Detecting anomalies in 2D topological phases

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(Wang, Levin, arXiv:1610.04624)

What this talk is about:

$$\eta_1 = \frac{1}{D} \sum_{a \in \mathcal{C}} d_a^2 \cdot e^{i\theta_a}$$

$$\eta_2 = \frac{1}{D} \sum_{a \in \mathcal{C}} d_a \cdot \mathcal{T}_a^2 \cdot e^{i\theta_a}$$

$$\eta_f = \frac{1}{\sqrt{2}D} \sum_{a \in \mathcal{C}_f} d_a \cdot \tilde{\mathcal{T}}_a^2 \cdot e^{i\theta_a}$$

Definition of time-reversal symmetric topological phase

2D gapped quantum many-body system with:

- (Unbroken) time reversal symmetry
- Anyon excitations

For now, focus on **bosonic** topological phases

Data

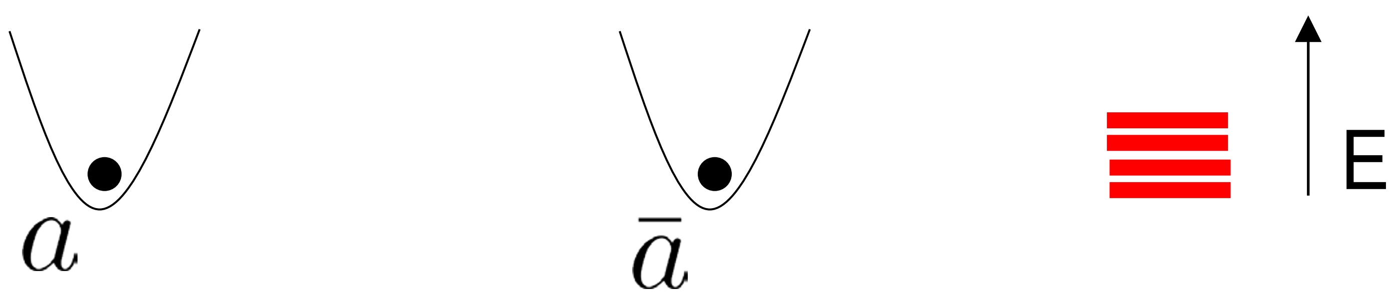
- Set of anyon types: $\mathcal{C} = \{a, b, c, ...\}$

- Fusion/braiding data: $d_a, N_{ab}^c, F_{abc}^d, R_{ab}^c, ...$

- Time-reversal permutation: $a \to \mathcal{T}(a)$

- Kramers degeneracy: If $\mathcal{T}(a) = a$, then

$$\text{either} \begin{cases} a \text{ is Kramers singlet: } \text{“}\mathcal{T}_a^2 = +1\text{”} \\ a \text{ is Kramers doublet: } \text{“}\mathcal{T}_a^2 = -1\text{”} \end{cases}$$



Some physical constraints

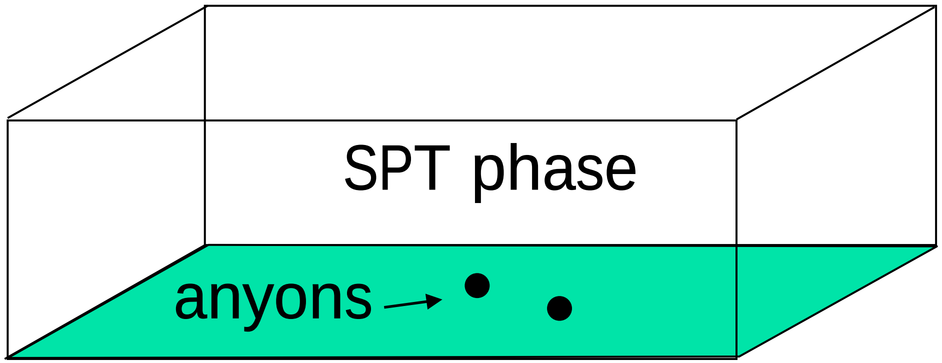
1. $R^{\mathcal{T}(c)}_{\mathcal{T}(a)\mathcal{T}(b)} R^{\mathcal{T}(c)}_{\mathcal{T}(b)\mathcal{T}(a)} = (R^c_{ab} R^c_{ba})^*$
2. $\mathcal{T}(\mathcal{T}(a)) = a$
3. $\mathcal{T}_1^2 = 1$
4. For Abelian phases,
   (a) $\mathcal{T}(a \times b) = \mathcal{T}(a) \times \mathcal{T}(b)$
   (b) $\mathcal{T}^2_{a \times b} = \mathcal{T}^2_a \mathcal{T}^2_b$

Note: Not a complete list

Anomalous topological phases

- Cannot be realized in strictly 2D system

- **Can** be realized on the surface of 3D SPT phase



(Wang, Senthil, 2013)

Example 1: "eTmT" phase

$$\mathcal{C} = \{1, e, m, em\} = \text{Toric code}$$

$$\theta_{e,m} = \pi; \qquad \theta_e = \theta_m = 0$$

$$\mathcal{T} \text{ perm.} = 1;$$

$$\boxed{\mathcal{T}_e^2 = \mathcal{T}_m^2 = -1}$$

(Vishwanath, Senthil, 2013)

Example 2: “eFmF” phase

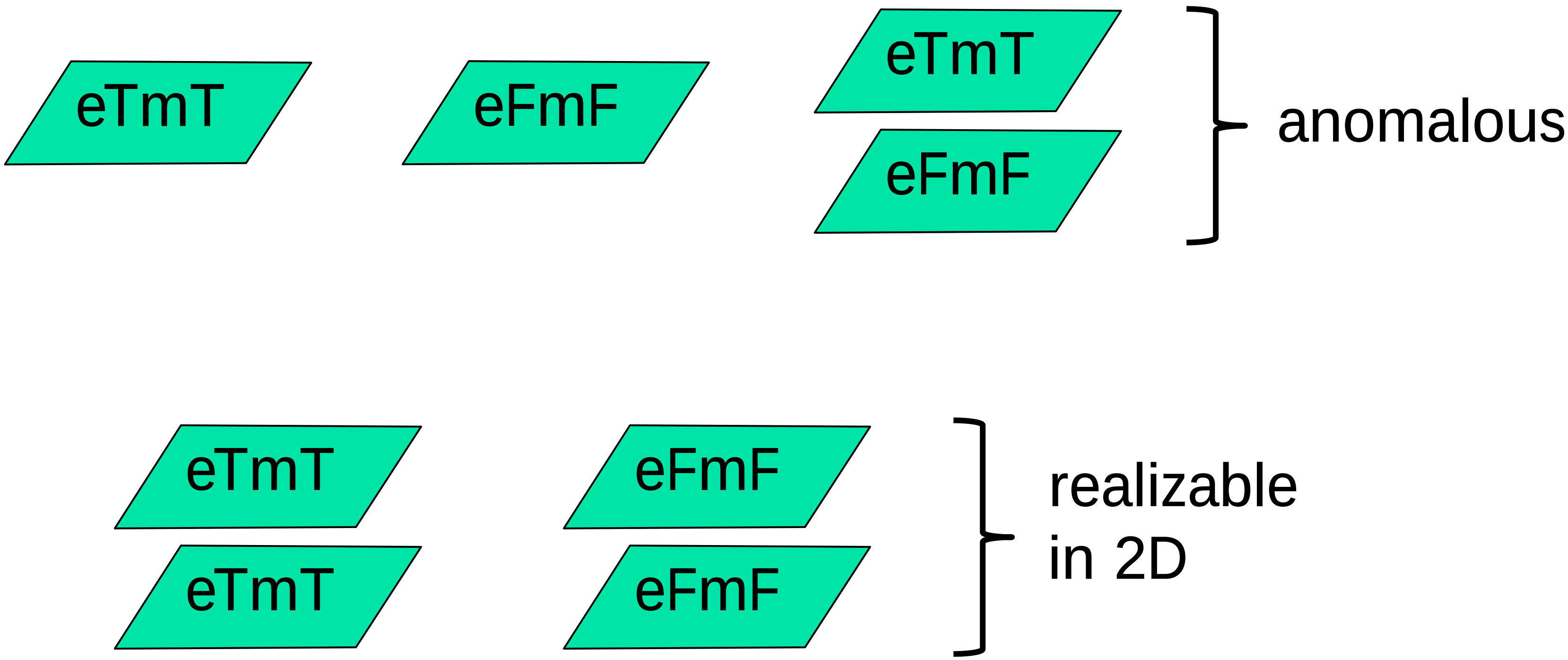
$$\mathcal{C} = \{1, e, m, em\}$$

$$\theta_{e,m} = \pi; \quad \boxed{\theta_e = \theta_m = \pi}$$

$$\mathcal{T} \text{ perm.} = 1;$$

$$\mathcal{T}_e^2 = \mathcal{T}_m^2 = 1$$

Behavior under stacking



$$\Longrightarrow \quad Z_2 \times Z_2 \text{ anomaly structure}$$

Two questions

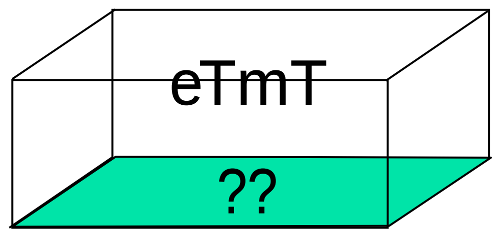
- How can we show that “eTmT” and “eFmF” are anomalous?

- How can we detect if a **general** top. phase carries an “eTmT’’ or “eFmF’’-type anomaly?

(Chen, Burnell, Vishwanath, Fidkowski, 2015)

Motivation

1. A step toward bulk/boundary correspondence



2. Unitary vs. anti-unitary symmetry

   (a) Unitary case: $$\nu(f,g,h,k) = R_{\omega(h,k),\omega(f,g)} F_{\omega(g,h),\omega(f,gh),\omega(fgh,k)} F^{-1}_{\omega(g,h),\omega(gh,k),\omega(f,ghk)} \times F_{\omega(f,g),\omega(h,k),\omega(fg,hk)} F^{-1}_{\omega(f,g),\omega(fg,h),\omega(fgh,k)} \times F_{\omega(h,k),\omega(g,hk),\omega(f,ghk)} F^{-1}_{\omega(h,k),\omega(f,g),\omega(fg,hk)}$$

   (b) Antiunitary case: **??**

Anomaly indicator for “eFmF”

$$\eta_1 = \frac{1}{D} \sum_{a \in \mathcal{C}} d_a^2 \cdot e^{i\theta_a}$$

$\boldsymbol{d_a}$: “quantum dimension” of a

$= \lim_{N\to\infty}(\text{GS degeneracy of } N \ a\text{'s})^{1/N}$

$\boldsymbol{e^{i\theta_a}}$: “topological spin” of a

$= \frac{1}{d_a} \sum_c d_c R^c_{aa}$

$$D = \sqrt{\sum_a d_a^2}$$

(Frohlich, Gabbiani, 1990)
(Kitaev, 2006)

Why is η_1 an "anomaly indicator"?

1. η_1 takes only two values: ± 1
2. $\eta_1 = -1 \implies$ top. phase is anomalous

Follows from general 2D relation:

$$\underbrace{\frac{1}{D}\sum_a d_a^2 e^{i\theta_a}}_{\eta_1} = e^{2\pi i c_-/8}$$

Indicator for "eTmT"?

Our proposal:

$$\eta_2 = \frac{1}{D} \sum_{a \in \mathcal{C}} d_a \cdot \mathcal{T}_a^2 \cdot e^{i\theta_a}$$

$$\mathcal{T}_a^2 = \begin{cases} 1, & \text{if } \mathcal{T}(a) = a, \text{ and } a \text{ is Kramers singlet} \\ -1, & \text{if } \mathcal{T}(a) = a, \text{ and } a \text{ is Kramers doublet} \\ 0, & \text{if } \mathcal{T}(a) \neq a \end{cases}$$

Conjecture:

1. η_2 takes only two values: ± 1

2. $\eta_2 = -1 \implies$ top. phase is anomalous

Example: Toric code

$$\eta_2 = \frac{1}{2}(1 + \mathcal{T}_e^2 + \mathcal{T}_m^2 - \mathcal{T}_e^2 \mathcal{T}_m^2)$$

η_2	$\mathcal{T}_m^2 = +1$	$\mathcal{T}_m^2 = -1$
$\mathcal{T}_e^2 = +1$	**1**	**1**
$\mathcal{T}_e^2 = -1$	**1**	**-1**

Realizable in 2D

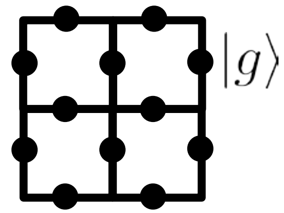
eTmT

(Kitaev, 2003)

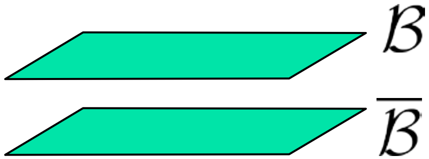
Evidence

1. $\eta_2 = 1$ for several large classes of strictly 2D systems:

   (a) Quantum double models with group G



   (b) Double layer systems: $\mathcal{B} \times \overline{\mathcal{B}}$



(Wang, Senthil,2013); (Fidkowski, Chen, Vishwanath, 2013)
(Metlitski, Fidkowski, Chen, Vishwanath, 2014); (Metlitski, 2015)

Evidence

2. $\eta_2 = -1$ for known anomalous phases.

(eTmT, gauged (T-Pfaffian)$_-$, SF×SF×SF×SF...)

3. η_2 is multiplicative under stacking:

$$\eta_2(\mathcal{C} \times \mathcal{C}') = \eta_2(\mathcal{C}) \cdot \eta_2(\mathcal{C}')$$

4. η_2 is invariant under anyon condensation transitions

Alternative formula for $\eta_1 \cdot \eta_2$ in Abelian case

$$\eta_1 \eta_2 = e^{i\theta_a}$$

where a is any anyon obeying:

$$e^{i\theta_{a,b}} = \mathcal{T}_b^2 \qquad \text{for all invariant anyons } b$$

Agrees with conjecture of Chen, Burnell, Vishwanath, Fidkowski (2015).

Fermionic case

Assume $\mathcal{T}^2 = P_f$:

Z_{16} anomaly $\leftrightarrow$ Z_{16} class. of topological s.c

What is corresponding indicator?

(Wang, Levin, arXiv:1610.04624)

Our proposal:

$$\eta_f = \frac{1}{\sqrt{2}D} \sum_{a \in \mathcal{C}_f} d_a \cdot \tilde{\mathcal{T}}_a^2 \cdot e^{i\theta_a}$$

$$\mathcal{T}_a^2 = \begin{cases} 1, & \text{if } \mathcal{T}(a) = a, \text{ and } a \text{ is Kramers singlet} \\ -1, & \text{if } \mathcal{T}(a) = a, \text{ and } a \text{ is Kramers doublet} \\ \pm i, & \text{if } \mathcal{T}(a) = a \times f \\ 0, & \text{otherwise} \end{cases}$$

local fermion

$$\tilde{\mathcal{T}}_a^2 = \begin{cases} -i\mathcal{T}_a^2, & \text{if } \mathcal{T}(a) = a \times f \\ \mathcal{T}_a^2, & \text{otherwise} \end{cases}$$

Conjecture:

1. η_f takes 16 values: $e^{\nu i\pi/8}$, $\nu = 0, ..., 15$

2. $\eta_f = e^{\nu\pi i/8} \implies$ top. phase lives on surface of index ν topological s.c.

(Fidkowski, Chen, Vishwanath, 2013)

Example: $SO(3)_{6,+}$

$$\mathcal{C} = \{1, f, s, \bar{s}\}$$

$$d_s = d_{\bar{s}} = 1 + \sqrt{2}; \quad \theta_s = -\theta_{\bar{s}} = \pi/2$$

$$\tilde{\mathcal{T}}_s^2 = -\tilde{\mathcal{T}}_{\bar{s}}^2 = 1$$

$$\eta_f = \frac{1}{\sqrt{16 + 8\sqrt{2}}}[1 + 1 + (1 + \sqrt{2})i + (1 + \sqrt{2})i]$$
$$= e^{3\pi i/8}$$

$\implies$ Anomalous phase with $\nu = 3$

How to prove the conjecture?

1. **Partition function approach:** Construct a 3+1D bulk TQFT for each 2+1D phase $\mathcal{C}$. Show that TQFT partition function obeys:

$$Z_{\mathcal{C}}(M) = \eta_2(\mathcal{C})$$

where M is some fixed closed (non-orientable) manifold.

2. **CFT approach:** Prove $\eta_2 = 1$ for time-reversal symmetric CFT.

Questions

1. Can we relate η_1, η_2 and η_f to measurable *bulk* quantities?

2. Are there other types of anomalies?