Isotropic Landau Levels of Relativistic and Non-Relativistic Fermions in 3D Flat Space

Yi Li and Congjun Wu

University of California, San Diego

Ref. 1. Three Dimensional Topological Insulators with Landau Levels, arXiv:1103.5422.
2. Isotropic High Dimensional Landau Levels of Dirac Fermions, arxiv:1108.5650.

Sept 29, KITP Topo-insulator workshop, UCSB

Collaborators:

Yue Ye (ITP, CAS, Beijing)

Ken Intriligater (UCSD)

Acknowledgement

Jorge Hirsch, Shou-Cheng Zhang,

Xi Dai, Luming Duan, Zhong Fang, Liang Fu, Kazuki Hasebe, Jiang-ping Hu, Nai Phuan Ong, Cenke Xu, Kun Yang

Outline

- **Introduction.**

- Isotropic 3D LLs of non-relativistic fermions from Aharanov-Casher coupling – strong TI insulators.

- Isotropic 3D LLs of Dirac fermions from non-minimal coupling.

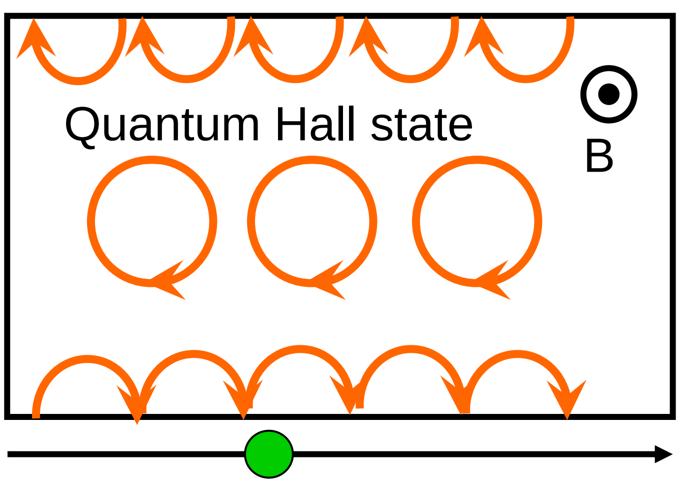
- Generalization to arbitrary dimensions.

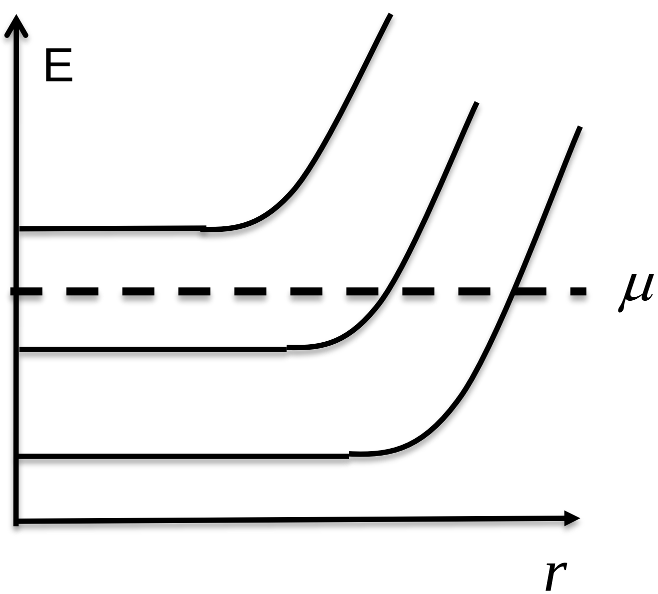
2D quantum Hall effect with LLs

- **2D Landau levels** in the external magnetic fields.

- Magnetic band-structure characterized by the topological TKNN (Chern) number.

- Chiral edge modes responsible for quantized transverse charge transport; stable against disorder and interactions.



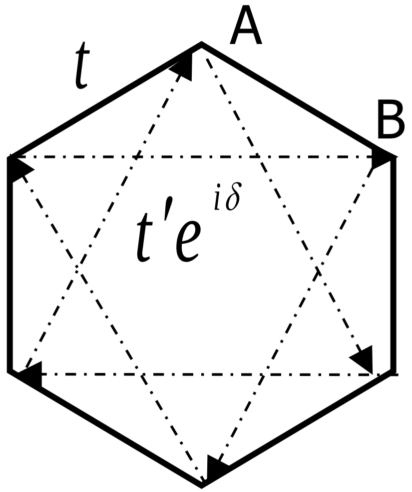


Quantum Anomalous Hall model without LLs

- Honeycomb lattice with complex-valued next-nearest neighbor hopping.

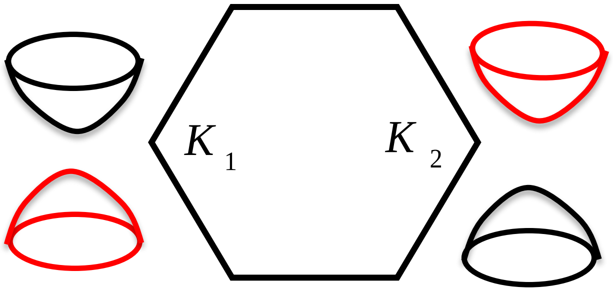
$$H_{NN} = -t\sum_{\vec{r}\in A}\{c^{+}(\vec{r}_A)c(\vec{r}_B) + h.c.\}$$

$$H_{NNN} = -\sum_{\vec{r}} t'\{e^{i\delta}c^{+}(\vec{r}_A)c(\vec{r}_A') + e^{i\delta}c^{+}(\vec{r}_B)c(\vec{r}_B') + h.c.\}$$



- Chern number $\nu = \pm 1$ if $\delta \neq 0, \pi$,

Mass changes sign at $K_{1,2}$.



F. D. M. Haldane, Phys. Rev. Lett. 61, 2015 (1988); X. L Qi, et al, PRB 74,85308 (2006)

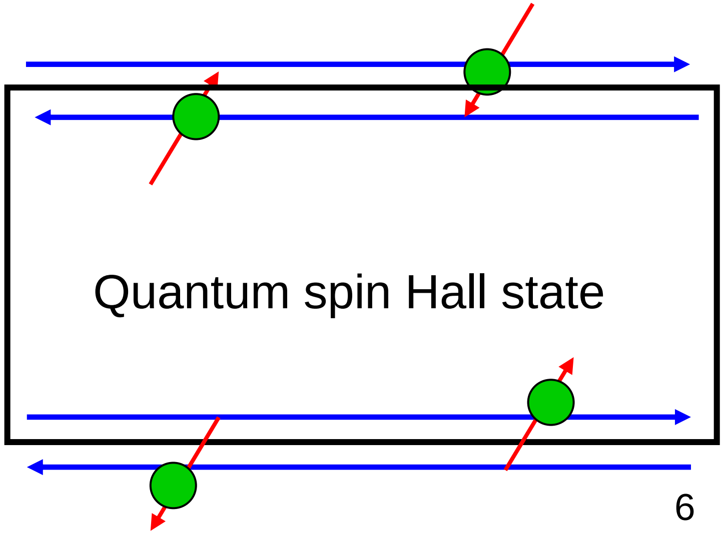
2D time-reversal invariant TIs with and without LLs

- The Kane-Mele model: two copies of Haldane model.

- Odd numbers of helical edge modes are stable against disorder; topological Z_2-index.

- Bernevig--Zhang model: LLs with opposite chiralities for spin up and down electrons. ---- fractional 2D TIs.

$$H = \frac{(\vec{P} - g\vec{A}\sigma_z)^2}{2M}, \quad \vec{A} = \vec{r} \times \hat{z}$$



- 2D TIs without LLs were predicted and realized in 2D HgTe/HgCdTe quantum wells.

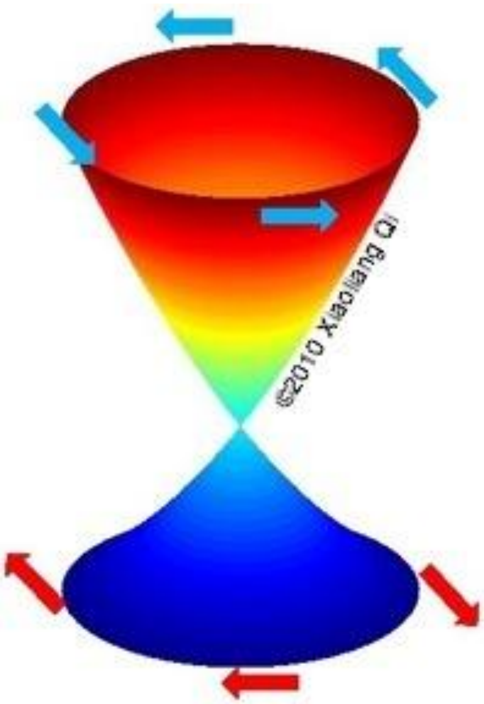
3D strong TIs without LLs

• Various 3D strong TIs based Bloch-wave band structures with non-trivial Z_2 index have been predicted and realized.

Bi_2Te_3, Bi_2Se_3, etc

IOP, Osaka, Princeton, Stanford, Tsinghua, Wuerzburg, etc

• Odd numbers of surface Dirac cones detected by ARPES, quantum oscillations, STM etc.



Motivation of 3D strong TIs with LLs ?

- **Question: can we construct 3D strong TIs based on LLs?**
Here we mean 3D isotropic LLs, not stacked 2D LL layers.

- LL wavefunctions are simple, explicit, and elegant.

LL in arbitrary-D flat space= **harmonic oscillator + spin-orbit coupling** → simple enough for the qual exam.

- Flat spectra + analytical properties may facilitate the study of high dimensional fractional TIs due to interactions (open).

- How to characterize the topo-properties within harmonic potentials, one of the simplest types of inhomogeneity? (open)

Pioneering Work: LLs on 4D-sphere ---Zhang and Hu

Science 294, 824 (2001).

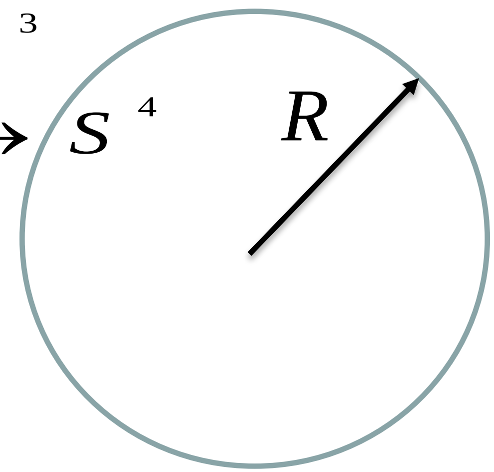
- Particles couple to the SU(2) gauge field on the S^4 sphere.

$$H = \frac{\hbar^2}{2MR^2} \sum_{1 \le a < b \le 5} \Lambda_{ab}^2, \qquad \Lambda_{ab} = x_a(-i\partial_b + A_b) - x_b(-i\partial_a + A_a)$$

- Second Hopf mapping. The spin value $I \propto R^2$.

$$x_a = \psi^+_\alpha \Gamma^a_{\alpha\beta} \psi_\beta, \qquad n_i = u^+_\alpha \sigma_{i,\alpha\beta} u_\beta$$

$$S^7 \xrightarrow{S^3} S^4$$



- Single particle LLLs

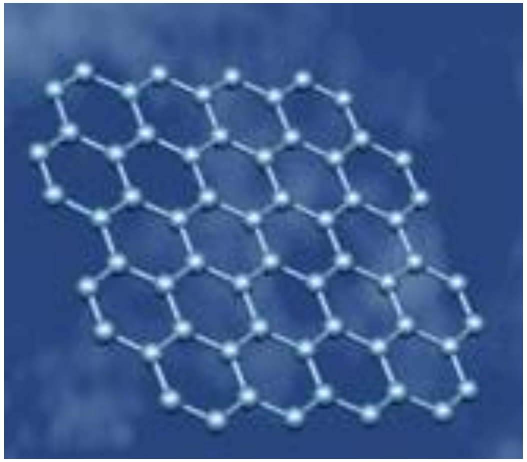
$$\langle x_a, n_i \mid m_1 m_2 m_3 m_4 \rangle = \psi_1^{m_1} \psi_2^{m_2} \psi_3^{m_3} \psi_4^{m_4}$$

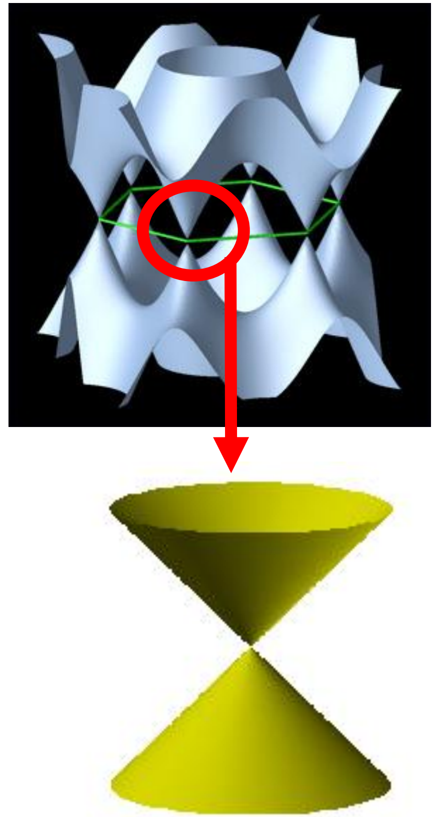
→ 4D integer and fractional TIs with time reversal symmetry

→ Dimension Reduction to 3D and 2D TIs (Qi, Hughes, Zhang)**.**

- 4D LLs in flat space – Elvang and Polchinski, 2002.

Quantum Hall Effect of Relativistic Fermions



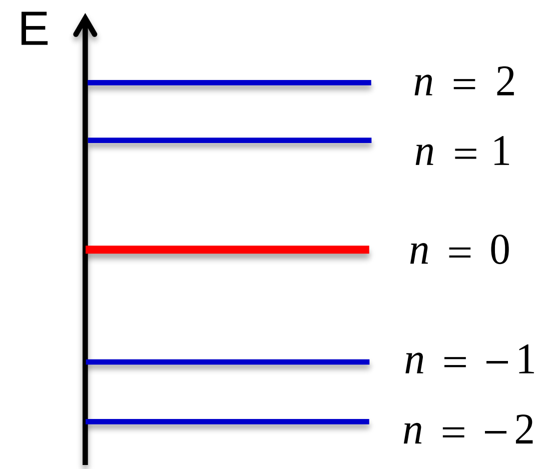


$E = v_0 | p |$ massless Dirac spectrum

$v_0 = 10^6$ m/s = $c/300$

G. Semenoff, Phys. Rev. Lett., 53, 2449 (1984);
Novoselov, Geim et al., Nature 438, 197 (2005);
Y. Zhang, P. Kim et al, Nature 438, 201,(2005)

Graphene Landau Levels



Generalize to 3D and above with spherical symmetry?

Outline

- **Introduction.**

  A brief review of Landau levels (LLs) and topological band states

- **Isotropic 3D LLs of non-relativistic fermions from Aharanov-Casher coupling – strong TI insulators.**

- Isotropic 3D LLs of Dirac fermions from non-minimal coupling.

- Possible realizations?

Review: 2D LLs in the symmetric gauge

$$H_{LL}^{2D} = \frac{1}{2M}(\vec{P} - \frac{e}{c}\vec{A})^2, \qquad \vec{A} = \frac{1}{2}\vec{B} \times \vec{r}$$

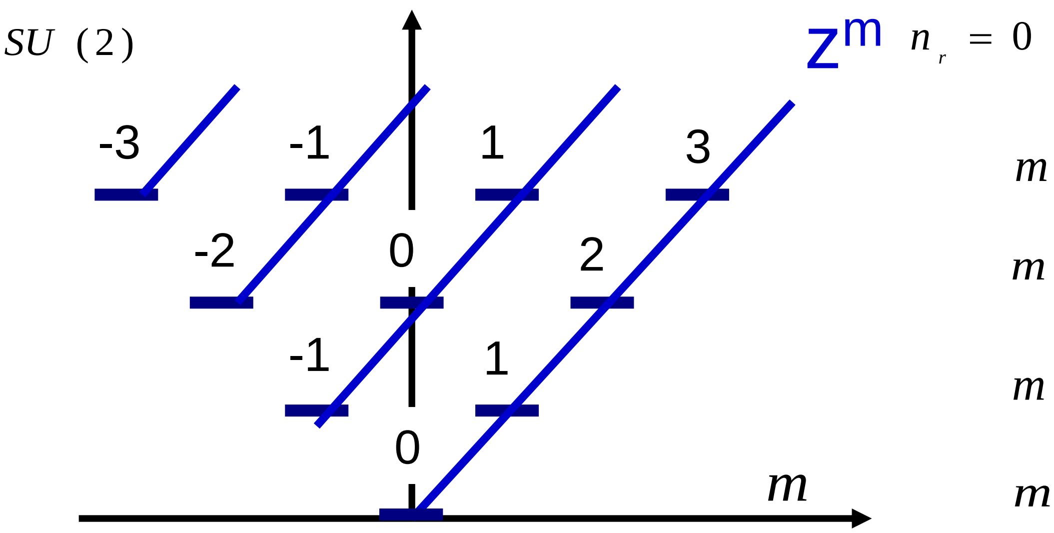
- 2D LL Hamiltonian = 2D harmonic oscillator (HO)+ orbital Zeeman.

$$H_{2D\,LL} = \frac{p^2}{2M} + \frac{1}{2}M\omega^2 r^2 \mp \omega L_z, \quad \omega = \frac{e|B|}{Mc}, \quad l_B = \sqrt{\frac{hc}{eB}}$$

- $H_{2D\,LL}$ has the same set of eigenstates as 2D HO.

Different organization leads to non-trivial topo-structure

$$E_{2D,HO}/(\hbar\omega) = 2n_r + |m| + 1$$



$m = \pm 3, n_r = 0;\ m = \pm 1, n_r = 1$

$m = \pm 2, n_r = 0;\ m = 0, n_r = 1$

$m = \pm 1, n_r = 0$

$m = 0, n_r = 0$

- When viewed horizontally, they are topologically trivial.

- When viewed along the diagonal line because $E_{Zeeman}/(\hbar\omega) = -m$, they become LLs.

- LLL wavefunctions. $\psi_{LLL} = z^m e^{-|z|^2/(2l_B^2)}, \quad z = x + iy, \quad m \geq 0.$

How to work in 3D? – Aharanov-Casher potential !!

- Replace the U(1) potential to the SU(2) gauge potential in 3D.

$$2D: \quad \vec{A} = \frac{1}{2} B \hat{z} \times \vec{r} \quad \Longrightarrow \quad 3D: \quad \vec{A}_{\alpha\beta} = \frac{1}{2} g \vec{\sigma}_{\alpha\beta} \times \vec{r}$$

- 3D LL Hamiltonian = 3D HO + spin-orbit coupling.

$$H_{LL}^{3D} = \frac{P^2}{2M} + \frac{1}{2} M \omega^2 r^2 - \omega \vec{\sigma}_{\alpha\beta} \cdot \vec{L}$$

$$= \frac{1}{2M} (\vec{P} - \frac{e}{c} \vec{A})^2 - \frac{M}{2} \omega^2 r^2$$

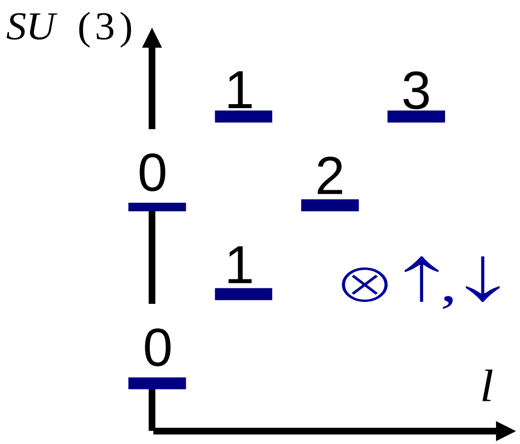
$$\omega = \frac{|eg|}{Mc}, \quad l_g = \sqrt{\frac{\hbar c}{|eg|}}.$$

- H_{LL}^{3D} has the same set of eigenstates of **3D HO in the eigen basis of j.**

- The full 3D rotational symm. + **time-reversal symm.**

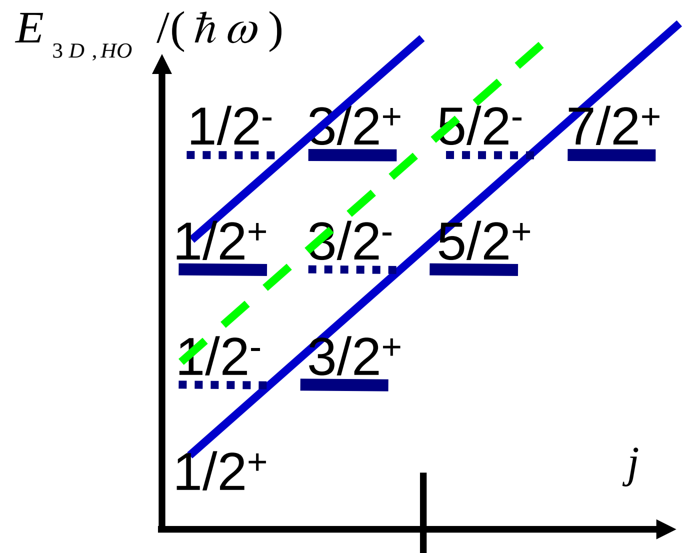
Constructing 3D Landau Levels from 3D HO Eigen-states

$$E_{3D,HO}/(\hbar\omega) = 2n_r + l + \frac{3}{2}$$



SOC : 2 helicity branches

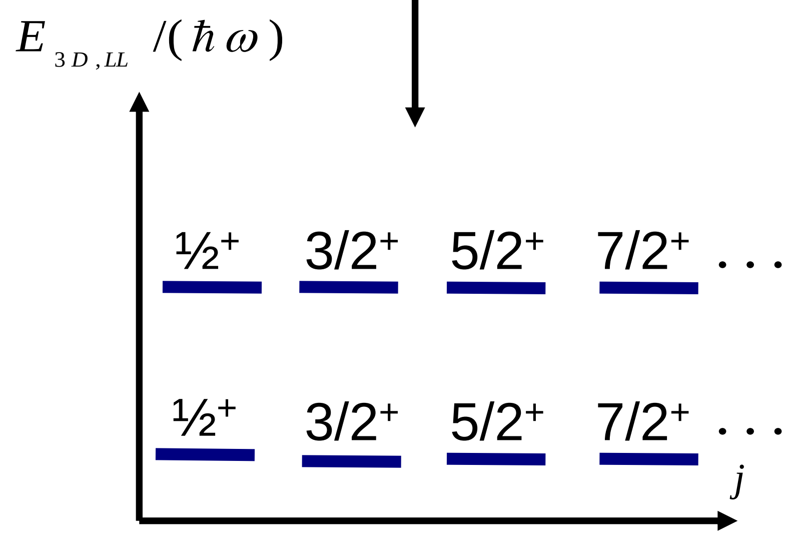
$$j_{\pm} = l \pm \frac{1}{2}.$$



$$H_{LL}^{3D} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 r^2 - \omega \vec{\sigma} \cdot \vec{L}$$

$$\vec{\sigma} \cdot \vec{L} = \begin{cases} l\hbar & \text{for } j_+ \\ -(l+1)\hbar & \text{for } j_- \end{cases}$$

Or: $$H_{LL}^{3D} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 r^2 + \omega \vec{\sigma} \cdot \vec{L}$$



3D LL wavefunctions

$$\psi_{n_r, j_+, l, j_z}(r, \Omega) = R_{n_r, l}(r) Y_{j_+, l, j_z}(\Omega)$$

- n_r : Landau level index

- $Y_{j_+, l, j_z}(\Omega)$: spin-orbit coupled spherical harmonics with the positive helicity.

$$Y_{j_+, l, j_z}(\Omega) = \begin{pmatrix} \sqrt{\dfrac{l + m_j + \frac{1}{2}}{2l+1}} Y_{l, m_j - \frac{1}{2}}(\Omega) \\ \sqrt{\dfrac{l - m_j + \frac{1}{2}}{2l+1}} Y_{l, m_j + \frac{1}{2}}(\Omega) \end{pmatrix}.$$

- The LLL wavefunctions:

$$\psi^{LLL}_{j_+, j_z}(r, \Omega) = r^l Y_{j_+, l, j_z}(\Omega) e^{-r^2/4 l_g^2}$$

The highest weight state in the 3D Landau Levels

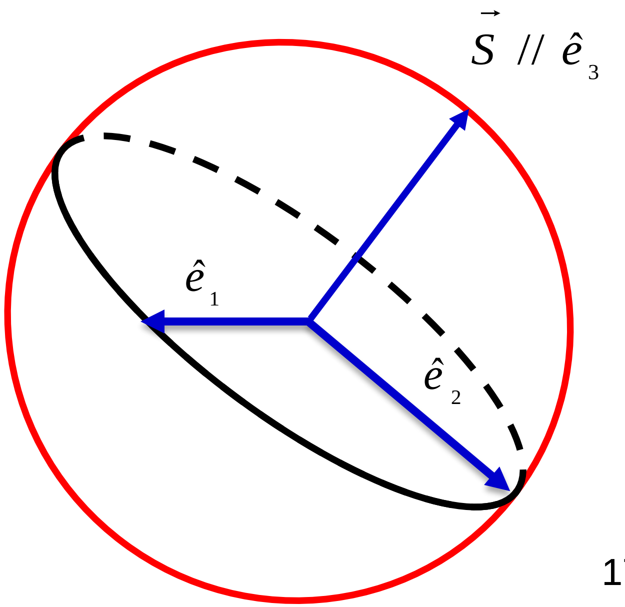
- The highest weight state $j_z = j_+$. Both L_z and S_z are conserved.

$$\psi^{LLL}_{j_+, j_z = j_+}(r, \Omega) = \begin{pmatrix} (x + iy)^l \\ 0 \end{pmatrix} e^{-r^2/4 l_g^2}$$

2D-like LLs with spin perpendicular to the plane of the orbital motion.

- The highest weight state as coherent states.

$$\psi^{LLL}_{j_+, \ high}(r, \Omega) = [(\hat{e}_1 + i\hat{e}_2) \cdot \vec{r}\,]^l \otimes \chi_{\hat{e}_3}$$



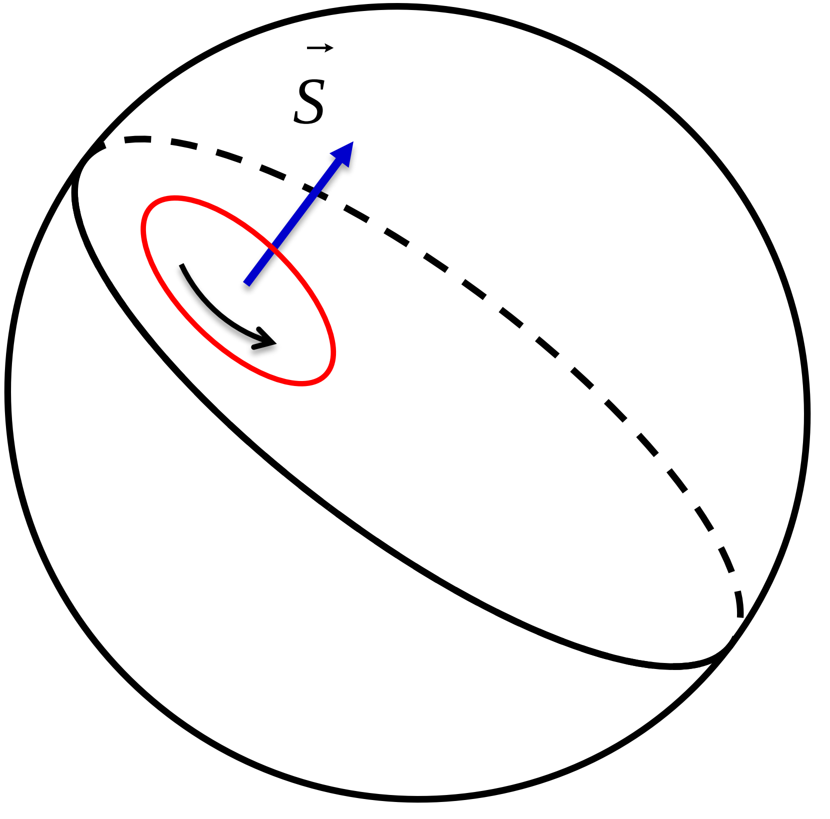
- The highest weight states form over-complete basis for all the j_z eigenstates.

Understanding the highest weight state from classical EOM

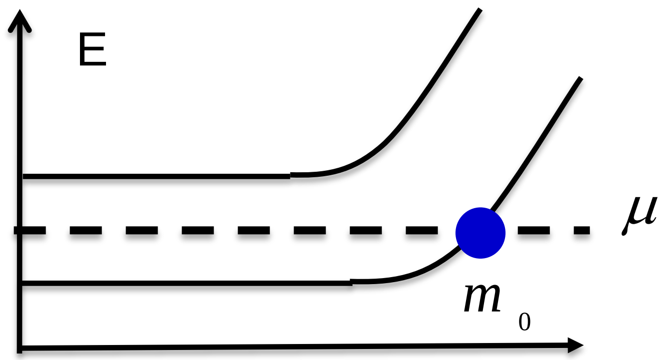
$$\dot{\vec{r}} = \frac{\vec{P}}{m} + \omega\, \vec{r} \times \frac{2\vec{S}}{\hbar}, \quad \dot{\vec{P}} = \omega\, \vec{P} \times \frac{2\vec{S}}{\hbar} - m\omega^2 r, \quad \dot{\vec{S}} = \omega \frac{2\vec{S}}{\hbar} \times \vec{L}.$$

- If we fix the direction of $\vec{S}$, and choose *r*, and *p* in the plane perpendicular to $\vec{S}$, then the motion is coplanar, which reduces to the 2D cyclotron motion. The plane of motion passes the center.

- Helical structure: we can rotate the motion plane and S together.



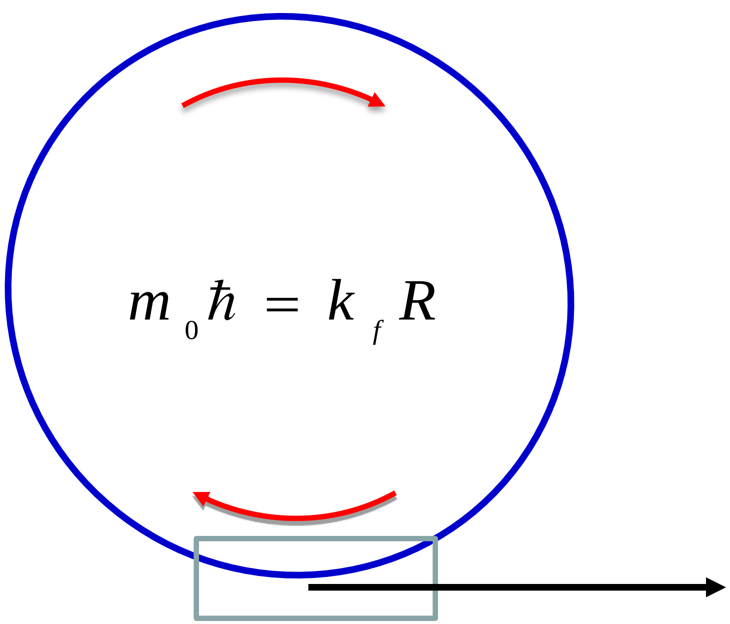
Review: chiral liquid of 2D QHE edge



Halperin, PRB, 25, 2185 (1982)

- Each LL contributes a branch of chiral edge modes.

- As *m* goes large, eigen-states are pushed to the open edge, and develop dispersion.



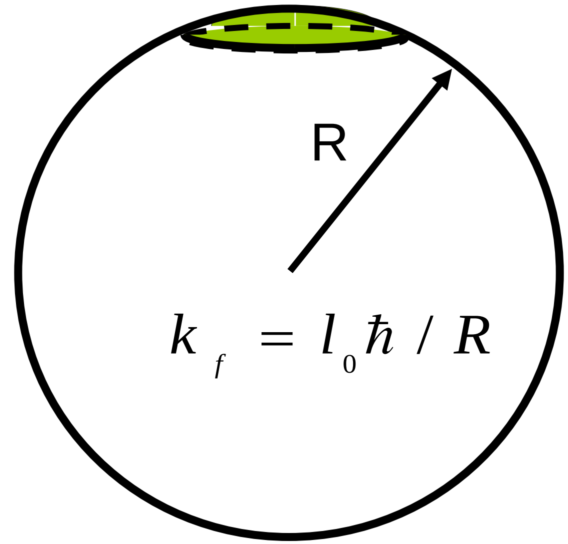
$$H^{2D}_{bulk} = \frac{p^2}{2M} + \frac{1}{2} M \cancel{\omega}^2 r^2 - \omega L_z .$$

$$\downarrow$$

$$H^{1D}_{edge} = \frac{\hbar^2 m^2}{2MR^2} - m\hbar\omega , \quad (m \sim m_0 > 0)$$

$$H^{line}_{edge}(k) \approx v_f (k - k_f)$$

Helical Surface Modes



Surface Effective Hamiltonian for the positive helicity branch

$$H^{3D}_{bulk} = \frac{p^2}{2M} + \frac{1}{2} M\omega^2 r^2 - \omega \vec{\sigma} \cdot \vec{L}$$

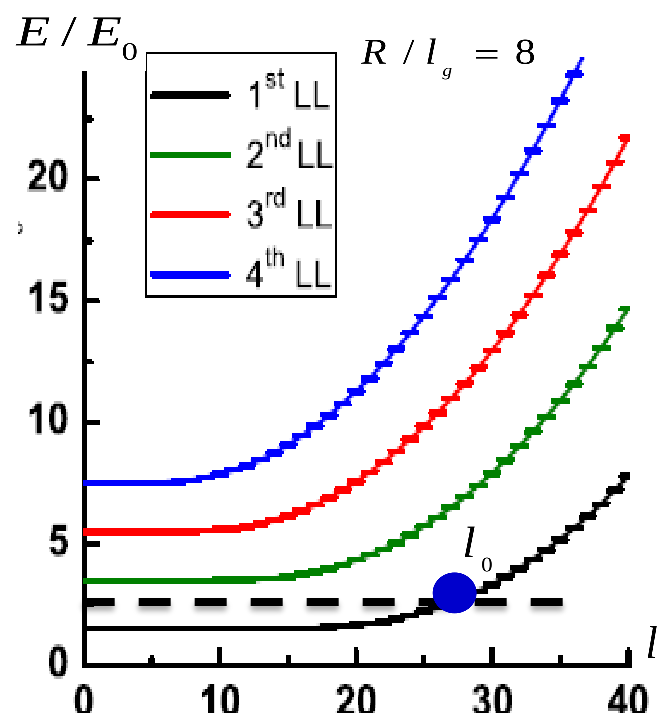
↓

$$H^{2D}_{surface} = \frac{\hbar^2 l(l+1)}{2MR^2} - \hbar \omega l$$

$$\vec{\sigma} \cdot \vec{L} = l\hbar = \vec{\sigma} \cdot (R\hat{e}_r \times \vec{p}) = R\hat{e}_r \cdot (\vec{p} \times \vec{\sigma})$$

↓

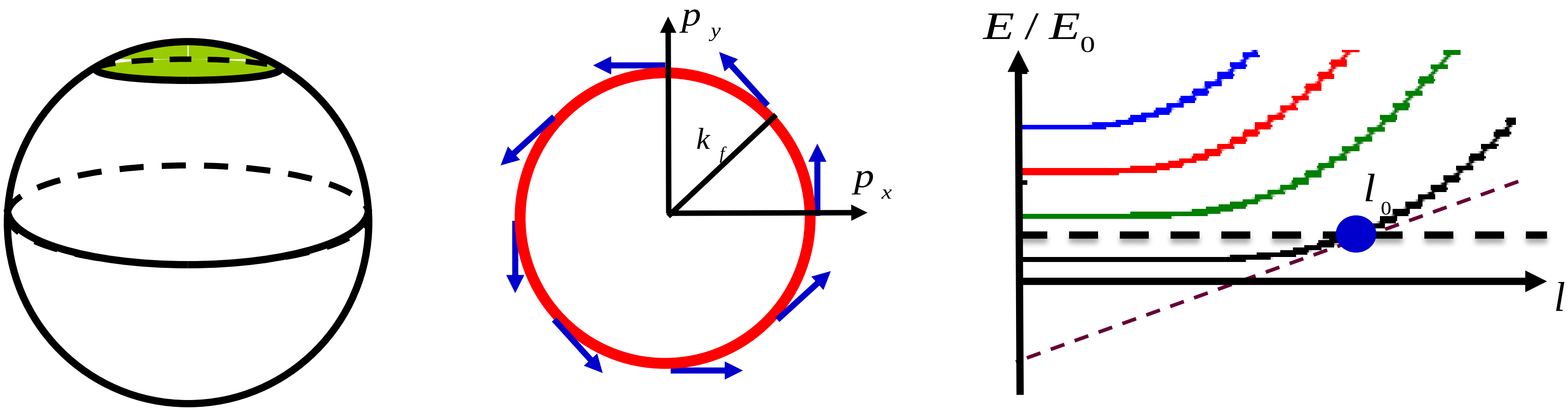
$$H^{2D}_{plane} = v_f (l - l_0)\hbar / R = v\hat{e}_r \cdot (\vec{p} \times \vec{\sigma}) - \mu$$



3D strong TI from Landau Levels

- Each LL contributes to one helical Fermi surfaces

Positive helicity branch $\vec{\sigma} \cdot \vec{L} = l\hbar$



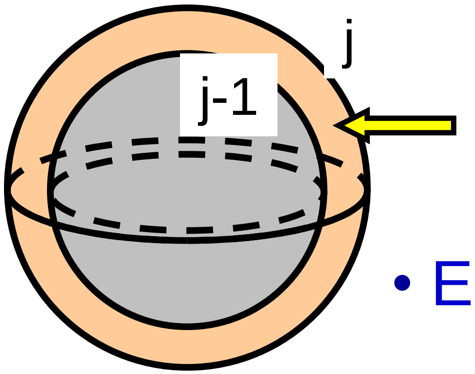
$$k_f = l_0 \hbar / R$$

- Strong Z_2 TI

Odd filling gives odd numbers of Dirac Fermi surface.

$$H^{2D}_{plane} = v \hat{e}_r \cdot (\vec{p} \times \vec{\sigma}) - \mu$$

Non-uniform Particle Density in 3 Dimensions



2j+1 degenerate states

- Estimation based on classic radius of LLL orbits.

$$r_l^{classic} \propto \sqrt{l}\, l_g \qquad r_{l+1}^{class} - r_l^{class} \propto \frac{l_g}{\sqrt{l}}$$

$$\rho(r) \approx \frac{2(l+1)}{4\pi r_l^2 \Delta r_l} \propto \sqrt{l}\, l_g^{-3} \sim r l_g^{-4}$$

- Exact calculation of particle density for filled LLLs.

$$\rho_{LLL}(r) = \frac{1}{\sqrt{2}}\left(\frac{2}{\pi l_g^2}\right)^{\frac{3}{2}} F(2, \frac{3}{2}, \frac{r^2}{l_g^2}) e^{-\frac{r^2}{l_g^2}} \xrightarrow{r \to \infty} \frac{r}{\pi l_g^4}$$

Outline

• Introduction.

A brief review of Landau levels (LLs) and topological band states

• 3D Isotropic LLs of non-relativistic fermions from Aharanov-Casher coupling – strong TI insulators.

• **Square root problem: 3D Isotropic LLs of Dirac fermions from non-minimal coupling.**

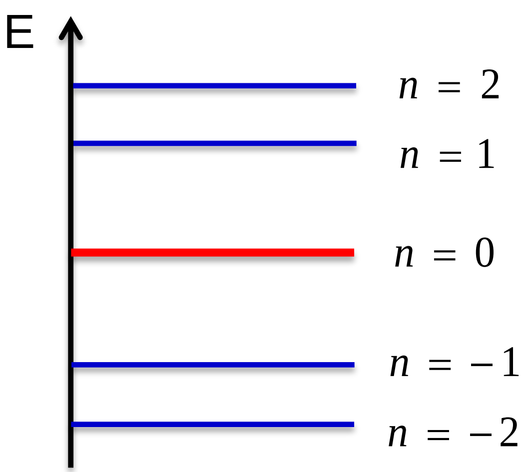
• Generalization to higher dimensions.

Review: 2D LL Hamiltonian of Dirac Fermions

$$H_{LL}^{2D} = v_F\{(p_x - \frac{e}{c}A_x)\sigma_x + (p_y - \frac{e}{c}A_y)\sigma_y\}, \qquad \vec{A} = \frac{1}{2}\vec{B}\times\vec{r}$$

- Rewritten in terms of complex combinations of phonon operators.

$$H_{LL}^{2D} = \frac{\sqrt{2}v_F\hbar}{l_B}\begin{pmatrix} 0 & i(a_x^+ - ia_y^+) \\ -i(a_x + ia_y) & 0 \end{pmatrix}, \qquad a_i = \frac{1}{\sqrt{2}}(\frac{x_i}{l_B} + i\frac{l_B}{\hbar}p_i), \quad i = x, y.$$



- LL dispersions: $E_{\pm n} = \pm\hbar\omega\sqrt{n}$

- Zero energy LL is a branch of half-fermion modes due to the chiral symmetry.

- Graphene QHE exhibits a pair of the above LL Hamiltonian.

3D LL: Dirac equation in **phase-space**

- Generalizing $\{1, -i\} \longleftrightarrow$ 2D harmonic oscillator operators $\{a_x, a_y\}$ to $\{-i\sigma_x, -i\sigma_y, -i\sigma_z\} \longleftrightarrow$ 3D harmonic oscillator operators $\{a_x, a_y, a_z\}$.

$$H_{LL}^{3D\ Dirac} = \frac{\hbar\omega}{2}\begin{pmatrix} 0 & i\vec{\sigma}\cdot\vec{a}^{+} \\ -i\vec{\sigma}\cdot\vec{a} & 0 \end{pmatrix} = \frac{l_0\omega}{2}\begin{pmatrix} 0 & \vec{\sigma}\cdot(\vec{p}+i\hbar\vec{r}/l_0^2) \\ \vec{\sigma}\cdot(\vec{p}-i\hbar\vec{r}/l_0^2) & 0 \end{pmatrix}$$

- This Lagrangian shows non-minimal Pauli coupling.

$$L = \overline{\psi}\{i\hbar(\gamma_0\partial_0 - v\gamma_i\partial_i)\}\psi + \frac{v\hbar}{l_g}\overline{\psi}\sigma_{0i}F^{0i}\psi, \quad \sigma_{0i} = -\frac{i}{2}[\gamma_0, \gamma_i], \quad F^{0i} = \frac{x^i}{l_g}.$$

- A related Hamiltonian was studied before under the name of Dirac oscillator, but its connection to LL and topological properties was not noticed. Benitez, et al, PRL, 64, 1643 (1990)

Reduce back to 2D

- If we only keep the σ_x and σ_y terms in the 3D Dirac LL Hamiltonian, it reduces to 2 copies of 2D Dirac LL Hamiltonian.

- They are time-reversal pairs, which can be considered as quantum spin Hall LLs of Dirac fermions.

$$H_{LL}^{2D\ Dirac} = \frac{\hbar\omega}{2}\begin{pmatrix} 0 & \sigma_x a_x^+ + \sigma_y a_y^+ \\ \sigma_x a_x + \sigma_y a_y & 0 \end{pmatrix} = v\begin{pmatrix} & & & a_x^+ - ia_y^+ \\ & & a_x^+ + ia_y^+ & \\ & a_x - ia_y & & \\ a_x + ia_y & & & \end{pmatrix}$$

$$= v\begin{pmatrix} & & & p_- + A_- \\ & & p_+ - A_+ & \\ & p_- - A_- & & \\ p_+ + A_+ & & & \end{pmatrix}$$

$$p_\pm = p_x \pm ip_y$$

$$A_\pm = A_x \pm iA_y$$

$$A_x = \frac{\hbar y}{l_0^2},\ A_y = -\frac{\hbar x}{l_0^2}$$

A square root problem: $\sqrt{H_{LL}^{3D,Schroedinger}} = H_{LL}^{3D\ Dirac}$

• The square of $H_{LL}^{3D\ Dirac}$ gives two copies of $(H_{LL}^{3D})_{\pm}$ with opposite helicity eigenstates.

$$\frac{(H_{LL}^{3D\ Dirac})^2}{\hbar\omega/2} = \frac{\vec{p}^{\,2}}{2M} + \frac{M}{2}\omega^2\vec{r}^{\,2} + \omega\begin{pmatrix} \vec{L}\cdot\vec{\sigma} + \frac{3}{2}\hbar & 0 \\ 0 & -(\vec{L}\cdot\vec{\sigma} + \frac{3}{2}\hbar) \end{pmatrix}$$

• LL solutions: dispersionless with respect to j. Eigen-states constructed based on non-relativistic LLs.

$$E_{\pm n_r}^{LL} = \pm\hbar\omega\sqrt{n_r},$$

$$\Psi_{\pm n_r; j, l, j_z}^{LL} = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_{n_r, j_+, l, j_z} \\ \pm i\psi_{n_r-1, j_-, l+1, j_z} \end{pmatrix}.$$

The zeroth LL:

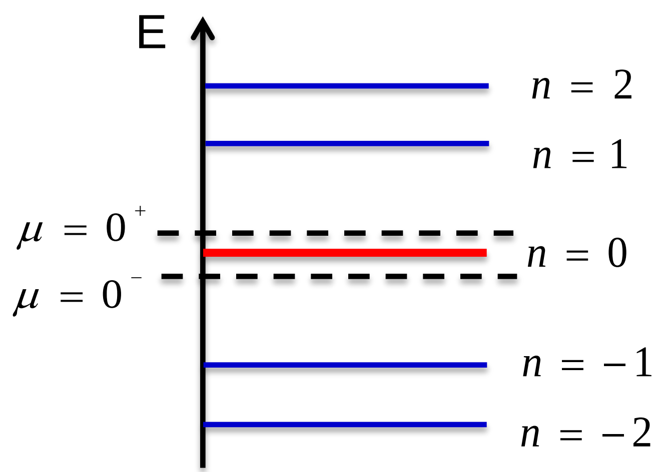
$$\Psi_{0; j, l, j_z}^{LL} = \begin{pmatrix} \psi_{j_+, l, j_z}^{LLL} \\ 0 \end{pmatrix}.$$

Zeroth LLs as half-fermion modes

• The LL spectra are symmetric with respect to zero energy, thus each state of the zeroth LL contributes $\pm$ ½- fermion charge depending on the zeroth LL is filled or empty.

• For the 2D case, the vacuum charge density is $j_0 = \pm \frac{1}{2}\frac{e^2}{h}B$, known as parity anomaly. G. Semenoff, Phys. Rev. Lett., 53, 2449 (1984).

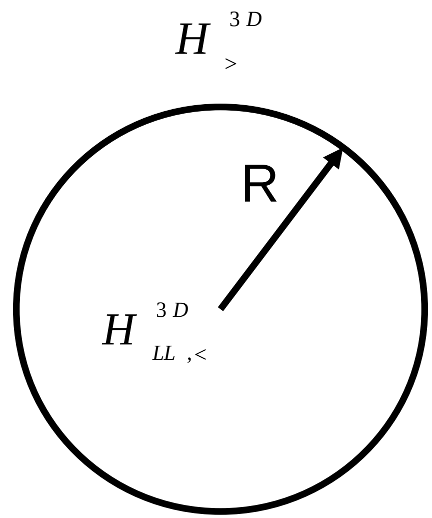
• For our 3D case, the vacuum charge density is plus or minus of the half of the particle density of the non-relativistic LLLs ---- "parity"-type anomaly?



Helical surface mode of 3D Dirac LL

- The mass of the vacuum outside $M \to +\infty$

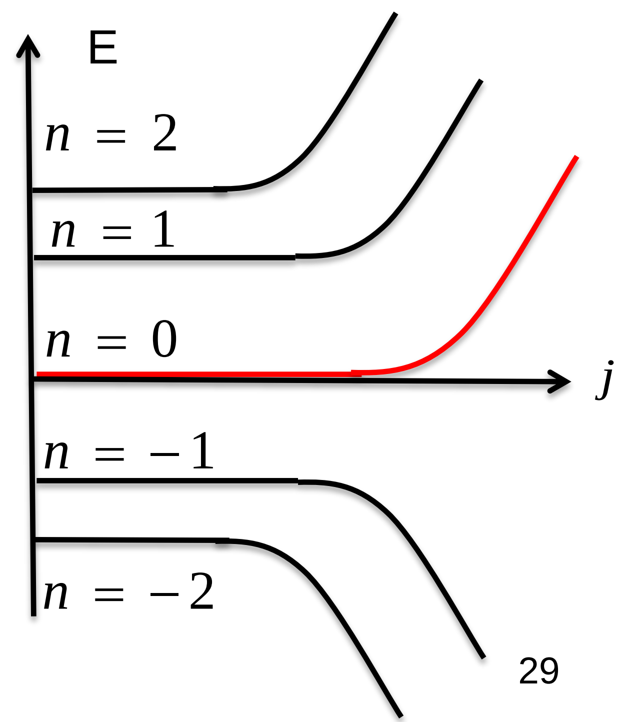
$$H_{<}^{3D} = H_{LL}^{3D} \qquad H_{>}^{3D} = \begin{pmatrix} M & \vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -M \end{pmatrix}$$



- Roughly, this is the square root problem of the open boundary problem of 3D non-relativistic LLs.

- Each surface mode for n>0 of the non-relativistic case splits a pair surface modes for the Dirac case.

- The surface mode of Dirac zeroth-LL of is singled out. Whether it is upturn or downturn depends on the sign of the vacuum mass.



Outline

• Introduction.

A brief review of Landau levels (LLs) and topological band states

• Isotropic 3D LLs of non-relativistic fermions from Aharanov-Casher coupling – strong TI insulators.

• Isotropic 3D LLs of Dirac fermions from non-minimal coupling.

• **Generalization to higher dimensions.**

Non-relativistic LLs in D-dimensions

• D-dimensional LL Hamiltonian = D-dimensional harmonic oscillator (HO)+ spin-orbit coupling.

• Generalizing the 3 2×2 Pauli matrices $\{\sigma^x, \sigma^y, \sigma^z\}$ to (2k+1) $2^k \times 2^k$ Γ-matrices $\Gamma_1^{(k)}, \ldots, \Gamma_{2k+1}^{(k)}$

$$H_{LL}^{D-\dim} = \frac{P^2}{2M} + \frac{1}{2} M \omega^2 r^2 \mp \omega \Gamma_{ij,\alpha\beta}^{(k)} \cdot L_{ij}$$

$$\Gamma_{ij}^{(k)} = -\frac{i}{2}[\Gamma_i^{(k)}, \Gamma_j^{(k)}], \quad L_{ij} = r_i p_j - r_j p_i, \quad i, j = 1, 2 .., D$$

• If D=2k+1, SO(D) has one fundamental spinor, H is irreducible.
• If D=2k, SO(D) has two fundamental spinors, H is reducible.

<u>LL Hamiltonian of Dirac Fermions in Arbitrary Dimensions</u>

- For odd dimensions.

$$H_{LL}^{D-\dim} = \frac{\hbar\omega}{2}\begin{pmatrix} 0 & i\Gamma_i^{(k)}\cdot a_i^{+} \\ -i\Gamma_i^{(k)}\cdot a_i & 0 \end{pmatrix}$$

- For even dimensions.

$$H_{LL}^{D-\dim} = \frac{\hbar\omega}{2}\begin{pmatrix} 0 & \pm a_{2k}^{+} + i\sum_{i=1}^{k}\Gamma_i^{(2k-1)}a_i^{+} \\ \pm a_{2k} - i\sum_{i=1}^{k}\Gamma_i^{(2k-1)}a_i & 0 \end{pmatrix}$$

Conclusions

• We generalize 2D LLs to 3 dimensions and above with the full rotational symmetry, including both non-relativistic and relativistic cases.

• The non-relativistic D-dimensional LL problem is a
D dimensional harmonic oscillator + spin-orbit coupling.

• The relativistic version is a square-root problem corresponding to Dirac equation with non-minimal coupling.

• Each filled LL contributes to a helical surface mode. For the 3D non-relativistic LLs, the system is a 3D TI if odd LLs are filled.

• Open questions: **interaction effects; experimental realizations; characterization of topo-properties with harmonic potentials**