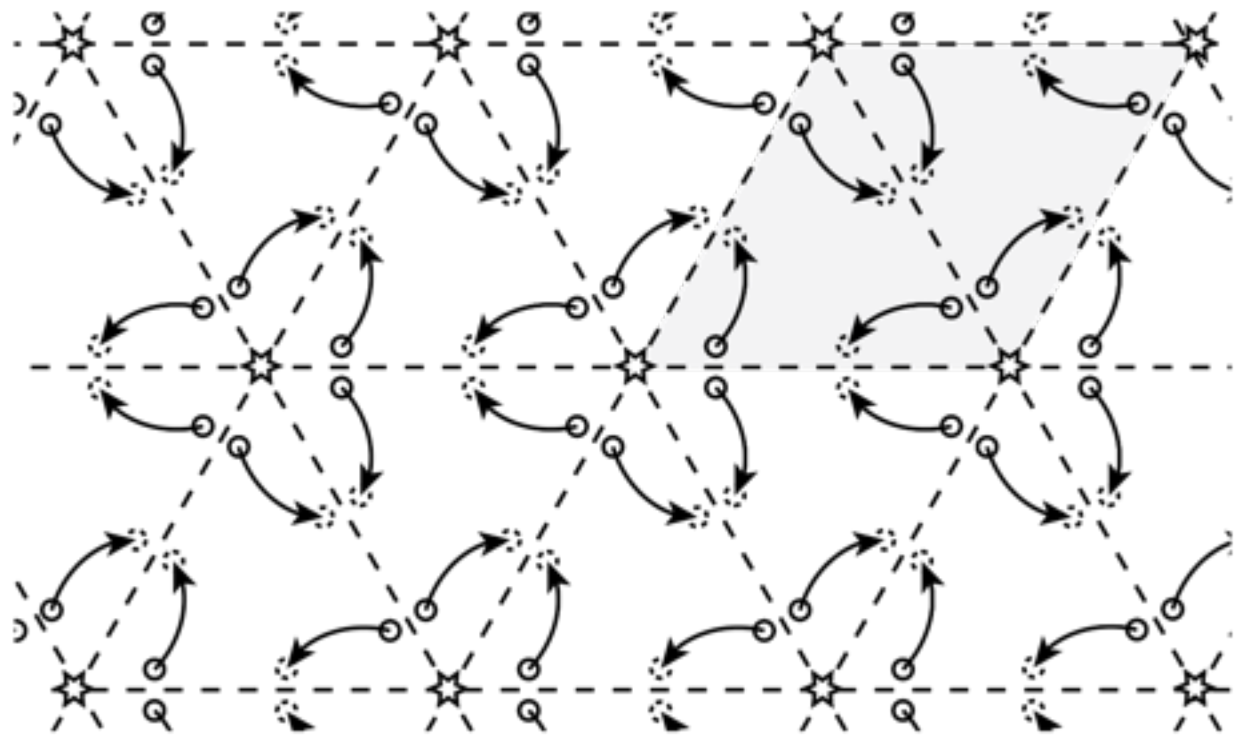
A “Lieb-Schultz-Mattis" constraint for space-groups



Mike Zaletel
Station Q Microsoft Research

KITP TopoQuant 2016


Adrian Po


Chao-Ming Jian


Haruki Watanabe

Adrian Po, et al., arXiv:161x.xxxxx

H. Watanabe, HC Po, A. Vishwanath & M. P. Zaletel, PNAS 112, 14551 (2015)

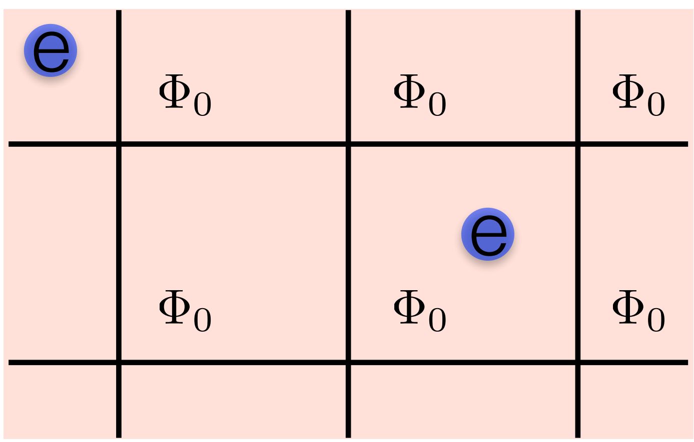

Ashvin Vishwanath

Cheng, Zaletel, Barkeshli, Vishwanath & Bonderson, 1511.02263 (2015)

Fractional filling & fractional excitations

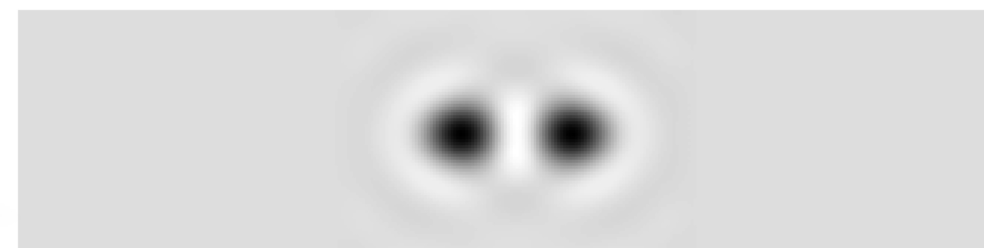
Fractional Quantum Hall Effect

p electrons per q flux quanta ($\nu = \dfrac{p}{q}$)



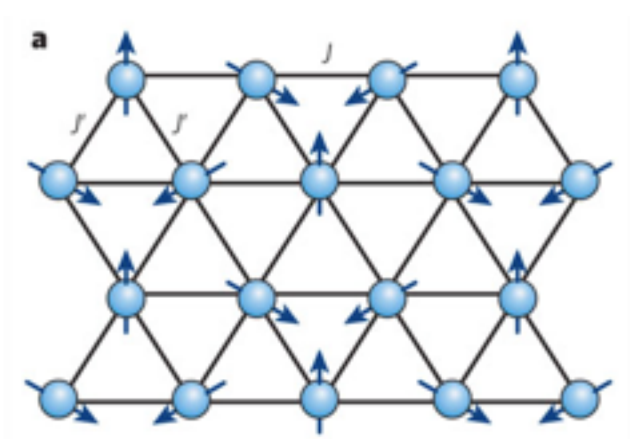
Emergence

Fractional excitations: charge $\dfrac{p}{q}$



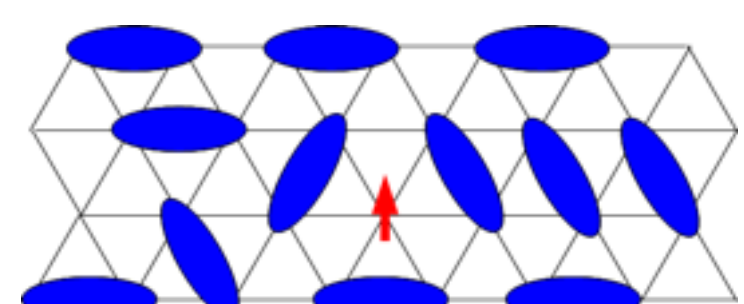
Frustrated magnetism

$S = \frac{1}{2}$ moment per unit cell



From L Balents, Nature 2010

Fractional excitations: spinons with $S = \frac{1}{2}$



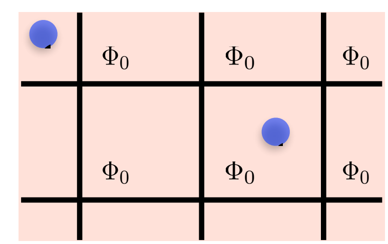
Lieb-Schultz-Mattis-type contraints

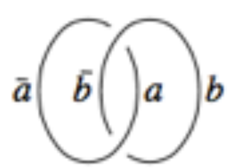
[Lieb, Schultz, Mattis 1963]

A constraint between the symmetry implementation in the UV
(e.g. electron filling, spin of magnetic moments)
and the emergent theory in the IR.

IR: fractional excitations

UV: fractional filling



$s_{ab} \sim$ 

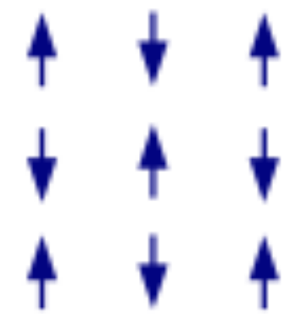
$$H = \frac{1}{2} \sum_{i<j} V(r_i - r_j) + H_K$$

Lieb-Schultz-Mattis theorem in two dimensions

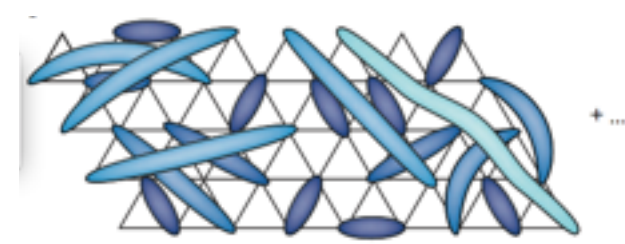
[Lieb, Schultz, Mattis 1963]
[Oshikawa, 1999; Hastings, 2005]

A magnet with half-integral spin / unit cell is either:

1. Symmetry broken



Spin liquid {
2. Gapless



3. Topologically ordered: anyonic excitations

~~Symmetric & short-range entangled (e.g. product state or SPT-phase)~~

LSM is a “no-go” for sym-SRE phases

Jian-Po-Watanabe-Vishwanath-Zaletel-Parameswaran-Turner-Arovas-Chen-Gu-Wen-Hastings-Oshikawa-Lieb-Schultz-Mattis Theorem(s):

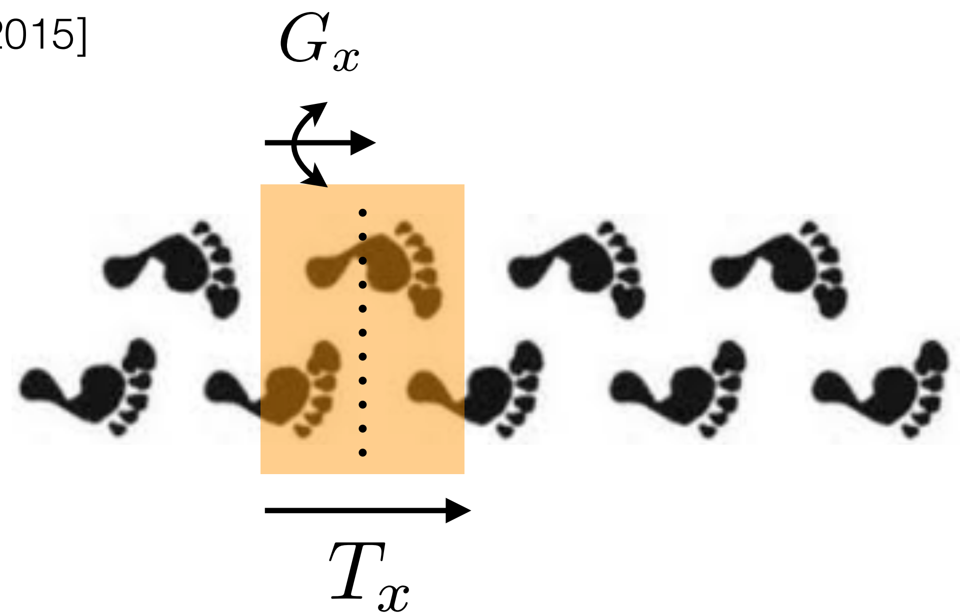
Wave 1: Translation only: “spin per unit cell”

[Lieb, Schultz, Mattis 1961; LSM - Affleck; Oshikawa, 1999; Misguich 2002; Hastings, 2003]

Wave 2: Non-symmorphic (glides and screws): “spin per reduced unit cell”

[Parameswaran et al. 2013; Watanabe, Po, Vishwanath & Zaletel PNAS 2015]

A magnet with integer spin per unit cell may still have a LSM-type no-go if there is half-integer spin in a “reduced” unit cell defined by glides & screws



Point group symmetries (rotation, reflection, inversion) were never used

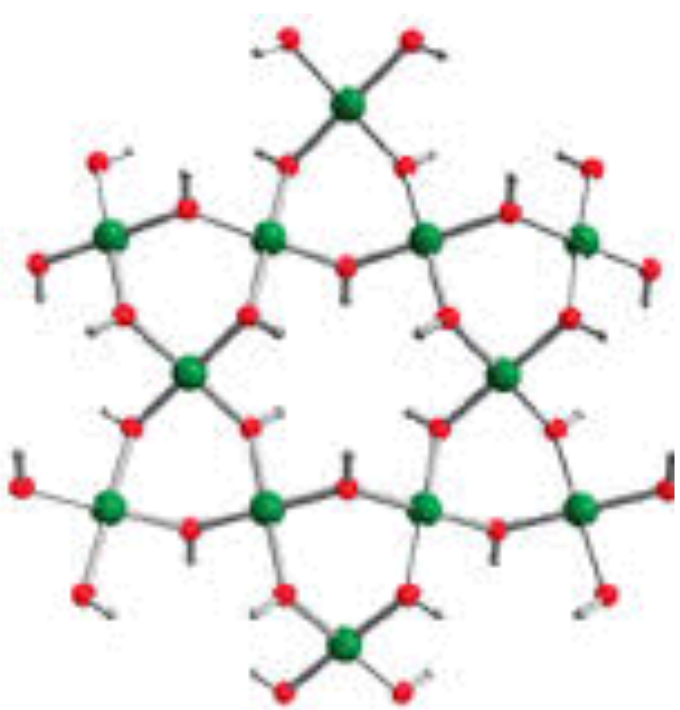
This talk: No-gos from the full space group

Starting point: the lattice of spins

Consider a magnet of spins symmetric under $\mathcal{S} \times SO(3)$

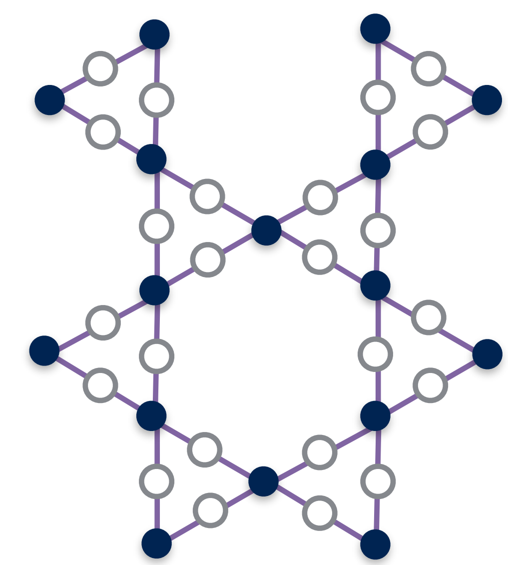
space group spin rotation

Each site carries either integer (○) or half-integer (●) spin



Forget the other microscopic details:
e.g. S = 1/2 is “same” as S = 3/2

Λ



Question: can the lattice Λ ever have a symmetric, short-range-entangled ground state?

Conjecture

A symmetric, short-range entangled ground state

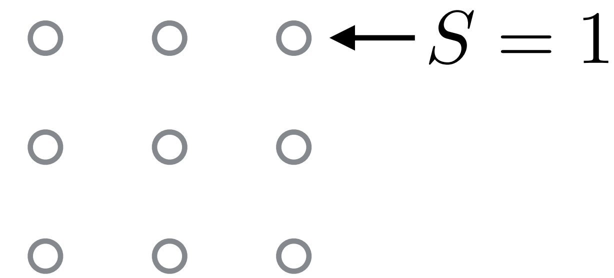
is possible only if Λ is “equivalent” to a “trivial” lattice.

need to define

“Trivial” lattice

A lattice is “trivial” if it only contains integer spins.

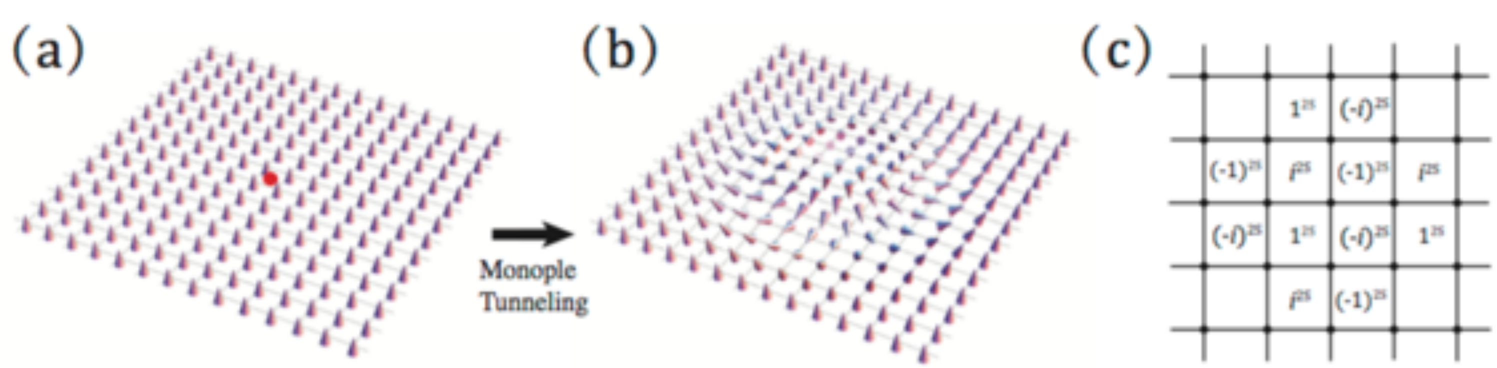
Example: spin-1 square lattice
with C_4 & mirror symmetries



There is no “obvious” symmetric SRE state here. AKLT construction requires S = [coordination number] / 2 = 2.

Haldane’s analysis of NLSM suggest monopoles see staggered flux $(i)^{2S}$

[Haldane 88; Read & Sachdev 90]



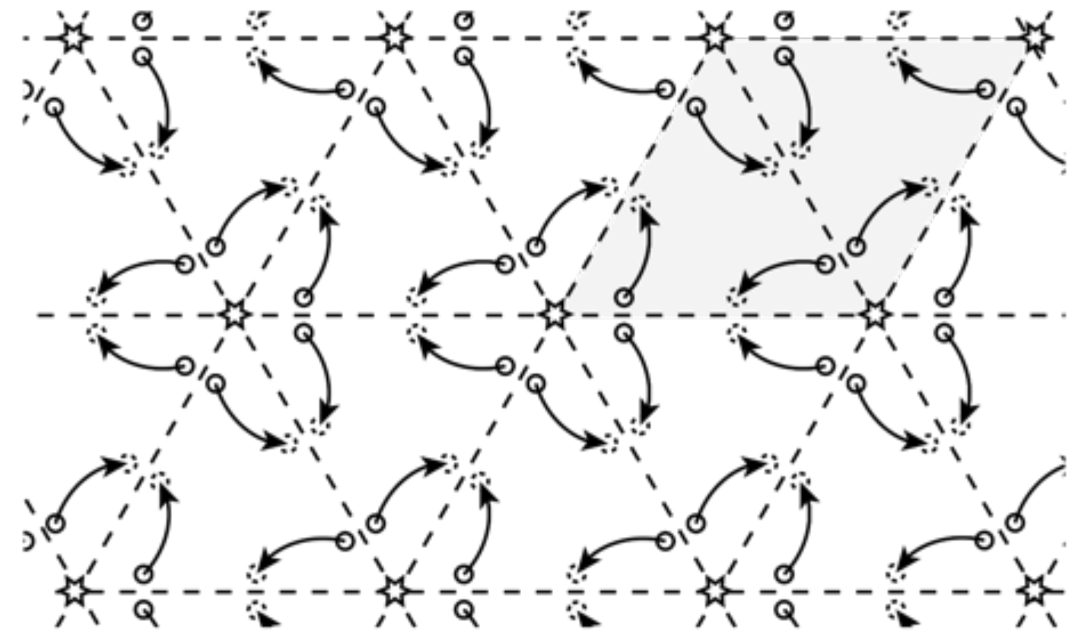
Nevertheless, a generalized AKLT construction, “tensor networks,” allow us to propose manifestly sym-SRE phases [CM Jian & Zaletel, PRB 93 2016]

“Equivalence” relation: $\Lambda \sim \Lambda'$

Rule 1: You can move spins around, so long as the symmetry is preserved throughout

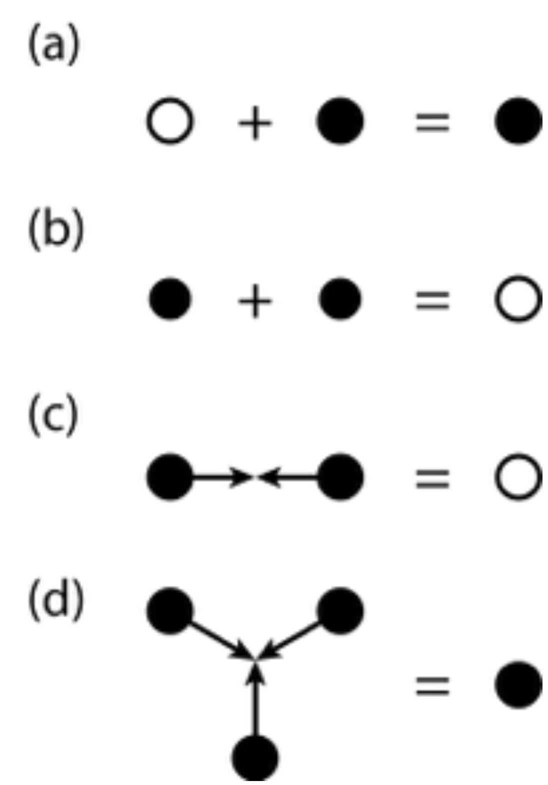
Example:

- Translations
- C_3 rotations
- Mirrors

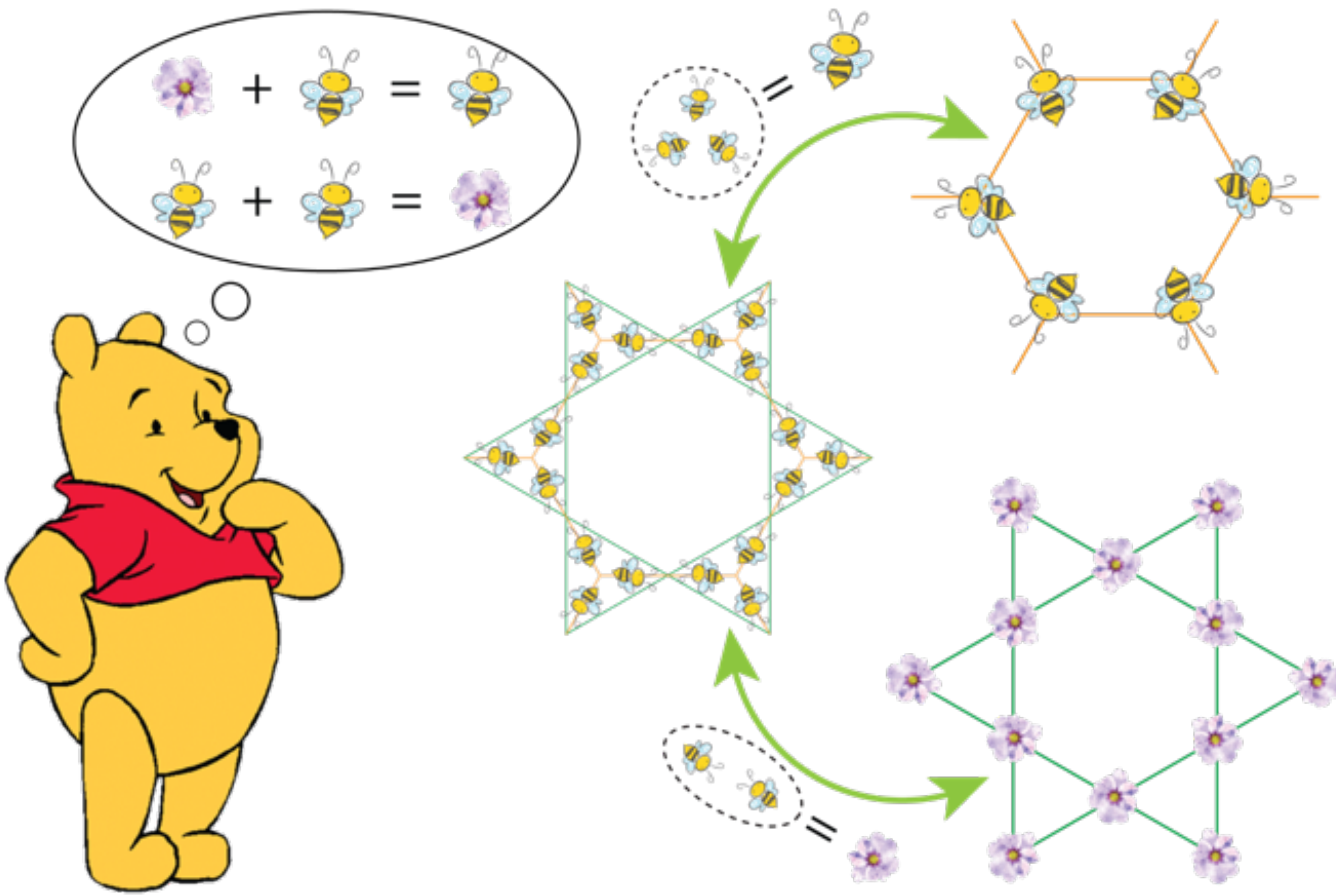


Rule 2: When spins collide, they “fuse” ($\mathbb{Z}_2$)

(coarse grained!)



Example: S=1/2 Honeycomb lattice



Half-integer honeycomb ~ integer kagome ~ “trivial”

No “obvious” AKLT-like state, recently shown there *is* a trivial S=1/2 honeycomb magnet ✔ [Kim, et al. PRB 94 2016]

Implies that graphene at charge neutrality *could* be in a completely insulating, symmetric, short-range entangled phase

"Lattice classification"

Given a space group $\mathcal{S}$, compute the inequivalent lattice types "$[\Lambda]_{\mathcal{S},\mathbb{Z}_2}$"

Forms an Abelian group under "stacking," with the empty lattice the identity.

Mike: "the way we were taught **not** to define equivariant homology in grad school."

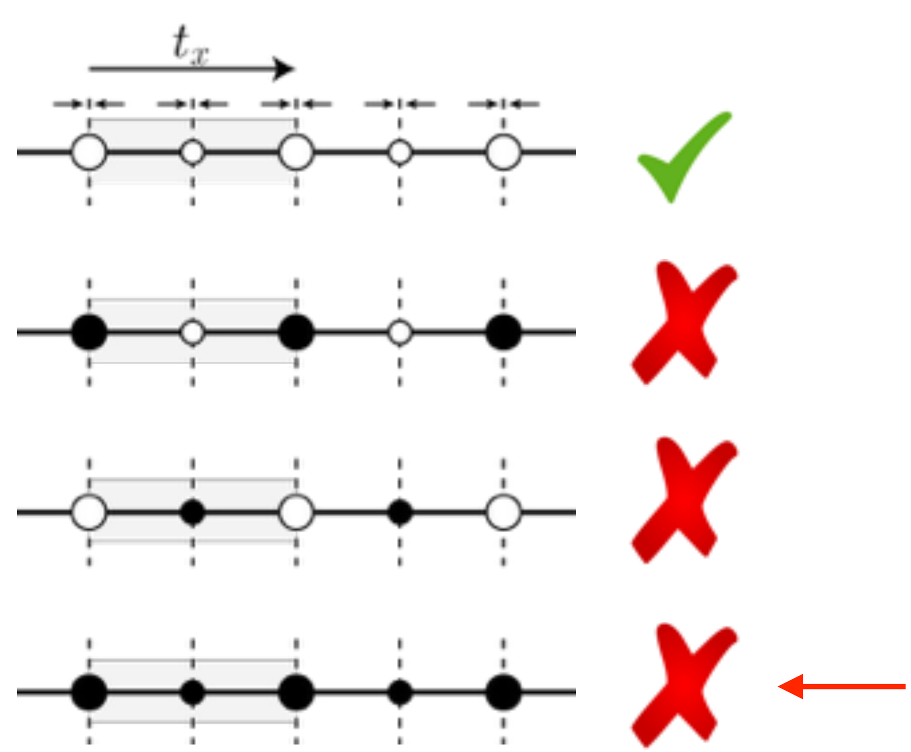
Conjecture: short-range-entangled only if $[\Lambda] = \mathbb{1}$

Example: 1D lattices with Translation & Reflection

Four inequivalent lattices:

$$[\Lambda] \in \mathbb{Z}_2 \times \mathbb{Z}_2$$

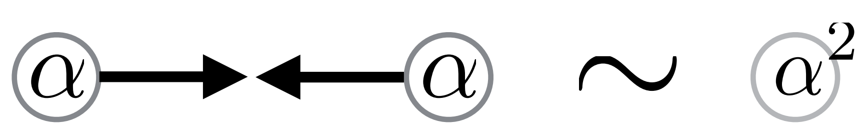
Invariant: is there half-integer spin on reflection plane?



Beyond LSM

Internal symmetry groups other than SO(3)

If the internal symmetry group is G , the role of “integer vs. half-integer” played by the *projective representation* of a site: $\alpha, \beta \in \mathcal{H}^2\left[G, U(1)\right]$



Fusion = abelian multiplication of projective classes

Lattice classification: The 17 2D space groups

For projective $\mathcal{H}^2\left[G, U(1)\right] = \mathbb{Z}_n$ reps.

Wallpaper group No.	Lattice homotopy
1,4,5	$\mathbb{Z}_n$
2,6	$\mathbb{Z}_n \times \mathbb{Z}_{\gcd(2,n)} \times \mathbb{Z}_{\gcd(2,n)} \times \mathbb{Z}_{\gcd(2,n)}$
3,8	$\mathbb{Z}_n \times \mathbb{Z}_{\gcd(2,n)}$
7,9	$\mathbb{Z}_n \times \mathbb{Z}_{\gcd(2,n)} \times \mathbb{Z}_{\gcd(2,n)}$
10,11	$\mathbb{Z}_n \times \mathbb{Z}_{\gcd(4,n)} \times \mathbb{Z}_{\gcd(2,n)}$
12	$\mathbb{Z}_n \times \mathbb{Z}_{\gcd(4,n)}$
13,14	$\mathbb{Z}_n \times \mathbb{Z}_{\gcd(3,n)} \times \mathbb{Z}_{\gcd(3,n)}$
15	$\mathbb{Z}_n \times \mathbb{Z}_{\gcd(3,n)}$
16,17	$\mathbb{Z}_n \times \mathbb{Z}_{\gcd(3,n)} \times \mathbb{Z}_{\gcd(2,n)}$

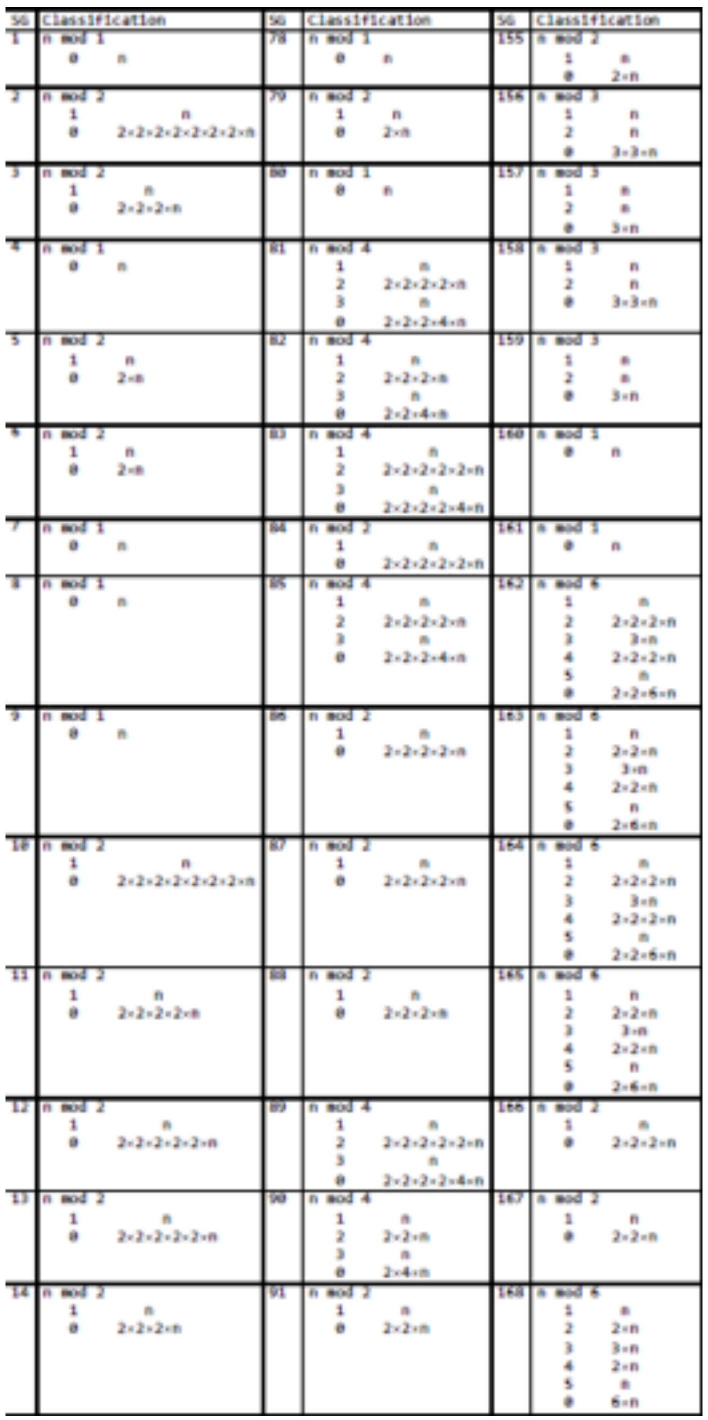
previous LSMs

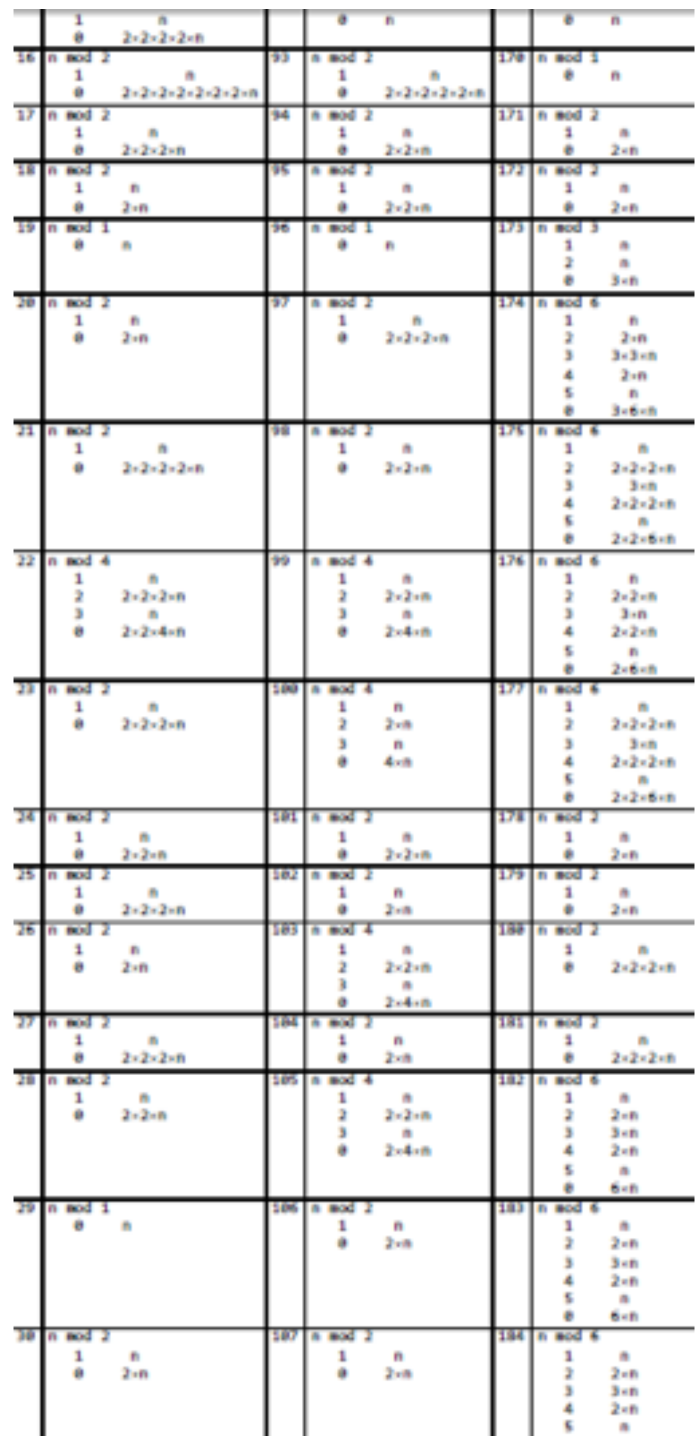
new constraints

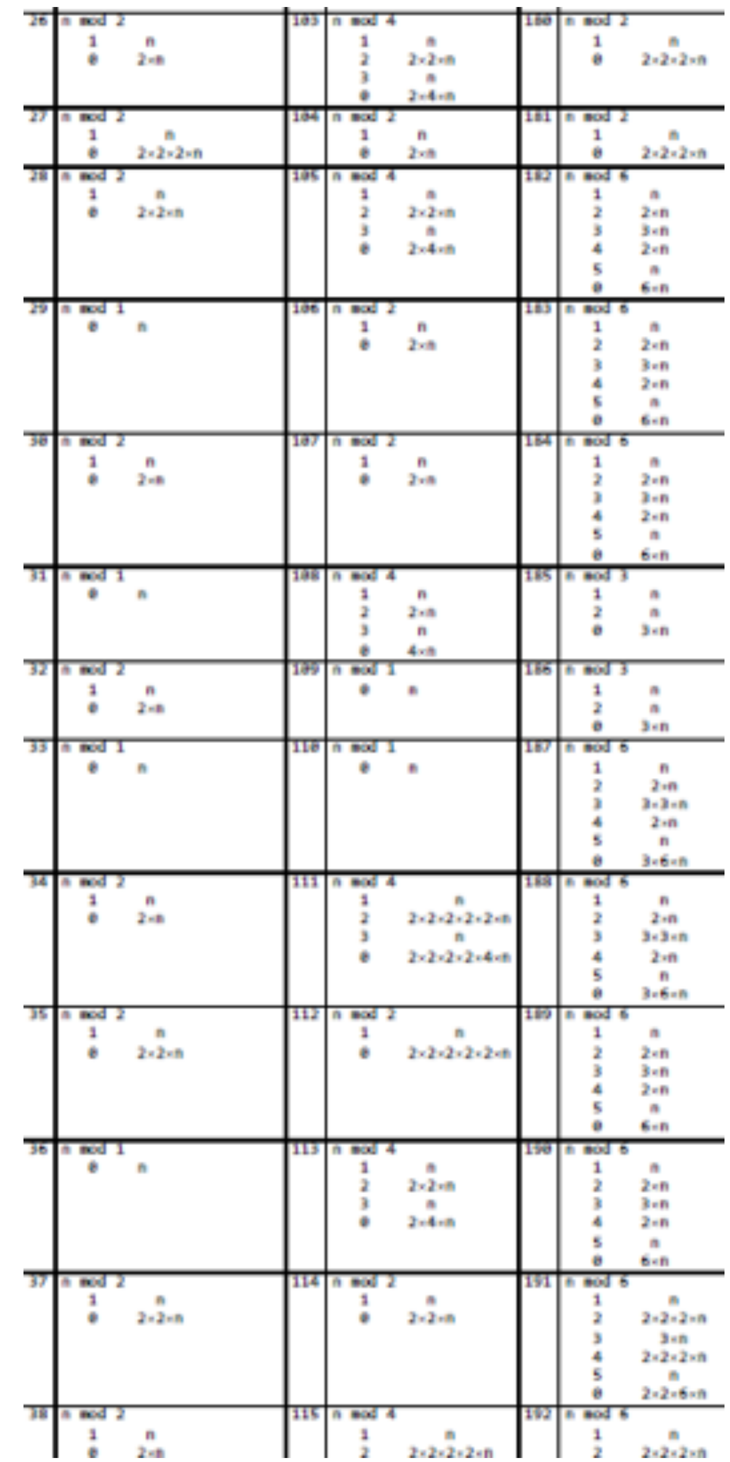
Non-trivial indices when the order of point group is commensurate with order of projective rep.

Lattice classification: 3D

230 space groups: luckily, can be automated using tables of “Wyckoff positions”

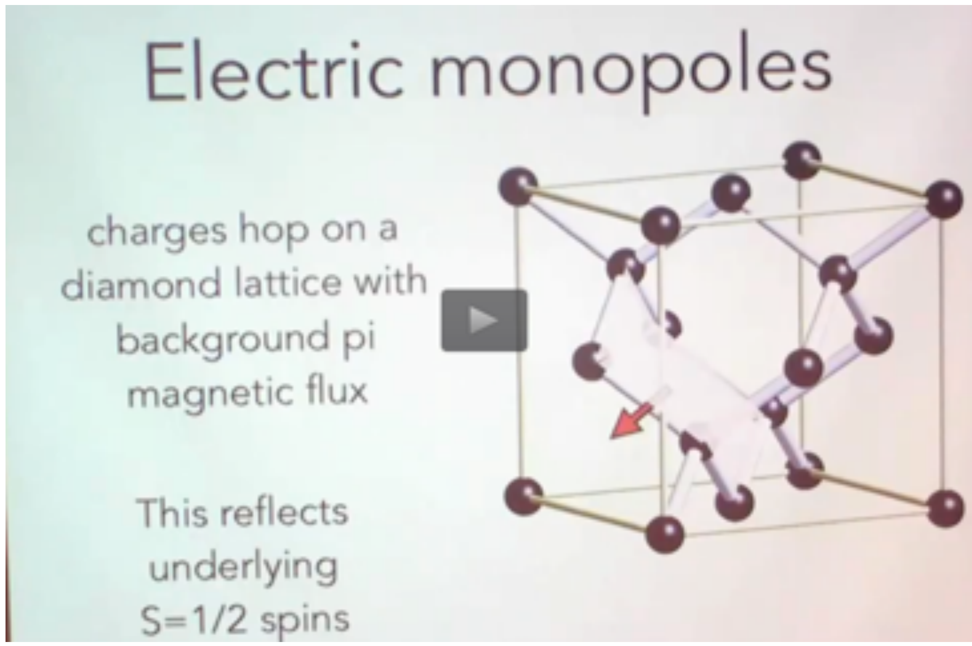


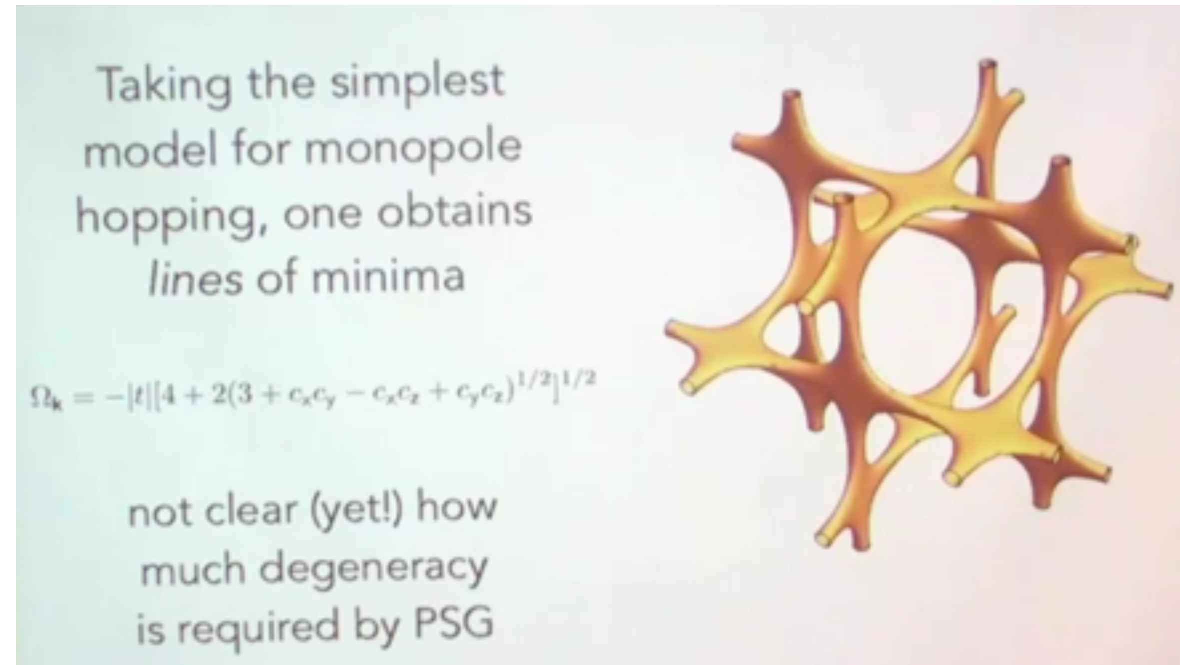




3D example

$Pr_2Zr_2O_7$





Space group No 227: Electron count is “trivial” for all previous “LSM”-bounds

$$[\Lambda]_{227,\mathbb{Z}_2} \in \mathbb{Z}_2^4$$

Lattice of Pr: non-trivial element,
(presumably) explains inevitable degeneracy

[caveat: spin-orbit coupled & we haven’t yet proved for pure time-reversal]

“Proof” in 2D

Conjecture: short-range-entangled only if $[\Lambda] = \mathbb{1}$

Step 1: A set of physical arguments which show a sym-SRE impossible if either:

phrased for SO(3)

1. Half-integer spin per “reduced unit cell” (LSM-like) [Watanabe, P, V, Z 2015]
2. Half-integer spin per “unit length of mirror line” [Watanabe, P, V, Z 2015]
3. Half-integer spin on a site with C_2 symmetry

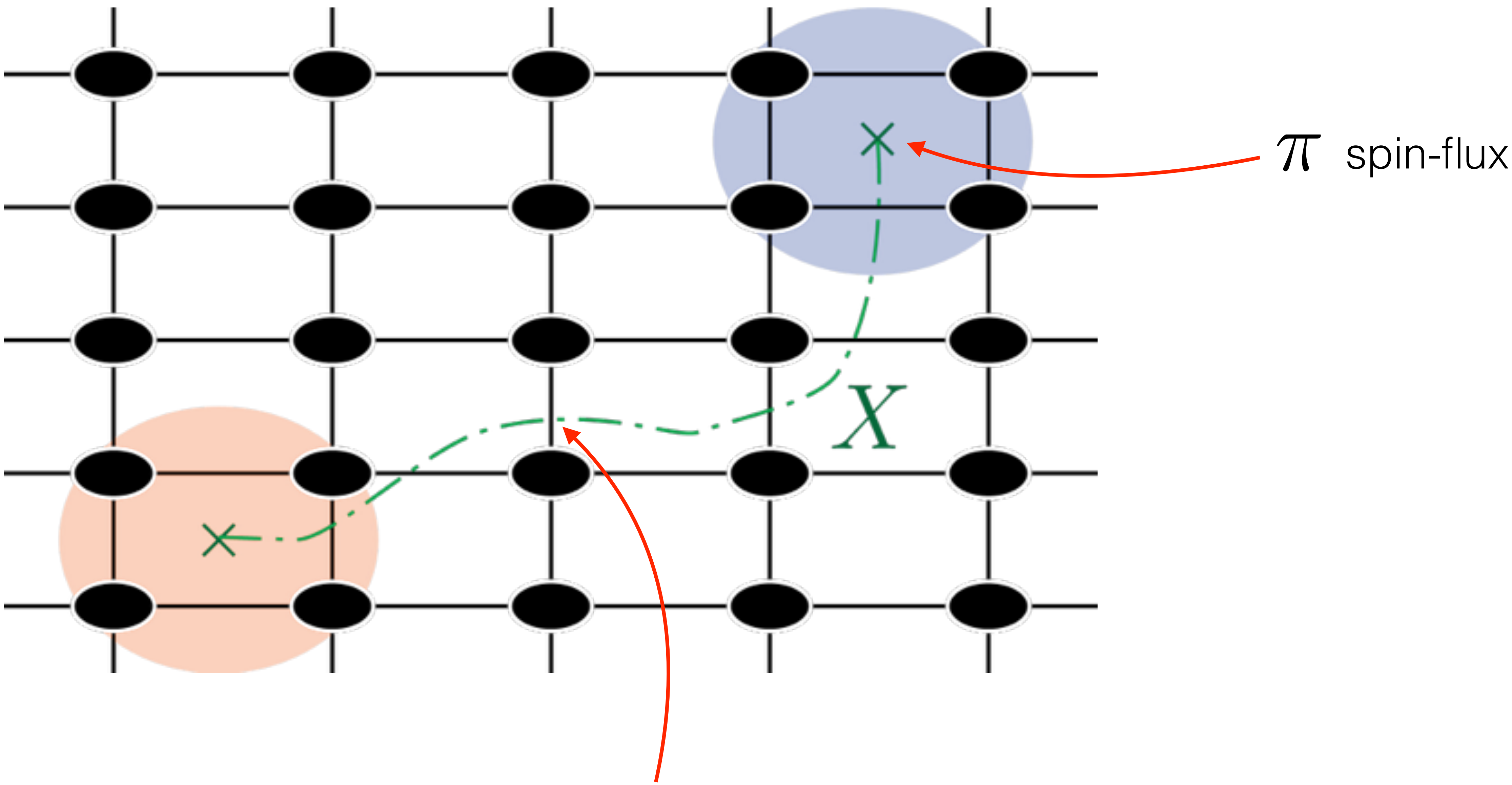
Step 2: Prove that the above physical arguments allow SRE phase if and only if $[\Lambda] = \mathbb{1}$

No-go for half-integer spin with C_2-site symmetry

Focus on the π- rotations, $1, X, Y, Z \in SO(3)$

Half-integer spin: $XZ = -ZX$

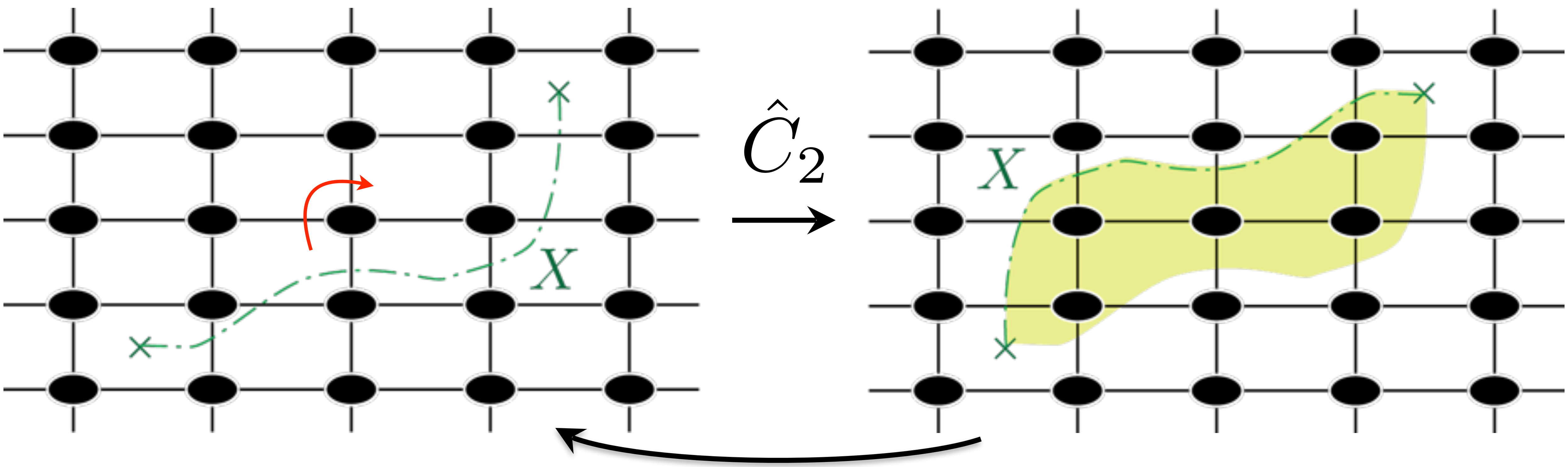
Tool: pair of C_2 related spin-flux on a torus



$$X_i X_j + Y_i Y_j + Z_i Z_j \to X_i X_j - Y_i Y_j - Z_i Z_j$$

What symmetries are there?

1) Rotation: $\hat{C}_2^{(X)}$



$$\prod_{j \in \square} X_j$$ "gauge transformation"

2) Internal symmetry: $Z = \prod_j Z_j$ $\quad X_j Z_j = -Z_j X_j$

$$\hat{C}_2^{(X)} Z = (-1)^{\mathrm{Vol}(\square)} Z \hat{C}_2^{(X)} = -Z \hat{C}_2^{(X)}$$ → ground state degeneracy

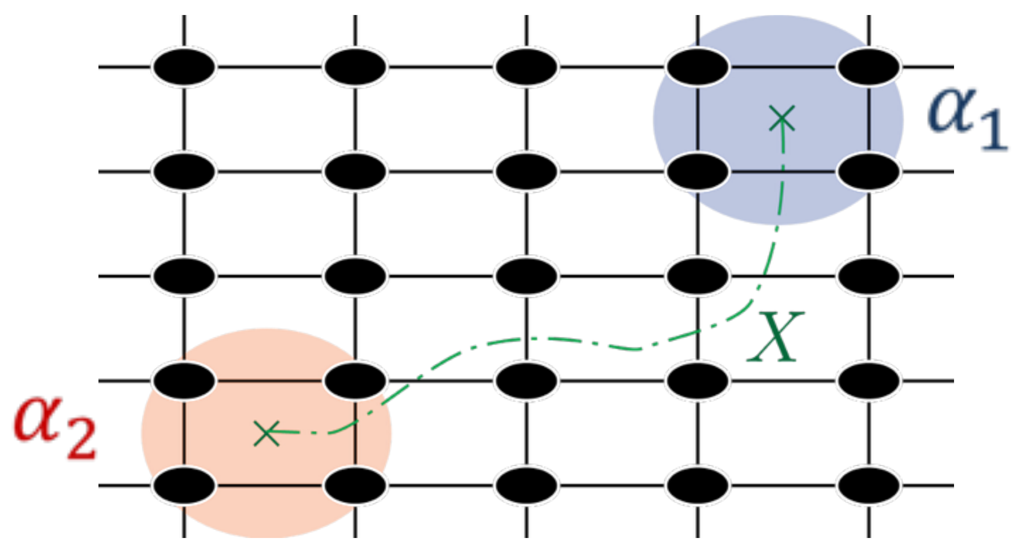
Why does this ground state degeneracy rule out SRE?

In some SPT phases, fluxes bind degeneracies - *GSD itself not a contradiction.*

"Decorated domain walls" [Chen, Lu, Vishwanath 2013]

Principle: in an SRE phase, all degeneracies are "localizable"

$d_1 d_2 = d$ -fold ground states: $\{|\alpha_1, \alpha_2\rangle : \alpha_1 = 1, \cdots d_1; \alpha_2 = 1, \cdots, d_2\}$

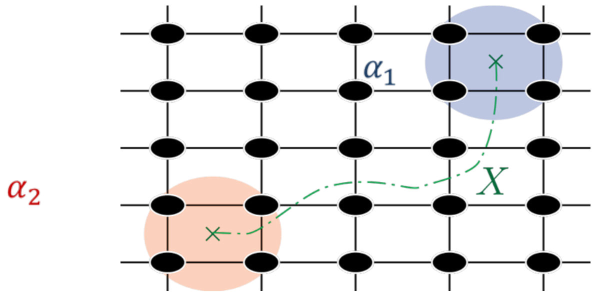


Density matrix near flux: $\rho_{12}^{\alpha_1 \alpha_2} = \rho_1^{(\alpha_1)} \otimes \rho_2^{(\alpha_2)} + \mathcal{O}(e^{-R/\xi})$

Density matrix away from flux: $\rho_{\overline{12}}^{\alpha_1 \alpha_2} = \rho_{\overline{12}}$

Very different than non-local "topological degeneracy" of non-Abelian anyons

Why does this ground state degeneracy rule out SRE?



Claim: it is impossible to realize $Z\hat{C}_2^{(X)} = -Z\hat{C}_2^{(X)}$ on the localized ground states $\{|\alpha_1, \alpha_2\rangle : \alpha_1 = 1, \cdots d_1; \alpha_2 = 1, \cdots, d_2\}$

Proof: Most general symmetry implementation on GSD:

$$d_1 = d_2 \quad \text{from } C_2$$

$$Z = Z_1 \otimes Z_2 \quad \text{from locality}$$

$$C_2^{(X)} = \mathrm{SWAP}_{12} \quad \text{without loss of generality: change of basis on '2'}$$

$$C_2^{(X)} Z C_2^{(X)^{-1}} = \mathrm{SWAP}_{12}(Z_1 \otimes Z_2) = Z_2 \otimes Z_1 \overset{?}{=} -Z_1 \otimes Z_2$$

This would require $Z_1 = \eta Z_2$, but

$$Z_1 \otimes Z_2 = \eta Z_2 \otimes Z_2 \neq -\eta Z_2 \otimes Z_2 = -Z_2 \otimes Z_1$$

Impossible to have required degeneracy in a SRE phase

“Proof” in 2D

Conjecture: short-range-entangled only if $[\Lambda] = \mathbb{1}$

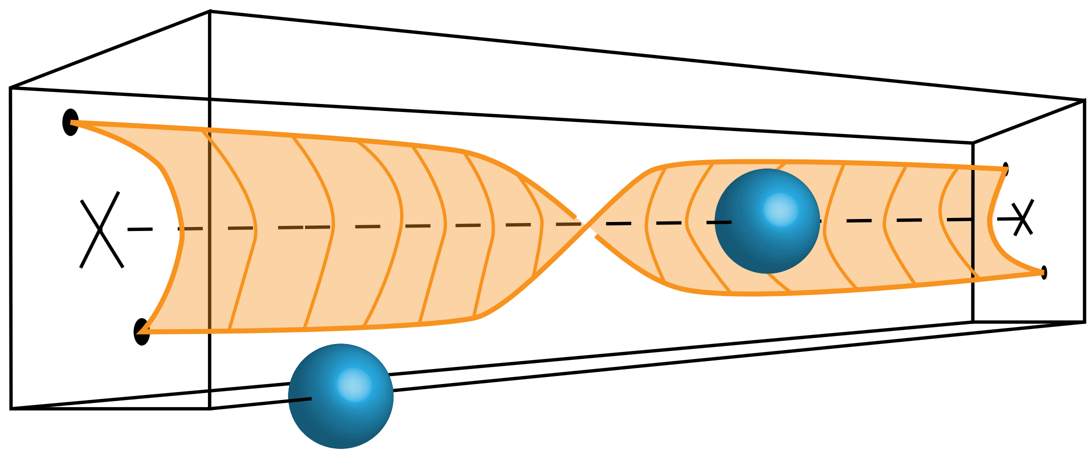
Step 1: A set of physical arguments which show a sym-SRE impossible if either:

phrased for SO(3)

1. Half-integer spin per “reduced unit cell” (LSM-like) [Watanabe, P, V, Z 2015]
2. Half-integer spin per “unit length of mirror line” [Watanabe, P, V, Z 2015]
3. Half-integer spin on a site with C_2 symmetry ✓

Step 2: Prove that the above physical arguments allow SRE phase if and only if $[\Lambda] = \mathbb{1}$

3D Conjecture: no complete proof yet



Manipulate flux-tubes on Bieberbach manifolds…

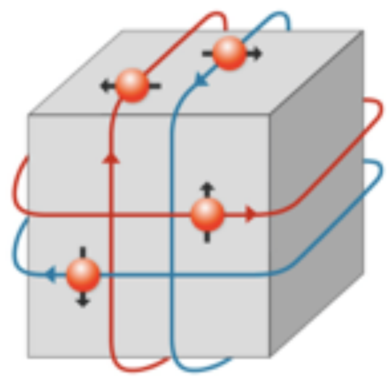
Many new constraints, but not yet complete

LSM constraints have deep connection to physics of topological insulators / SPT phases

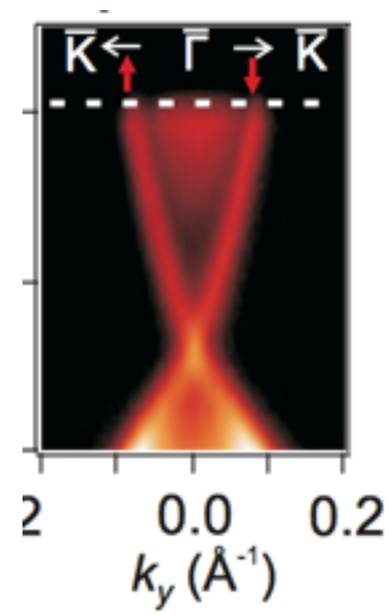
[Meng, Zaletel, Barkeshli, Vishwanath & Bonderson arXiv:1511.02263]

3D Topo Insulator:

[Hasan & Kane 2010;
Moore & Molenkamp 2010]



Bulk is insulating



[from Xia et al 2008]

Surface observed to be gapless
(odd # Dirac cones)

Robust with U(1) and time-reversal

With interactions
surface of an SPT is either:

symmetry broken
gapless
or
fractionalized insulator
“anomalous surface topological order”

[Callan & Harvey 1985; Barkeshli et al. 2014;
X. Chen et al. 2015; Fidkowski Vishwanath 2015;
Wang, Lin, Levin 2015]

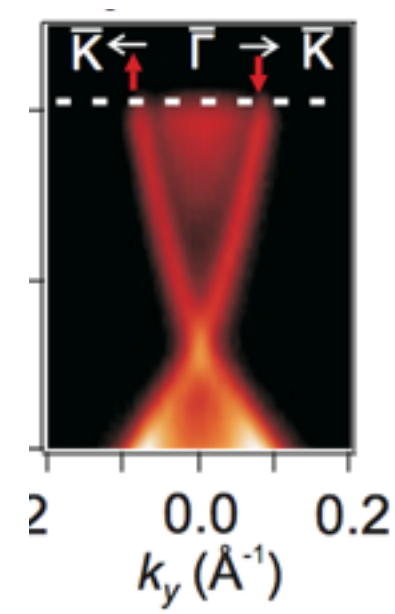
2D $S = 1/2$ magnet:

LSM - Theorem:

symmetry broken
gapless
or
fractionalized

Surface of 3D SPT:

symmetry broken
gapless
or
fractionalized



[from Xia et al 2008]

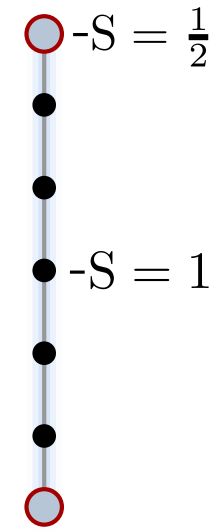
But one is a surface, and one isn't!

How to view the S=1/2 magnet as a surface

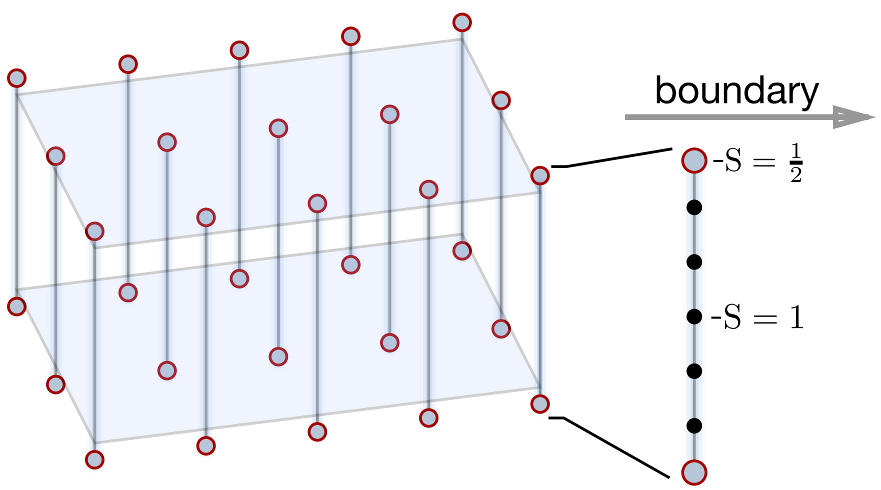
[Cheng, Zaletel, Barkeshli, Vishwanath & Bonderson, 1511.02263 (2015)]

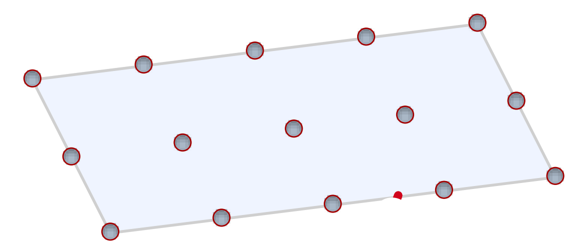
Spin-1 chain has
Spin-1/2 edge states

The "Haldane" / "AKLT" phase



Pack together Haldane chains into a 3D crystal:





2D Spin-1/2 magnet

=

3D "weak" interacting *bosonic* SPT

[Fu, Kane, Mele 2007; X Chen et al; Song et al 2016]

Technical trick to leverage knowledge of SPTs

How to view the S=1/2 magnet as a surface

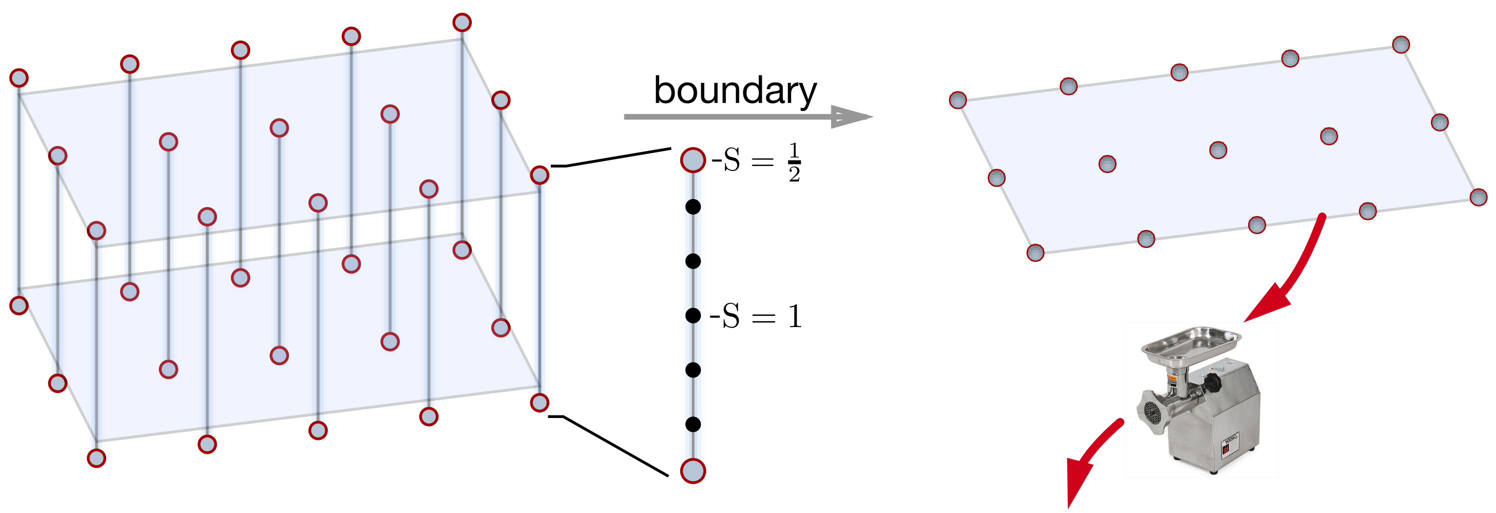
[Cheng, Zaletel, Barkeshli, Vishwanath & Bonderson, 1511.02263 (2015)]

Bulk-boundary correspondence between 3D SPTs - Anomalous Surface order

LSM-type constraints for 2D magnets

Weak 3D SPT phase

Gapped 2D $\mathrm{S} = \frac{1}{2}$ magnet



$$\Omega_{\text{weak}} \in \mathcal{H}^4\left[\text{translation} \times \mathrm{SO}(3)_{\text{int}}, \mathrm{U}(1)\right] = \mathscr{O} \in \mathcal{H}^4\left[\text{translation} \times \mathrm{SO}(3)_{\text{int}}, \mathrm{U}(1)\right]$$

For translation symmetry, this gives the most general constraint.

Example: the Double Semion model forbidden in an S=1/2 magnet [Zaletel, Vishwanath 2015]

Lattice classification: the boundary of M Hermele's talk?

(with D. Else & M. Hermele)

Wallpaper group No.	Lattice homotopy
1,4,5	$\mathbb{Z}_n$
2,6	$\mathbb{Z}_n \times \mathbb{Z}_{\gcd(2,n)} \times \mathbb{Z}_{\gcd(2,n)} \times \mathbb{Z}_{\gcd(2,n)}$
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7,9	$\mathbb{Z}_n \times \mathbb{Z}_{\gcd(2,n)} \times \mathbb{Z}_{\gcd(2,n)}$
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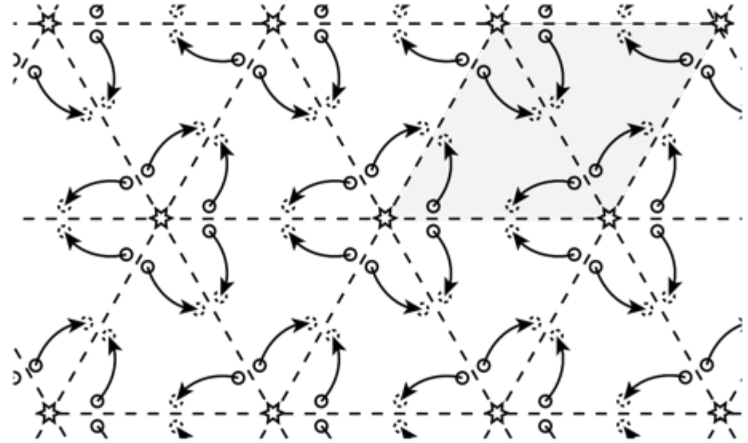
15	$\mathbb{Z}_n \times \mathbb{Z}_{\gcd(3,n)}$
16,17	$\mathbb{Z}_n \times \mathbb{Z}_{\gcd(3,n)} \times \mathbb{Z}_{\gcd(2,n)}$

? ⟷

Anomaly classes of space-group SETs?

Thanks!



Adrian Po

Chao-Ming Jian

Haruki Watanabe

Adrian Po, et al., arXiv:161x.xxxxx

H. Watanabe, HC Po, A. Vishwanath & M. P. Zaletel, PNAS 112, 14551 (2015)

Ashvin

Cheng, Zaletel, Barkeshli, Vishwanath & Bonderson, 1511.02263 (2015)