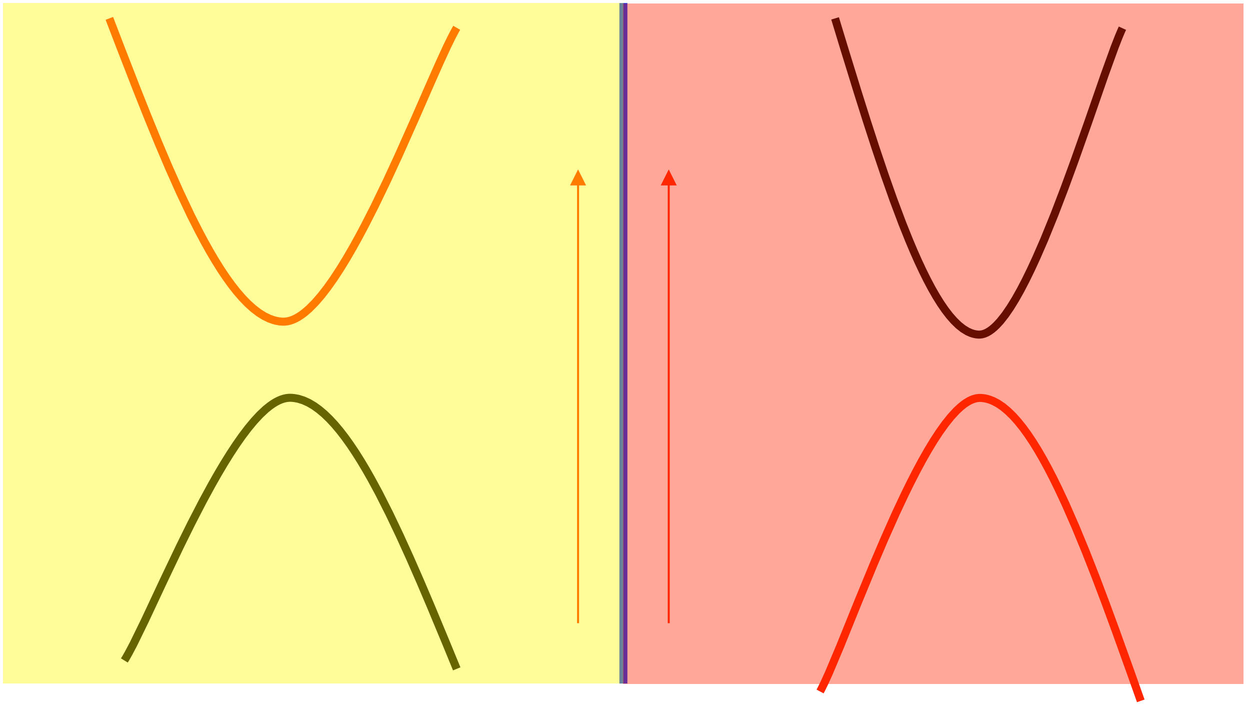


$$H = cp_x\sigma_x + m_1c^2\sigma_z \mid H = cp_x\sigma_x - m_2c^2\sigma_z$$

Zero-energy mode: $\Psi(x) = \sqrt{\frac{c}{\hbar}\frac{m_1m_2}{m_1+m_2}}\begin{pmatrix} i \\ 1 \end{pmatrix} e^{-m_i c|x|/\hbar}$

2D example: Chiral Bound State carrying 1 e^2/h conductance

$+m_1$



$-m_2$

$$H = p\cdot\sigma + m_1\sigma_z \,|\, H = p\cdot\sigma - m_2\sigma_z$$

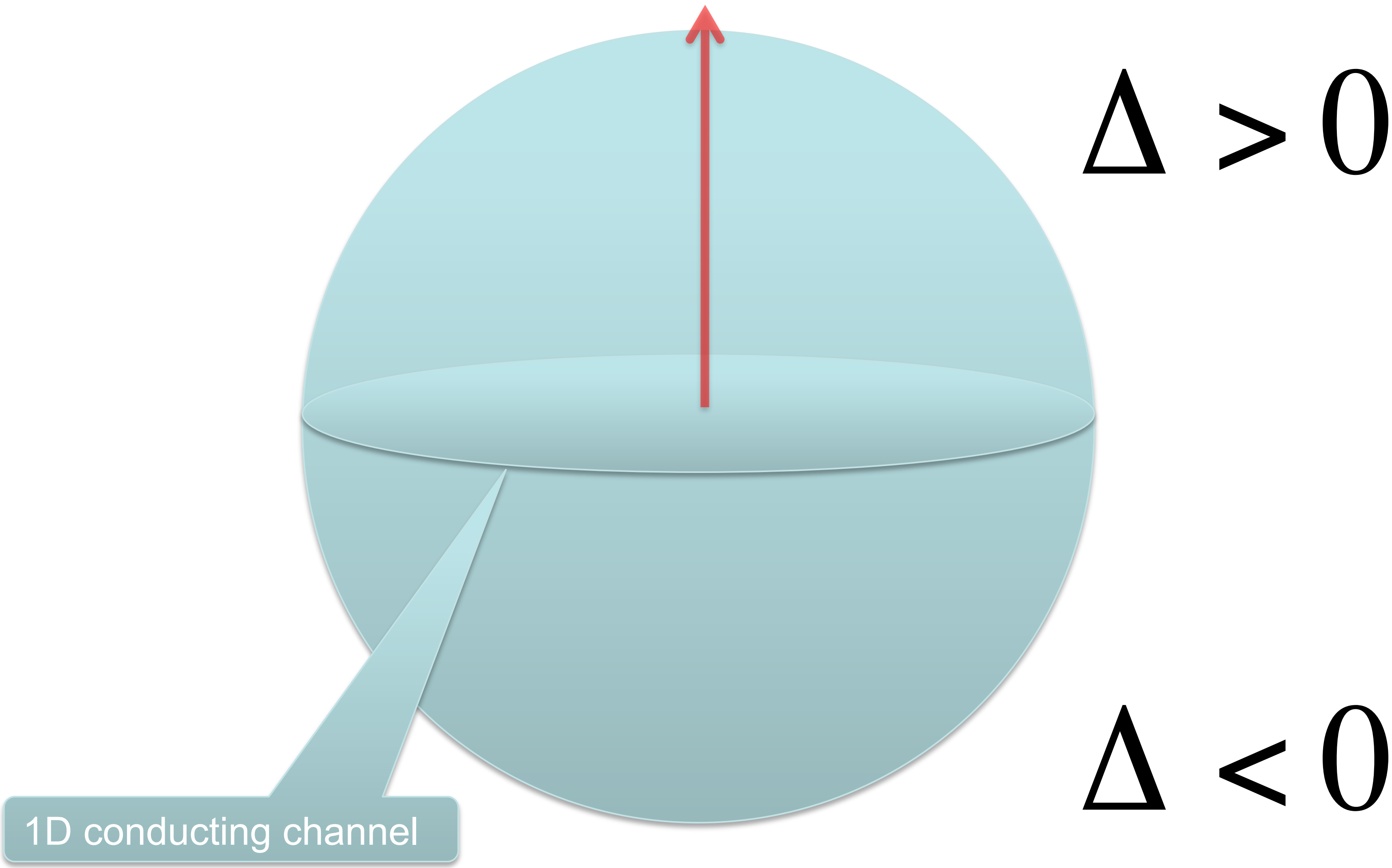
Chiral mode:

$$\Psi(x,y) \propto \begin{pmatrix} i \\ 1 \end{pmatrix} e^{-(\lambda|x| + ik_y y)/\hbar}$$

Dispersion:

$$E = v\hbar k_y$$

A Topological Insulating Ball in a Zeeman Field



1D Quantum Conductance

The current for a quantum wire

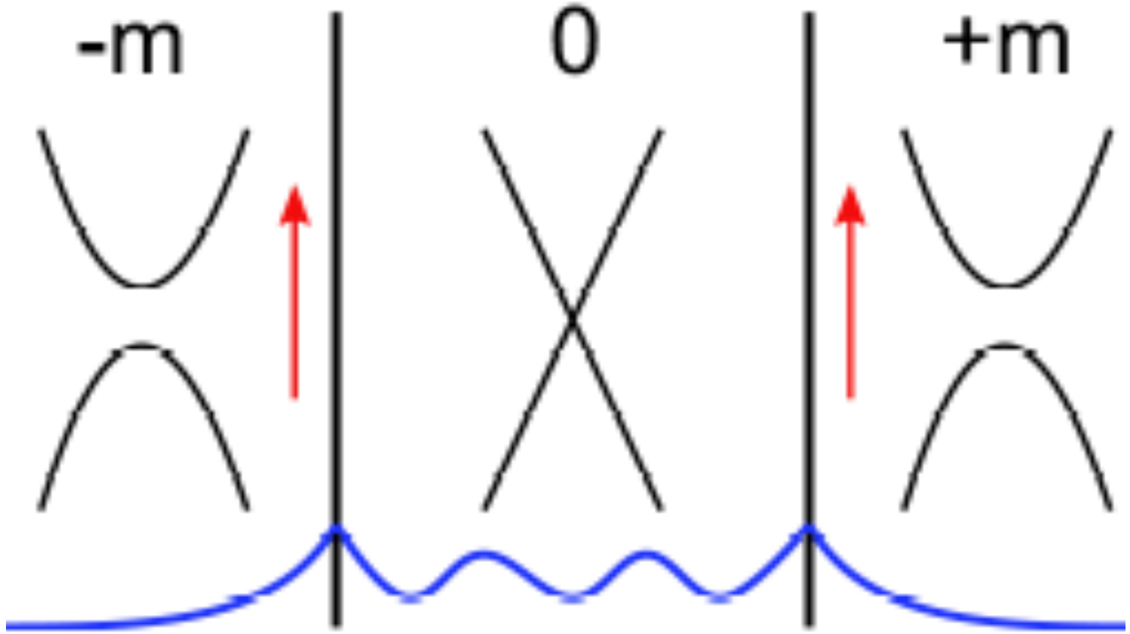
$$I = evD(\mu_L - \mu_R)$$

v: the effective velocity
D: the density of the states
μ-μ: the potential difference between two ends

$$v = \frac{\partial E(k)}{\hbar \partial k}; D = \frac{\delta n}{\delta E} = \frac{\partial k / 2\pi}{\partial E(k)}; \mu_L - \mu_R = eV$$

$$G = I / V = e^2 / h$$

Separated by conducting area



3D Model for TI

$$H = v_F \hbar k \cdot \alpha + (M - Bk^2)\beta$$

$$\alpha_i = \sigma_x \otimes \sigma_i ; \beta = \sigma_z \otimes \sigma_0$$

The Zeeman field: $\Delta_z \sigma_z \otimes \sigma_0$

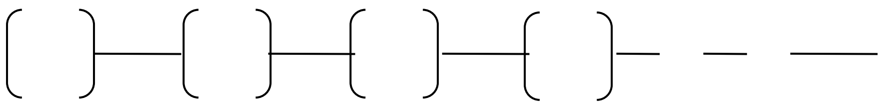
Take a tight binding model by mapping the continuous model on a cubic lattice.

$$k_i \leftarrow \frac{1}{a} \sin k_i a$$

$$k_i^2 \leftarrow \frac{4}{a^2} \sin^2 \frac{k_i a}{2} = \frac{2}{a^2}\left(1 - \cos k_i a\right)$$

Calculation of surface Green's function using the transfer matrix

The system is finite or semi-infinite.



G_{00}

H_{00} and H_{01} are matrix elements of the Hamiltonian between layer Bloch states

$$G_{00}(\omega) = (\omega - H_{00} - H_{01}\, T(\omega))^{-1}$$

$$T(\omega) = (\omega - H_{00} - H_{01}\, T(\omega))^{-1} H_{01}^{+}$$

Transfer matrix T is calculated iteratively until self-consistency is achieved

Lopez Sancho and Rubio, Phys. F: Met. Phys. 14, 1205(84).

Surface local density of states is given by

$$\rho(k) = -\frac{1}{2} Tr[\operatorname{Im} G_{00}(k)]$$

Current Density

$$\langle j_x(y) \rangle = ie \int_{k_x} Tr(v_x(y,k_x) G^{<}(y,k_x)$$

Surface States
In a Zeeman Field

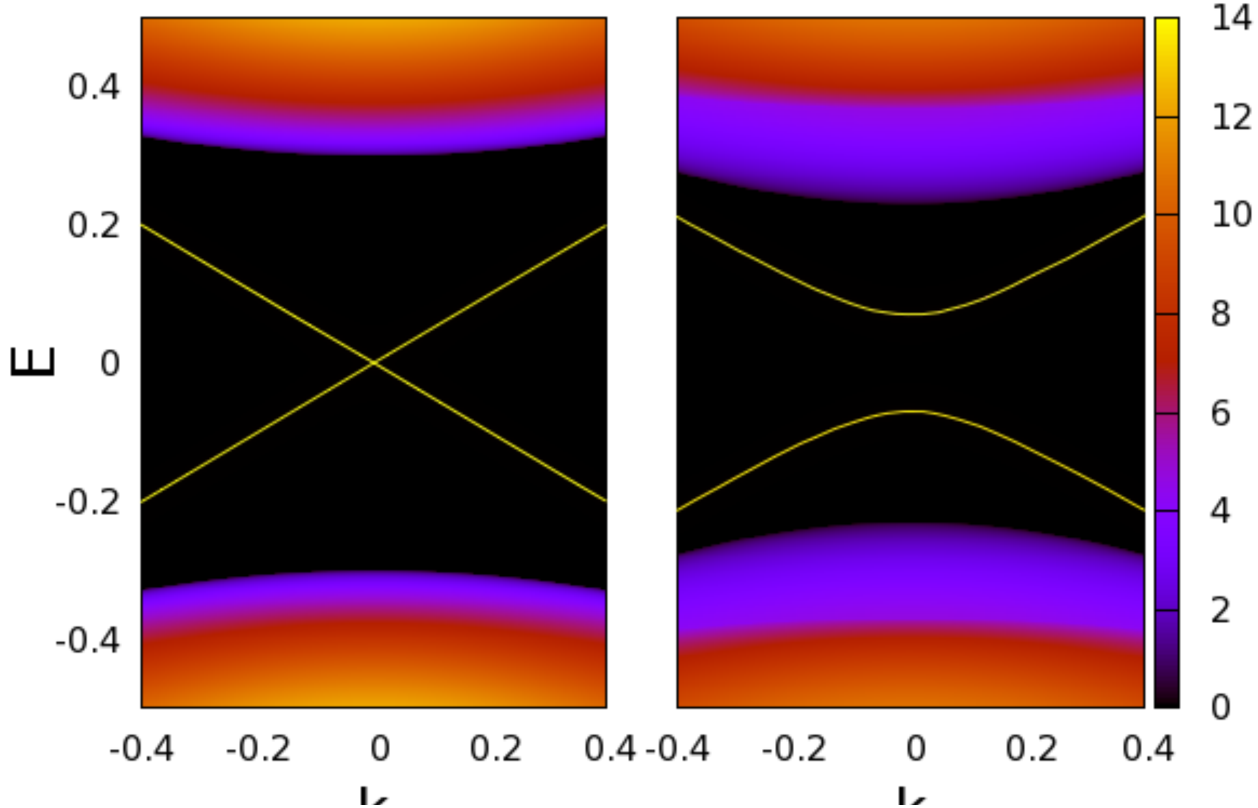


FIG. 2: (Color online) Local density of state on an infinite xy surface of a semi-infinite 3D system. (left) gapless single Dirac cone of the surface state; (right) gap opening by application of a Zeeman splitting term. The model parameters are $A = 0.5$, $B = 0.25$, $M = 0.3$, and $\Delta_z = 0.07$.

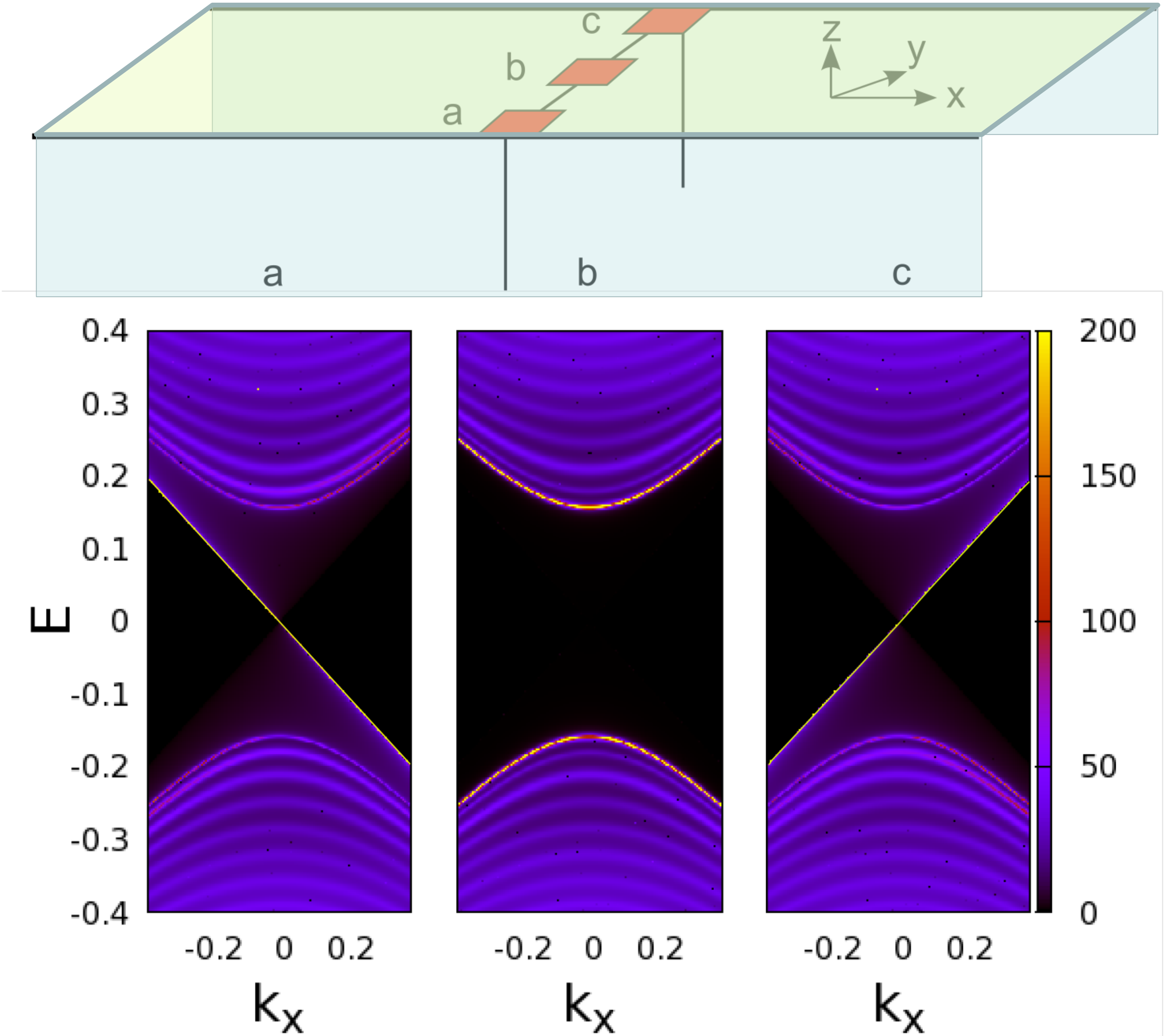
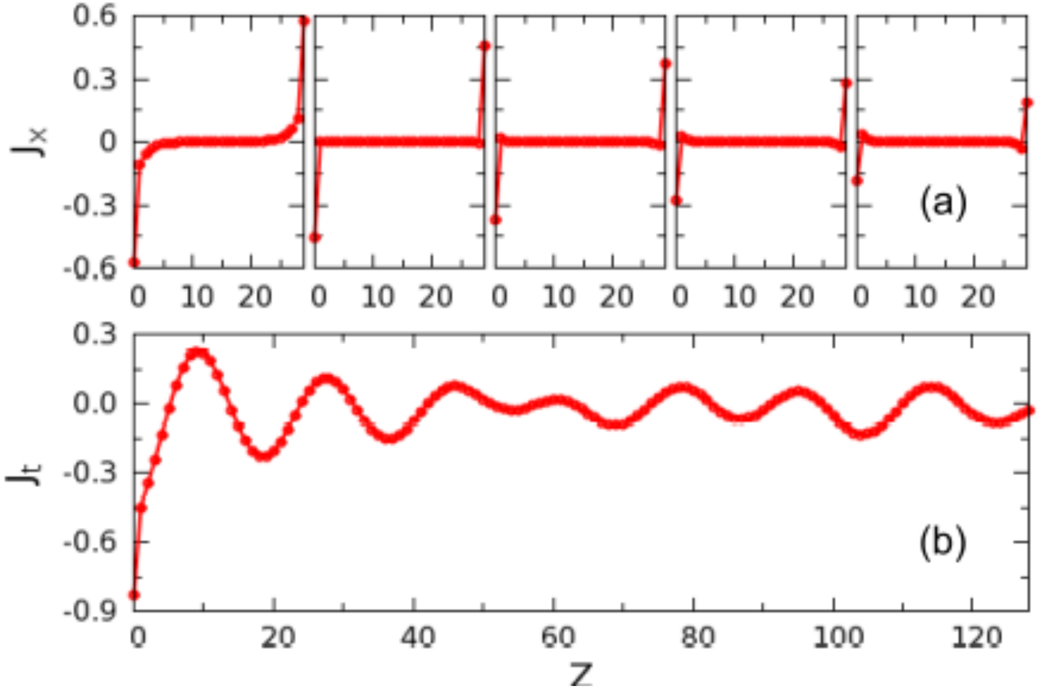


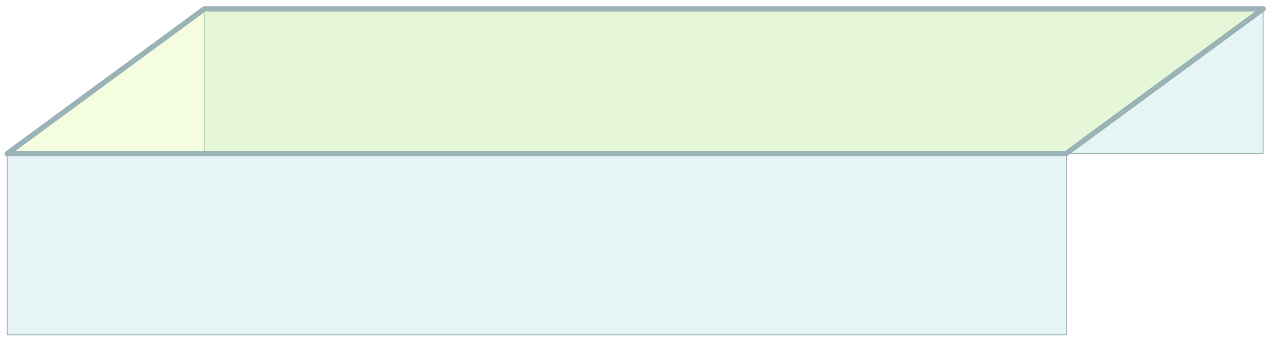
FIG. 3: (Color online) LDOS on the top surface of a structure that is infinite in X, finite in Y and semi-infinite in Z direction. Sampling is taken correspondingly in a, b and c regions as illustrated in the upper panel. $\Delta_z = 0.15$, $M = 0.4$, and $L_y = 30a$ (a is the lattice space).

Current Distribution at the Top and Side Surfaces

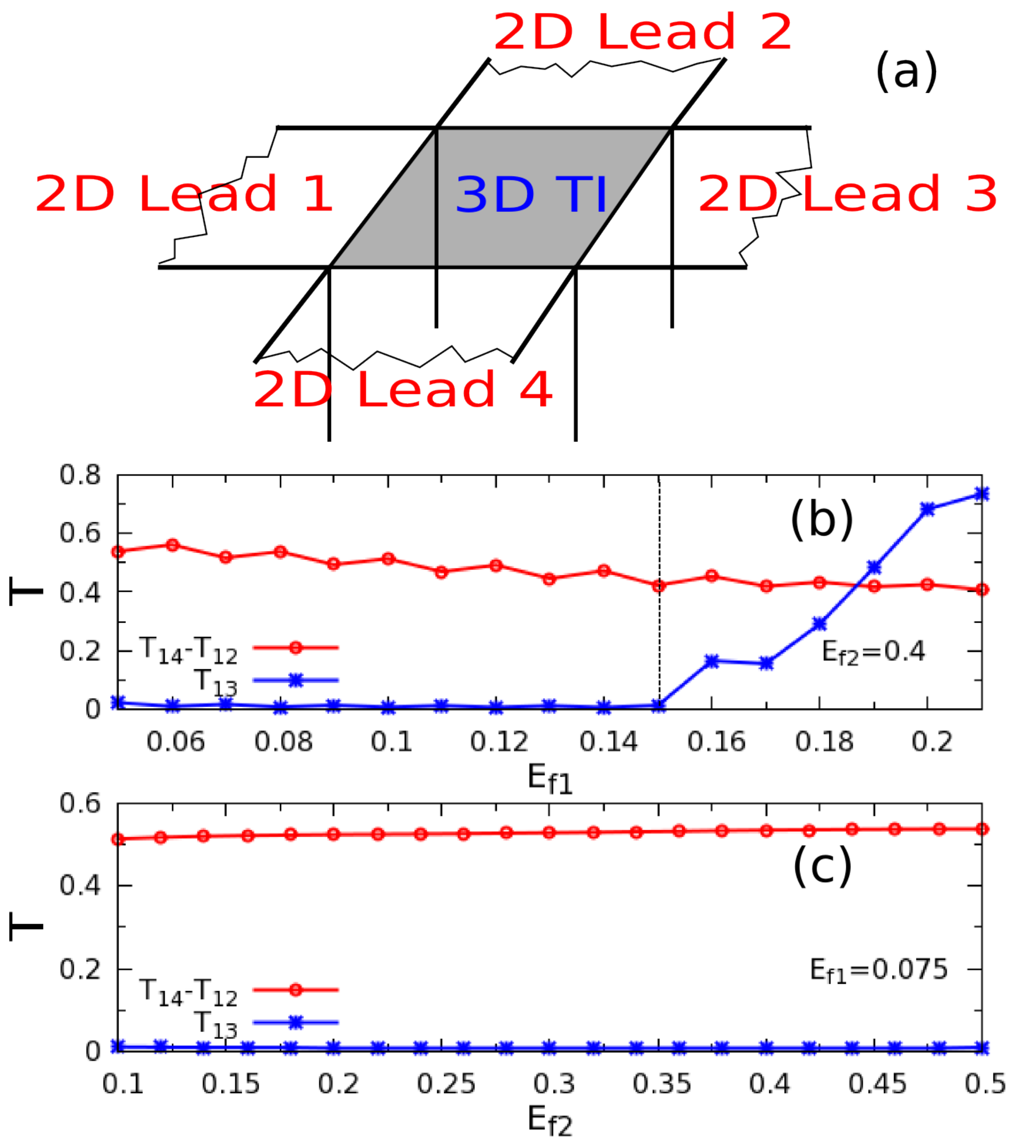


Current density distribution in several top layers of lattices

Current density distribution in the side surface

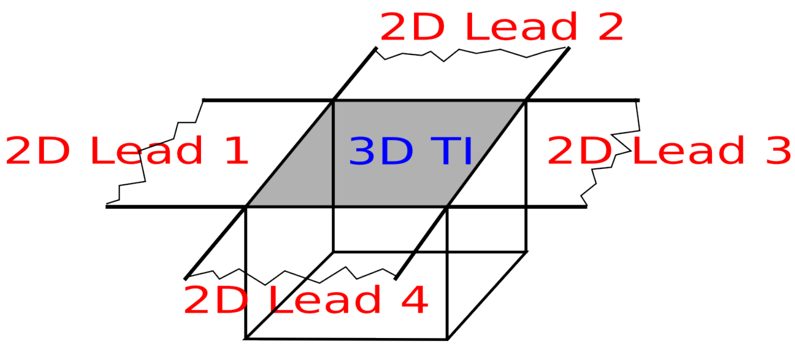


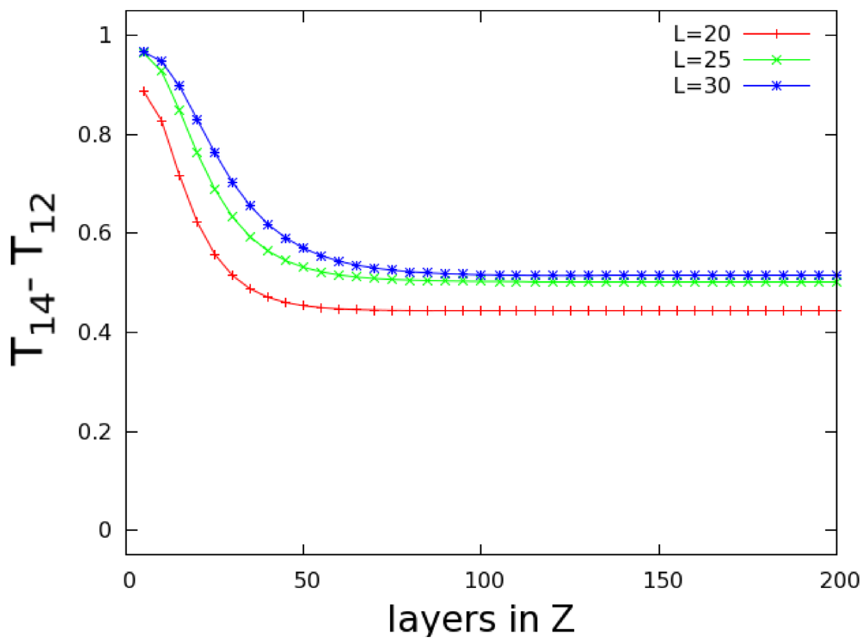
Transmission Coefficients in a Four-Terminal Device



(a) Schematic illustration of the 3D device with 2D semi-infinite metallic leads, the sample is semi-infinite in Z direction, the top surface size is 30×30; (b) Transmission coefficients of the 4-terminal device, E_{f2} is fixed, the dashed line indicates the gap position; (c) E_{f1} is fixed. $\Delta z = 0.15$, $M = 0.4$.

Thickness Dependence of the Difference





(upper) Schematic illustration of the 3D device with 2D semi-infinite metallic leads, the sample has finite thickness in Z direction, the top surface size is L×L; (lower) Transmission coefficients $T_{14} - T_{12}$ of the 4-terminal device as a function of the sample thickness in Z. $\Delta z = 0.15$, $M = 0.4$, $E_{f1} = 0.001$, $E_{f2} = 0.04$.

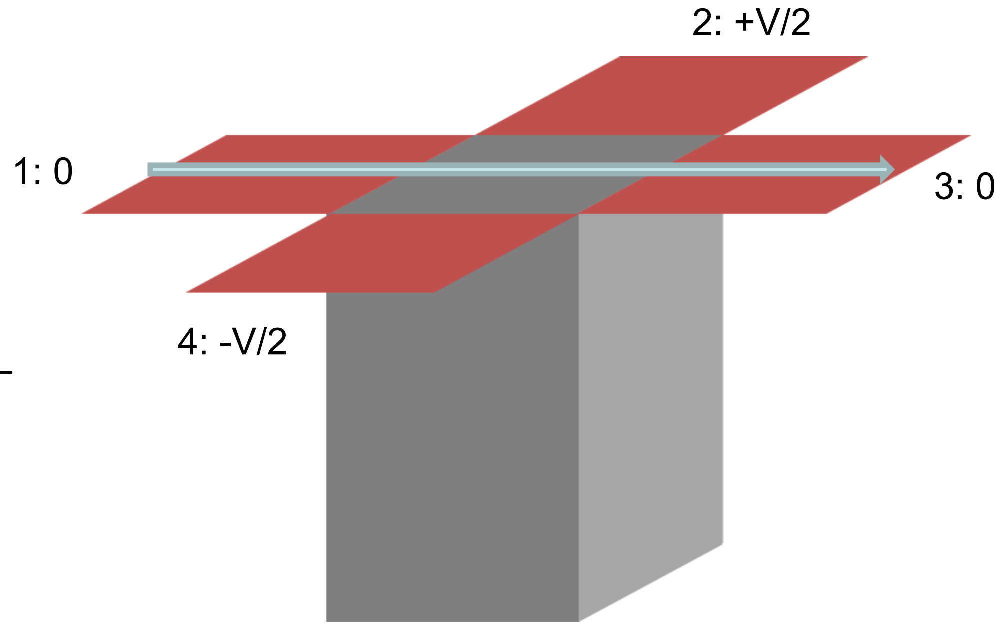
Measurement

The Landauer-Buttiker formula

$$I_i = \sum_{j=1}^{4} T_{ij} V_j - \sum_{j=1}^{4} T_{ji} V_i$$

The Hall conductance

$$G_c = I_{13} / V_{24} = \frac{1}{2}\left(T_{12} - T_{14}\right)\frac{e^2}{h}$$



Half Quantized?

IQHE: chiral edge state (n=1)

$$T_{12} = T_{23} = T_{34} = T_{41} = 1$$

$$T_{21} = T_{32} = T_{43} = T_{14} = 0$$

$$T_{13} = T_{31} = T_{24} = T_{42} = 0$$

$$T_{12} - T_{41} = 1$$

$$G_c = \frac{1}{2}\left(T_{12} - T_{14}\right)\frac{e^2}{h} = \frac{1}{2}\frac{e^2}{h}$$

Surface states in Zeeman field: surface-edge state

$$T_{12}, T_{23}, T_{34}, T_{41} \approx (? =)\delta + 1/2$$

$$T_{21}, T_{32}, T_{43}, T_{14} \approx \delta$$

$$T_{13}, T_{31}, T_{24}, T_{42} \approx 0$$

$$T_{12} - T_{14} \approx 1/2$$

$$G_c = \frac{1}{2}\left(T_{12} - T_{14}\right)\frac{e^2}{h} \approx \frac{1}{4}\frac{e^2}{h}$$

In this setup, the Hall conductance is ¼ for the surface states of 3D topological insulators. It is ½ for IQHE.

Q: Can we measure one half-quantized Hall conductance?

A: Yes, the surface-edge state carries one half of quantum conductance.
But……..

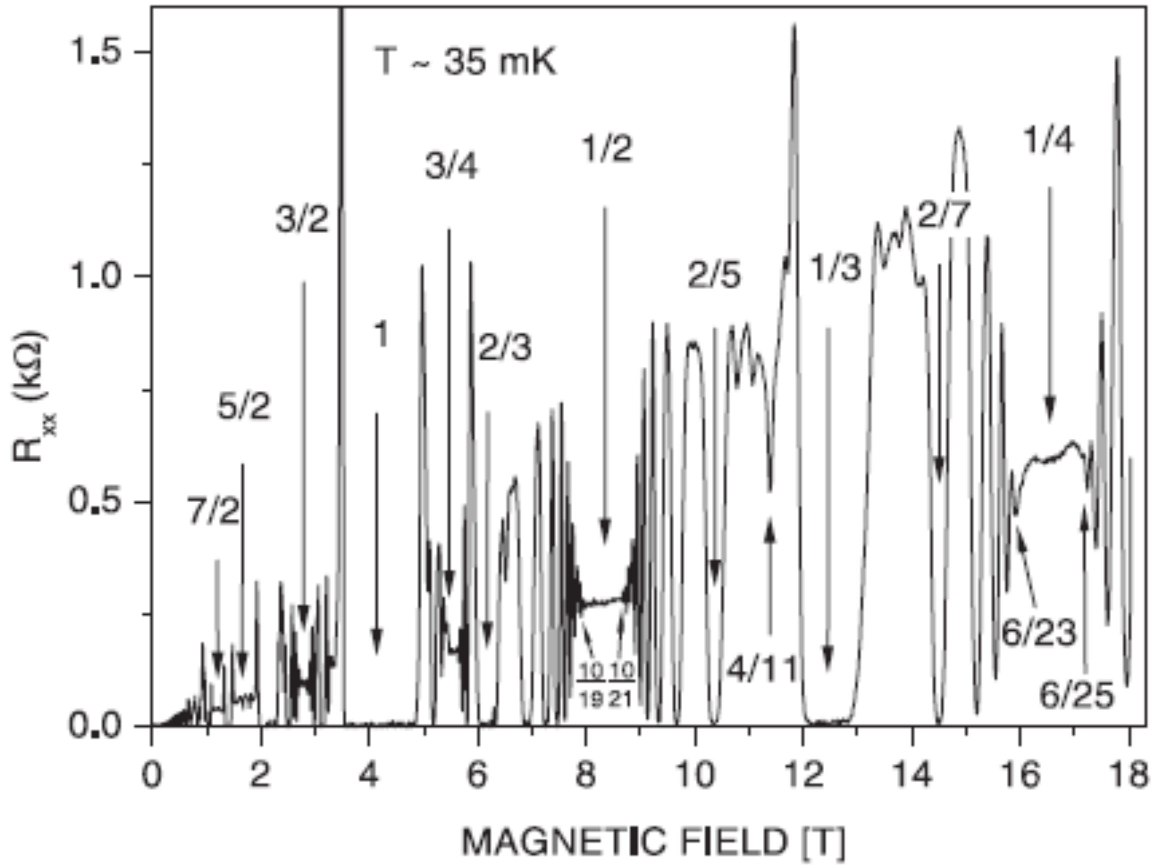


Fig. 1.3. The magnificent FQHE skyline. Diagonal resistance as a function of the magnetic field for a two-dimensional electron system with a mobility of 10 million cm^2/V s. A FQHE or an IQHE state is associated with each minimum. Many arrows only indicate the positions of filling factors (for example, 1/2, 1/4, etc.) and have no FQHE associated with them. Source: W. Pan, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, *Phys. Rev. Lett.* **88**, 176802 (2002). (Reprinted with permission.)

Fractional Quantum Hall Effect

Fractional Charge, Composite Fermions and Edge States

Tsui et al, 82; Laughlin, 82; Jain, 88

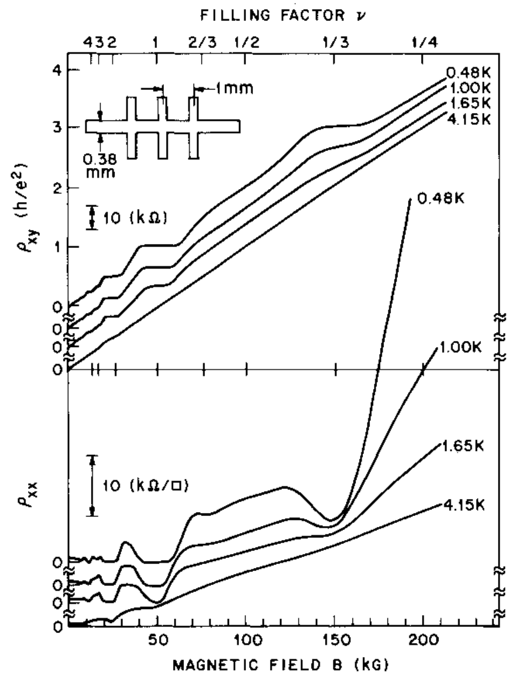


FIG. 1. ρ_{xy} and ρ_{xx} vs B, taken from a GaAs-Al$_{0.3}$-Ga$_{0.7}$As sample with $n = 1.23\times10^{11}/\mathrm{cm}^2$, $\mu = 90\,000\ \mathrm{cm}^2/\mathrm{V\,sec}$, using $I = 1\ \mu\mathrm{A}$. The Landau level filling factor is defined by $\nu = nh/eB$.

D.C. Tsui, H.L. Stormer, and A.C. Gossard, Two-Dimensional Magnetotransport in the Extreme Quantum Limit

Phys. Rev. Lett. **48**, 1559 (1982)

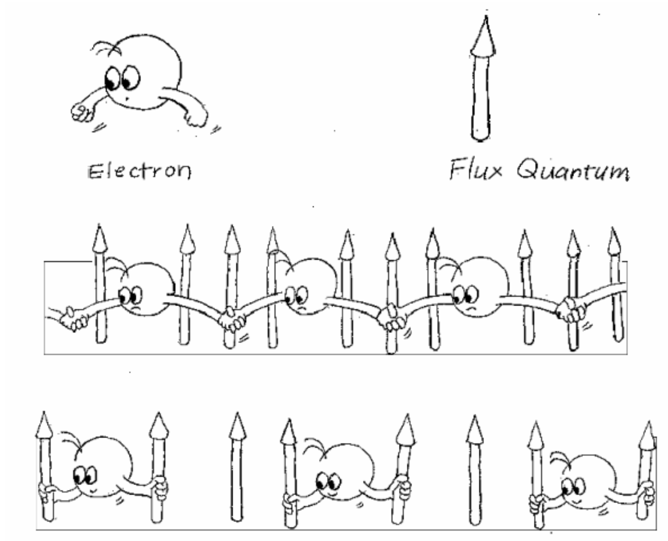
$$\rho_{xy} = -\frac{h}{\nu e^2} \qquad (\nu = \frac{1}{3}, \frac{2}{5}, \frac{3}{7} etc)$$

Laughlin wavefunction (1982)

$$\left[\prod_{N\geqslant i\geqslant j\geqslant 1} (z_i - z_j)^n\right] \prod_{k=1}^{N} \psi(z_k)$$

Fractional charge: e/n

Jain's composite fermion theory (1988)



METAMORPHOSIS OF INTERACTING ELECTRONS INTO FREE COMPOSITE FERMIONS