Fractional topological insulators

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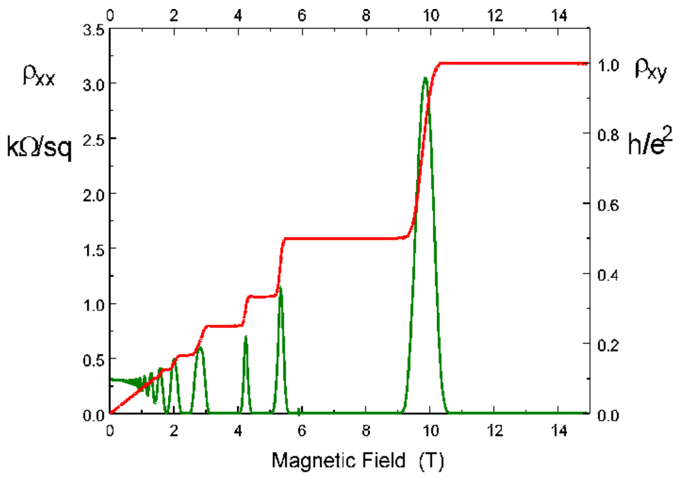
Outline:

1. The context, the goal and the results
2. Construction of an exactly solvable model for a fractional topological insulator
3. The properties of the fractional topological insulator, particularly the stability of its gapless edge modes

The context: Topological phases of matter

The quantum Hall effect



- The Hall conductivity as a topological quantum number $\sigma_{xy} = \frac{e^2}{h}\nu$
- Protected as long as the bulk energy gap does not close
- Gapless modes at the edges

Interactions enable the Fractional Quantum Hall Effect – fractionalized charges, anyonic statistics, topological ground state degeneracy etc. etc.

Quantized spin Hall effect - a generalization of the QHE to systems that are symmetric under time reversal

A useful toy model – two copies of the IQHE

Red electrons experience a magnetic field B

Green electrons experience a magnetic field -B

ν=odd per each color – a topological insulator

Gapless edge protected by time reversal symmetry

ν=even per each color – a trivial insulator, no protection

These are examples of 2D non-interacting topological/trivial insulators.

Will interactions introduce new fractionalized states?

Three dimensional (3D) non-interacting topological insulators

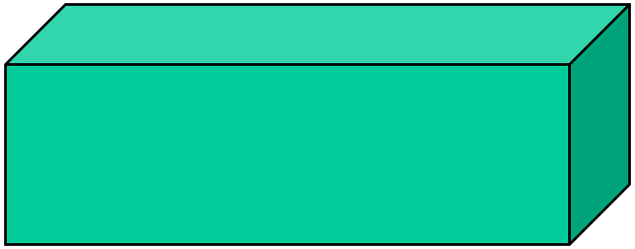
• Gapless edge modes are now Dirac cones on each of the surfaces

• The Dirac cone cannot be gapped without breaking of time reversal symmetry.

• When time reversal symmetry is broken on the surface, a quantum Hall state forms, with half-integer Hall conductivity.

• When a finite thin solenoid is inserted into the 3D bulk, charges of $\pm e/2$ are bound to its ends (“charge-monopole binding”).

Will interactions induce new fractionalized states (FQHE on the surface, “fractional charge-monopole binding” ?)

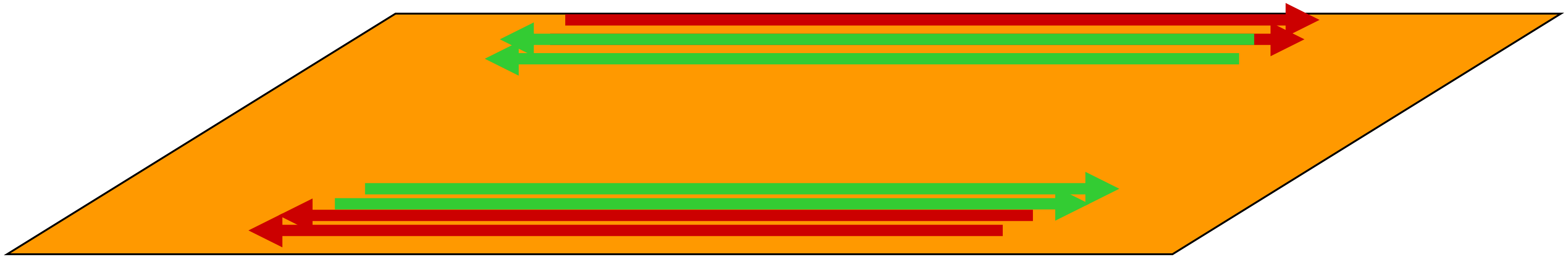


The answer: Interactions may give rise to fractionalized topological insulators

The first example (Bernevig et al)

Red electrons at a fraction ν

Green electrons at a fraction $-\nu$



The question – can the edge states be gapped out without breaking time reversal symmetry ?

The answer is determined by the parity of ν/e^*:

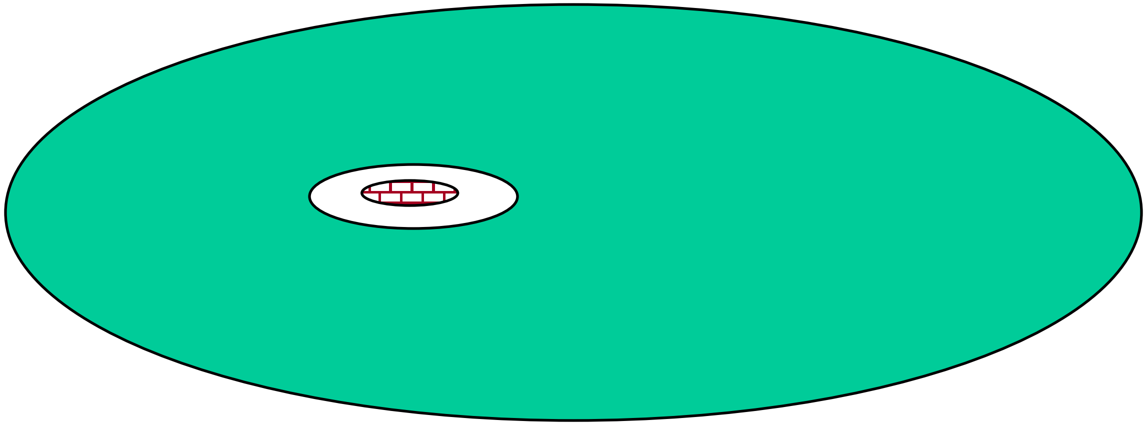
Even –Yes

Odd - No (Levin & Stern)

Not directly generalizable to three dimensions

Creating fractional charges:

1. The quantum Hall example – by inserting flux and using the Hall conductance



2. The quantum spin Hall effect – a simple modification

But – 3D ?

A different route is needed.

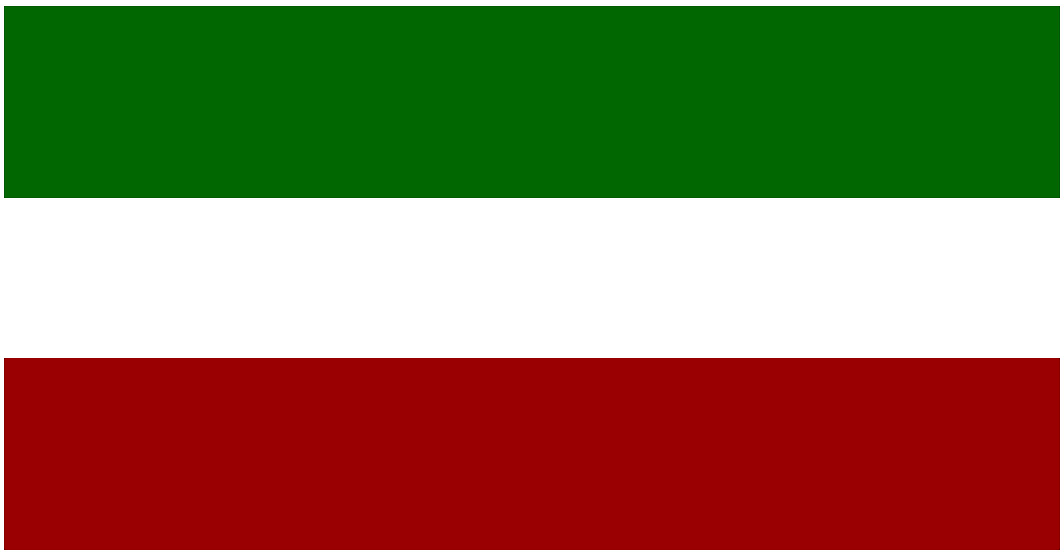
Our method:

• constructing an Hamiltonian of electrons on a lattice, where the low energy degrees of freedom are non-interacting fermions of fractional charge q_F.

• The fermions may be put in a metallic state, a quantum Hall state, a topological insulator state.

• At high energy, there are bosonic excitations of two types, carrying charge and flux. There is a mutual fractional statistics between the two. The charge is e^*. The ratio q_F / e^* is an integer. The bosonic spectrum is gapped, and the gap does not close at the edges.

Spectrum:



bosonic spectrum
Charge 2e
Fractional excitations

Fermionic spectrum -
Bloch band of fermions
of Charge q_F

The fermions can be put into metallic states (partially filled band), fractional quantum Hall states (fractionally charged but no low energy fractional statistics).

Our focus – topological insulator states.

Properties of the fractional topological insulator:

2D ($\nu=\pm 1$ for the two spin directions):

1. Quantum spin Hall effect with e replaced by q_F.

2. The spectrum of an annulus is periodic with respect to an Aharonov-Bohm flux, with a period that is consistent with the Byers-Yang theorem for an electron charge

3. Edge modes are stable for q_F / e^* odd, unstable for q_F / e^* even (same criterion as in the earlier example).

Properties of the fractional topological insulator – three dimensions:

1. Bulk is gapped, edge has a Dirac cone of fractionally charged fermions

2. In a weak magnetic field, the surface exhibits fractional quantum Hall effect, with e replaced by q_F

3. Charge-monopole binding:
   a. A monopole-antimonopole pair of Dirac monopoles is "confined"
   b. An unconfined pair requires monopoles larger by a factor of e/e^*.
   c. For those monopoles, the bound charge is

      $e^* / 2$ when q_F/e^* is odd.

      zero when q_F/e^* is even.

Properties of the fractional topological insulator – three dimensions (cont.):

4. Surface modes are stable for q_F / e^* is odd.

5. We do not know whether the modes are stable when q_F / e^* is even.

The ratio q_F / e^* is significant also in 3D.

How is the model constructed?

1. Constructing a model Hamiltonian with fractionally charged excitations that are bosons

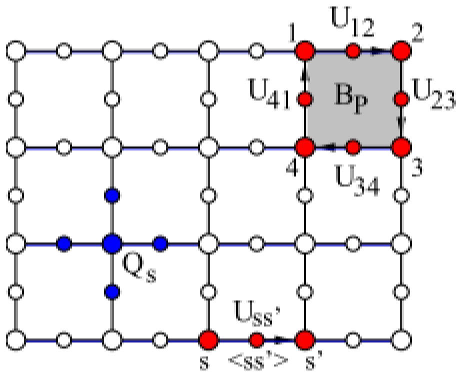
Senthil & Motrunich

Kitaev's toric code

2. Creating fermions that are fractionally charged by gluing an electron to fractionally charged bosons

3. Creating a topological insulator out of these fractionally charged fermions

Step 1: The bosonic part



The building blocks:

1. Square lattice
2. Bosons of charge 2e (pairs of electrons) may live on lattice sites *s* and on link sites *ss'*
3. The Hamiltonian includes a charging term and a hopping term

$$H = V_1 \sum Q_s^2 - V_2 \sum \left(B_p + B_p^+ \right)$$

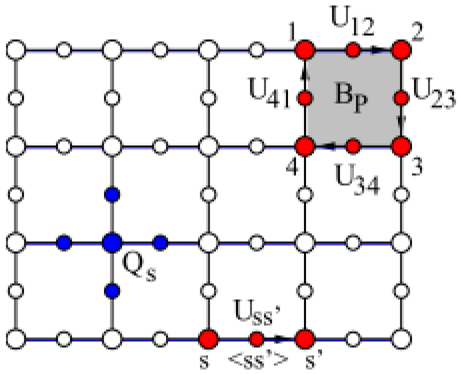
4. Exact solvability requires
   Number of commuting terms = number of degrees of freedom

The paradigm of a bosonic Hamiltonian is

$$H = E_c \sum_{sites} n_s^2 - E_J \sum_{n.n.sites} \cos\left(\theta_i - \theta_j\right)$$

Too many terms!

Instead:



$$H = V_1 \sum_{sites} Q_s^2 - V_2 \sum_{plaquettes} \left(B_p + B_p^+\right)$$

Charging terms only on sites, not on links!

$$Q_s = mn_s + \alpha_s \sum_{s'} n_{ss'}$$

Ground state – all Q_s 's are zero

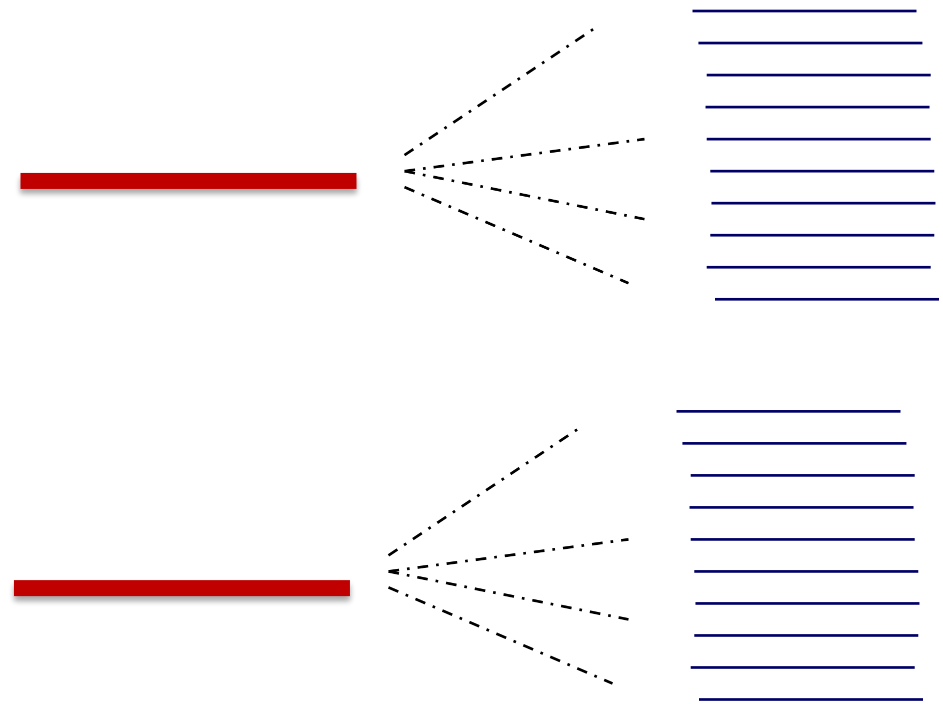
Fractional charge:
A fractional expectation value for n_s when $q_s = 1$

Fixing all Q_s does not fix all charges. The spectrum consists of highly degenerate subspaces.

The hopping term should split the degeneracy of the ground state subspace of the charging term, without mixing states from different subspaces.
And – it should have a discrete spectrum.

all Q_s =0 except one

all Q_s =0



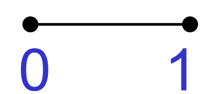
$$H = V_1 \sum_{sites} Q_s^2 - V_2 \sum_{plaquettes} \left(B_p + B_p^+ \right)$$

To get a discrete spectrum:

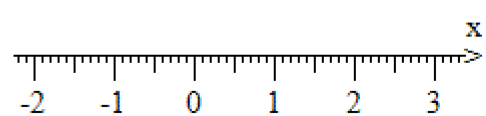
1. B_p should be unitary
2. It should satisfy $(B_p)^m = 1$ eigenvalues $\exp\left(i \frac{2\pi j}{m} \right)$

Hopping terms we know and love:

1. Fermion hopping: $c_i^+ c_j$ - vanishes when applied twice



2. Boson hopping $b_i^+ b_j \Rightarrow \exp\left[i\left(\theta_i - \theta_j \right) \right]$



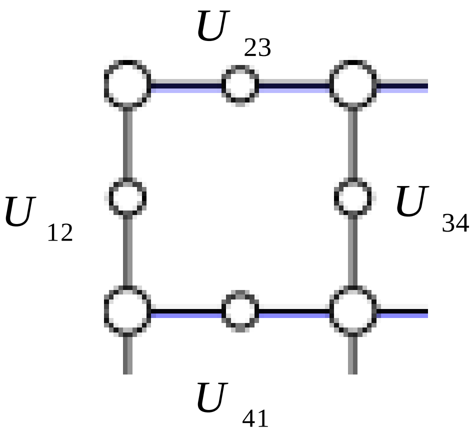
None of these gives a discrete spectrum

Our solution:

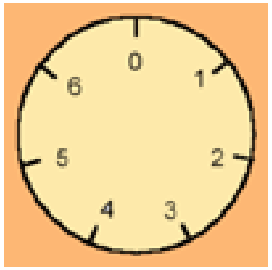
Two basic properties:

1. The hoppoing involves an entire plaquette

$$B_p = U_{12} U_{23} U_{34} U_{41}$$



$U_{ss'}$ changes q_s by +1 and $q_{s'}$ by -1

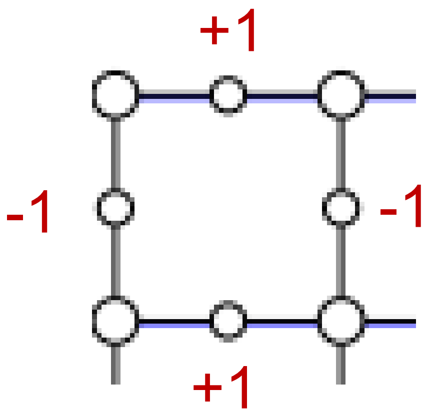


2. For the link variables –

- Only m possible states
- Hopping is ±1 on the “modulo” scale.

2. For the site variables – all states are allowed.

3. The site variables supply/absorb bosons to guarantee conservation of charge.



$$B_p = U_{12} U_{23} U_{34} U_{41}$$

m=2: $U_{ss'} = b_s^+ b_{ss'} + b_{ss'}^+ b_{s'}$

First consequence of the modular structure:

$(B_p)^m = 1$ discrete spectrum $\exp\left[i 2\pi j / m\right]$

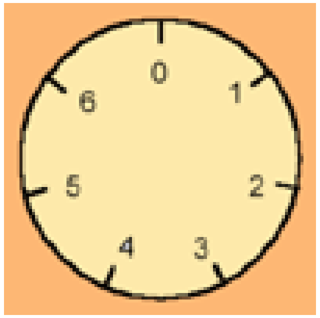
The Hamiltonian: $$H = V_1 \sum_{sites} Q_s^2 - V_2 \sum_{plaquettes} \left(B_p + B_p^+\right)$$

has the spectrum:

$$E\left(q_s, j_p\right) = V_1 \sum_{sites} q_s^2 - 2V_2 \sum_{plaquettes} \cos\left(2\pi \frac{j_p}{m}\right)$$

- q_s , j_p are integers
- The spectrum is gapped. Ground state has $q_s = j_p = 0$.
- Charge excitations $q_s = 1$. Flux excitations $j_p = 1$.

Second consequence of the modularity of $n_{ss'}$: all allowed values are equally probable, for all eigenstates (Z_m symmetry).



$\langle n_{ss'} \rangle$ is the same for all values of q_s , j_p

$$q_s = mn_s + \sum_{s'} n_{ss'}$$

A change of 1 in q_s changes the expectation value of n_s by 1/m

An excitation that carries a fractional charge 2e/m (both 2D and 3D)

Coupling to a static magnetic field

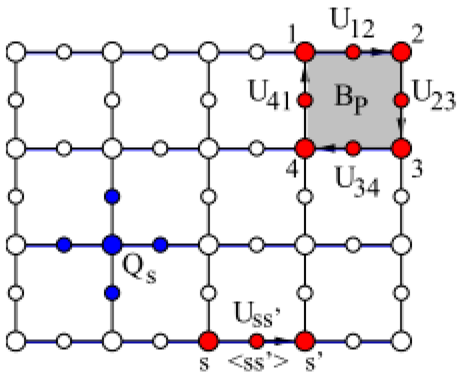
$$E(q_s, j_p) = V_1 \sum_{sites} q_s^2 - V_2 \sum_{plaquettes} \cos 2\pi \frac{\left(j_p + \frac{2e\Phi_p}{hc} \right)}{m}$$

- Exciting j_p amounts to a flux excitation of hc/2e flux quantum.
- When a charge of 2e/m goes around a vortex of h/2e, a phase of $2\pi/m$ is accumulated – fractional mutual statistics

- In three dimensions, flux particles become flux loops, that have fractional mutual statistics with the charge quasi-particles
- Ground state degeneracy on a torus, as needed. (Swingle et al., Maciejko et al.)
- No gapless edge modes (as in the toric code)

Step 2: Binding electrons to fractional bosonic charges

- Electrons live on lattice sites only
- Two spin directions
- They affect (for now) only the charging term



$$H = V_1 \sum_{sites} Q_s^2 \quad \Rightarrow \quad V_1 \sum_{sites} \left[Q_s - k \sum_{\sigma} n_{e,\sigma} \right]^2$$

The presence of an electron on a lattice site enforces k charge excitations on that same site, making the total charge $e\left(1 + \frac{2k}{m}\right)$

Step 3: making the composite object hop "in one piece"

If an electron hops "alone", a charging energy will be paid

$$V_1 \sum_{sites} \left[Q_s - k \sum_{\sigma} n_{e,\sigma} \right]^2$$

To avoid that energy cost, Q_s must be changed by ±k in the site from/to which the electron hops. The operator that "moves" a fractional charge *2e/m* from *s* to *s*' is $U_{ss'}$.

$$H_{hopping} \sim c^{+}_{i,\sigma} U^{k}_{ij} c^{+}_{j,\sigma'} \equiv d^{+}_{i,\sigma} d_{j,\sigma'}$$

So far, low energy fermionic part and a high energy fractionalized bosonic part.

Do the fermions have a fractional charge?

Couple to an electro-magnetic field by the Peirles substitution

$$c_i^+ c_j \to c_i^+ c_j e^{ie\int_i^j A \cdot dl} \qquad b_i^+ b_j \to b_i^+ b_j e^{i2e\int_i^j A \cdot dl}$$

The hopping part

$$H_{hopping} \sim d_{i,\sigma}^+ d_{j,\sigma'} e^{iq_F \int_i^j A \cdot dl} \qquad q_F = e\left(1 + \frac{2k}{m}\right)$$

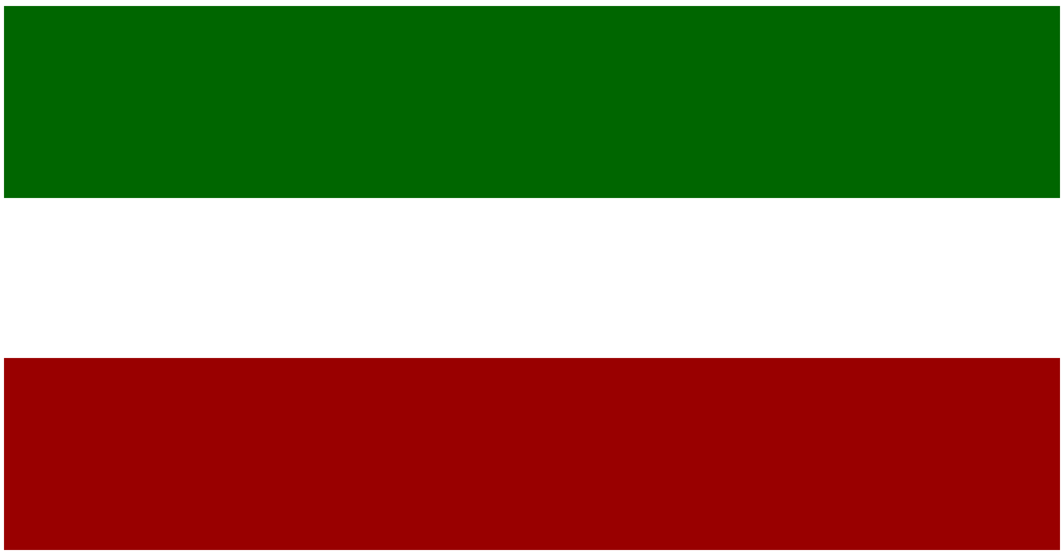
The bosonic part

$$E(q_s, j_p) = V_1 \sum_{sites} q_s^2 - V_2 \sum_{plaquettes} \cos\left(\frac{2\pi}{m}\left(j_p + \frac{2e\Phi_p}{hc}\right)\right)$$

The correct Byers-Yang periodicity is guaranteed by the bosonic part. Adding $\Phi_0 / 2$ and changing j by -1 brings the system back to the initial energy

$$E\left(q_s, j_p\right) = V_1 \sum_{sites} q_s^2 - V_2 \sum_{plaquettes} \cos\left(\frac{2\pi}{m}\left(j + \frac{2e\Phi}{\Phi_0}\right)\right)$$

So, we have



bosonic spectrum
Charge 2e

Fermionic spectrum -
Bloch band of fermions
of Charge q_F

The fermions can be put into metallic states (partially filled band), fractional quantum Hall states (fractionally charged but no low energy fractional statistics).

Our focus – topological insulator states.

2D: Quantum Spin Hall effect of the fractionalized fermions.

Quantized spin Hall conductance $\frac{e}{2\pi} \Rightarrow \frac{q_F}{2\pi}$

Two types of fractional charges:

Fermionic – of charge q_F

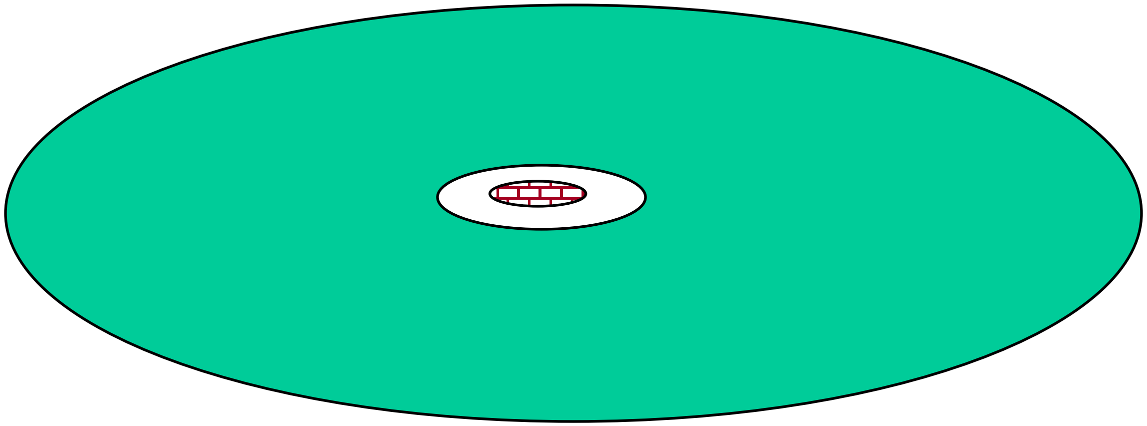
"Bosonic"/mixed – of charge $e^* = $ $2e/m$ for m even

e/m for m odd

One pair of gapless counter-propagating edge states.

Are they protected by time reversal symmetry?

Flux insertion argument – the general idea (Fu&Kane)



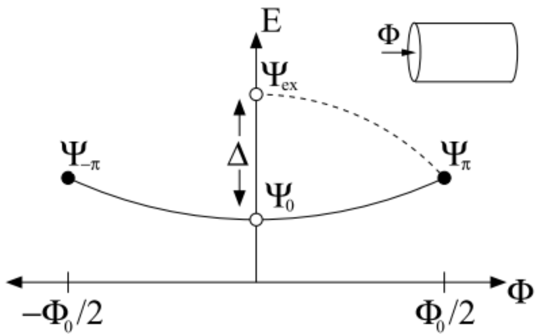
For $\Phi = \pm n\Phi_0/2$ there are pairs of states that are degenerate due to time reversal symmetry.

Their splitting at zero flux will be small (~1/L)

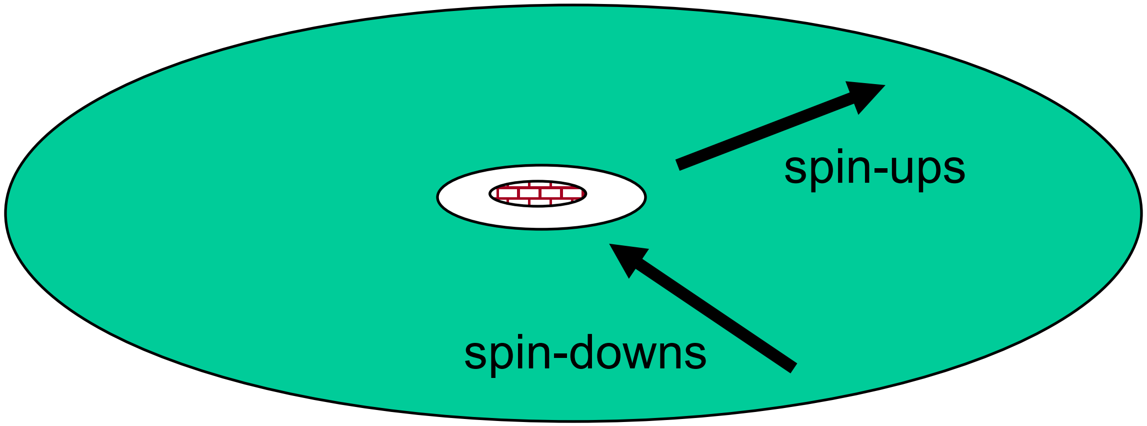
→ Indication for a gapless edge mode

If the degeneracy cannot be lifted in a time-reversal-symmetric way

→ protection



Flux insertion argument – Non-interacting QSHE (Fu&Kane)

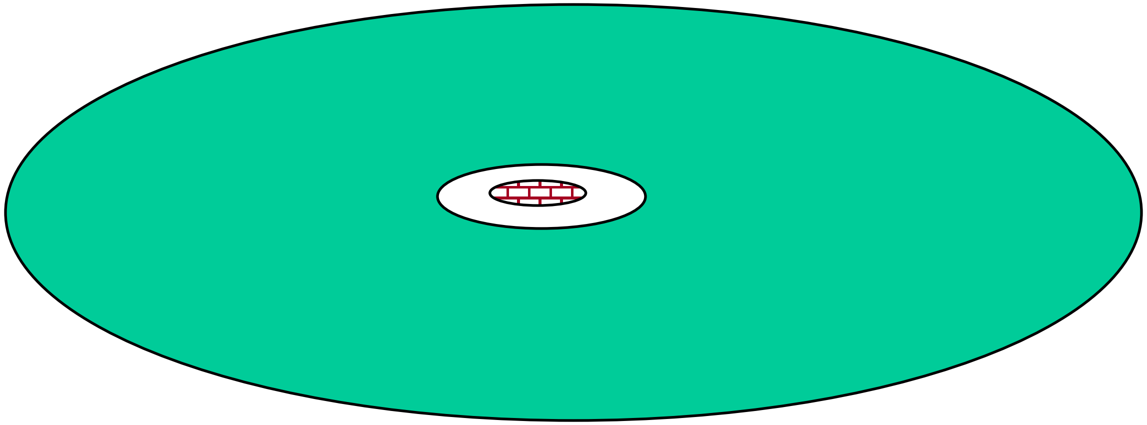


Turn on a $\pm \Phi_0/2$ in the hole.

- A spin imbalance of $\pm\sigma_{sH}$ (integer number) is created on each edge.
- Two states that are degenerate in energy and time reversed of one another.
- The degeneracy may be lifted in a time-reversal-symmetric way only if σ_{sH} is even.

For the fractional case, a complication – two types of degeneracies

Reminder: FQHE disk (say, ν=1/3)

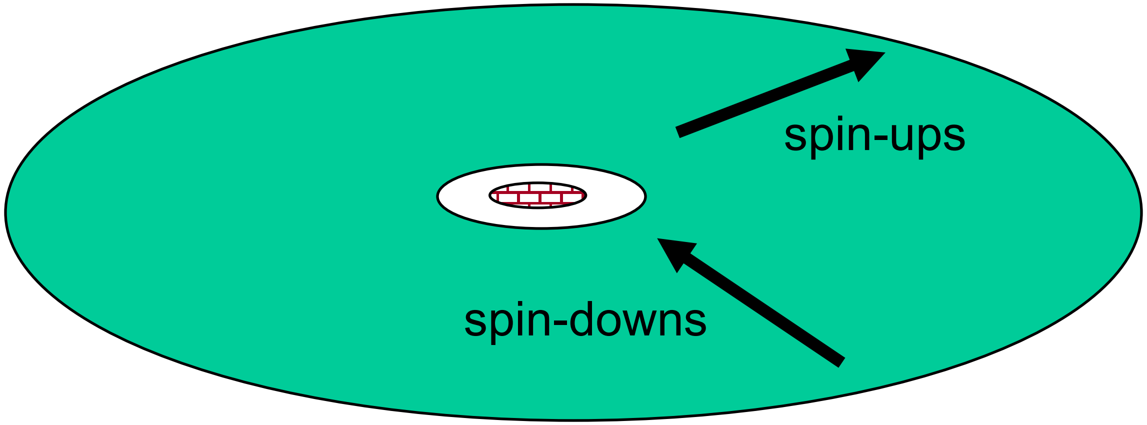


• Degeneracy associated with different topological sectors, not with the existence of a gapless mode.

• May be removed only by tunneling of fractional charges between interior and exterior.

This is not what we are interested in.

We should focus on excitations where the edges are uncoupled



- We should insert the flux needed to bring each edge back to the topological sector from which it started.
- The number of flux quanta needed for that is $1/2e^*$.
- The imbalance is then σ_{sH}/e^*. It is the parity of this number that determines the protection of the gap.
- parity of σ_{sH}/e^* ⟺ parity of q_F/e^*

By this argument, if q_F / e^* is odd, the gapless modes are protected as long as time reversal symmetry and charge conservation are maintained.

When this ratio is even, there is no protection.

Can the modes be gapped then?

We look for an edge perturbation that is

a. Charge conserving

b. Possibly strong ("relevance is irrelevant")

d. symmetric to time reversal

And gaps the edge

A microscopic calculation

The field theory of the edge modes

$$K_{ij}\partial_t\phi_i\partial_x\phi_j + V_{ij}\partial_x\phi_i\partial_x\phi_j + \varepsilon_{\mu\nu}A_\mu t_i\partial_\nu\phi_i$$

(Wen…)

The Integer case:

$\nu = 1$

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \vec{\Phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \qquad t = \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ 1 \end{pmatrix}$$

$\nu = 2$

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \vec{\Phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}$$

Our case is more complicated

Boson-electron binding

bosons

$$K = \begin{pmatrix} 0 & m & -k & -k \\ m & 0 & 0 & 0 \\ -k & 0 & 1 & 0 \\ -k & 0 & 0 & -1 \end{pmatrix} \qquad t = \begin{pmatrix} 0 \\ 2e \\ e \\ e \end{pmatrix}$$

fermions

Perturbations: $\exp\left[i\Theta(\vec{l}\,) \right] \equiv \exp\left[i\vec{l} K \vec{\Phi} \right]$

l is an integer vector

$l = \begin{pmatrix} 3 & 3 & -1 & -5 \end{pmatrix}$

To conserve charge $\vec{l} \cdot \vec{t} = 0$

The presence of the bosons allows more perturbations as compared with the case of a 2D $\nu=1$ quantum spin Hall state. Two important ones:

1. Breaking a boson into two electrons (one of each spin direction)
2. Backscattering an even number of electrons together with creating flux quanta

The combination of these two perturbations gaps the edge of the 2D system in the case of even q_F / e^*.

Three dimensions:

- The same construction gives a fractional topological insulator
- FQHE on the surface, with $\sigma_{xy} = q_F^2 / 2$

- Unique charge-monopole binding (not to be detailed here)
- Surface modes are protected by a modified flux insertion argument (the Corbino disk is replaced by a Corbino donut, a thickened torus) when q_F / e* is odd.
- We do not know how to gap the surface states when they are not protected.

Summary:

1. Fractional topological insulators are possible in 2D and in 3D.
2. Explicit construction using the "gluing" of fractionalized excitations of a bosonic system with electrons. The resulting composite particles are stable, are fermions, and carry a fractional charge.
3. The crucial component – Z_m symmetry of the link bosons
4. The gapless modes of these fractional topological insulators are protected by time reversal symmetry when q_F / e^* is odd, not protected when it is even in 2D, and unknown when it is even in 3D.