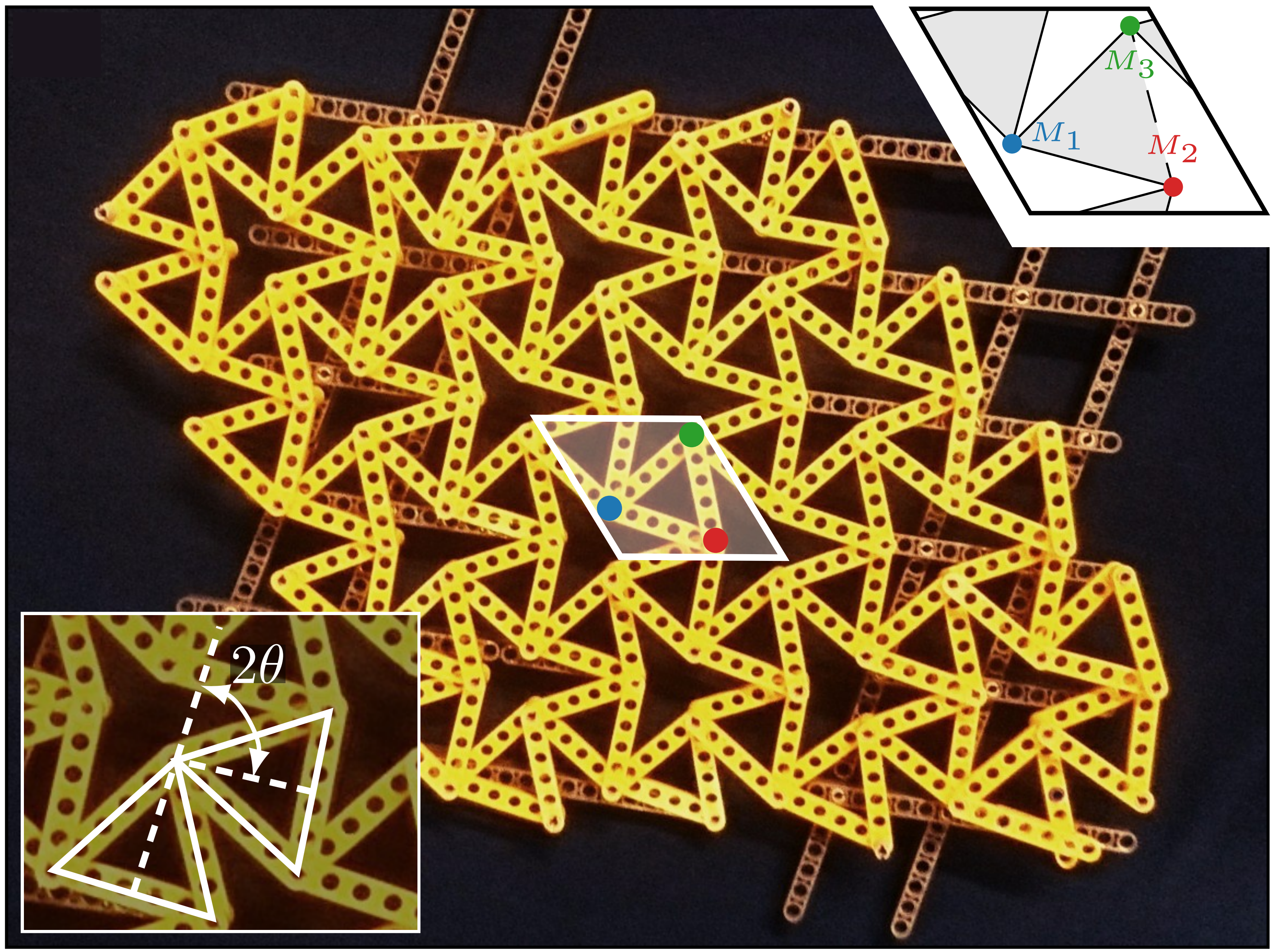
Dualities and non-abelian mechanics

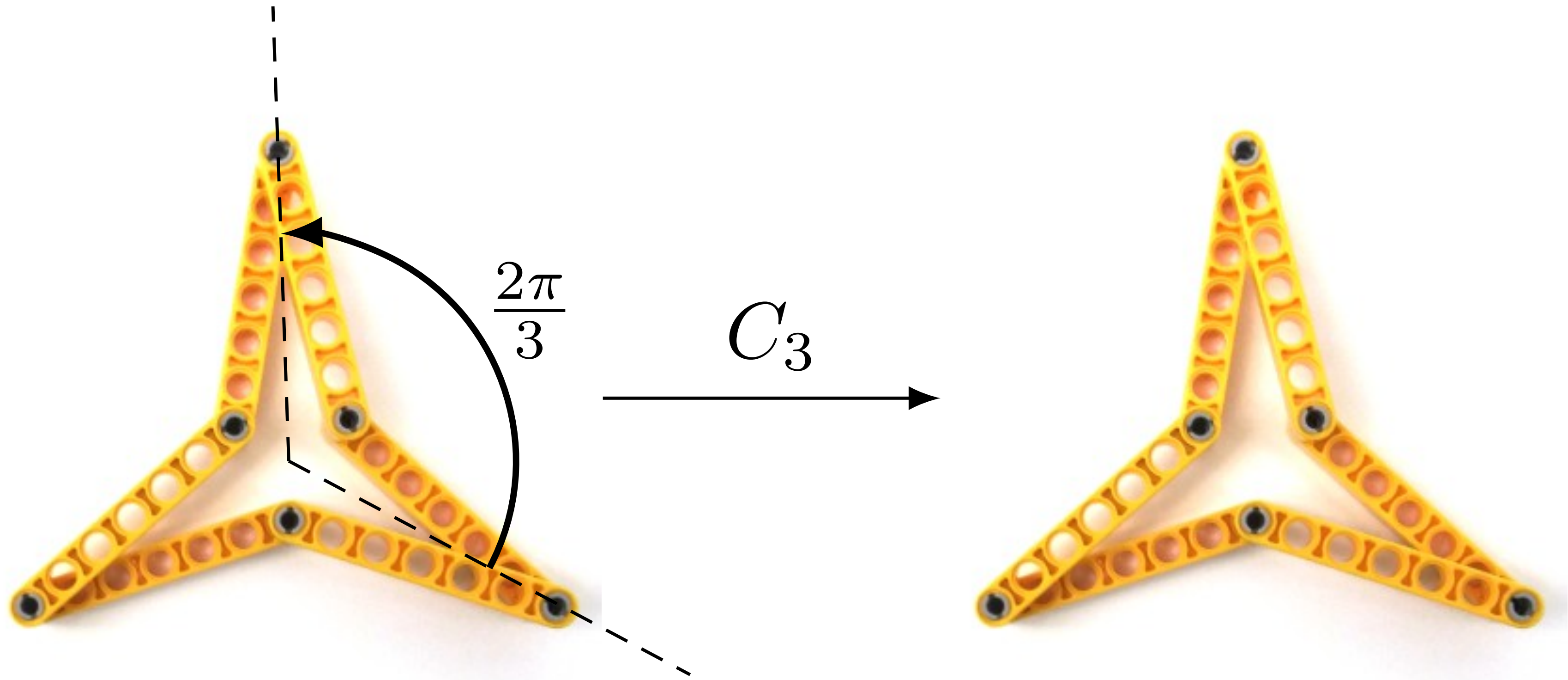
Vincenzo Vitelli
UChicago



M. Fruchart, Y. Zhou, V. Vitelli, Nature (in press), arXiv:1904.07436

What’s a symmetry ?

What's a symmetry ?



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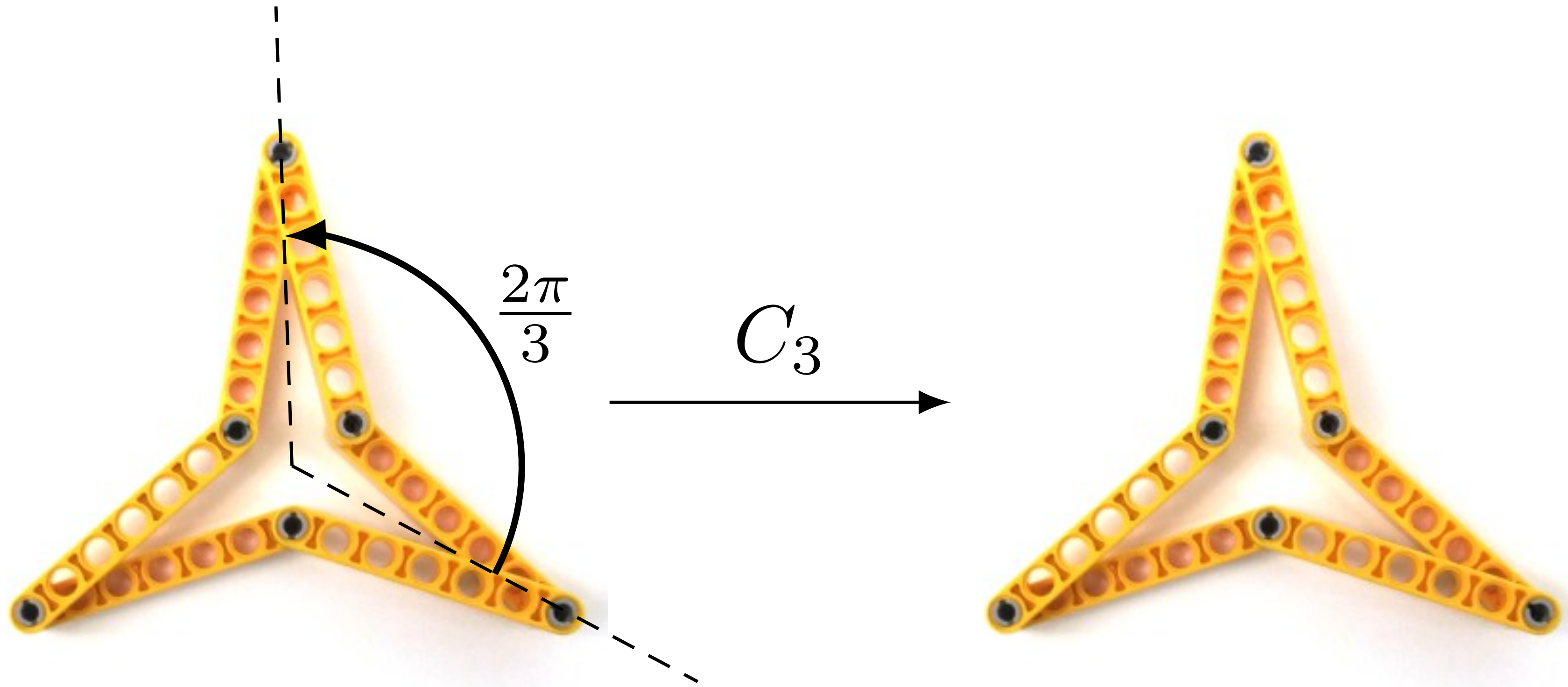
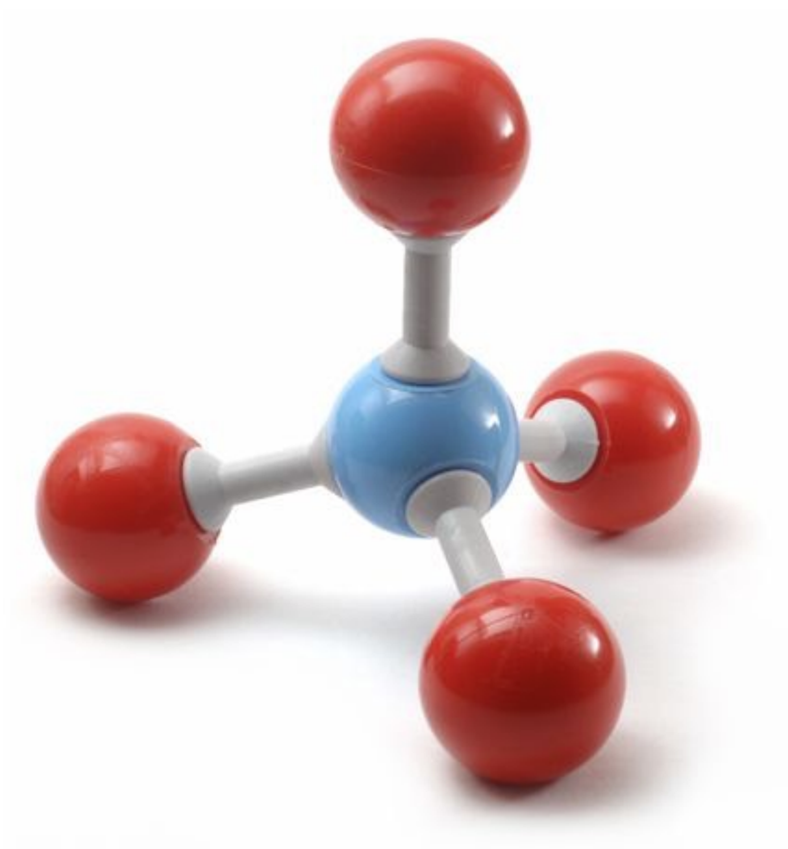


$T\Big($  $\Big) =$ 

Symmetries are useful

Symmetries are useful

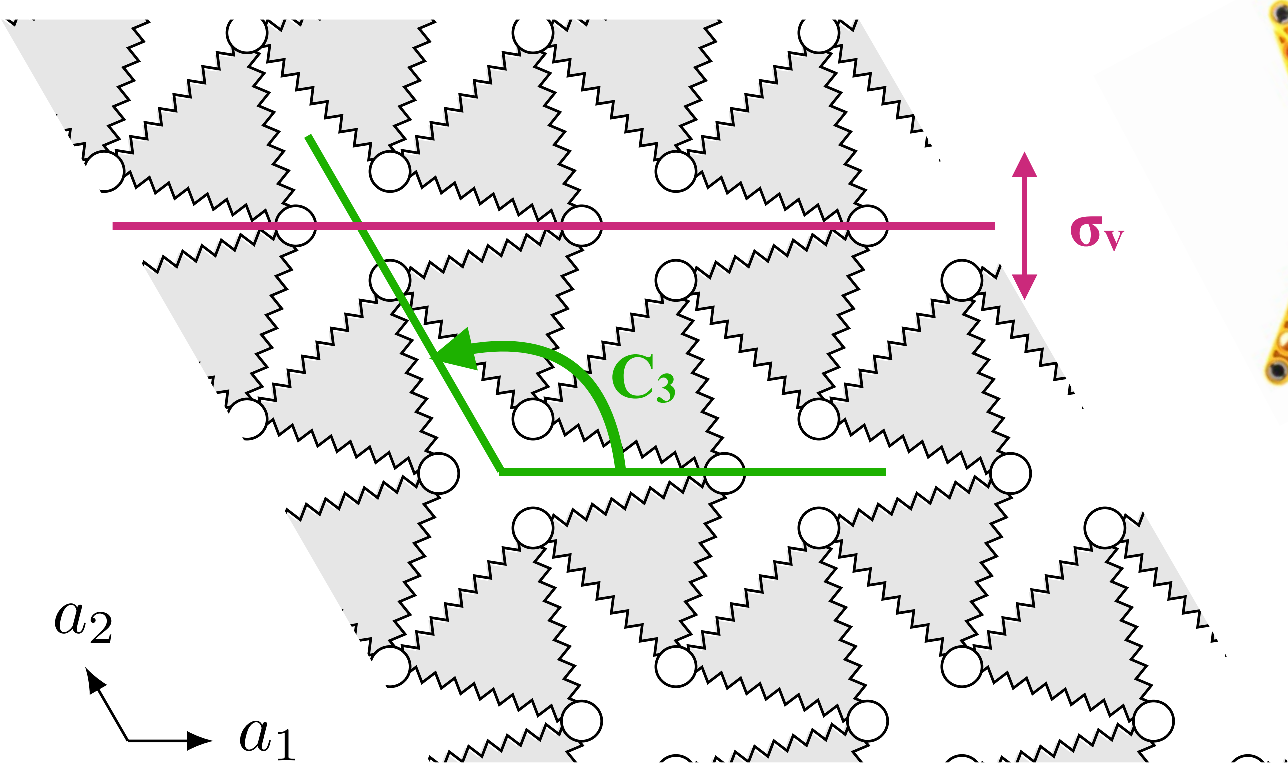


Character table

$\Im_d$	E	$8C_3$	$3C_2$	$6S_4$	$6\sigma_d$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
F_1	3	0	-1	1	-1
F_2	3	0	-1	-1	1
Σ	8	-1	0	0	0

Symmetries determine the degeneracies of vibrational modes

Symmetries are useful



point group		$c_{((ij)(k\ell))}$
1	C_1	6
2	C_2	6
m	C_s	4
2mm	C_{2v}	4
4	C_4	4
4mm	C_{4v}	3
3	C_3	2
3m	C_{3v}	2
6	C_6	2
6mm	C_{6v}	2

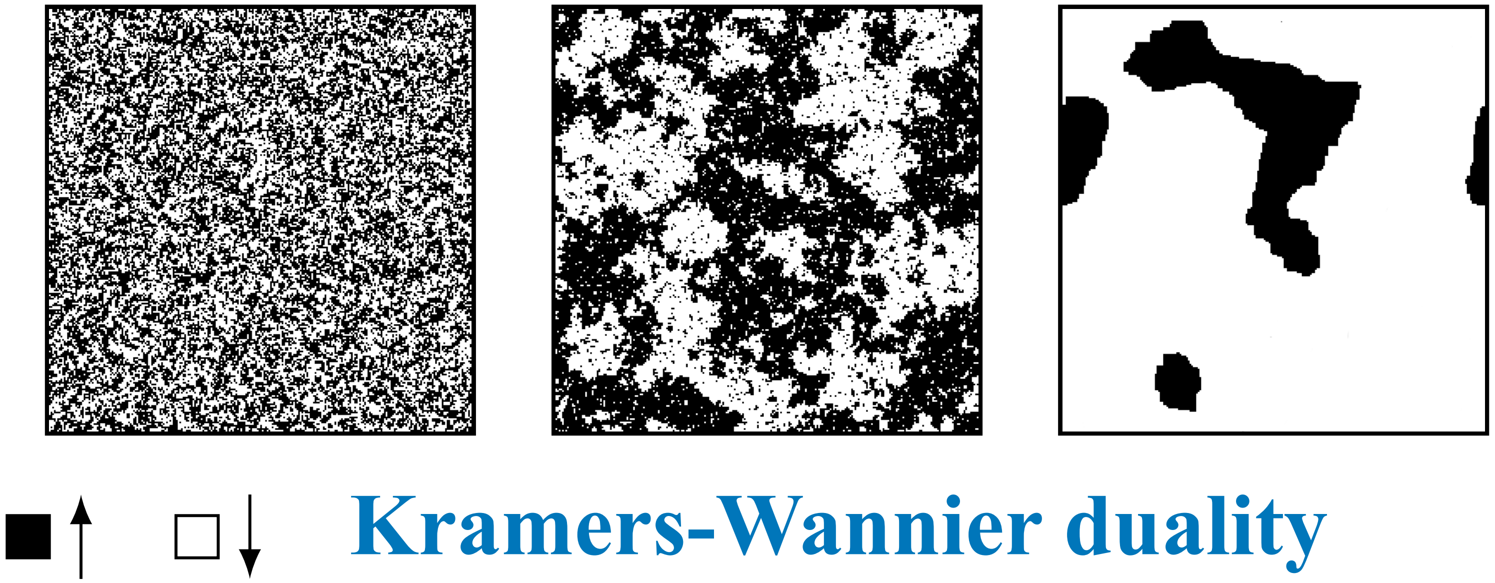
point group C_{3v} (3m)

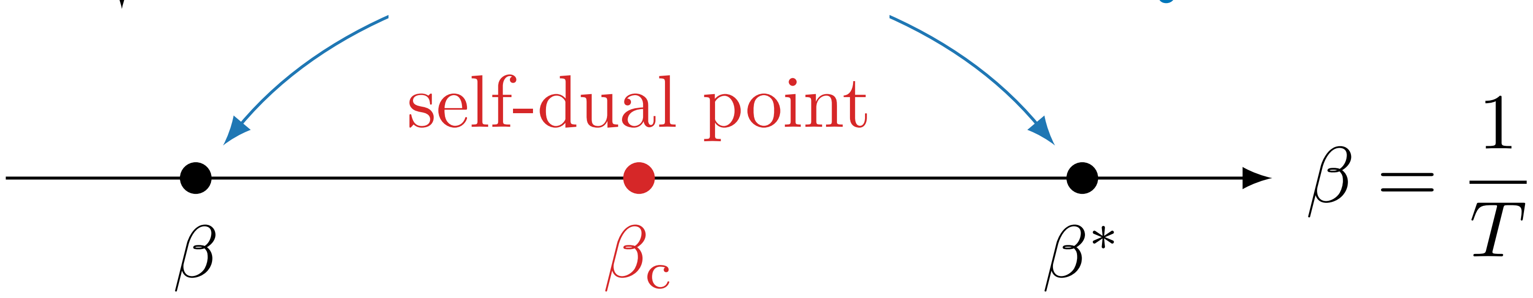
C_3 rotation + σ_v mirror

Symmetries determine the number of independent elastic moduli

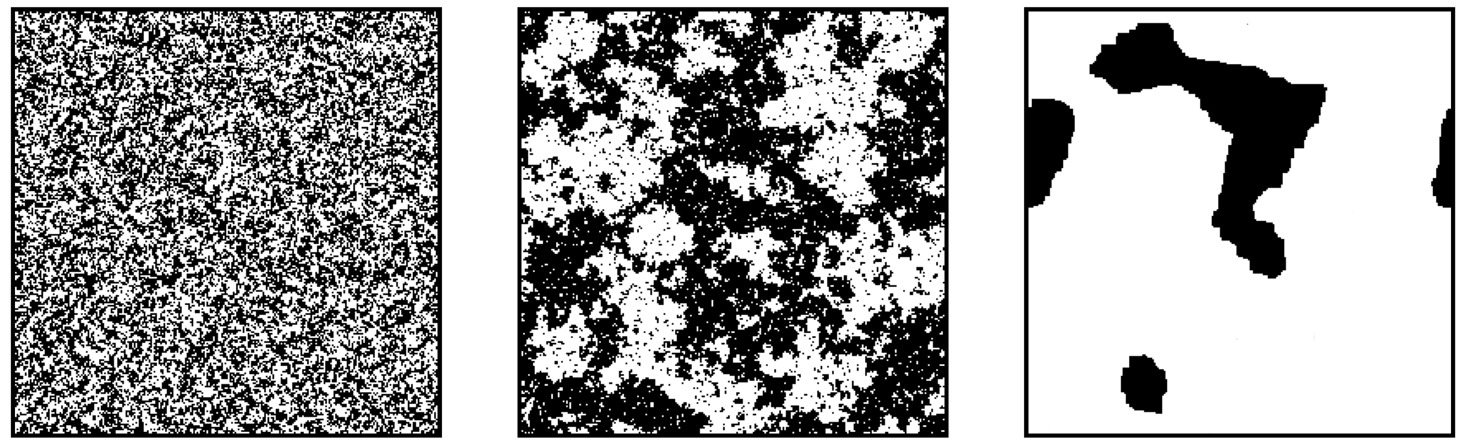
What’s a duality ?

What's a duality ?





What's a duality ?



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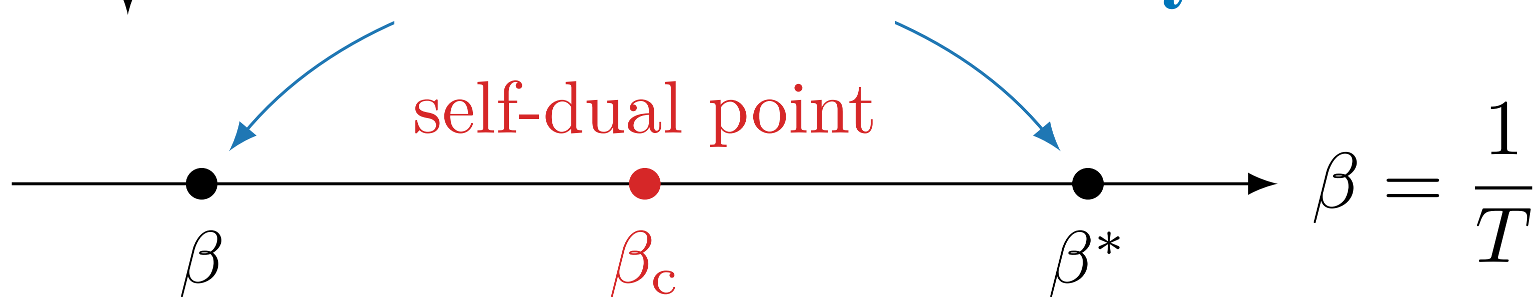
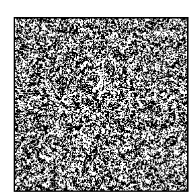
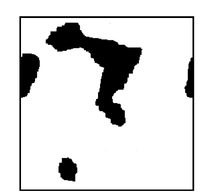
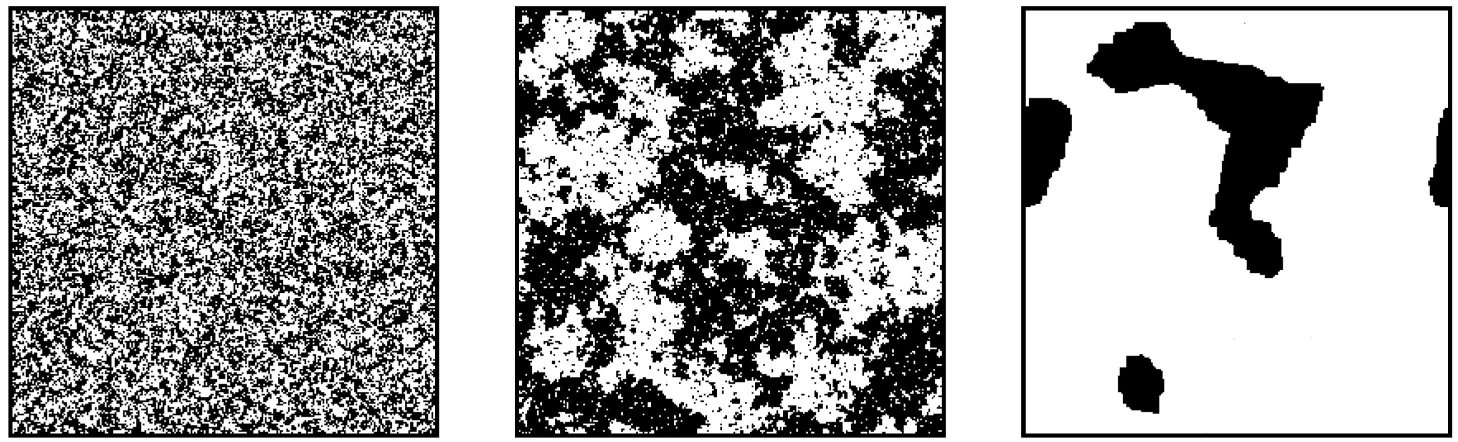
Kramers-Wannier duality



$T($  $) =$ 

The self-dual point



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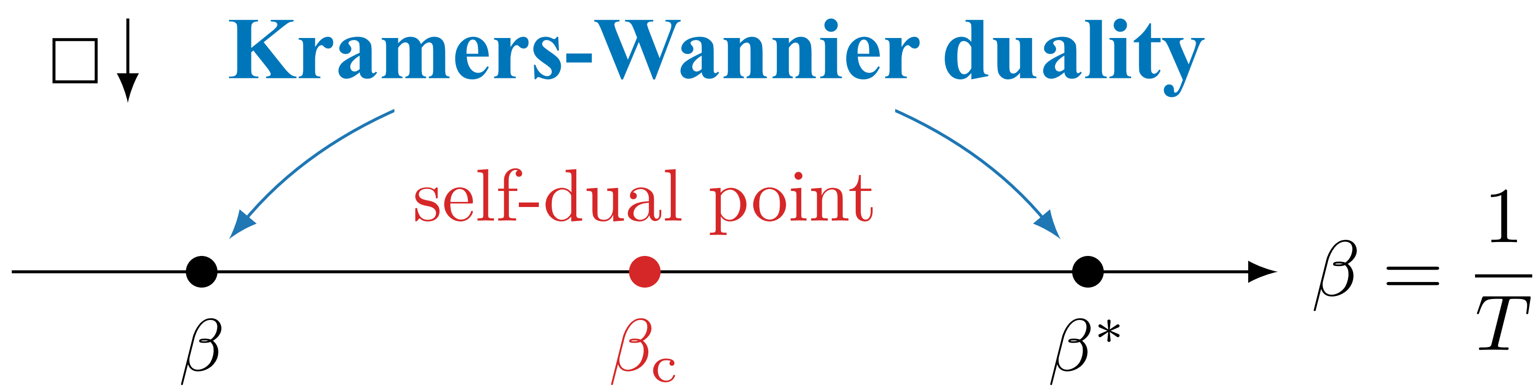
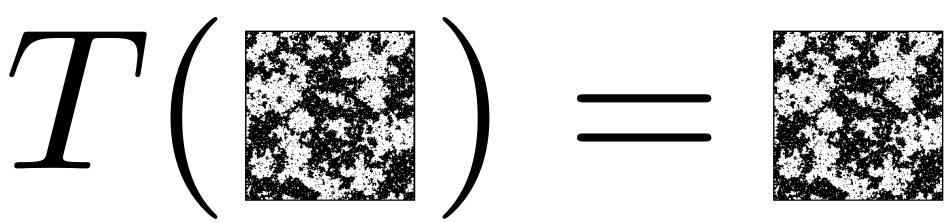
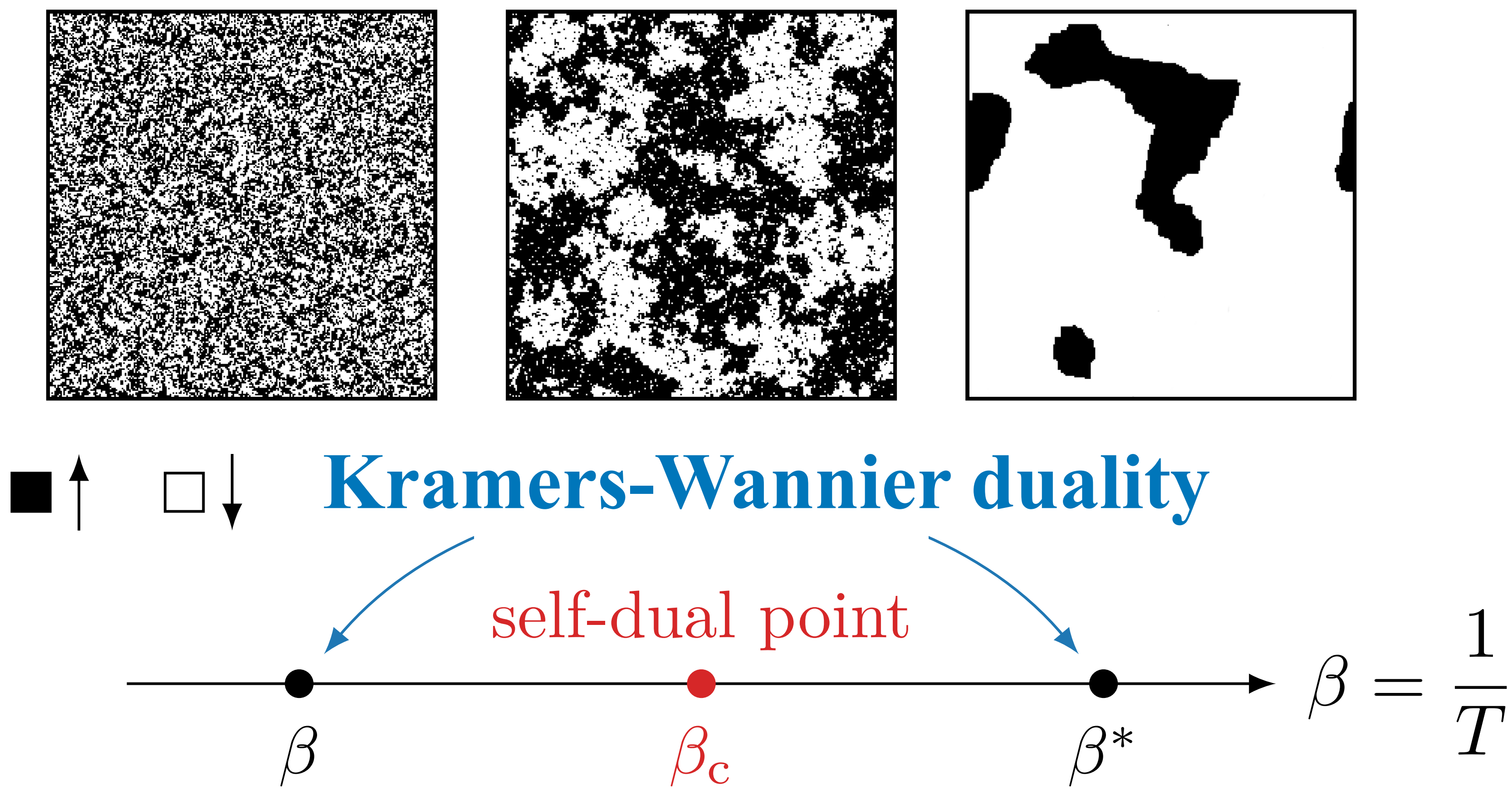
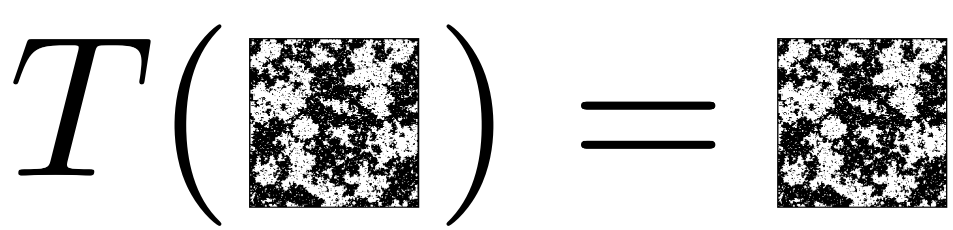




The self-dual point

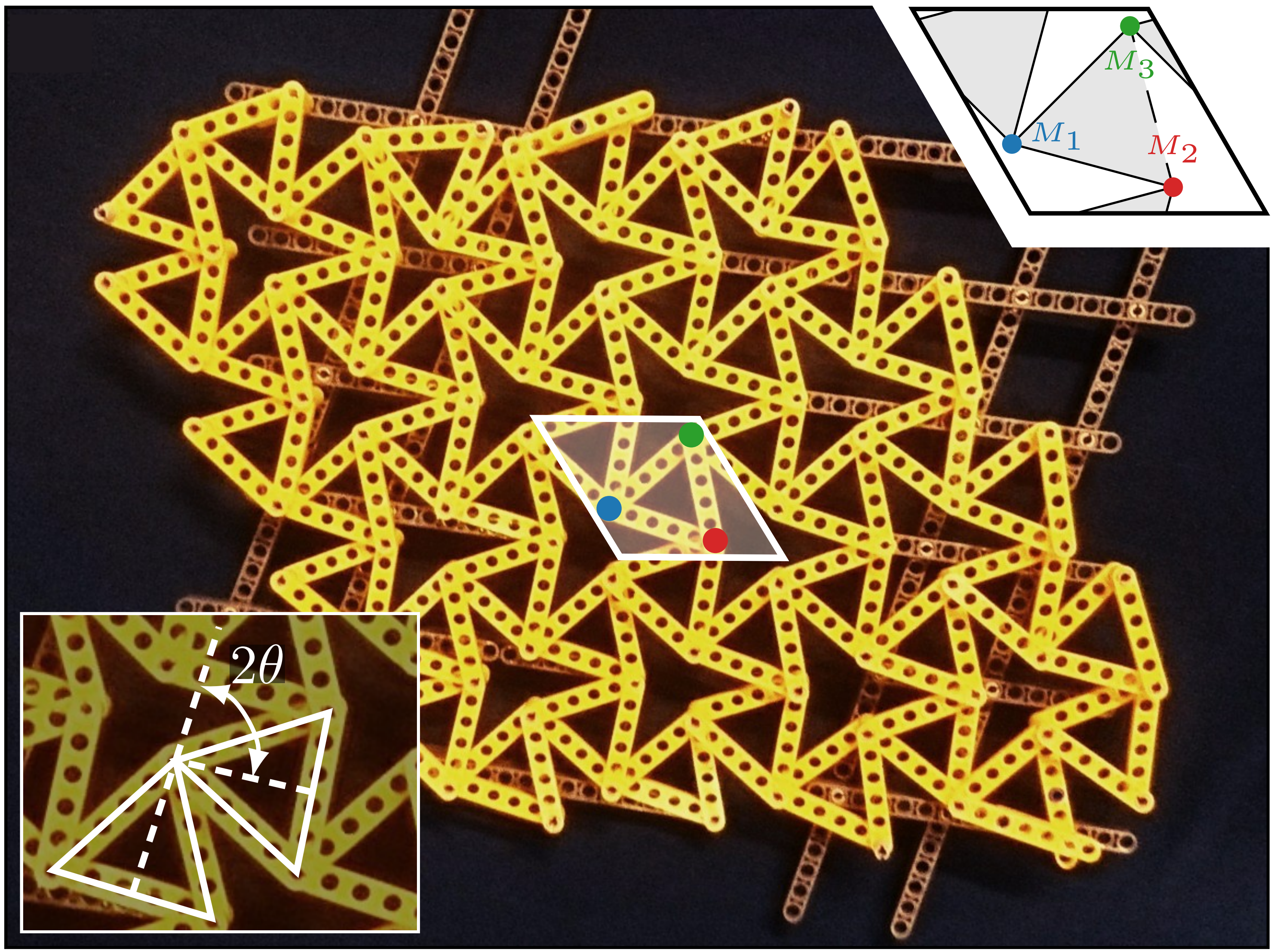




Self-duality is an emergent symmetry

Can you engineer self-dualities
for materials design ?

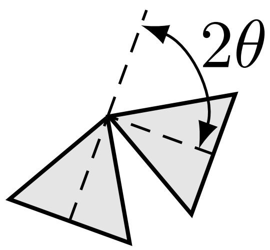
The twisted Kagome lattice



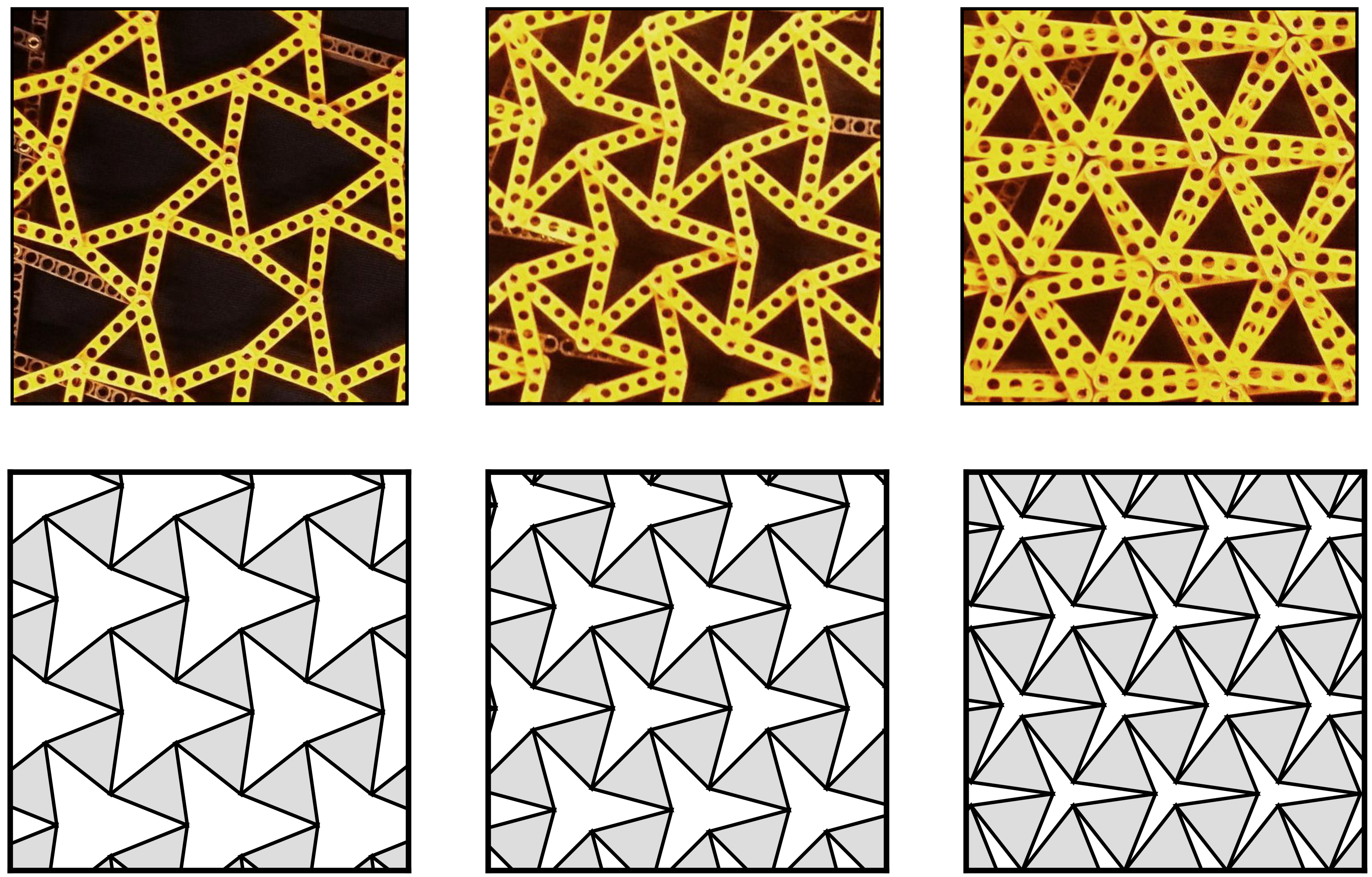
Twisting angle θ : a geometric control parameter

M. Fruchart, Y. Zhou, V. Vitelli, Nature (in press), arXiv:1904.07436

A family of twisted Kagome lattices

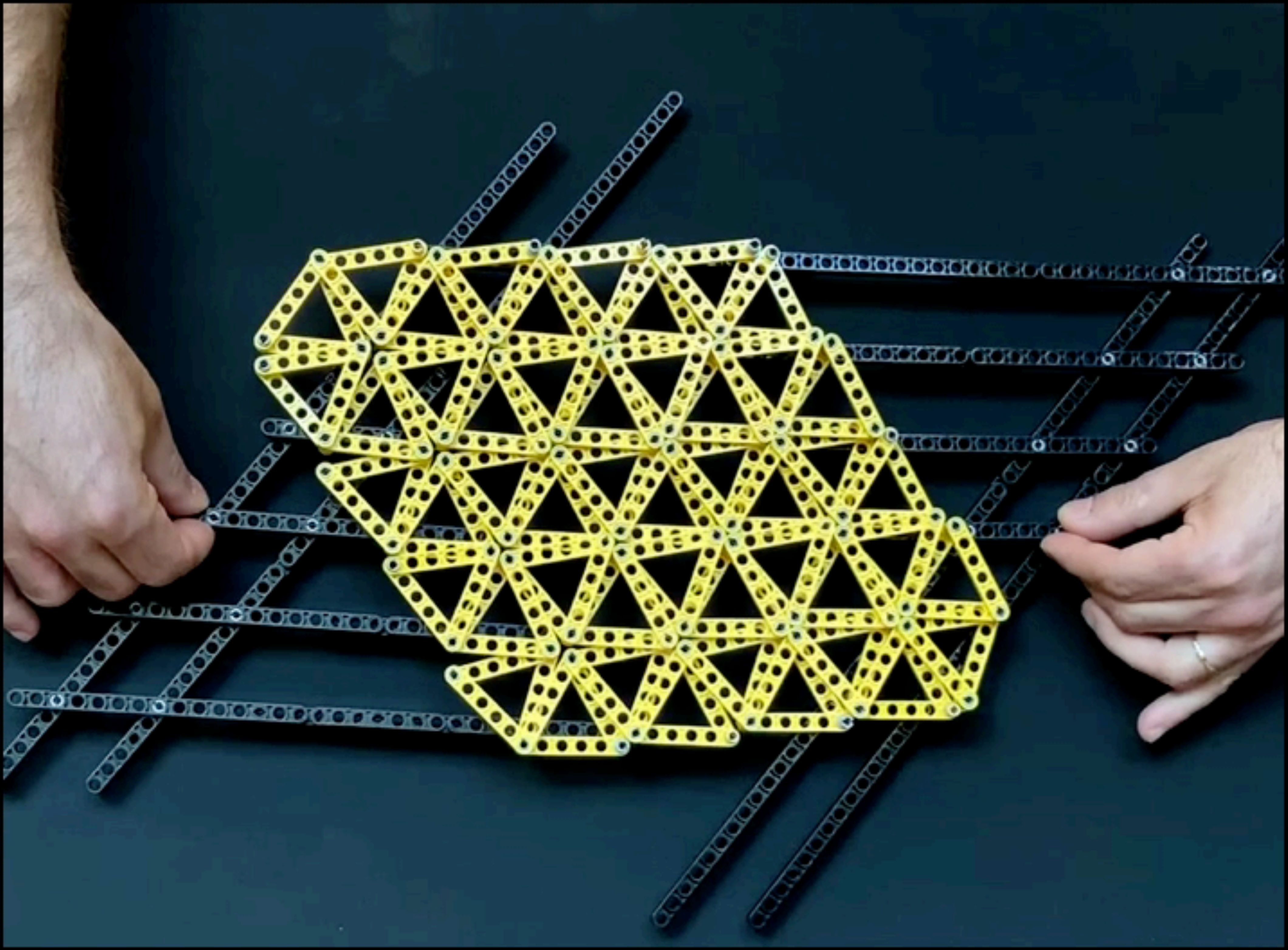


$\theta = \theta_c - \Delta\theta$ $\qquad \theta_c = \frac{\pi}{4}$ $\qquad \theta^* = \theta_c + \Delta\theta$



Guest Hutchinson mode J. Mech. and Phys. Solids 51, 383 (2003).

M. Fruchart, Y. Zhou, V. Vitelli, Nature (in press), arXiv:1904.07436



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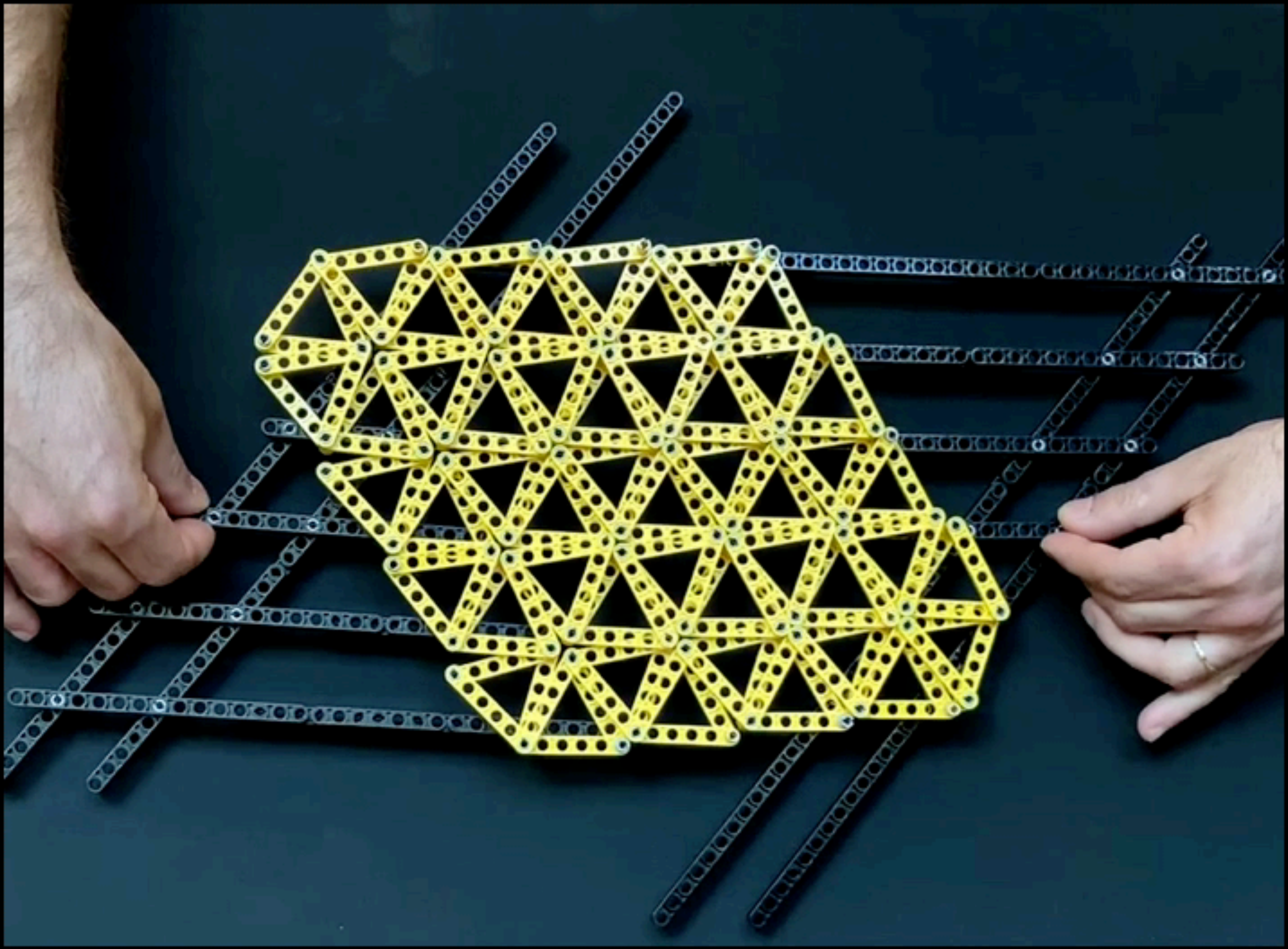

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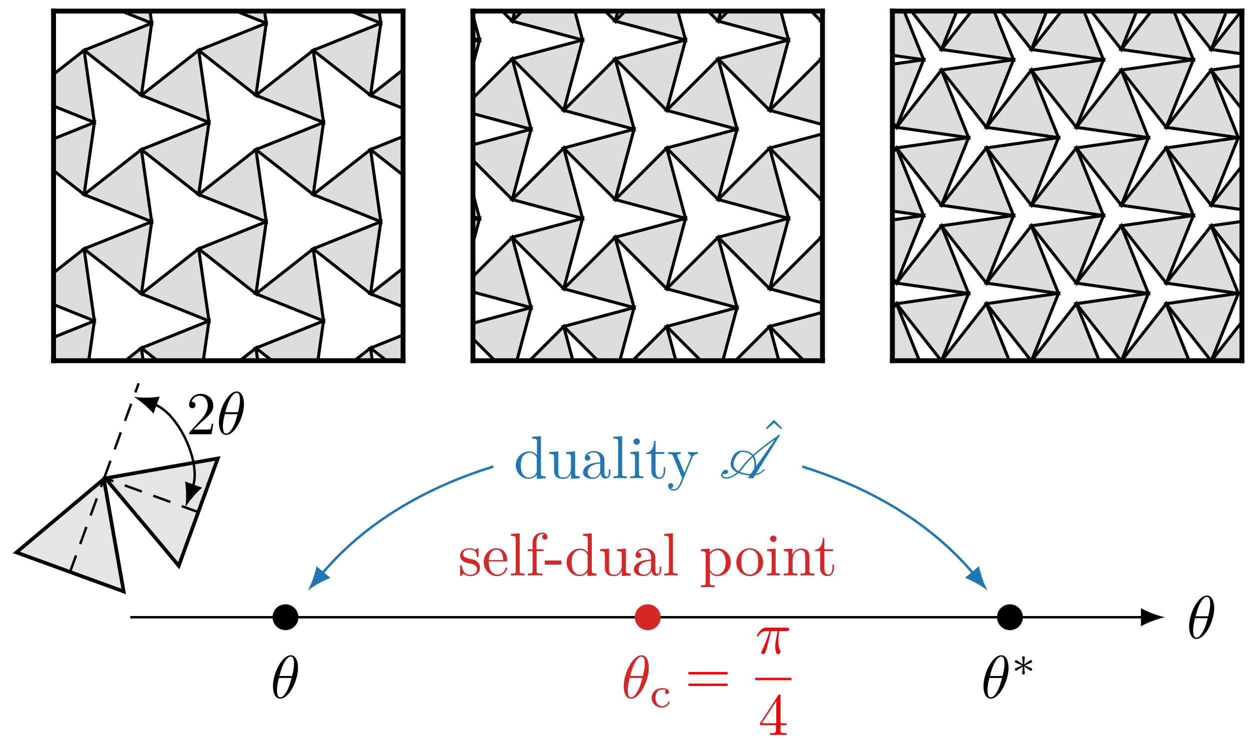
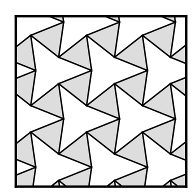
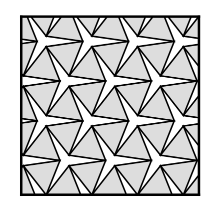
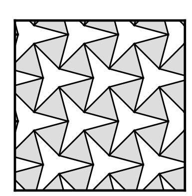
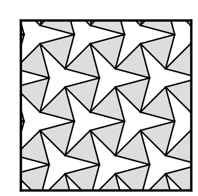
A duality



$$\theta^* = 2\theta_c - \theta$$

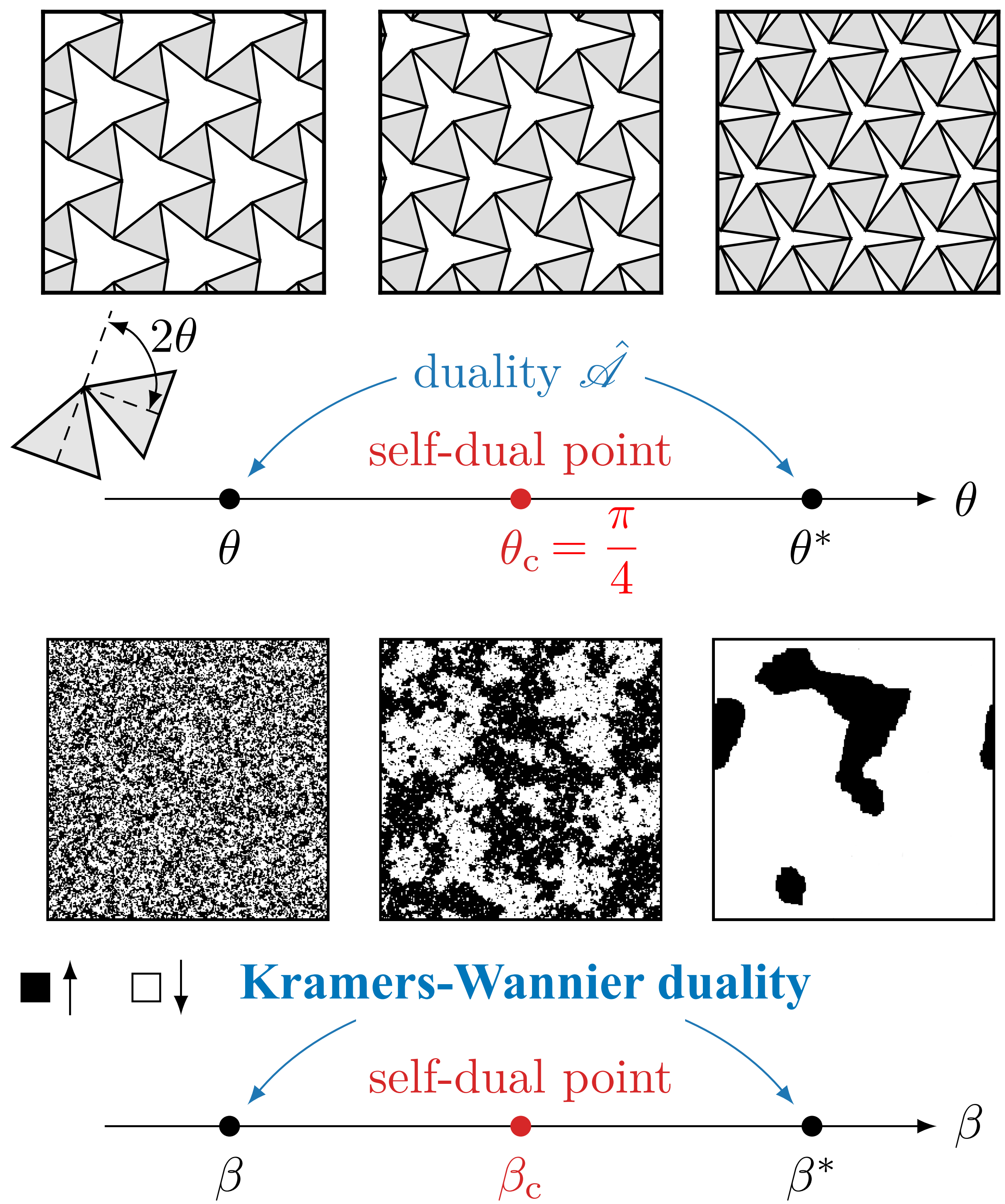
$T($  $) =$ 

duality

$T($  $) =$ 

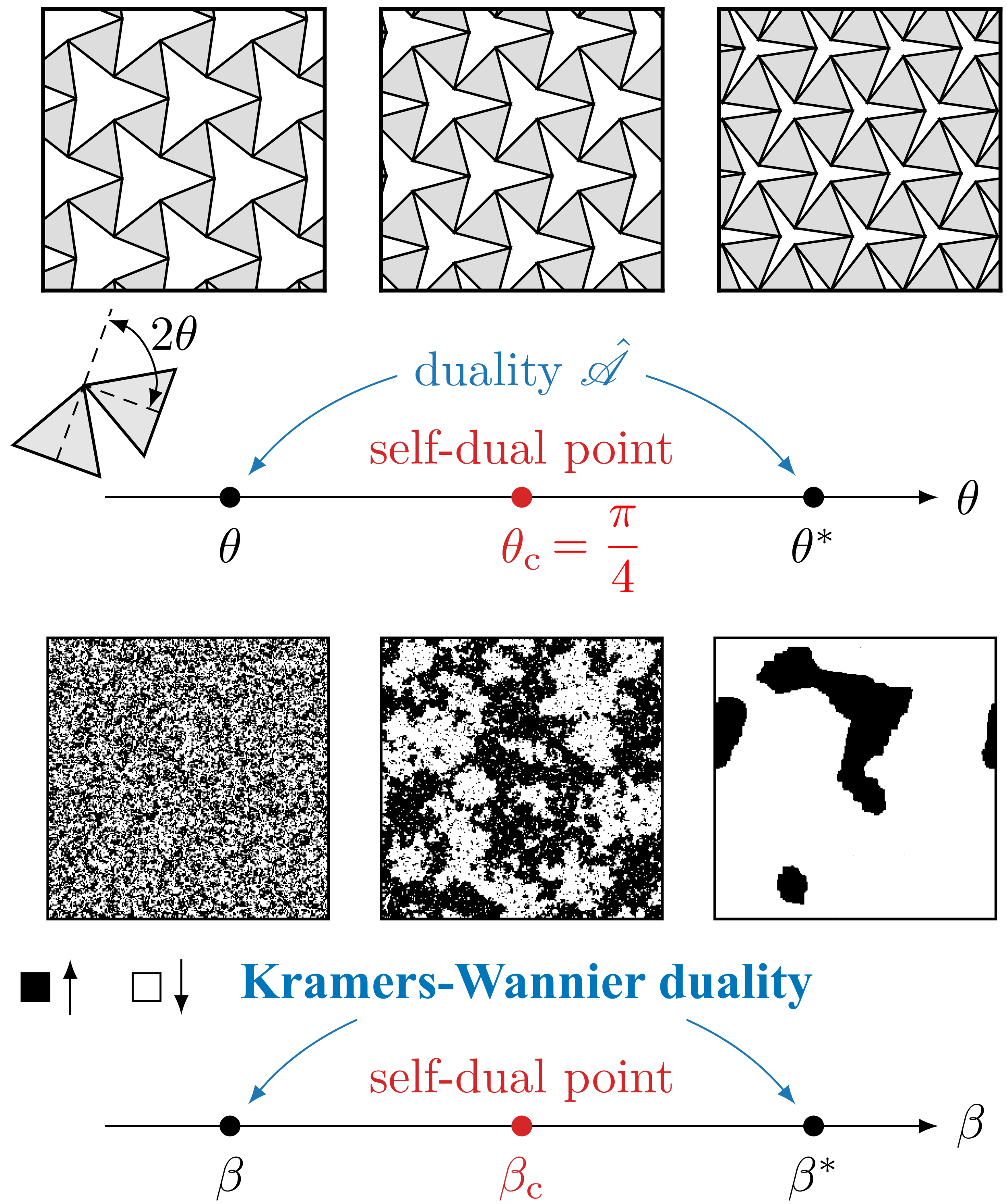
self-duality

A duality



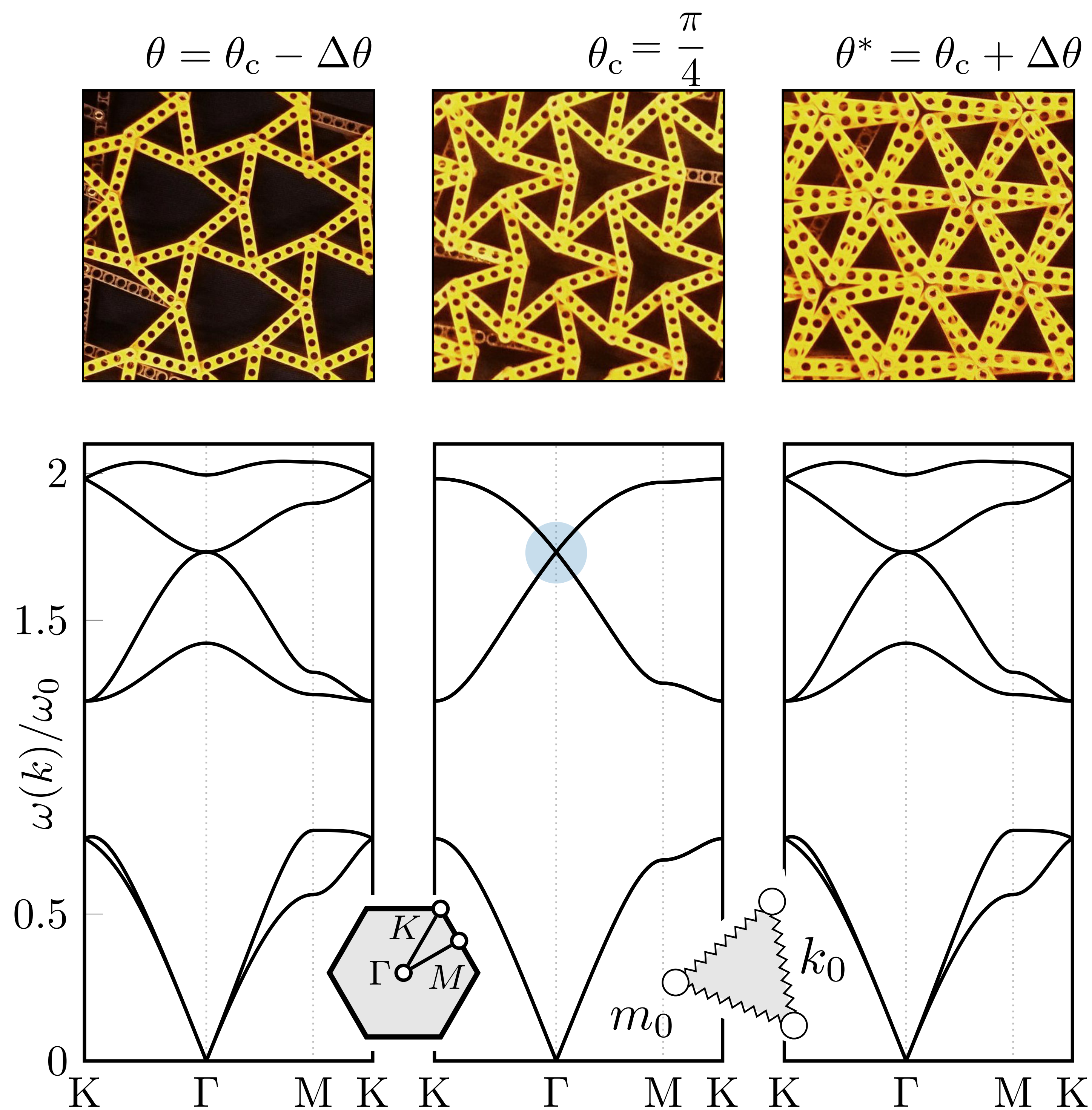
2θ
duality $\hat{\mathscr{A}}$
self-dual point
θ
θ
$\theta_c = \frac{\pi}{4}$
θ^*
Kramers-Wannier duality
self-dual point
β
β
β_c
β^*

A duality



What properties are constrained by the duality transformation ?

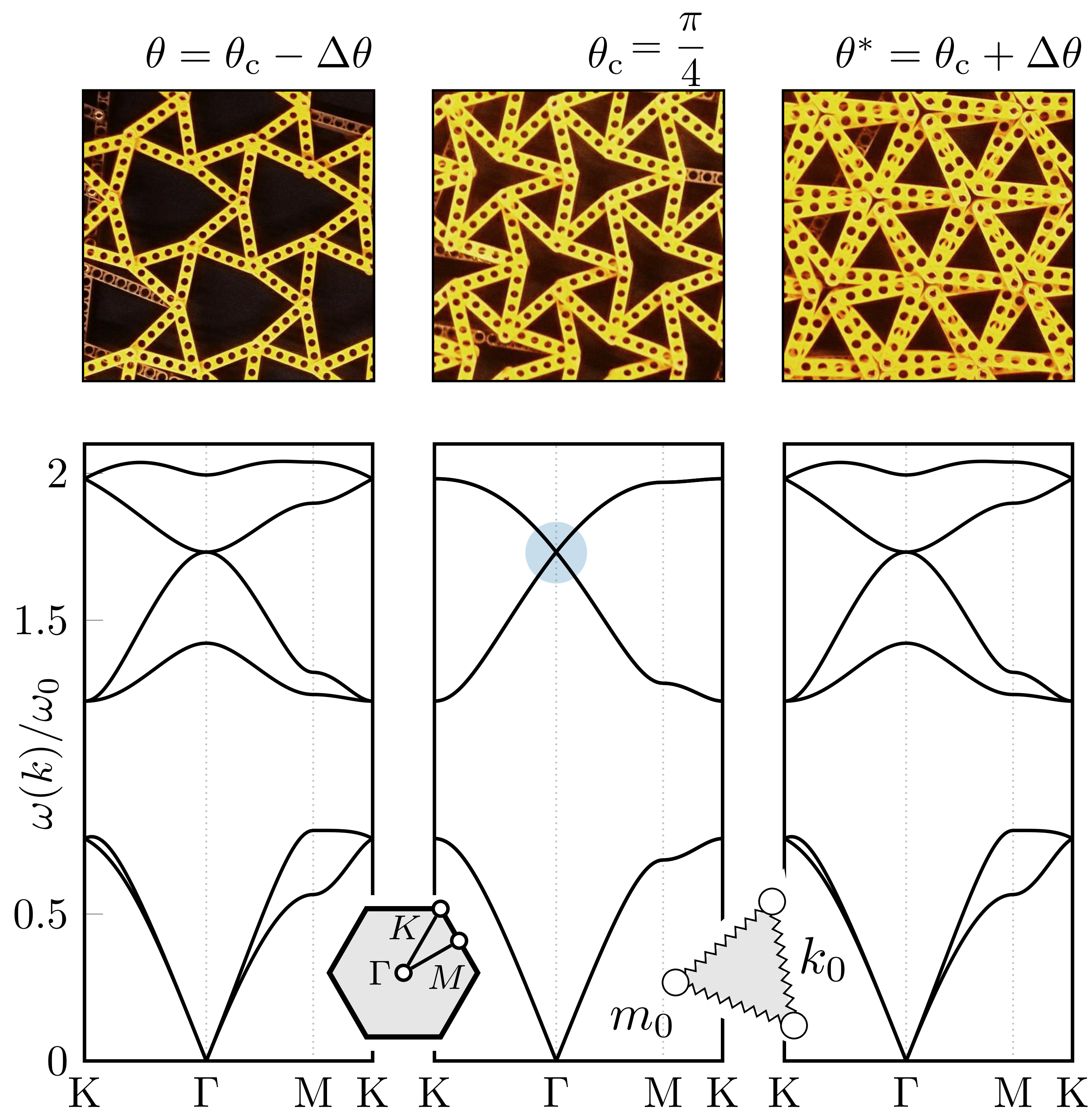
Duality I: the vibrational spectrum



$$\theta^* = 2\theta_c - \theta$$

$$\omega_0 = \sqrt{\frac{k_0}{m_0}}$$

Duality I: the vibrational spectrum



$$\theta^* = 2\theta_c - \theta$$

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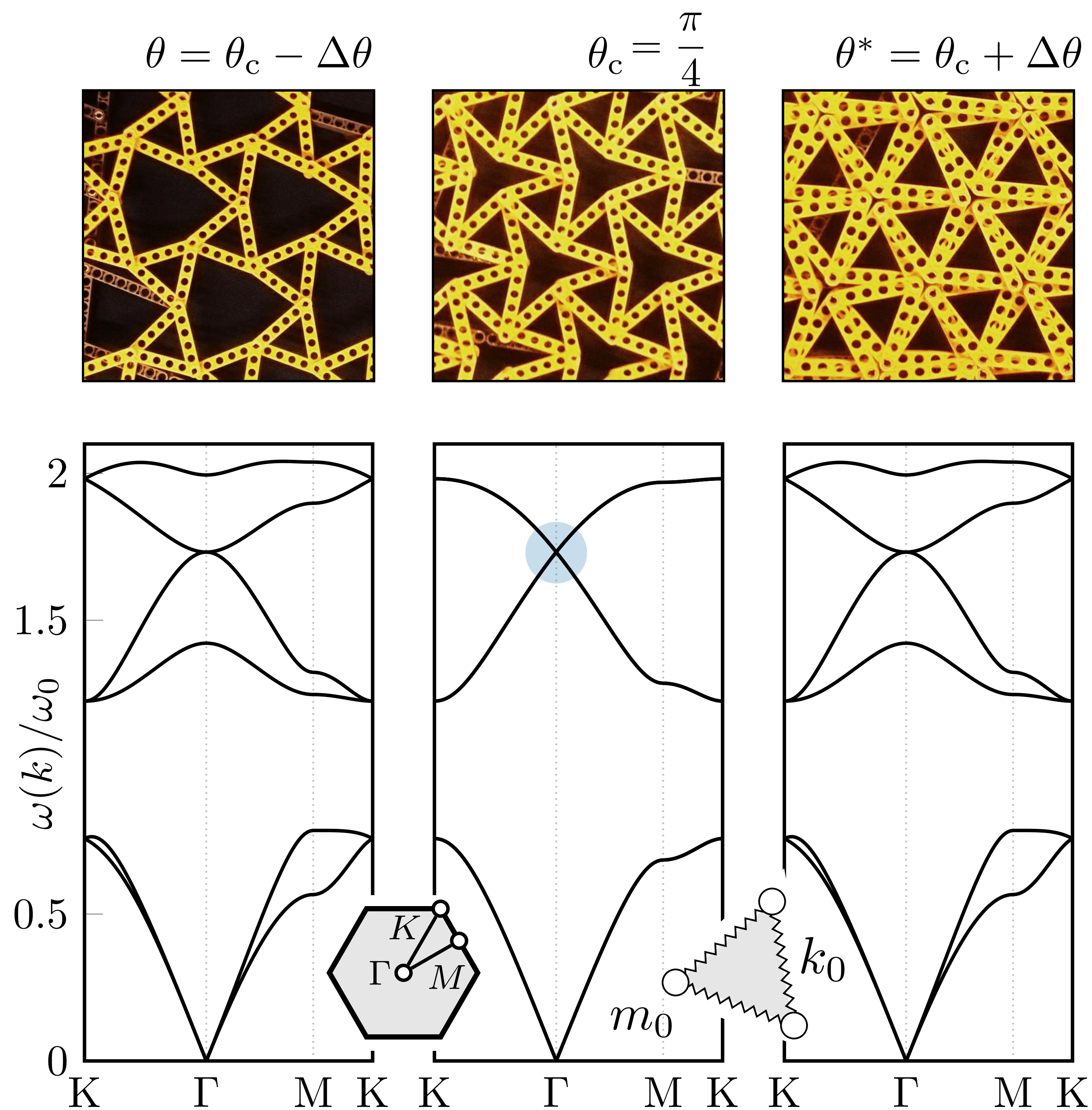
Identical spectra above and below θ_c

Duality I: the vibrational spectrum

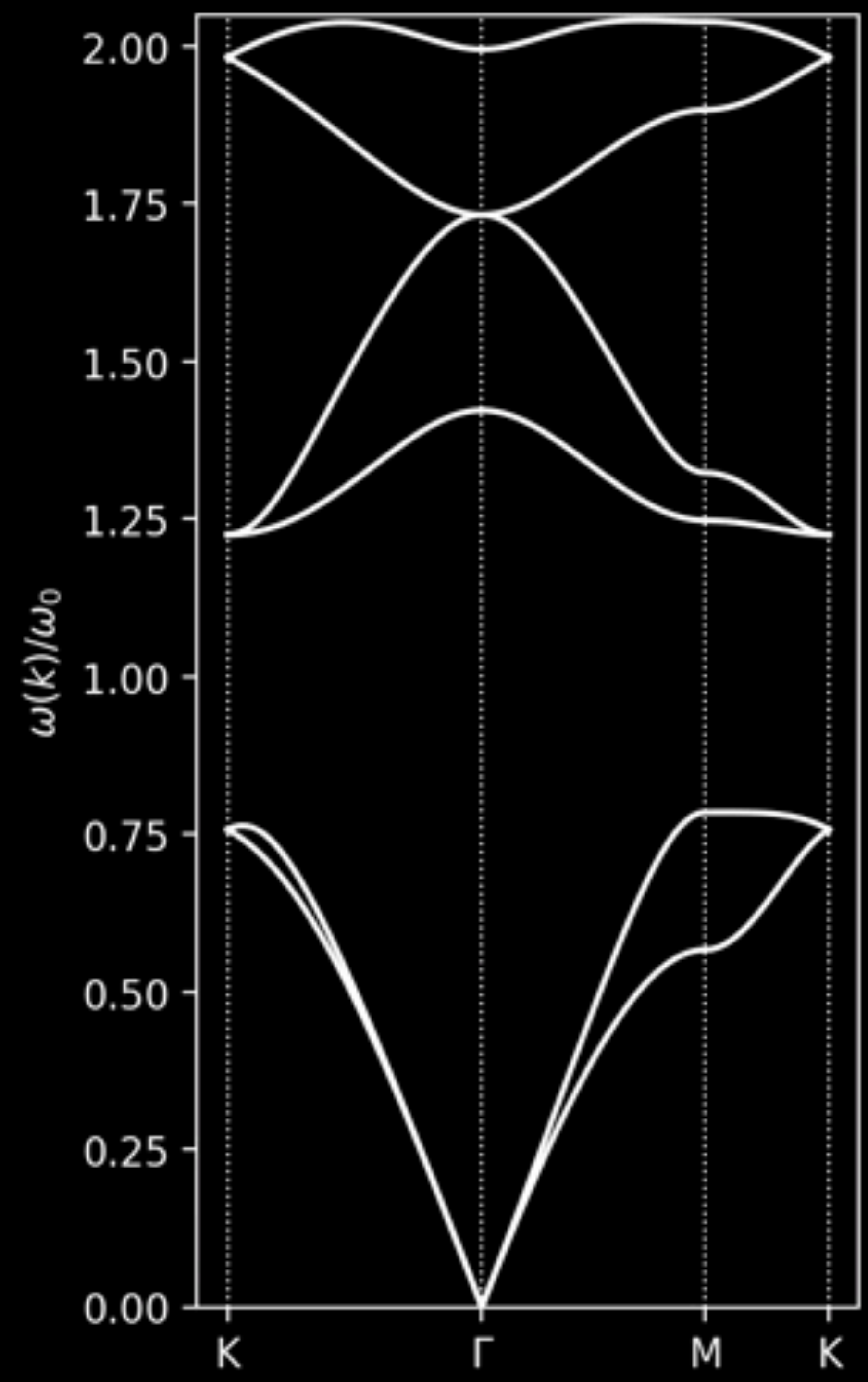


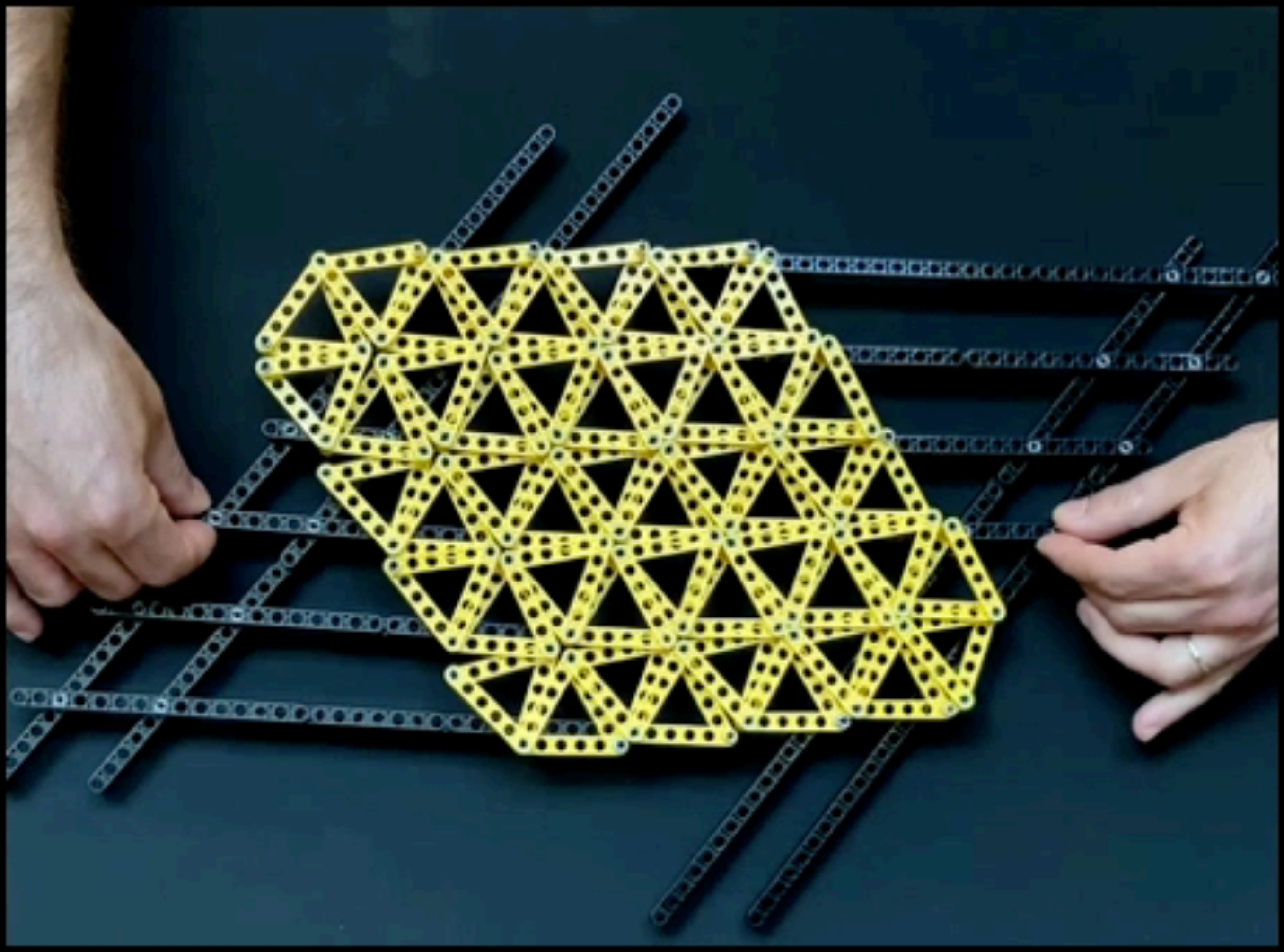
$$\theta^* = 2\theta_c - \theta$$

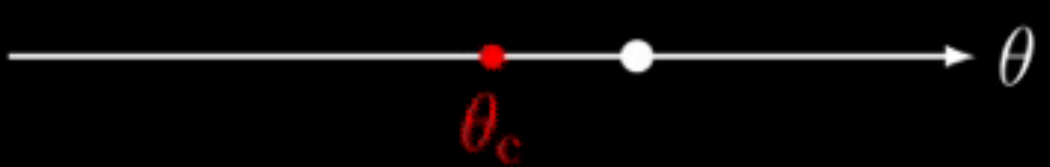
$$\omega_0 = \sqrt{\frac{k_0}{m_0}}$$

Identical spectra above and below θ_c

Double degeneracy for all wave vectors at θ_c







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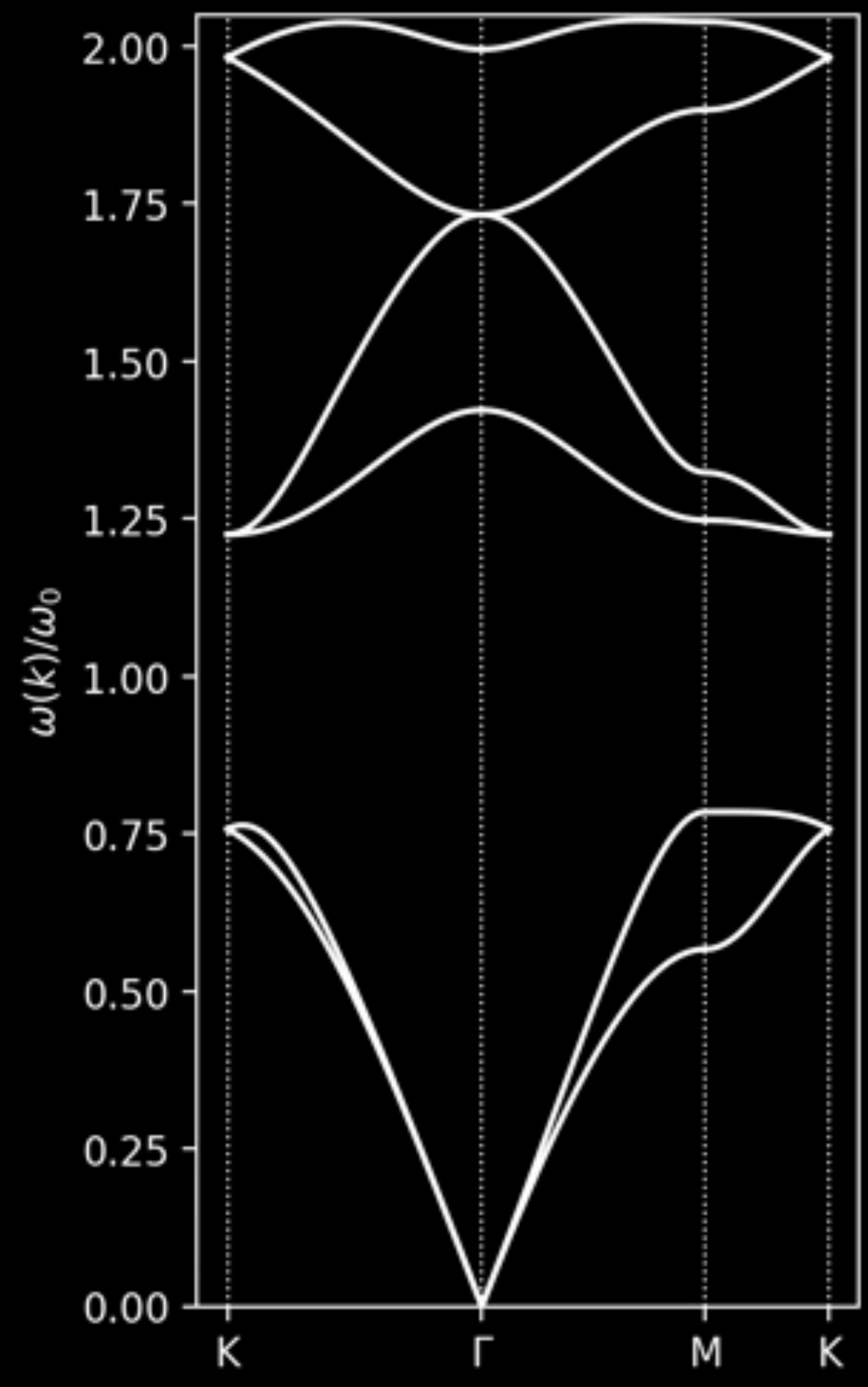

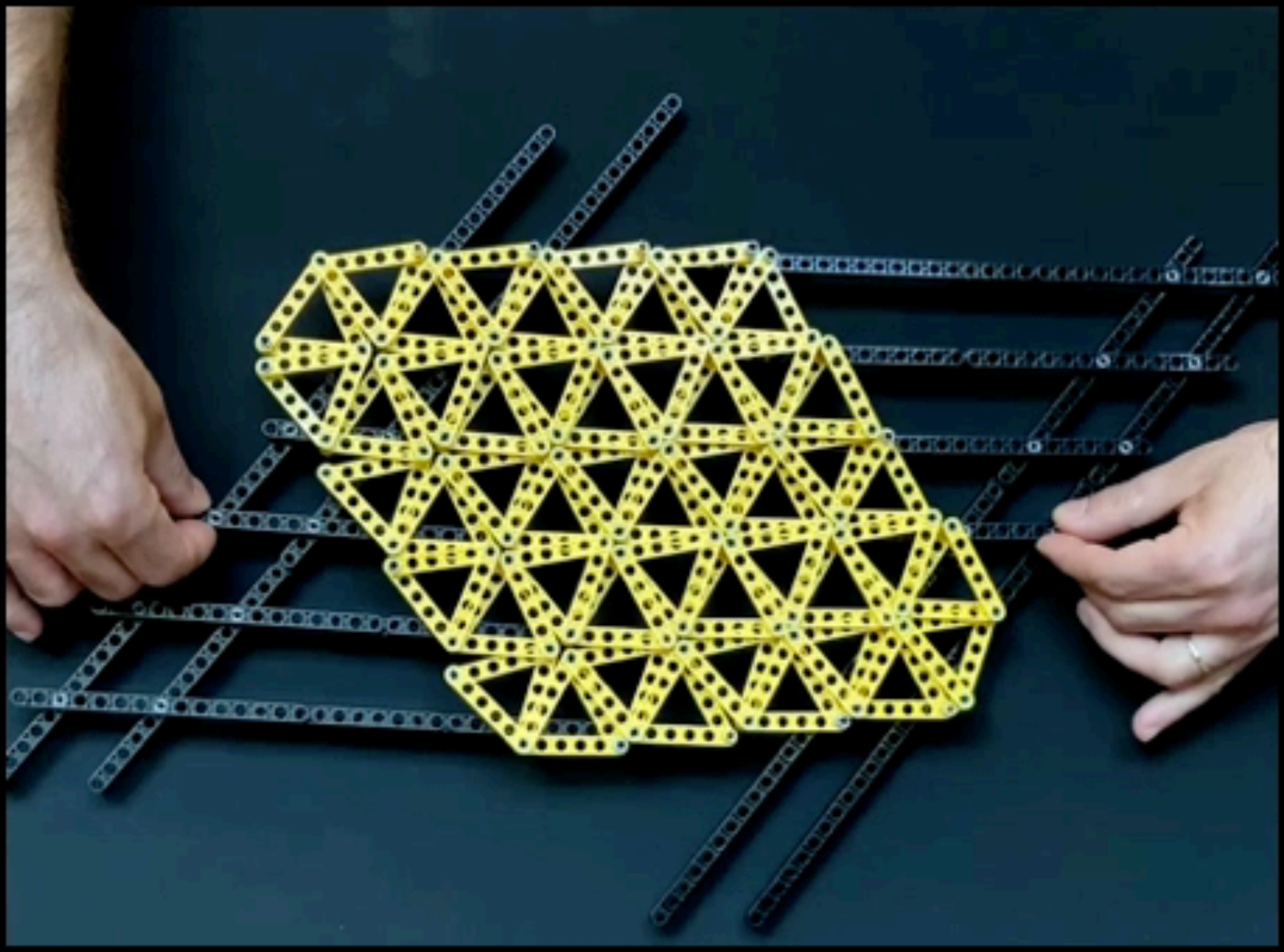

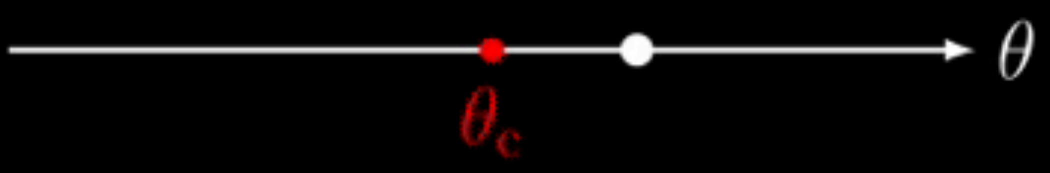

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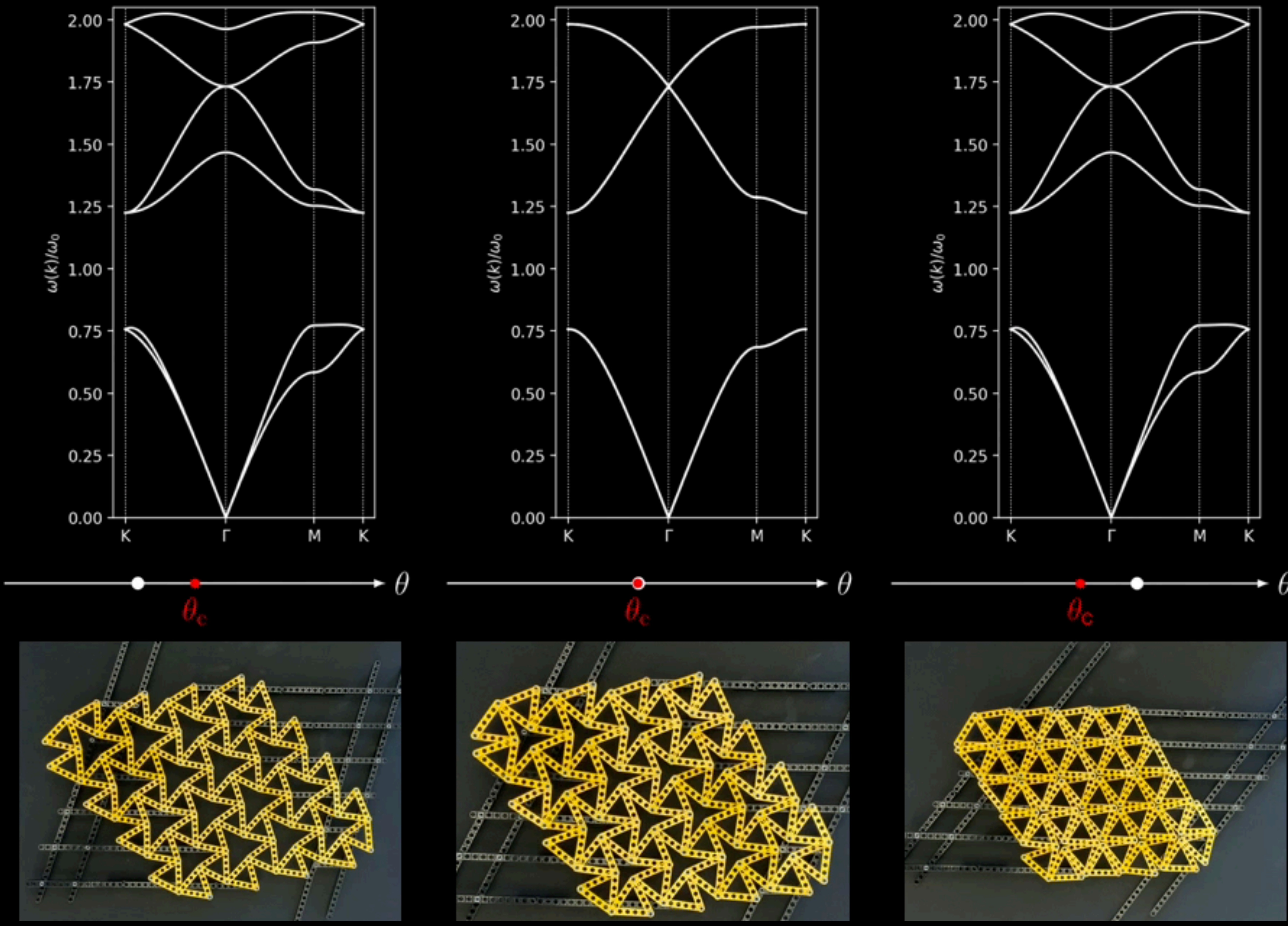

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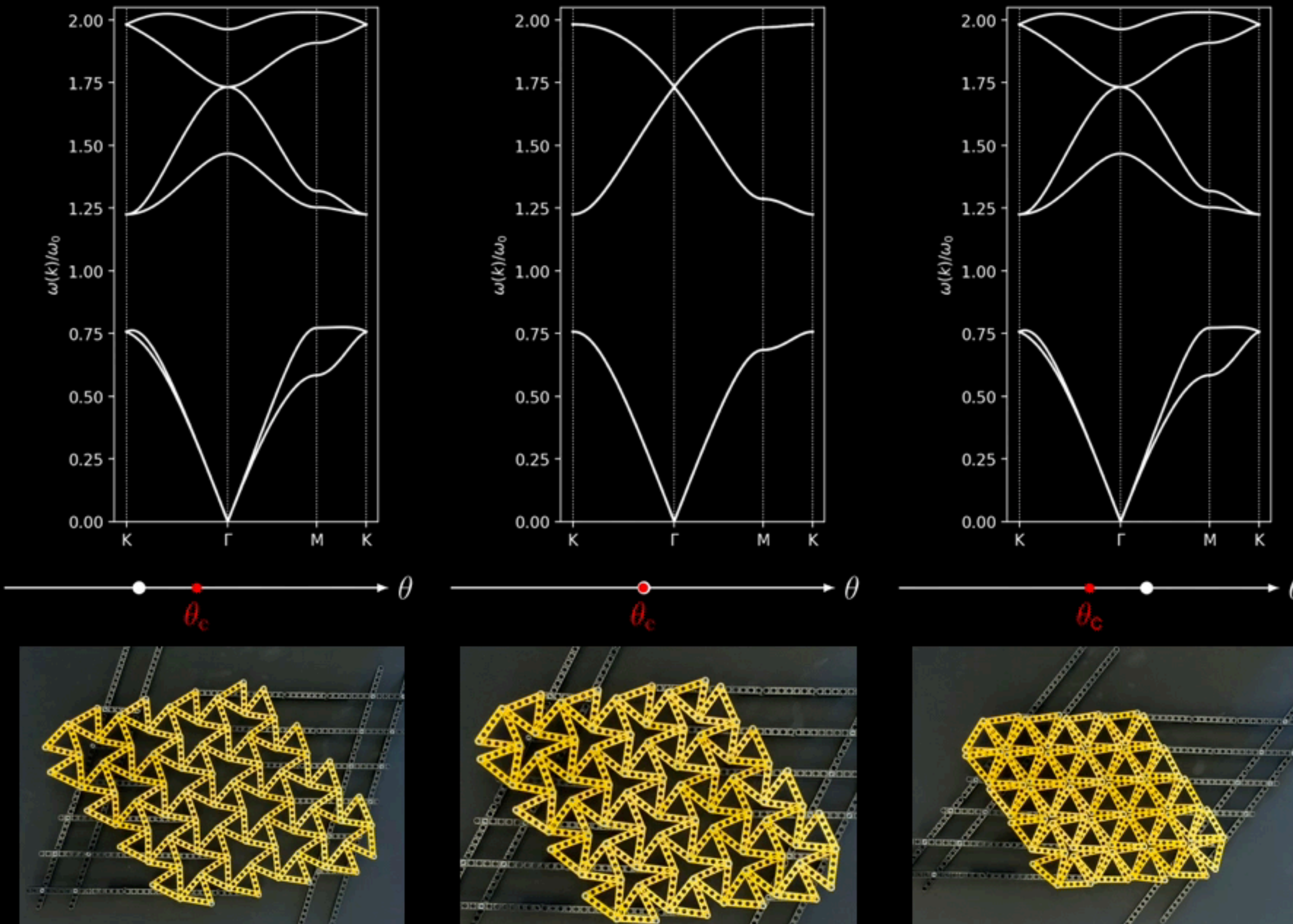

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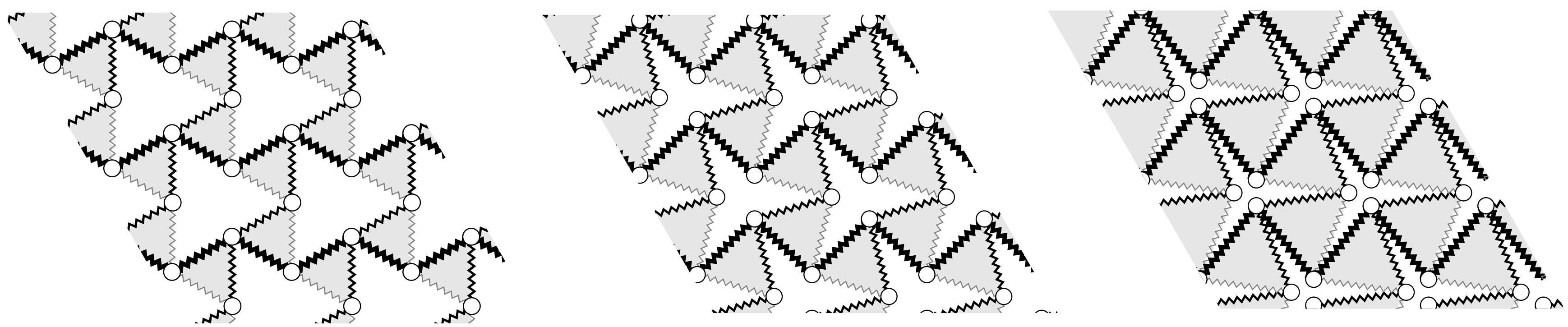
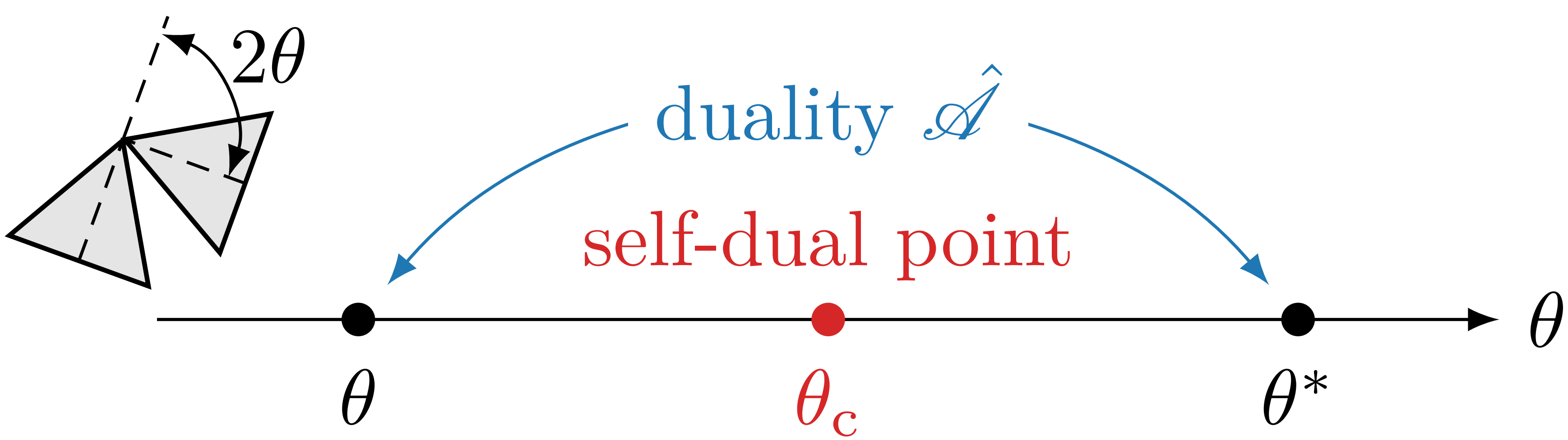
Duality II: elastic moduli

all spatial
symmetries
are broken

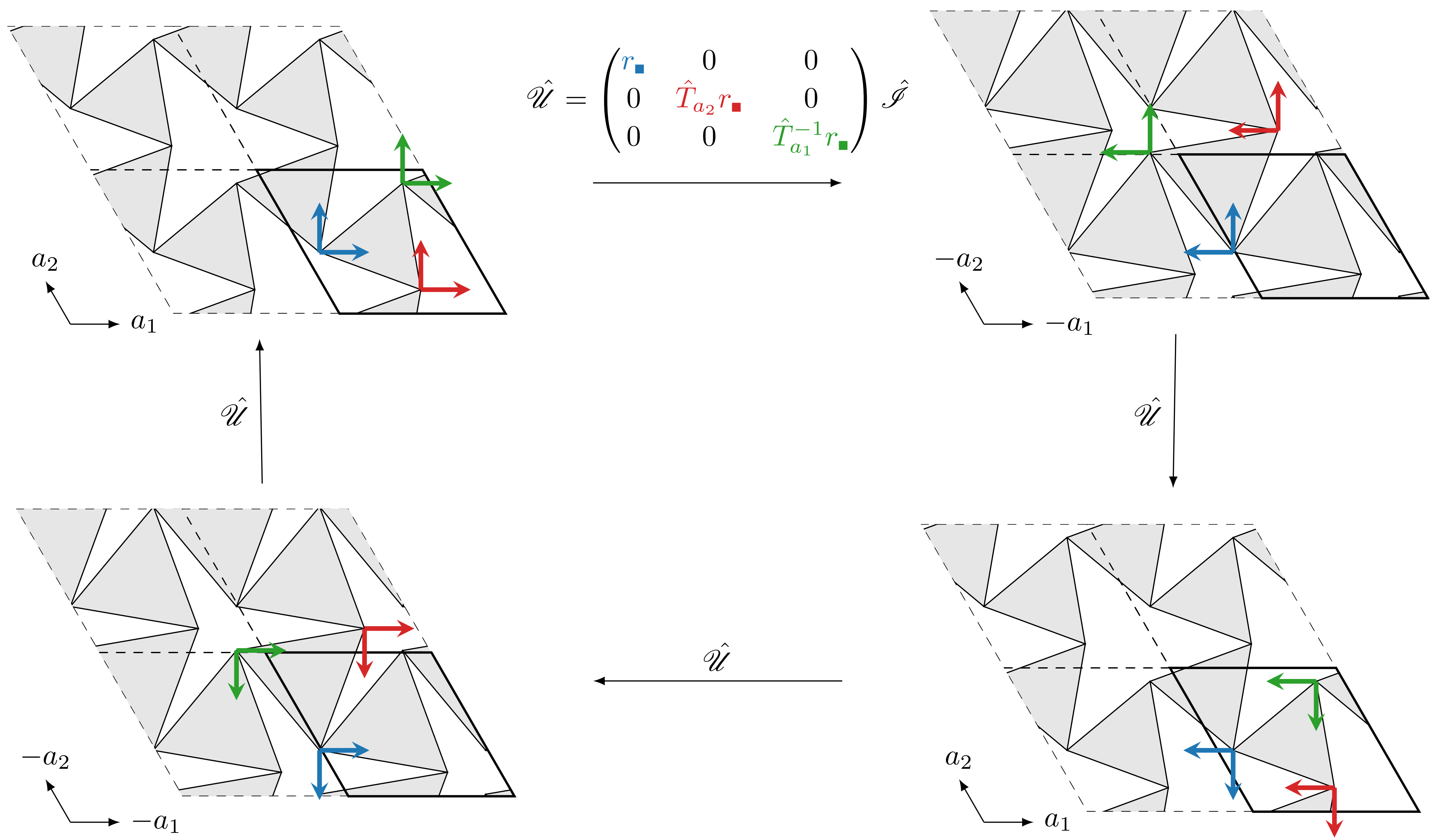




3 independent moduli	**1 modulus**	3 independent moduli

Point group does not change with θ but number of moduli does

The duality operator $\hat{\mathscr{U}}$

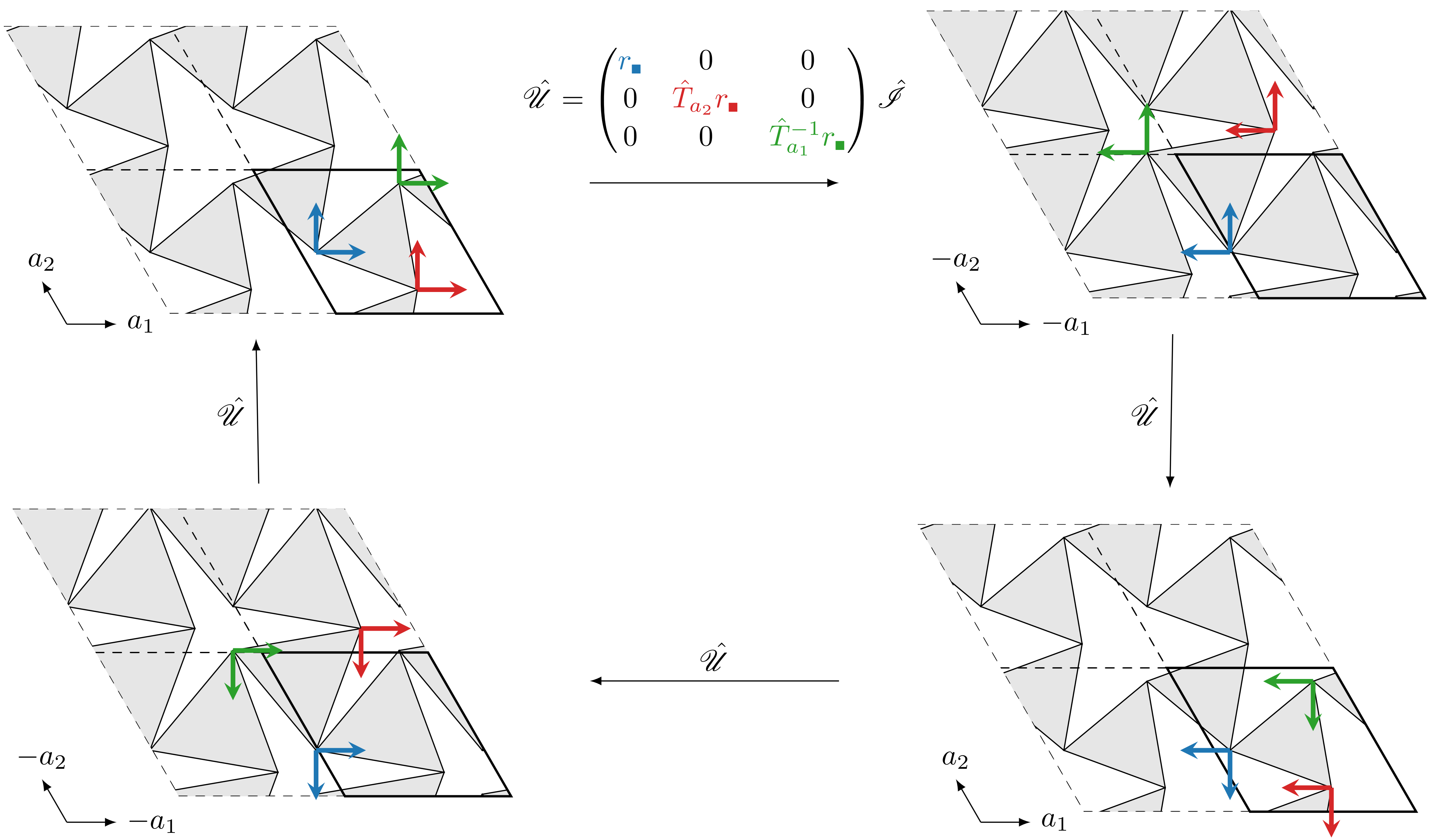


unitary

dynamical matrix
in momentum space

$$\mathscr{U}(k) D(\theta^*, -k) \mathscr{U}^{-1}(k) = D(\theta, k)$$

An anti-unitary operator $\mathscr{A}$



$$\Theta D(\theta, k)\Theta^{-1} = D(\theta, -k)$$

$$\mathscr{A}(k) = \mathscr{U}(k)\Theta$$

$$\mathscr{A}(k)^2 = -\mathrm{Id}$$

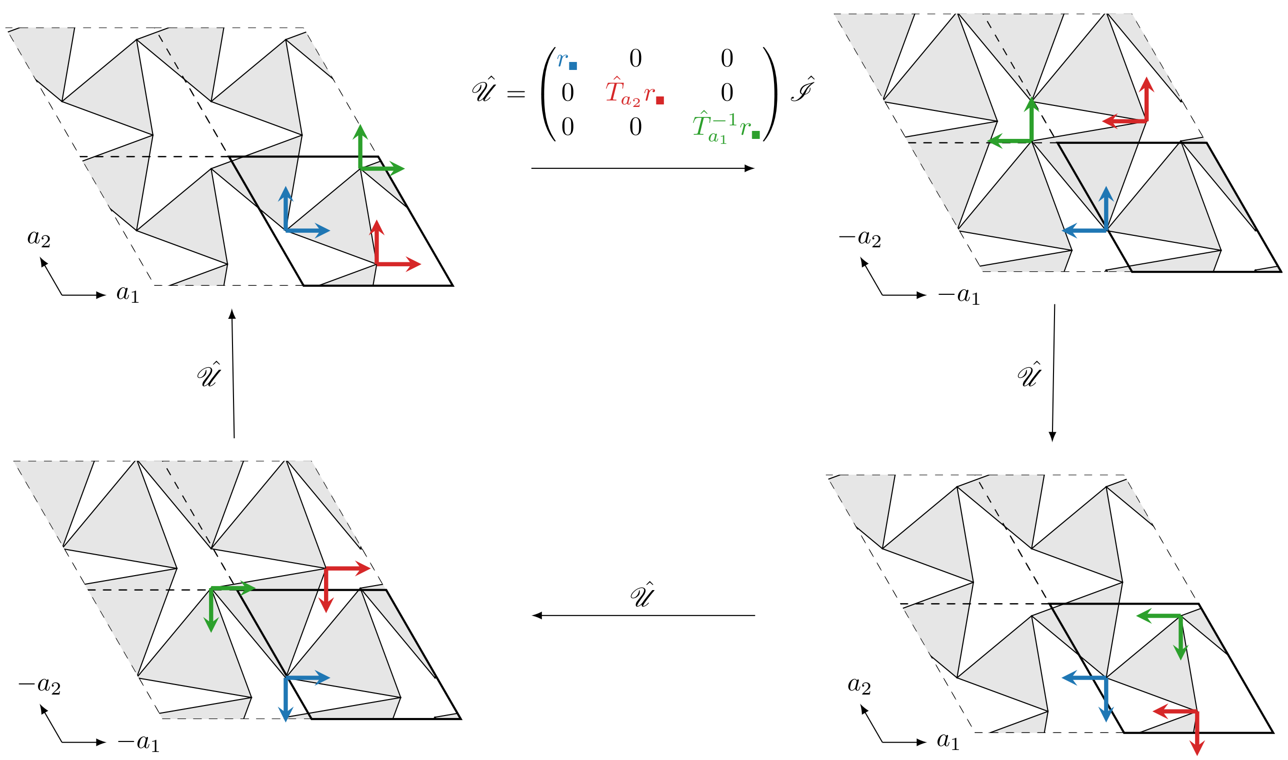
unitary

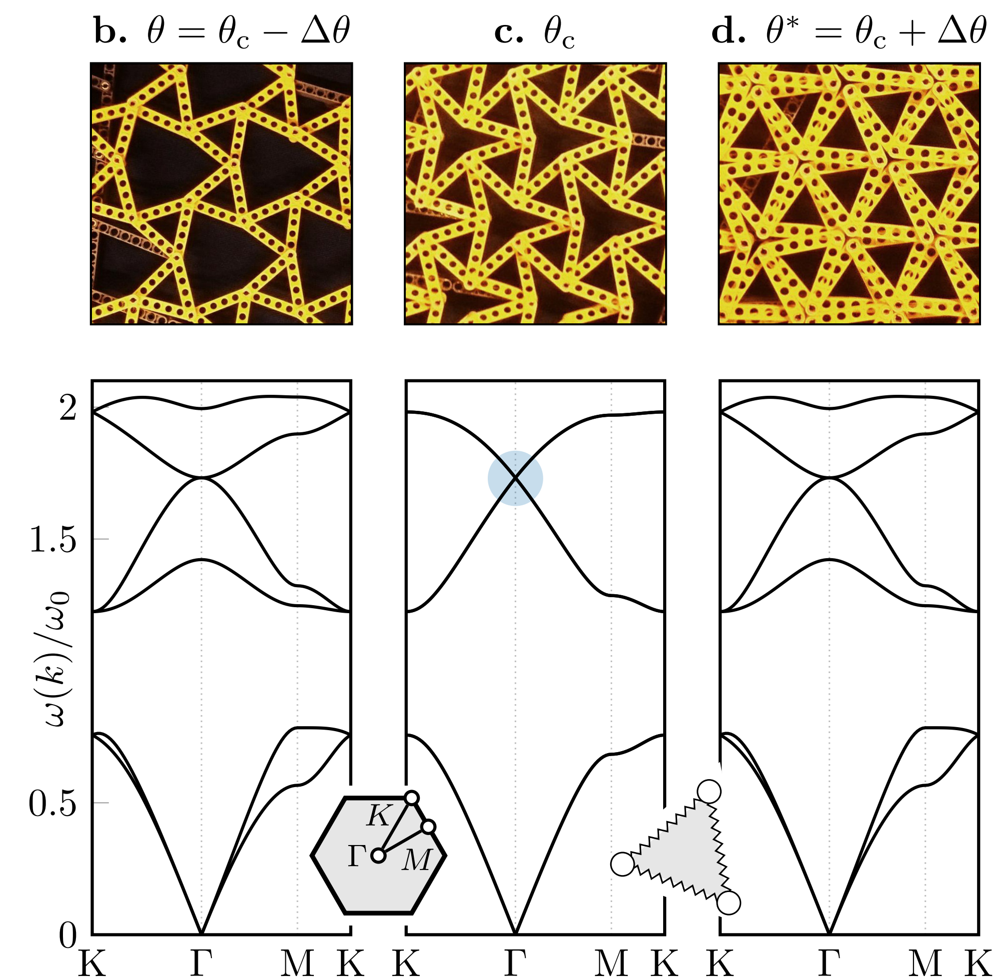
dynamical matrix
in momentum space

$$\mathscr{U}(k)D(\theta^*, -k)\mathscr{U}^{-1}(k) = D(\theta, k)$$

A mechanical Kramers theorem

without fermionic time-reversal symmetry





$$\Theta D(\theta, k)\Theta^{-1} = D(\theta, -k)$$

$$\mathscr{A}(k) = \mathscr{U}(k)\Theta$$

$$\mathscr{A}(k)^2 = -\mathrm{Id}$$

anti-unitary

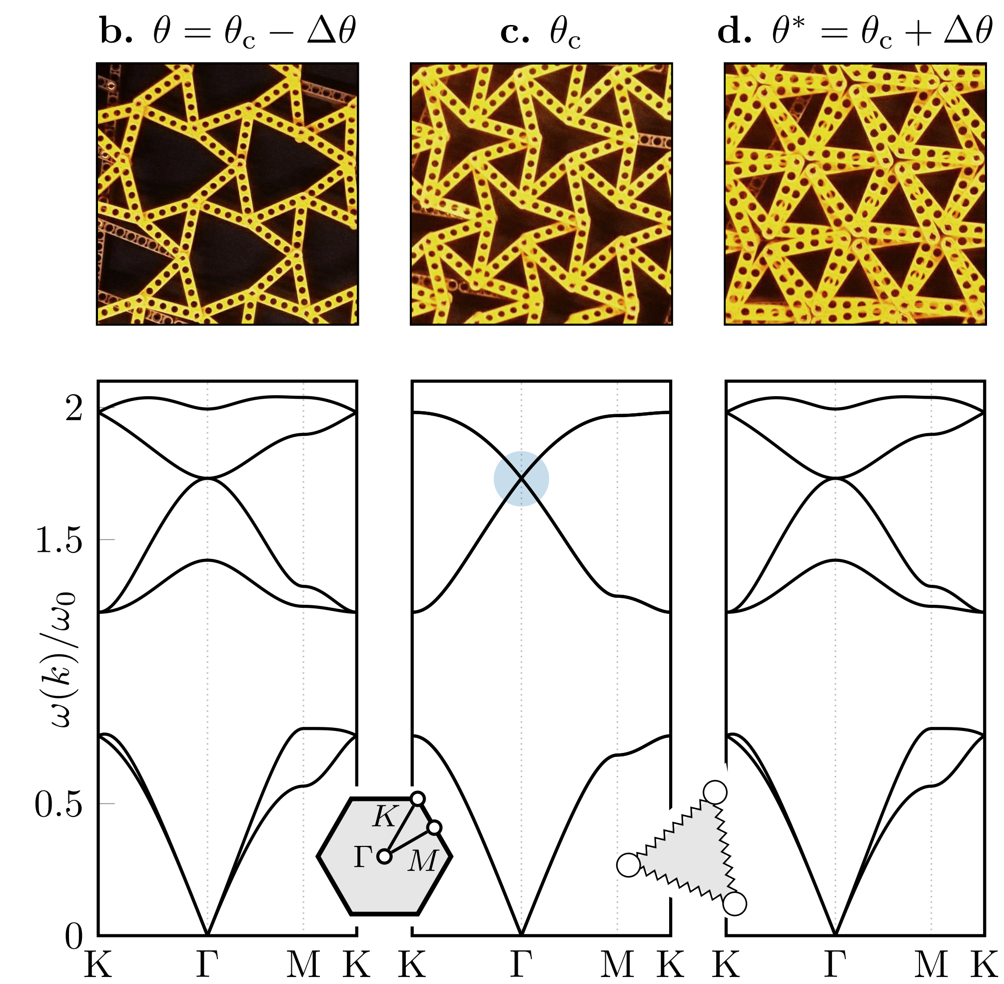
dynamical matrix

$$\mathscr{A}(k)D(\theta^*, k)\mathscr{A}^{-1}(k) = D(\theta, k)$$

A mechanical Kramers theorem

without fermionic time-reversal symmetry

band structure two-fold degenerate
for **all wavevectors at** θ_c



commutes with dynamical matrix at self-dual point θ_c

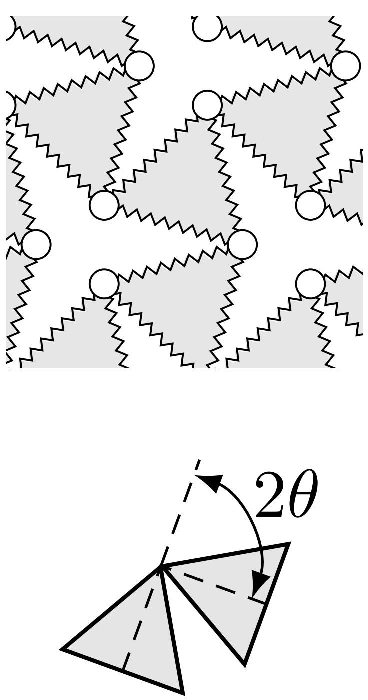
dynamical matrix

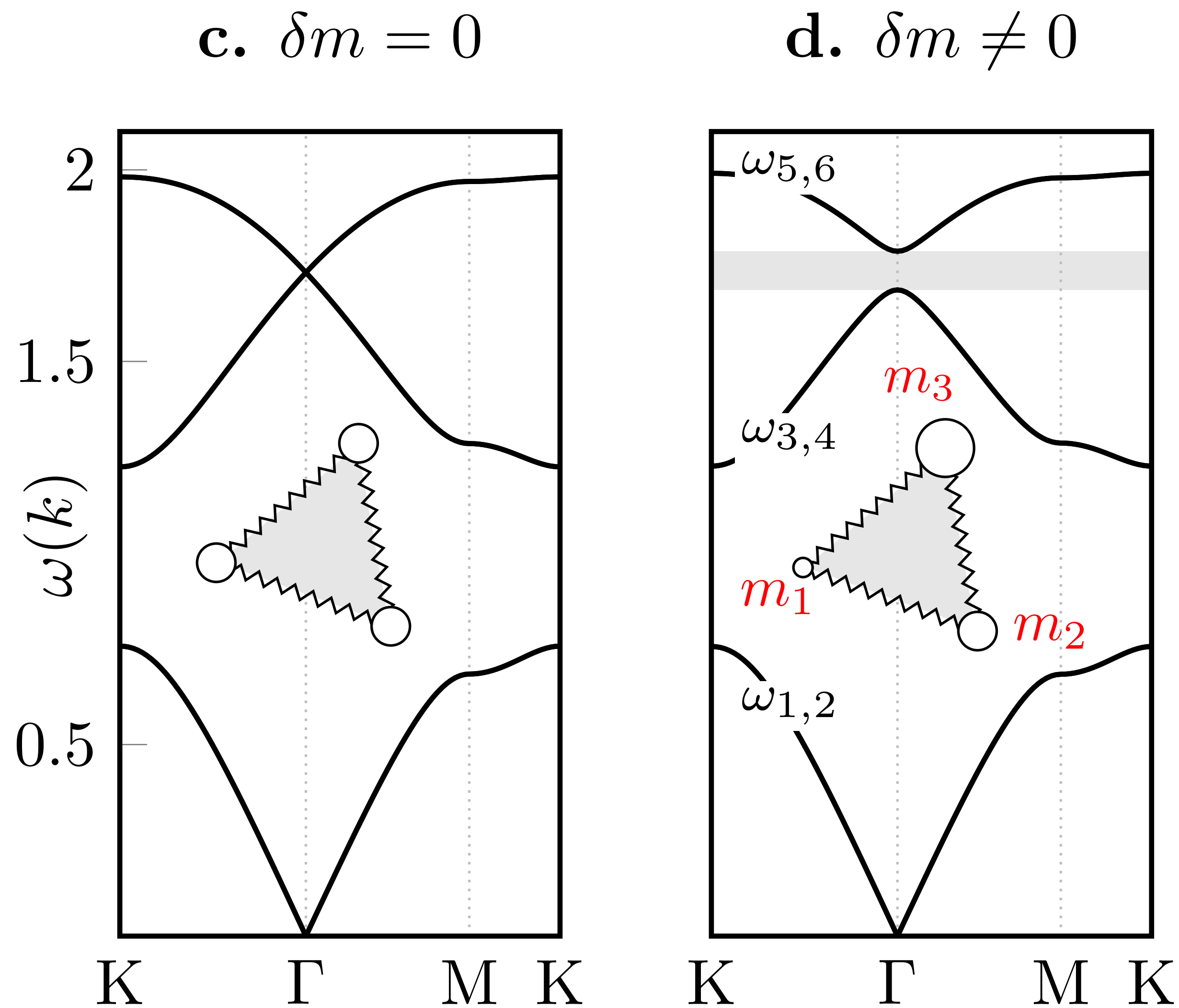
$$\mathscr{A}(k) = \mathscr{U}(k)\Theta$$

$$\mathscr{A}(k)^2 = -\mathrm{Id}$$

$$\mathscr{A}(k)D(\theta_c\,, k)\mathscr{A}^{-1}(k) = D(\theta_c\,, k)$$

Gapping the double Dirac cone

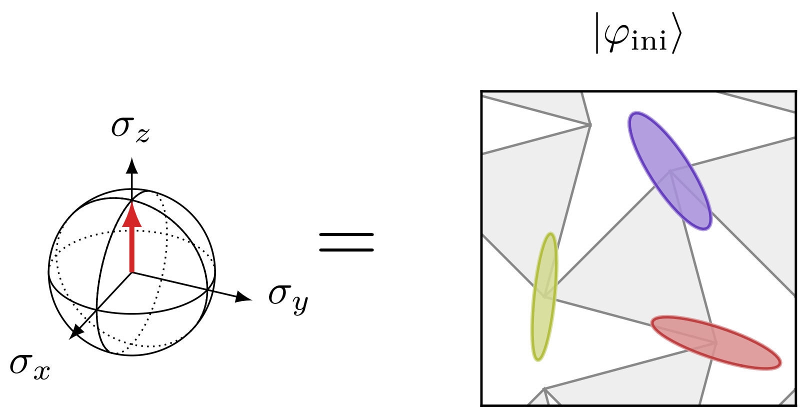


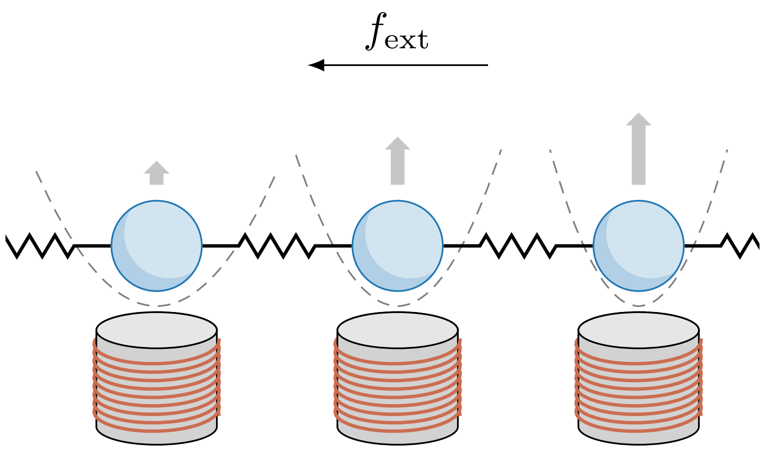


$$(m_1, m_2, m_3) = (1 - \delta m, 1, 1 + \delta m)$$

to get (adiabatic) geometric phases

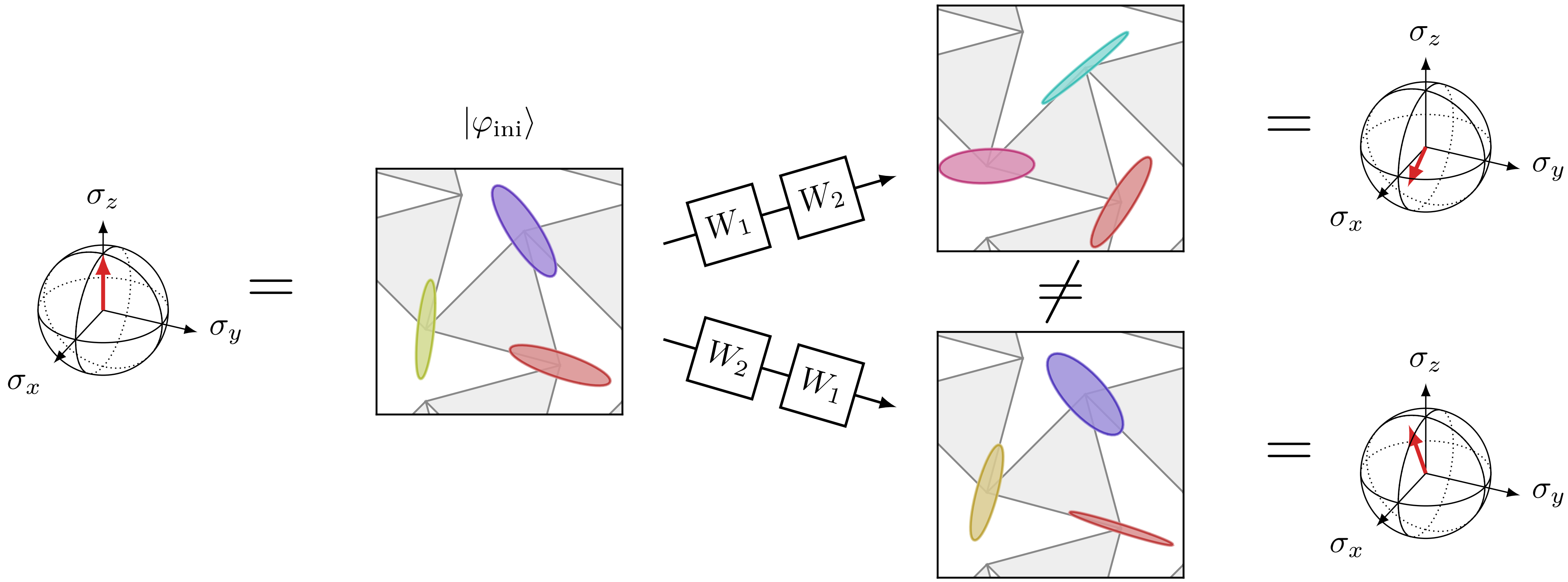
Mechanical spins





Semiclassical evolution of wave-packet

Non-abelian geometric phases



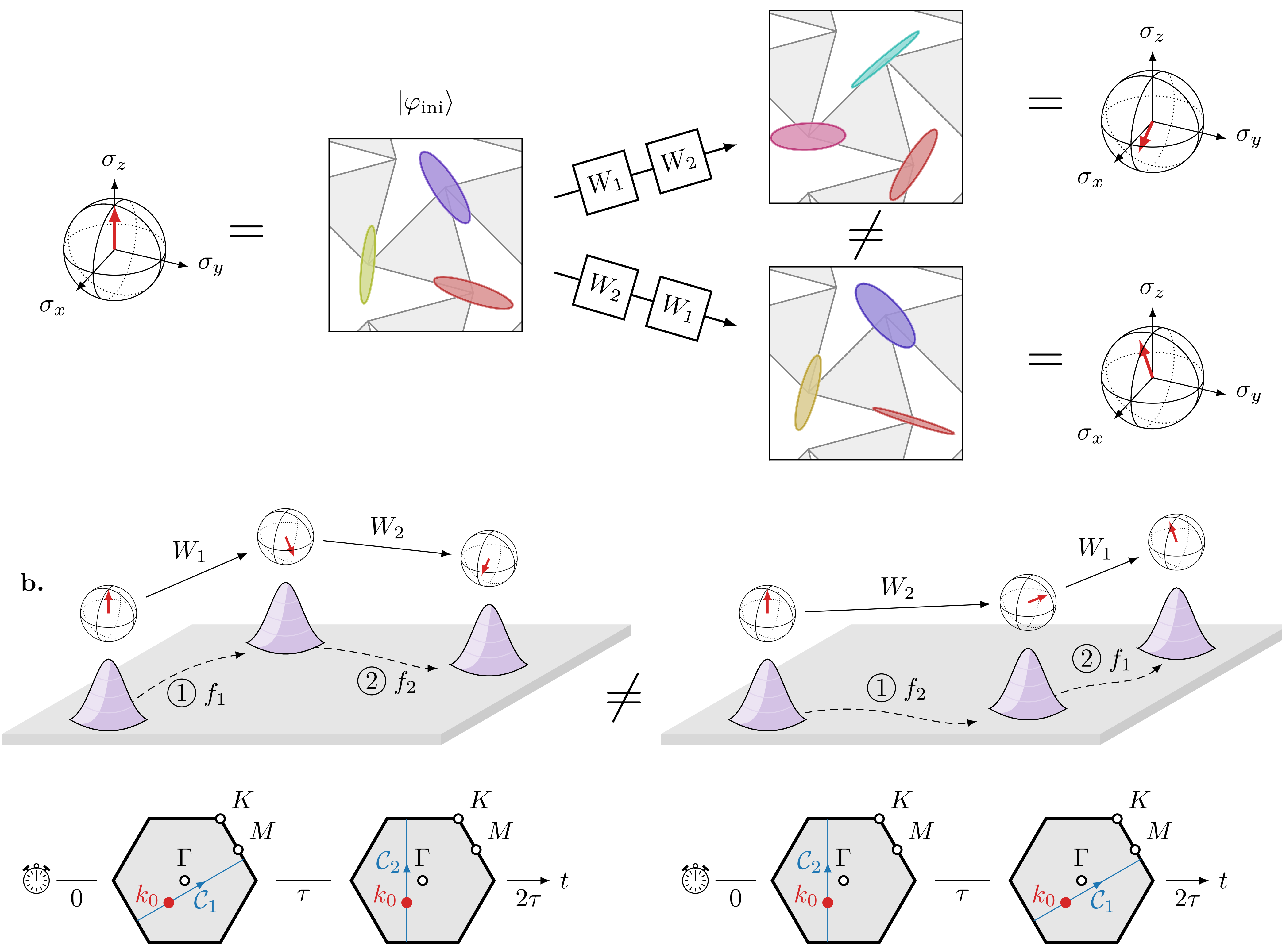
$$W_1 W_2 \neq W_2 W_1$$

Matrix Berry connection

Wilson loop

$$W(\mathcal{C}) = \mathcal{P}\exp\left(-\int_{\mathcal{C}} A\right)$$

Mechanical spintronics



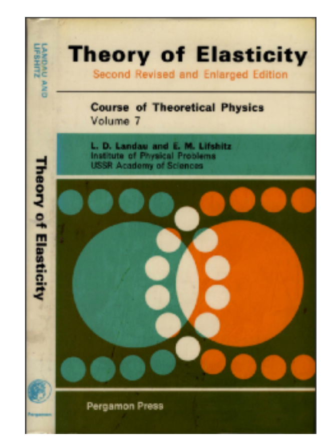
On the fly manipulations of mechanical spins

What is elasticity ?

Linear elasticity

Hooke's law $$\sigma_{ij} = K_{ijmn} u_{mn}$$

Stress — Stiffness Tensor — Strain



Independent entries of stiffness tensor are static elastic moduli

What is elasticity ?

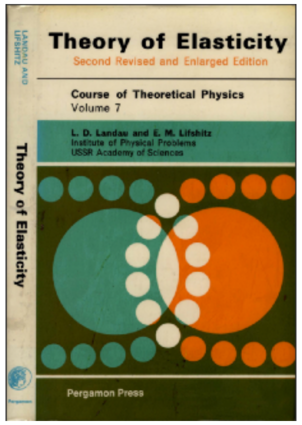
Hooke's law $$\sigma_{ij} = K_{ijmn} u_{mn}$$

Stress

Stiffness Tensor

Strain

$$K_{ijmn} = K_{mnij}$$



Where does this symmetry come from?

What is elasticity ?

Hooke's law $$\sigma_{ij} = K_{ijmn} u_{mn}$$

$$\sigma_{ij} = \frac{\partial f}{\partial u_{ij}}$$

If $$f = \frac{1}{2} K_{ijmn} u_{ij} u_{mn}$$ **Elastic Energy density**

$$K_{ijmn} = K_{mnij}$$

What is elasticity ?

Hooke's law

$$\sigma_{ij} = K_{ijmn} u_{mn}$$

$$\sigma_{ij} = \frac{\partial f}{\partial u_{ij}}$$

If

break microscopic energy conservation

$$\cancel{f = \frac{1}{2} K_{ijmn} u_{ij} u_{mn}}$$

Elastic Energy density

Scheibner, Souslov, Banerjee, Surowka, Irvine, Vitelli, *arXiv:1902.07760*

Odd elasticity

Hooke's law

$$\sigma_{ij} = K_{ijmn} u_{mn}$$

$$\sigma_{ij} = \frac{\partial f}{\partial u_{ij}}$$

If

break microscopic energy conservation

$$\cancel{f = \frac{1}{2} K_{ijmn} u_{ij} u_{mn}}$$

Elastic Energy density

$$K^0_{ijmn} = -K^0_{mnij}$$

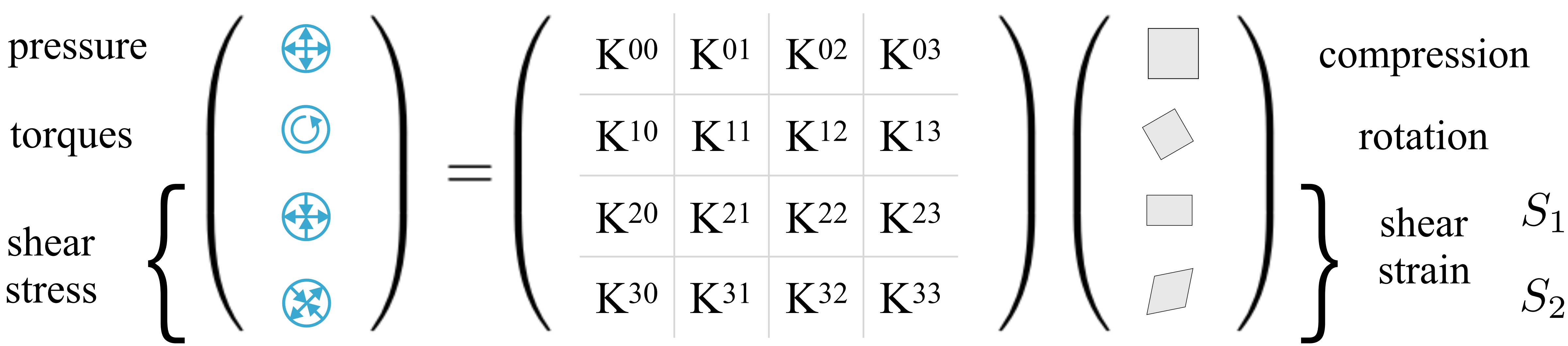
Scheibner, Souslov, Banerjee, Surowka, Irvine, Vitelli, *arXiv:1902.07760*

Visual representation of the stiffness tensor

Hooke's law $\sigma_{ij} = K_{ijmn} u_{mn}$

$$\sigma^a = K^{ab} u^b$$

2D



Number of independent entries gives the number of elastic moduli

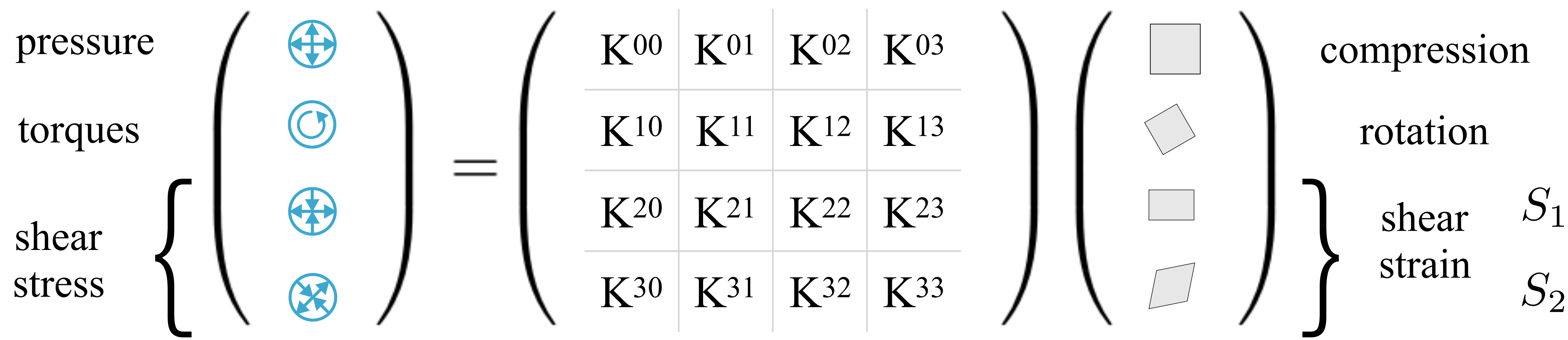
Scheibner, Souslov, Banerjee, Surowka, Irvine, Vitelli, *arXiv:1902.07760*

The stiffness tensor

Hooke's law $\sigma_{ij} = K_{ijmn} u_{mn}$

$$\sigma^a = K^{ab} u^b$$

2D



Number of independent entries gives the number of elastic moduli

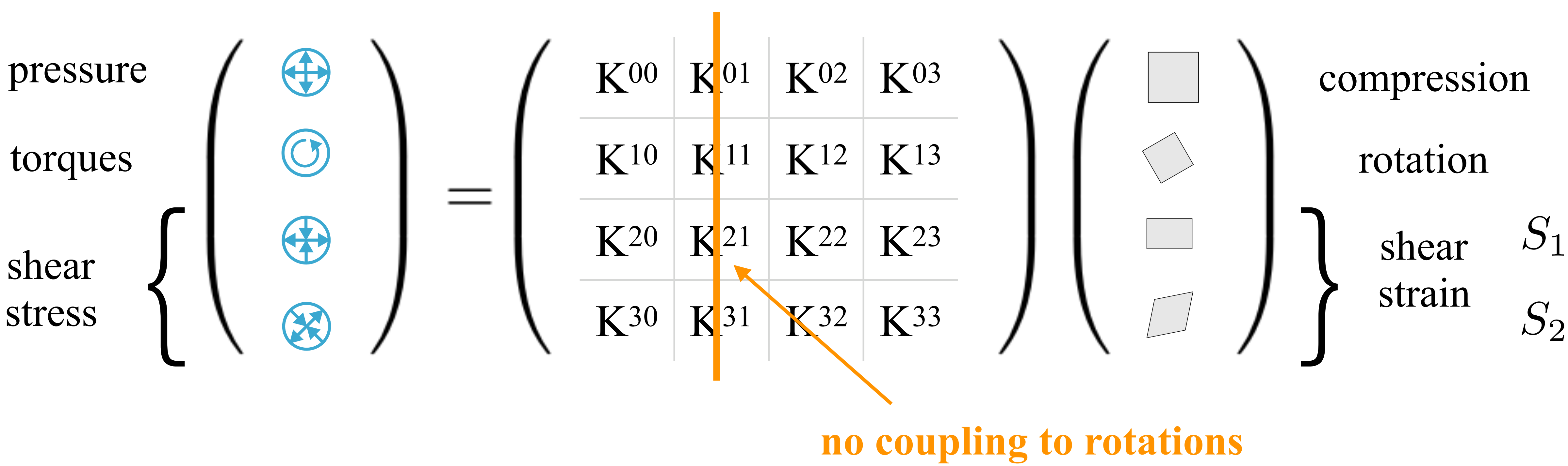
Scheibner, Souslov, Banerjee, Surowka, Irvine, Vitelli, *arXiv:1902.07760*

The stiffness tensor

Hooke's law $\sigma_{ij} = K_{ijmn} u_{mn}$

$$\sigma^a = K^{ab} u^b$$

2D



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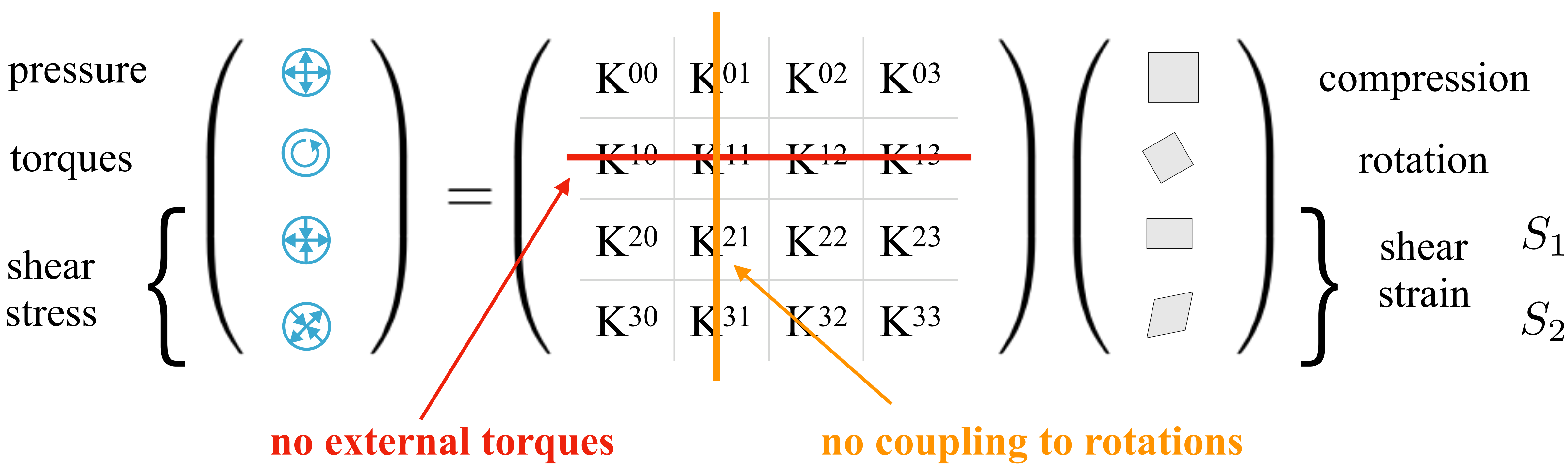


Number of independent entries gives the number of elastic moduli

Scheibner, Souslov, Banerjee, Surowka, Irvine, Vitelli, *arXiv:1902.07760*

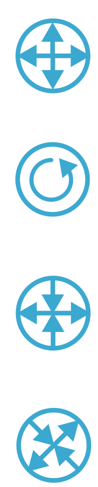
The stiffness tensor with energy conservation

Hooke's law $\sigma_{ij} = {\color{red}K_{ijmn}} u_{mn}$

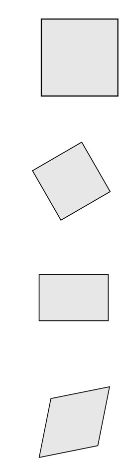
energy conservation

$$\sigma^a = K^{ab} u^b \qquad K^{ab} = K^{ba}$$

pressure
torques
shear stress



$$= \begin{pmatrix} K^{00} & 0 & K^{02} & K^{03} \\ 0 & 0 & 0 & 0 \\ K^{20} & 0 & K^{22} & K^{23} \\ K^{30} & 0 & K^{32} & K^{33} \end{pmatrix}$$



compression
rotation
shear strain S_1 S_2

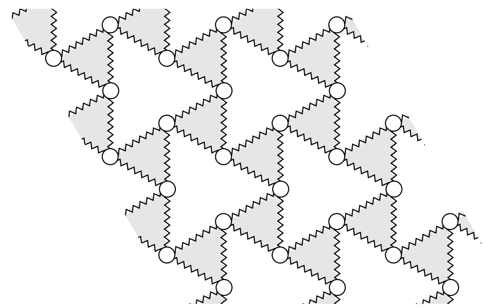
Only 6 independent coefficients remain

Scheibner, Souslov, Banerjee, Surowka, Irvine, Vitelli, *arXiv:1902.07760*

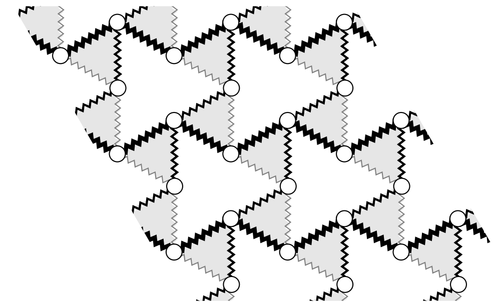
Computation of elastic moduli: Kagome lattice

assume

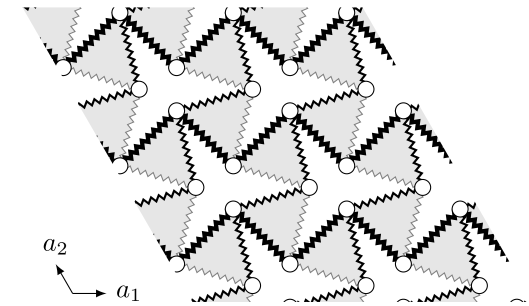
$$K^{ab} = K^{ba}$$



any θ



$\theta \neq \theta_c$



$\theta = \theta_c$

point group: C_{3v} | C_1 | C_1

K:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & K^{22} & 0 \\ 0 & 0 & 0 & K^{22} \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & K^{22} & K^{23} \\ 0 & 0 & K^{23} & K^{33} \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & K^{22} & 0 \\ 0 & 0 & 0 & K^{22} \end{pmatrix}$$

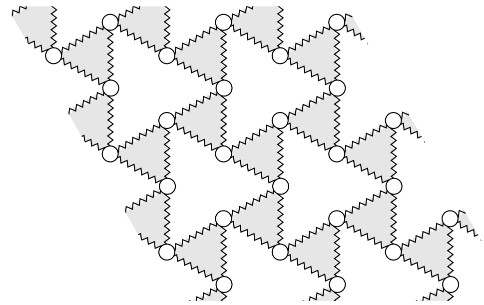
actual v.s. naive expectation of number of moduli: **1** v.s. 2 | **3** v.s. 6 | **1** v.s. 6

Standard point group analysis misses constraints from duality

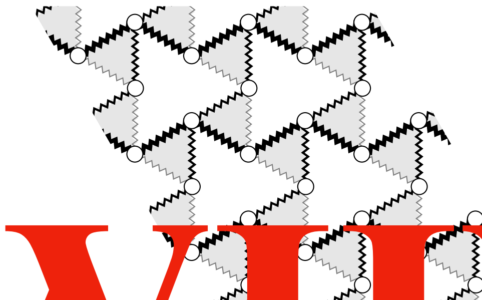
Computation of elastic moduli: Kagome lattice

assume

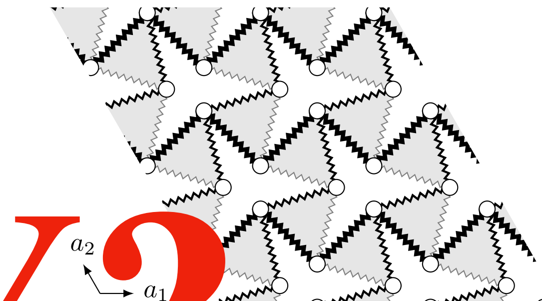
$$K^{ab} = K^{ba}$$



any θ



$\theta \neq \theta_c$



$\theta = \theta_c$

WHY?

point group: C_{3v} | C_1 | C_1

$$K: \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & K^{22} & 0 \\ 0 & 0 & 0 & K^{22} \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & K^{22} & K^{23} \\ 0 & 0 & K^{23} & K^{33} \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & K^{22} & 0 \\ 0 & 0 & 0 & K^{22} \end{pmatrix}$$

actual v.s. naive expectation of number of moduli: **1** v.s. 2 | **3** v.s. 6 | **1** v.s. 6

Standard point group analysis misses constraints from duality

Dualities and elastic moduli

$$D(\theta, q) = \mathscr{U}(q) D(\theta^*, -q) \mathscr{U}^{-1}(q)$$

$$K(\theta) = V K(\theta^*) V^\dagger$$

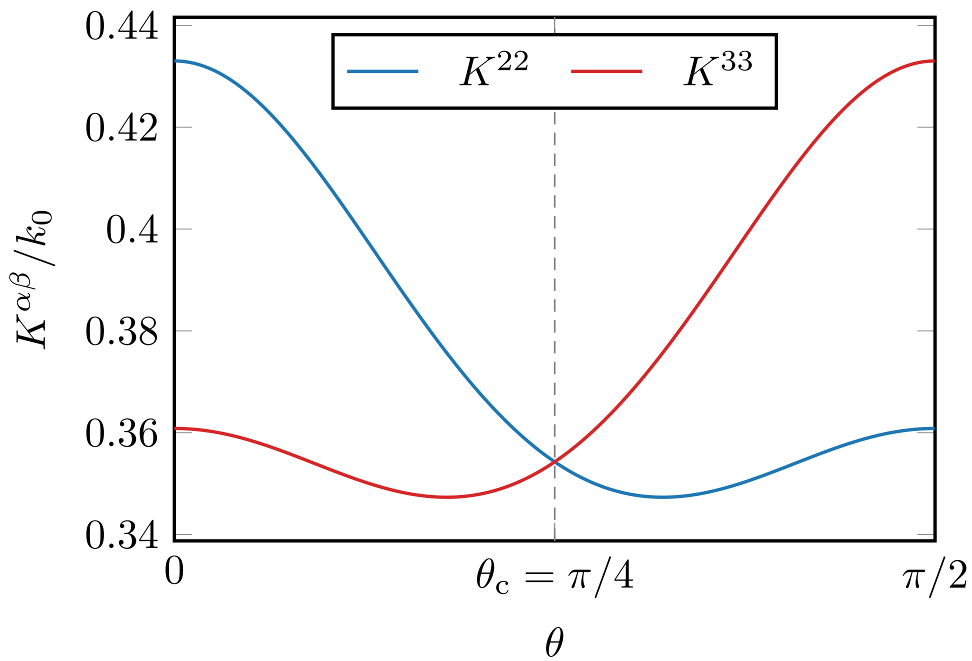
$$K(\theta) = \begin{pmatrix} K^{00} & 0 & K^{02} & K^{03} \\ 0 & 0 & 0 & 0 \\ K^{02} & 0 & K^{22} & K^{23} \\ K^{03} & 0 & K^{23} & K^{33} \end{pmatrix} \qquad V K(\theta^*) V^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & K^{00} & K^{03} & -K^{02} \\ 0 & K^{03} & K^{33} & -K^{23} \\ 0 & -K^{02} & -K^{23} & K^{22} \end{pmatrix}$$

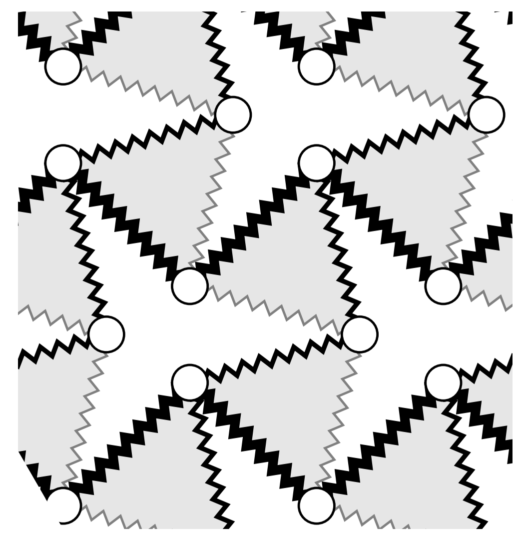
these must vanish!

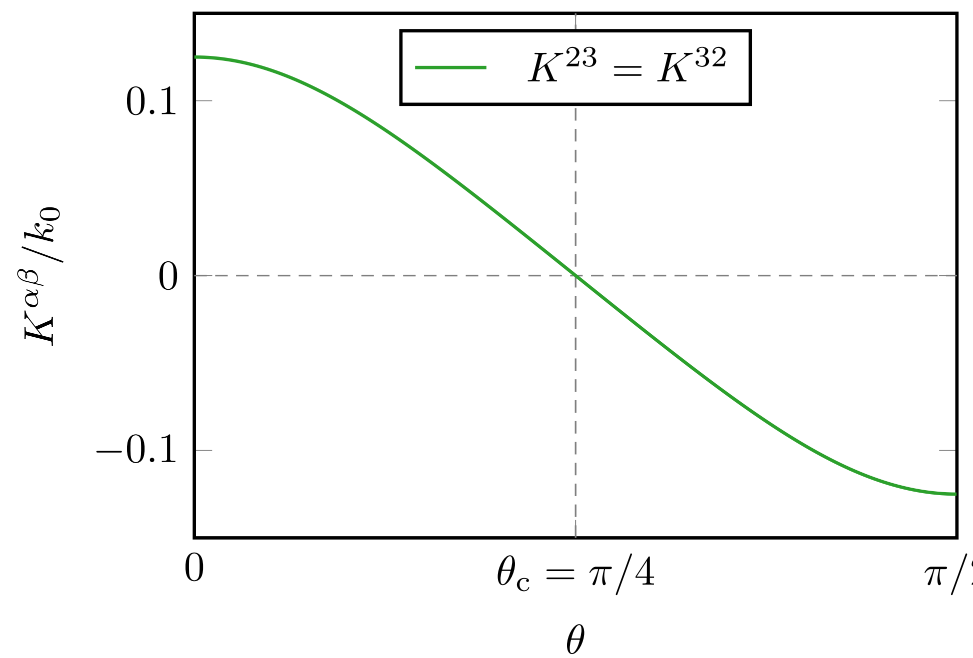
M. Fruchart and V. Vitelli **in preparation**

here we have $V = \sigma_3 \otimes \mathrm{i}\sigma_2$

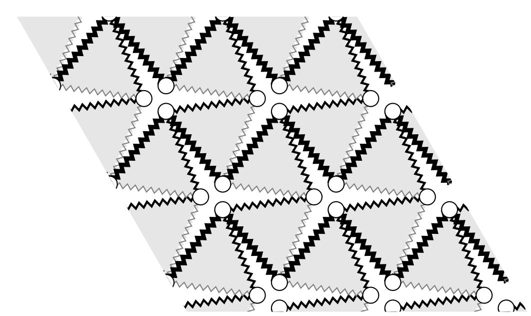
Dualities and elastic moduli: Kagome lattice







$$K(\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & K^{22}(\theta) & K^{23}(\theta) \\ 0 & 0 & K^{23}(\theta) & K^{33}(\theta) \end{pmatrix}$$



$$K(\theta_c) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}$$

duality constrains the elastic moduli
for all θ : only shear moduli

at the self-dual point
the elastic tensor is **isotropic**

Duality constraint acts as an emergent symmetry

Odd moduli

$$\sigma_{ij} = K_{ijmn} u_{mn}$$

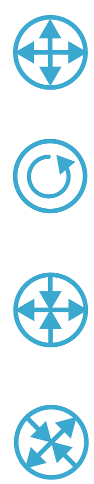
energy ~~conservation~~

$$\sigma^a = K^{ab} u^b \qquad K^{ab} \neq K^{ba}$$

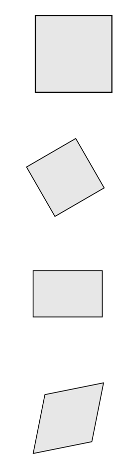
pressure

torques

shear stress



$$= \begin{pmatrix} K^{00} & 0 & K^{02} & K^{03} \\ 0 & 0 & 0 & 0 \\ K^{20} & 0 & K^{22} & K^{23} \\ K^{30} & 0 & K^{32} & K^{33} \end{pmatrix}$$



compression

rotation

shear strain S_1 S_2

9 independent coefficients remain

Scheibner, Souslov, Banerjee, Surowka, Irvine, Vitelli, *arXiv:1902.07760*

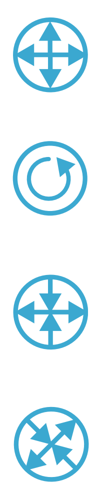
Odd moduli: isotropy and angular momentum conservation

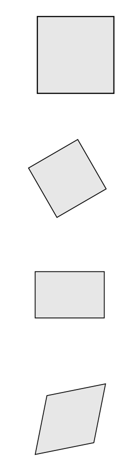
$$\sigma_{ij} = K_{ijmn} u_{mn}$$

$$\sigma^a = K^{ab} u^b$$

~~energy conservation~~

$$K^{ab} \neq K^{ba}$$



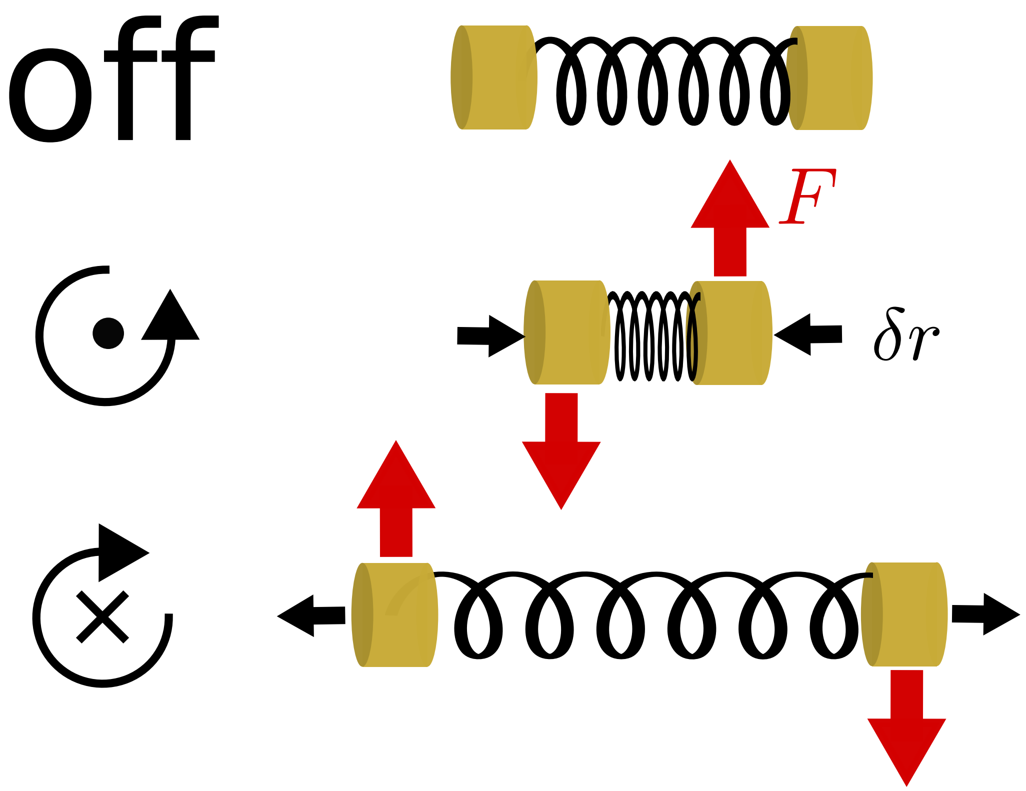


$$\begin{array}{l} \text{pressure} \\ \text{torques} \\ \text{shear stress} \end{array} \quad \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & K^0 \\ 0 & 0 & -K^0 & \mu \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \quad \begin{array}{l} \text{compression} \\ \text{rotation} \\ \text{shear strain} \; S_1, \, S_2 \end{array}$$

1 odd coefficient remains: Hall modulus analogous to Hall viscosity

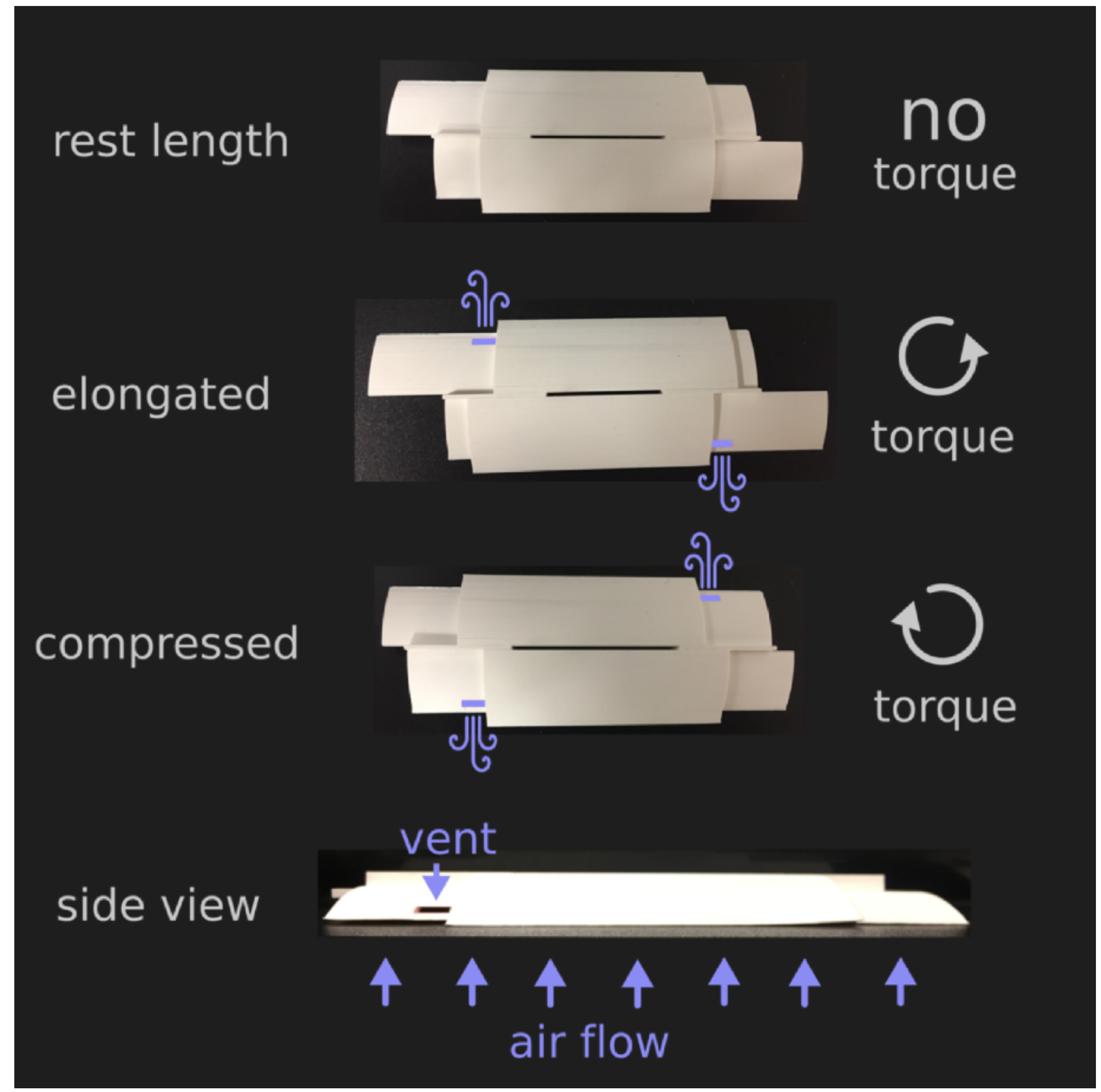
Scheibner, Souslov, Banerjee, Surowka, Irvine, Vitelli, *arXiv:1902.07760*

Microscopic model: active bonds



Active

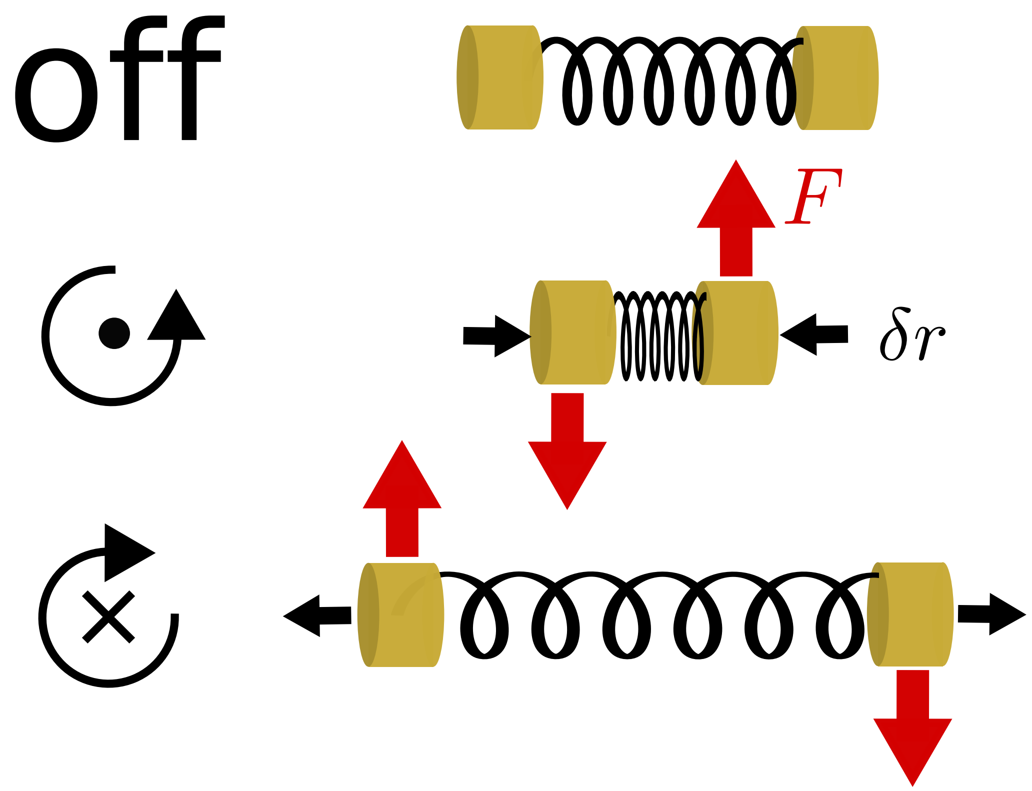
$$\mathbf{F} = -(k\hat{\mathbf{r}} + k^o\hat{\boldsymbol{\varphi}})\delta r$$

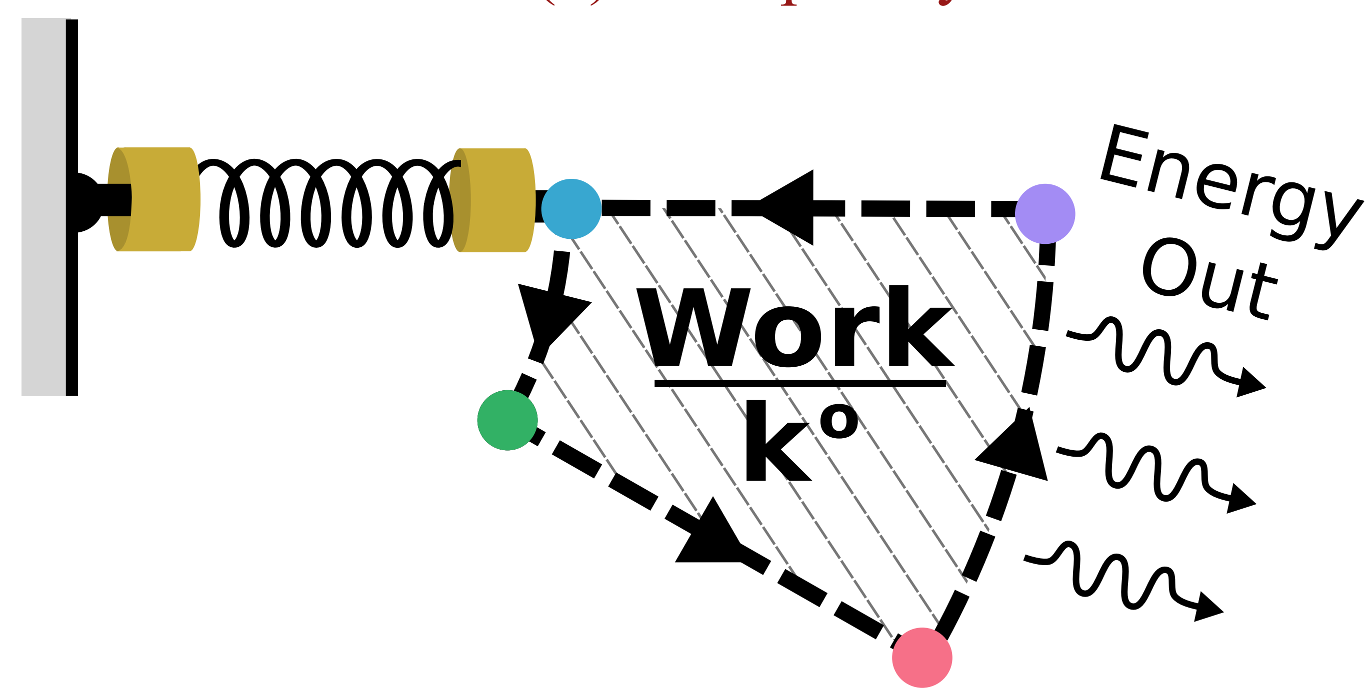


Compression/elongation induce active torques

Microscopic model: active bonds

Beam violates:
(1) Energy Conservation
(2) Reciprocity





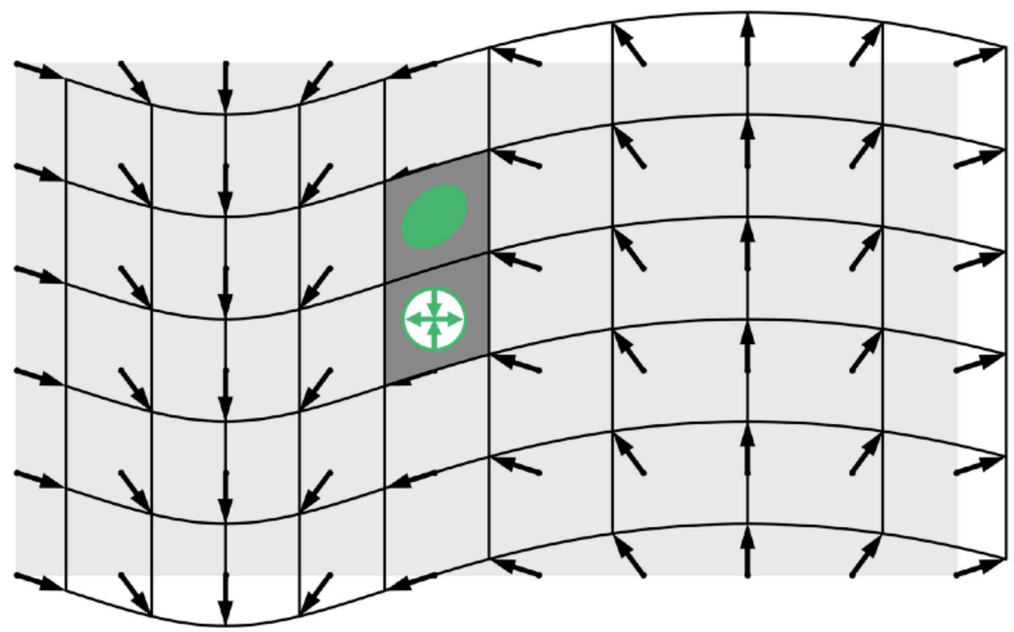
$$\mathbf{F} = -(k\hat{\mathbf{r}} + k^o\hat{\boldsymbol{\varphi}})\delta r$$

$$\nabla \times F \neq 0$$

Active bonds are microscopic engines that harvest energy around loops

Odd elastodynamics

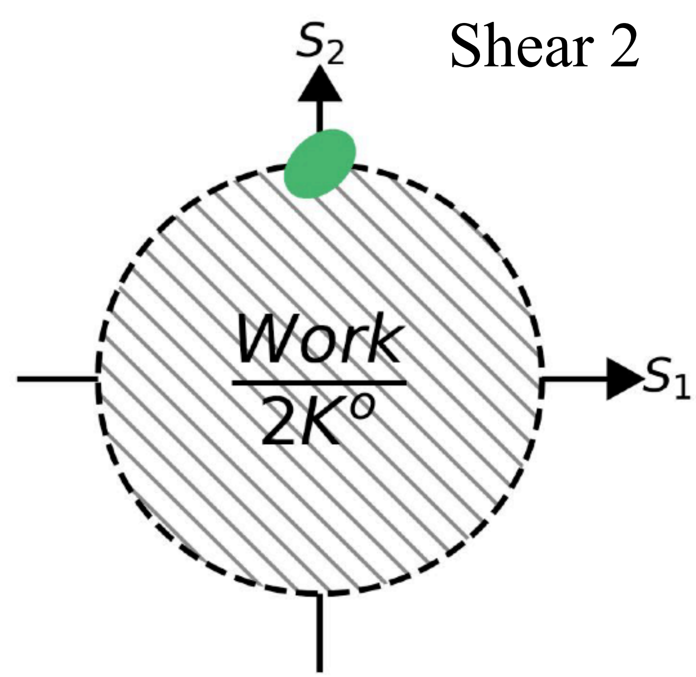
Odd Wave



$$\eta \partial_t u_i = \partial_j \sigma_{ij}$$

$$\sigma_{ij} = K^0_{ijmn} u_{mn}$$

Elastic engine cycle powers the wave



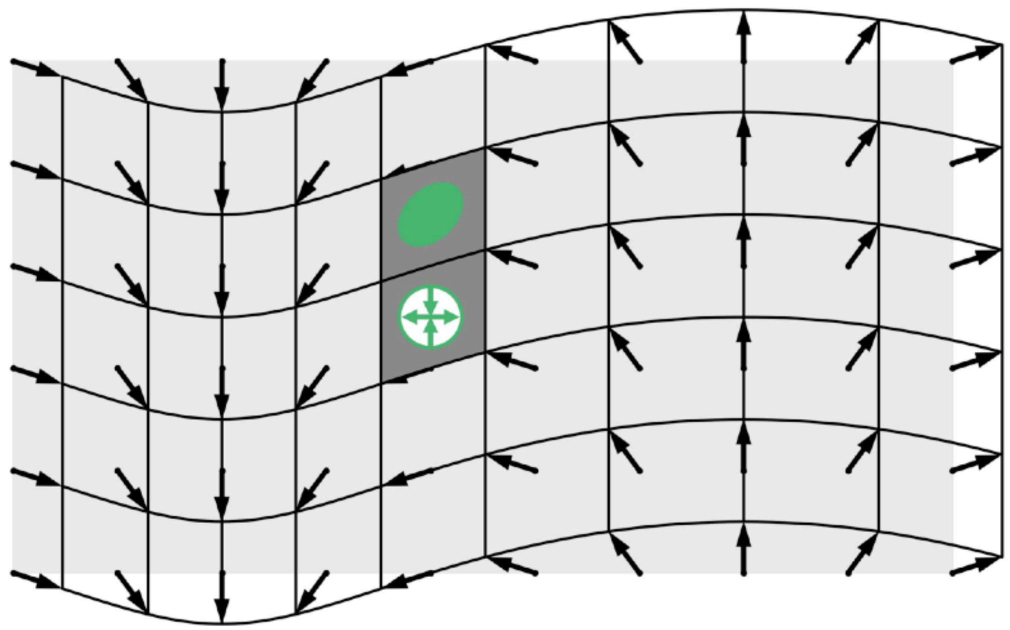
Shear 2

Shear 1

Active phonons propagate in over damped media

Odd elastodynamics

Odd Wave



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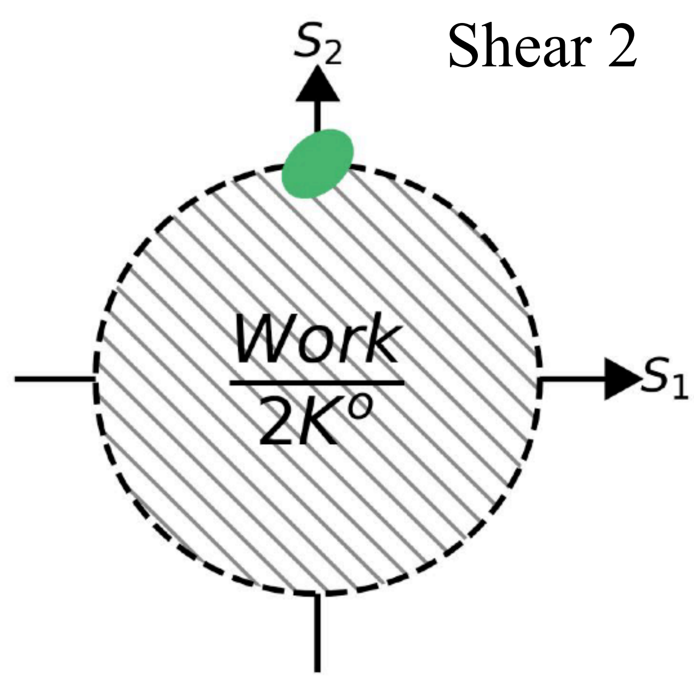
Elastic engine cycle powers the wave

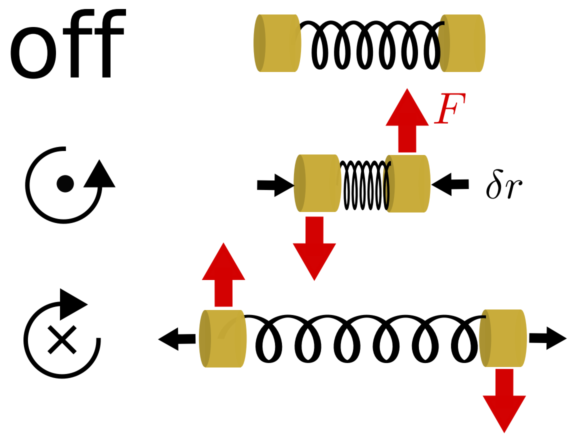


Shear 2

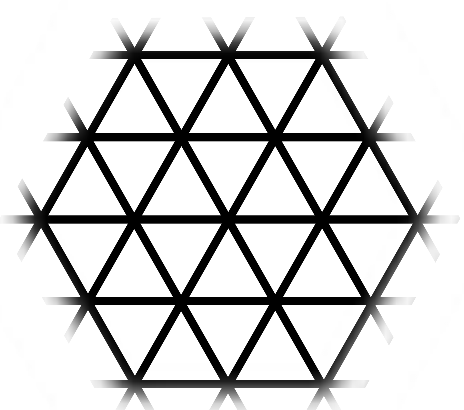
Shear 1

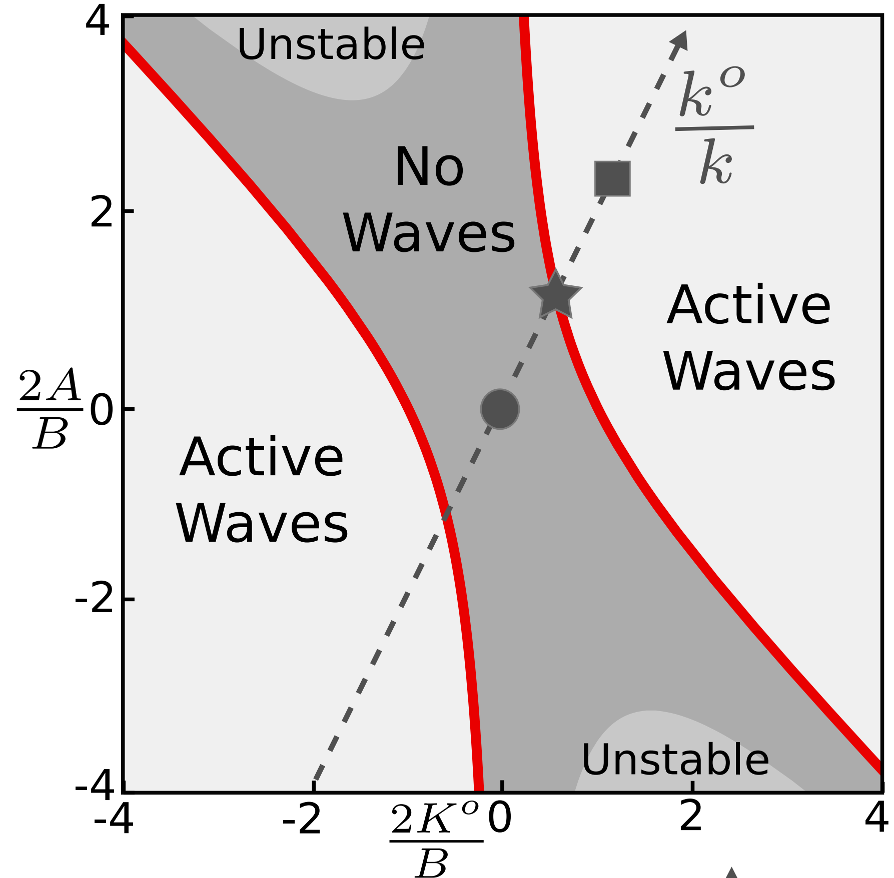
Active phonons propagate in over damped media

Phonons in non-Hermitian mechanics

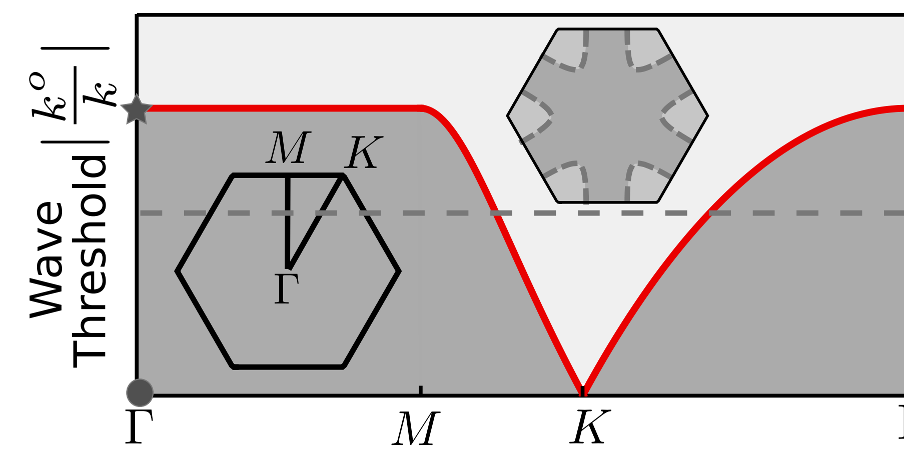


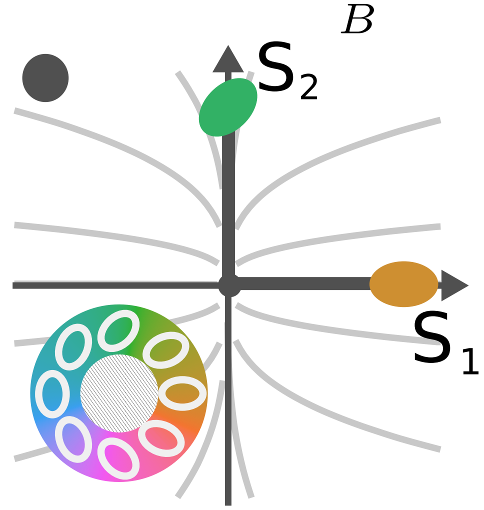
$$\mathbf{F} = -(k\hat{\mathbf{r}} + k^o\hat{\boldsymbol{\varphi}})\delta r$$

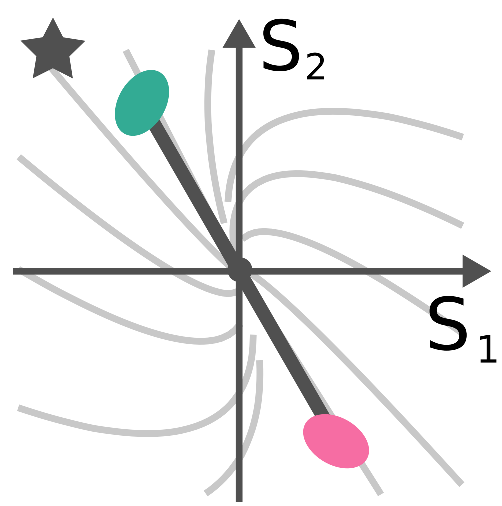


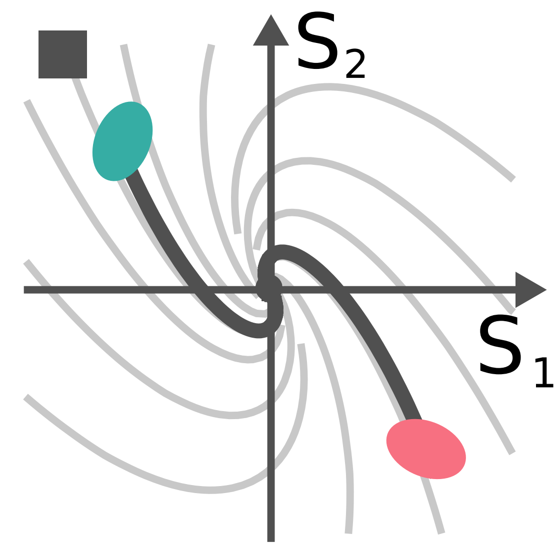


Non-Hermitian
dynamical matrix



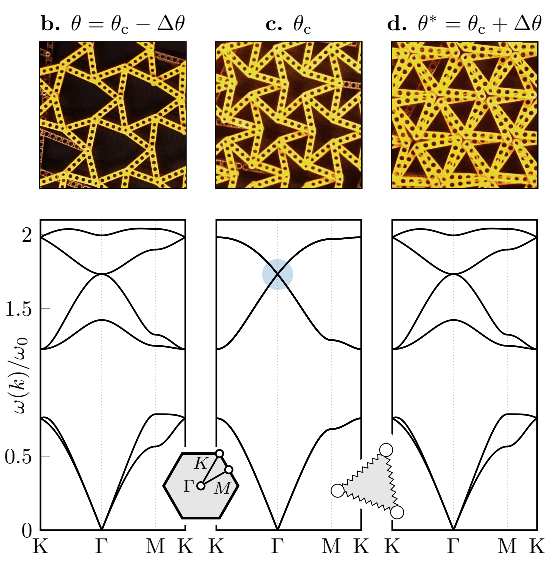


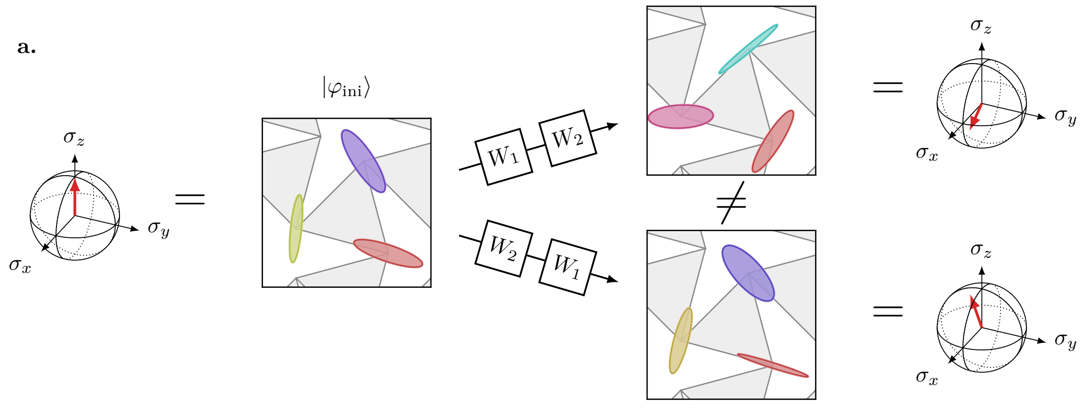


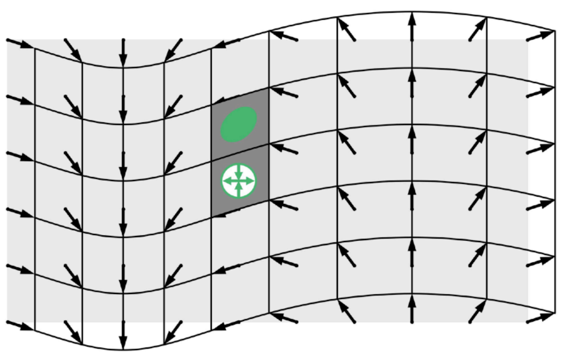


Exceptional Point

Visual summary







Dualities
and symmetries

Non-abelian
sound

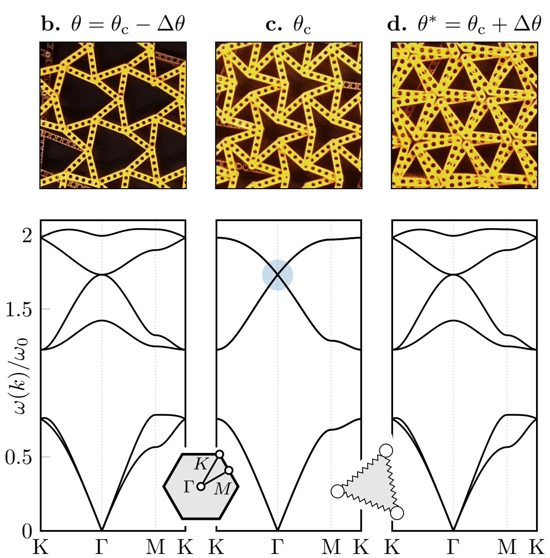
Non-Hermitian
mechanics

Thanks

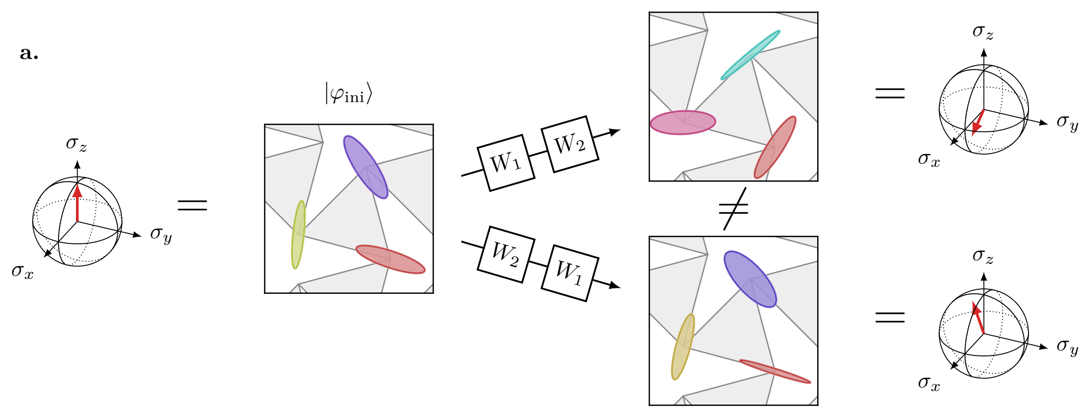
M. Fruchart, Y. Zhou, V. Vitelli, Nature (in press), arXiv:1904.07436

Scheibner, Souslov, Banerjee, Surowka, Irvine, Vitelli, *arXiv:1902.07760*

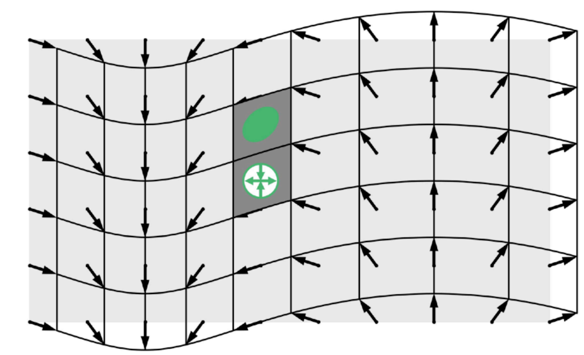
Visual summary



Dualities
and symmetries



Non-abelian
sound



Non-Hermitian
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