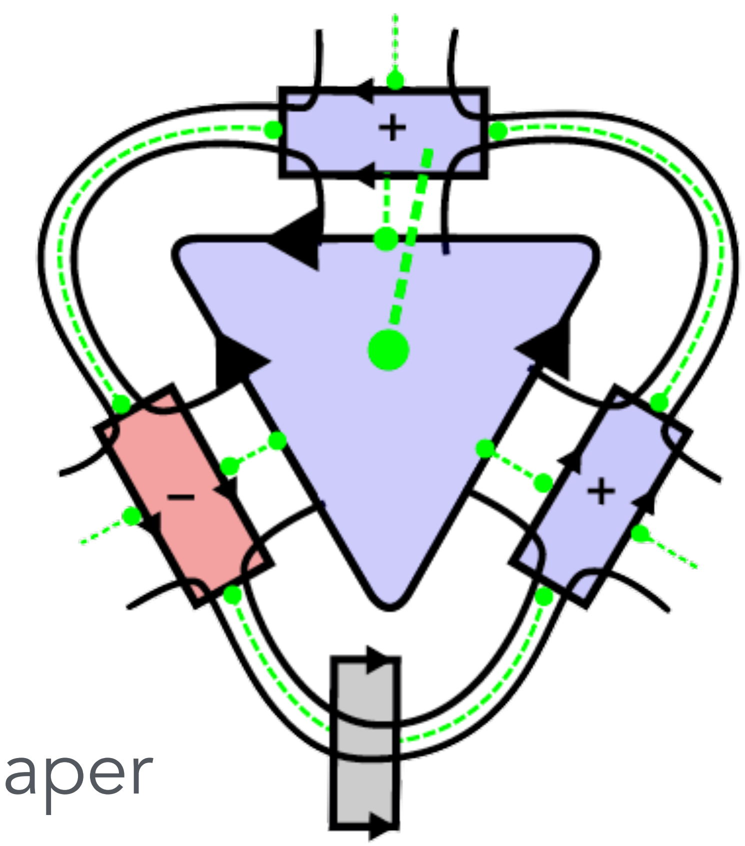
Fermionic topological quantum states as tensor networks



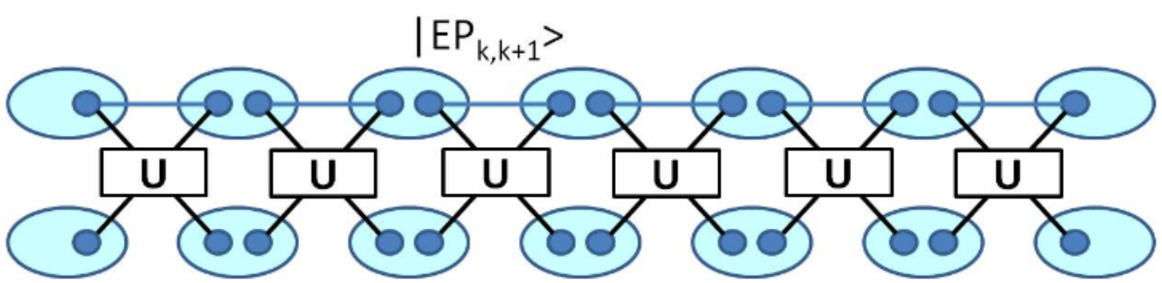
Jens Eisert, Freie Universität Berlin
Joint work with Carolin Wille and Oliver Buerschaper

Symmetry, topology, and quantum phases of matter: From tensor networks to physical realizations, KITP program, December 2016

What this talk is about?

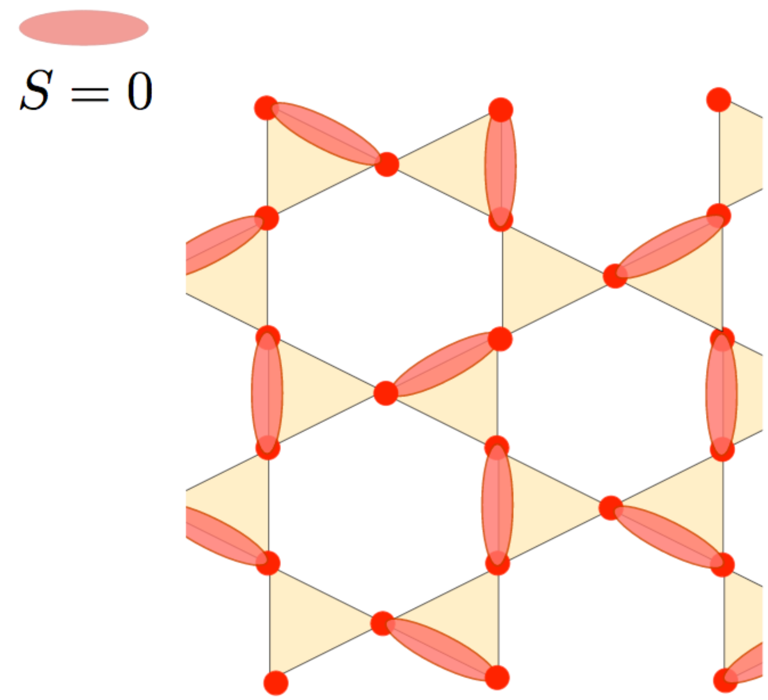
- Topological and symmetry protected quantum phases of matter

- Can methods like tensor networks be useful in modelling these phases?



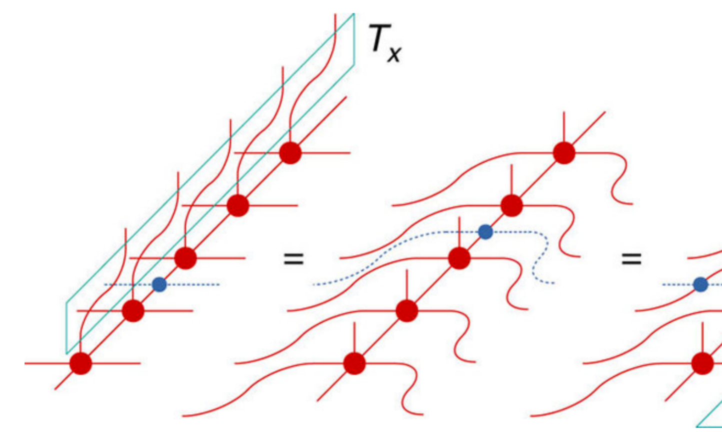
Classification of phases in 1D

Pollmann, Turner, Berg, Oshikawa, Phys Rev B 81, 064439 (2010)
Chen, Gu, Wen, Phys Rev B 83, 035107 (2011)
Schuch, Perez-Garcia, Cirac, Phys Rev B 84, 165139 (2011)



RVB spin liquids as tensor networks

Schuch, Poilblanc, Cirac, Perez-Garcia, Phys Rev B 86, 115108 (2012)



Shadows of anyons

Haegeman, Zauner, Schuch, Verstraete, Nature Comm 6, 8284 (2015)

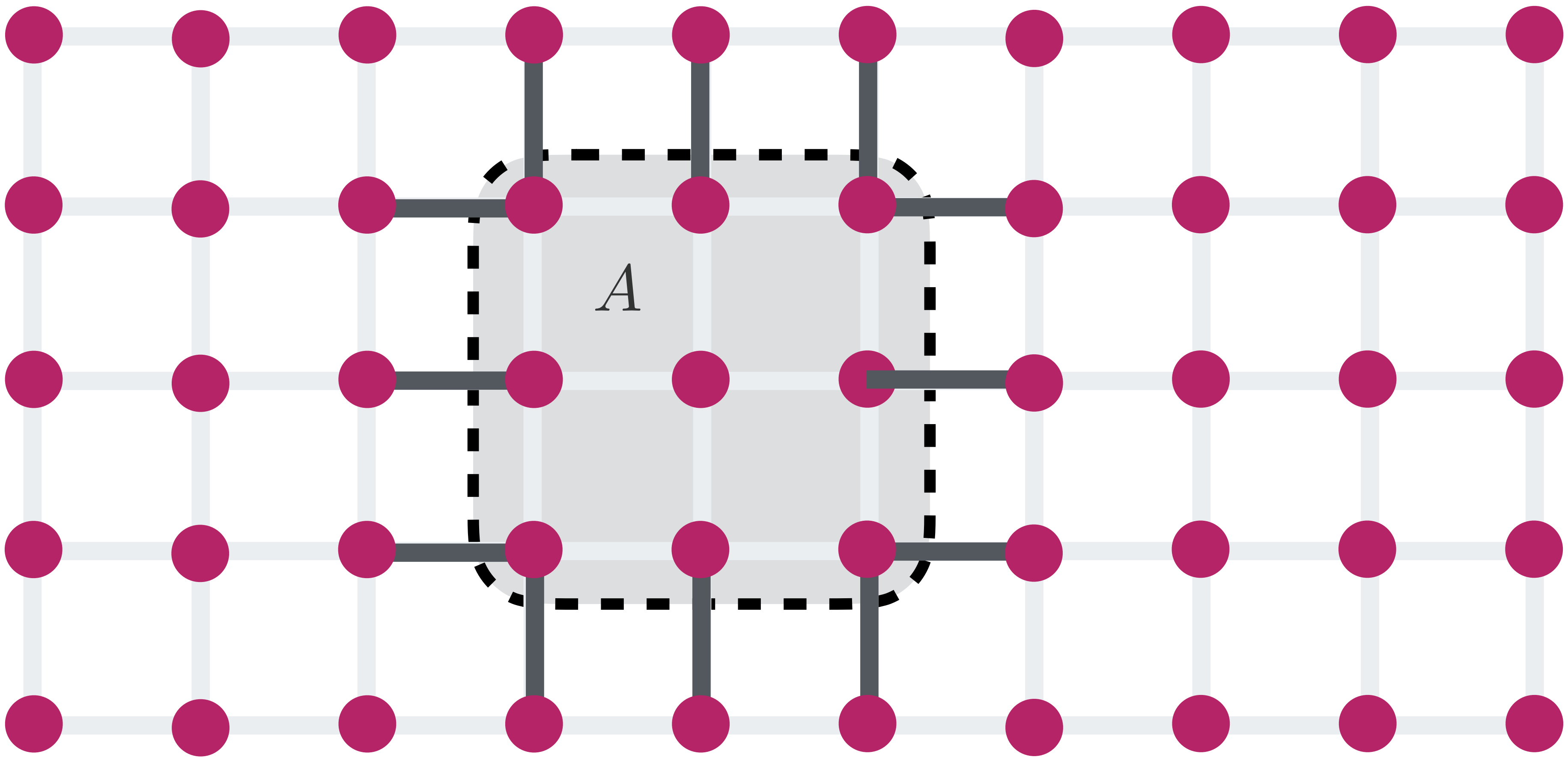
- **New twist:** Put emphasis on quantum states, not so much Hamiltonians, which are reinserted in the picture by means of parent Hamiltonians

What this talk is about?

- Work on phases of matter using tensor networks exclusively on bosons/spins
- How about systems having a fermionic component?
- An attempt:
  - Area laws for entanglement entropies and tensor networks
  - Topological order in PEPS
  - MPO-injective PEPS
  - Fermionic PEPS

Area laws for the entanglement entropy and tensor network states

Area laws for entanglement entropies



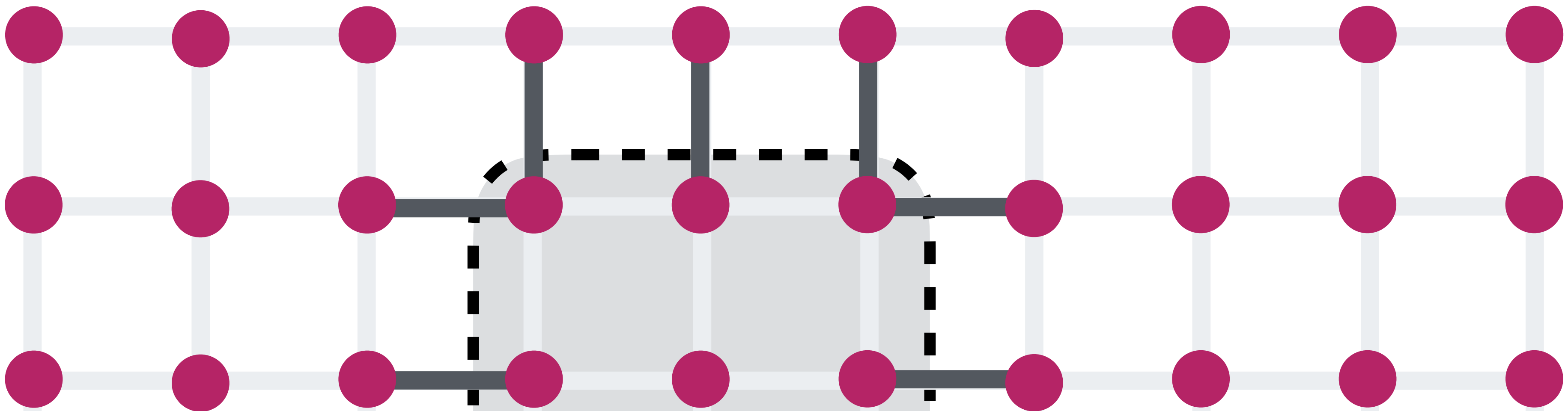
- **Area law** for the entanglement entropy $S(\rho_A)$:

$$S(\rho_A) = O(|\partial A|)$$

- Scale like boundary area, not volume: Much less entangled than possible!

Eisert, Cramer, Plenio, Rev Mod Phys 82, 277 (2010)
Hastings, JSTAT, P08024 (2007)

Area laws for entanglement entropies



- **Theorem:** Area laws hold true for

  1. arbitrary gapped models in 1D
  2. free bosonic and fermionic gapped Hamiltonians in any D
  3. Stabiliser Hamiltonians

- **Topological** entanglement entropy γ

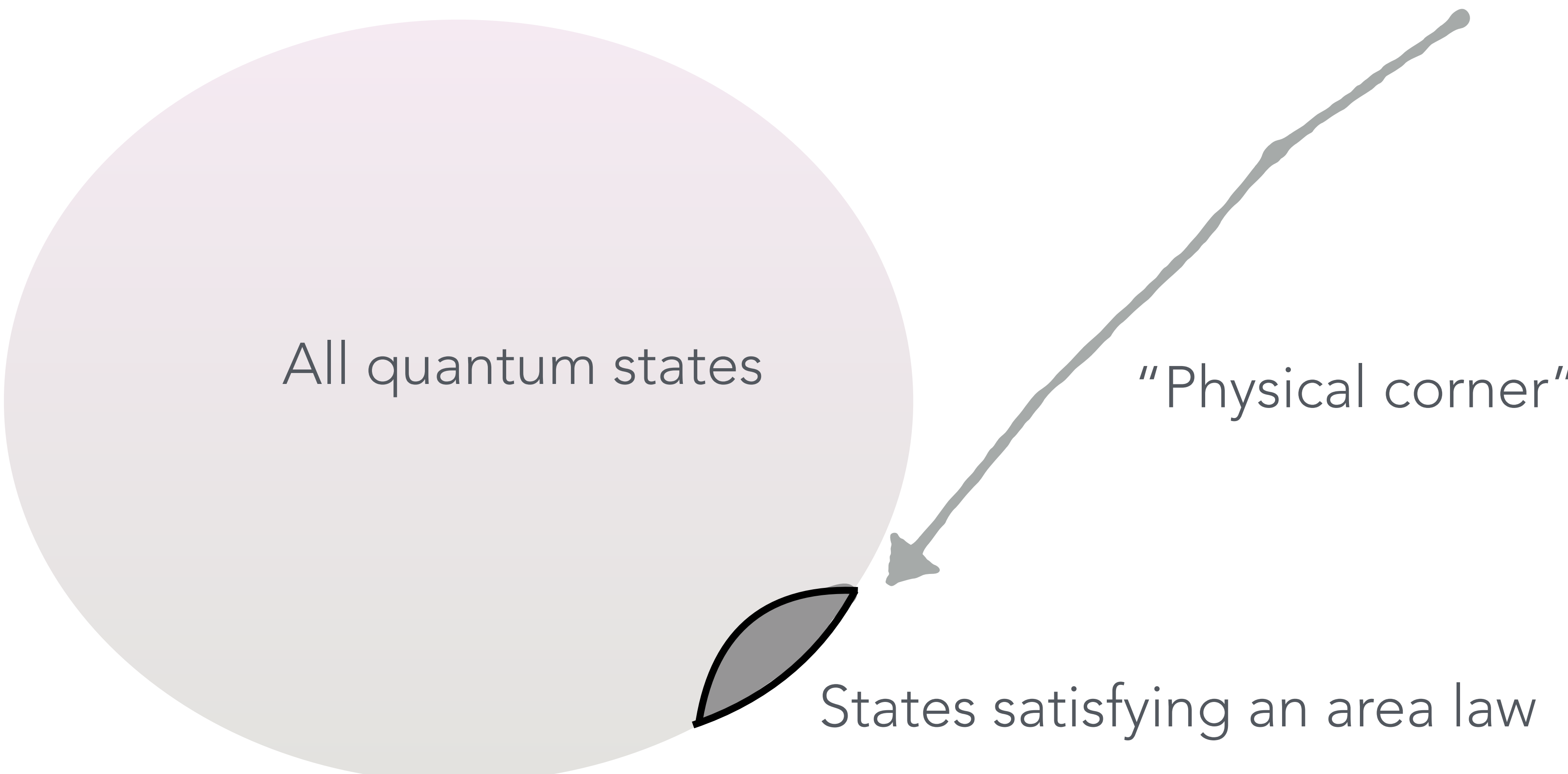
$$S(\rho_A) = \alpha|\partial A| - \gamma + O(|\partial A|^{-\beta})$$

Kitaev and J. Preskill, Phys Rev Lett 96, 110404 (2006)
Levin and Wen, Phys Rev Lett 96, 110405 (2006)
Bauer, Cincio, Keller, Dolfi, Vidal, Trebst, Ludwig, Nature Comm 5, 5137 (2014)

- **Are**
- Sca ... le!

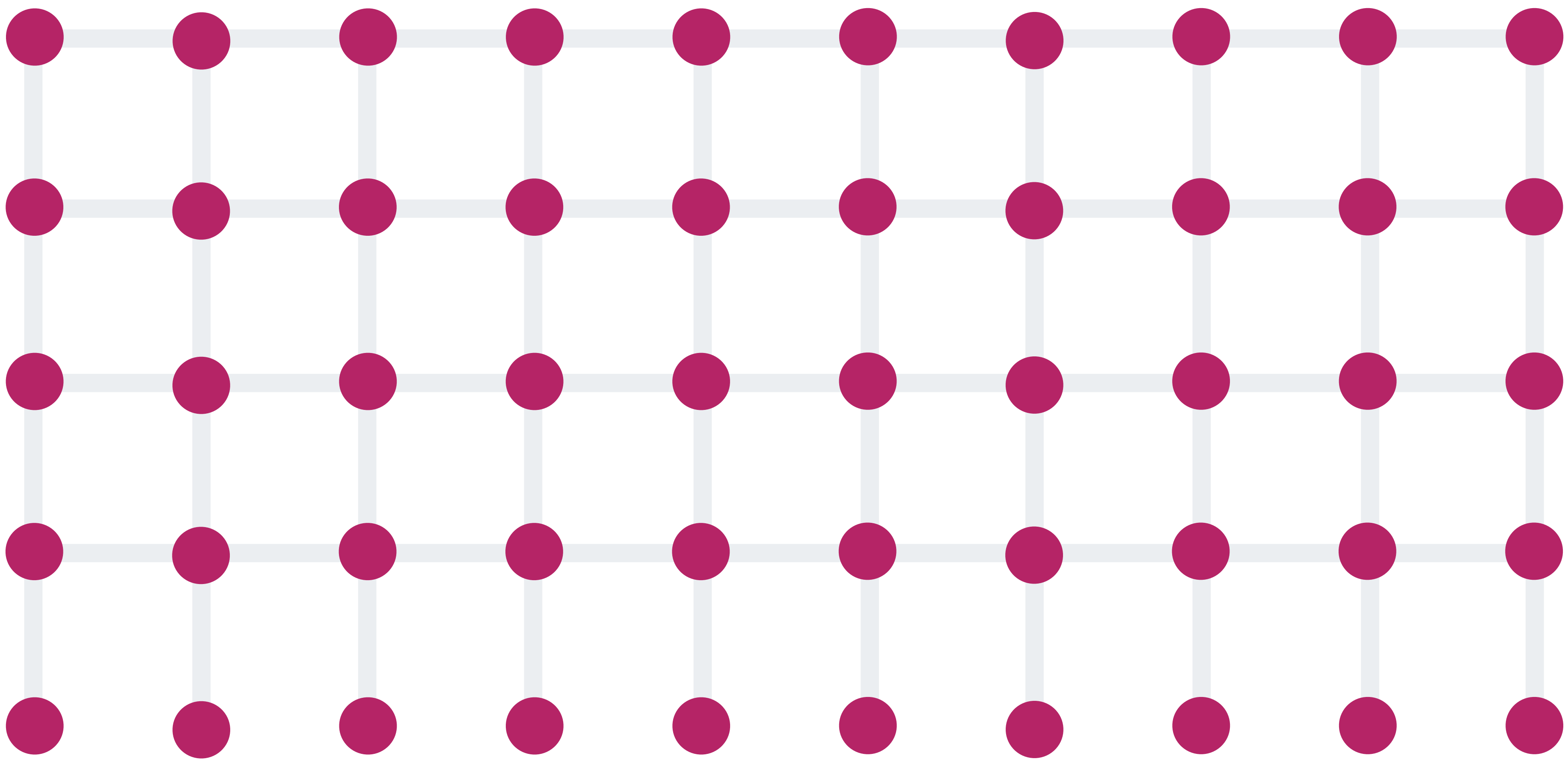
Eisert, Cramer, Plenio, Rev Mod Phys 82, 277 (2010)
Hastings, JSTAT, P08024 (2007)

- **Area law** fo
- Scale like b



- **Entanglement** captures "essential degrees of freedom, hugely removes redundancy"
- Can this be used to largely "**parametrise" states**?

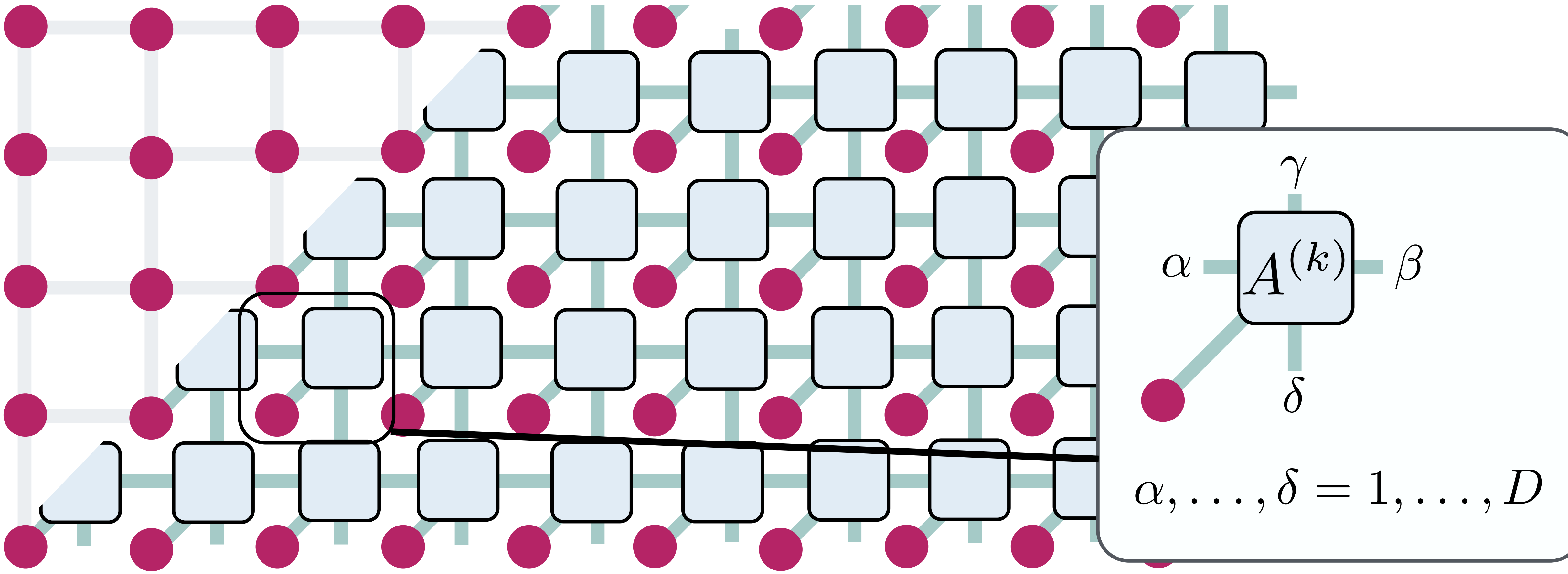
Projected entangled pair states



- Equip lattice system with **local tensor structure**
- Projected entangled pair states (PEPS) in 2D

Verstraete, Cirac, cond-mat/0407066
Martin-Delgado, cond-mat/9610196

Projected entangled pair states



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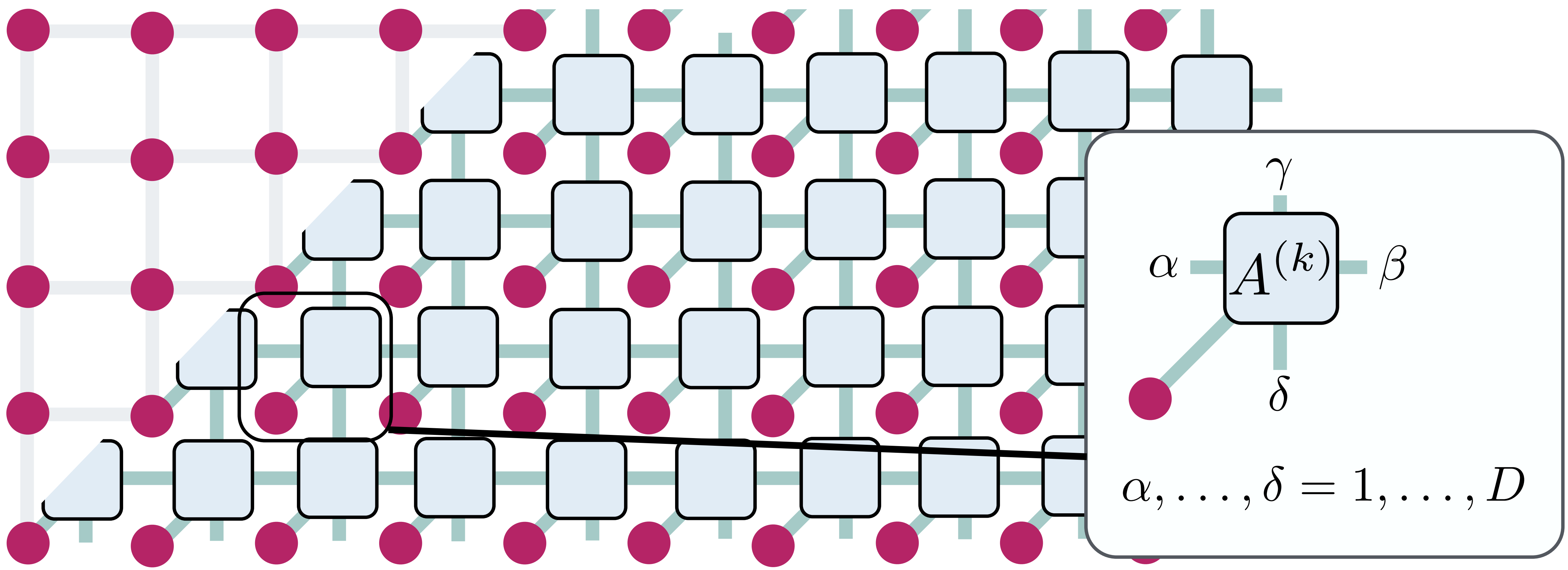
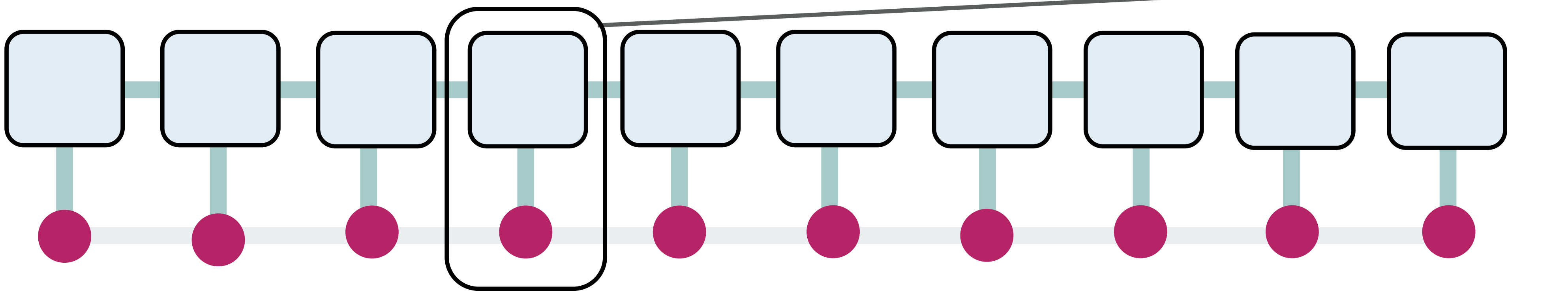
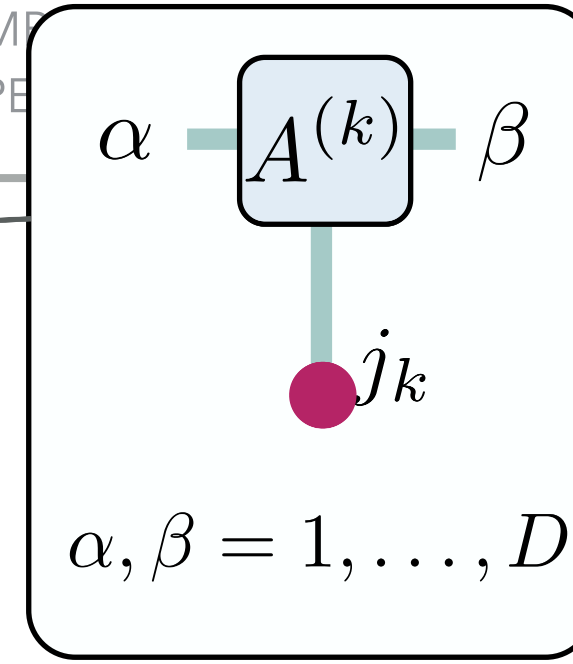


- Virtual indices of "bond dimension" D
- **Drastic** reduction of parameters, down to $O(n^2 d D^4)$
- In TI, a **single tensor** captures Hamiltonian and all global state properties

Verstraete, Cirac, cond-mat/0407066
Martin-Delgado, cond-mat/9610196

Matrix product states



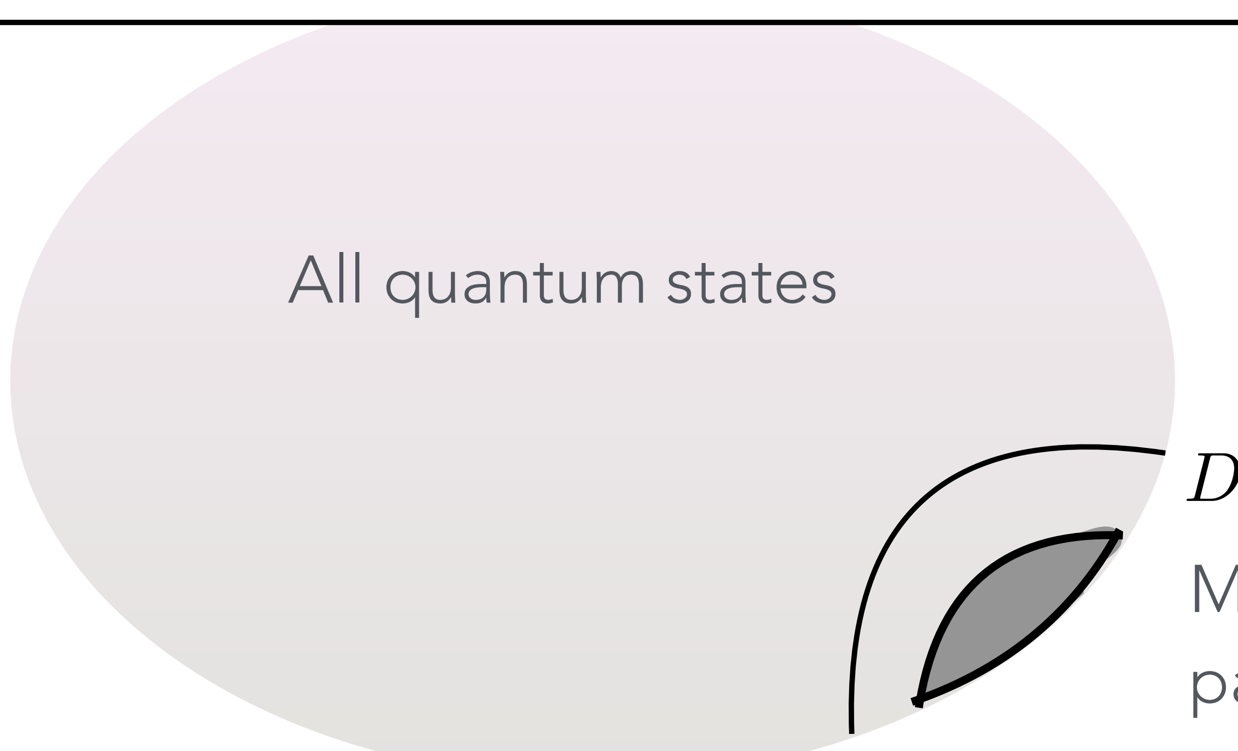


- Matrix product states (MPS)
- At basis of powerful DMRG (density-matrix renormalisation group)

White, Phys Rev Lett 69, 2863 (1992)

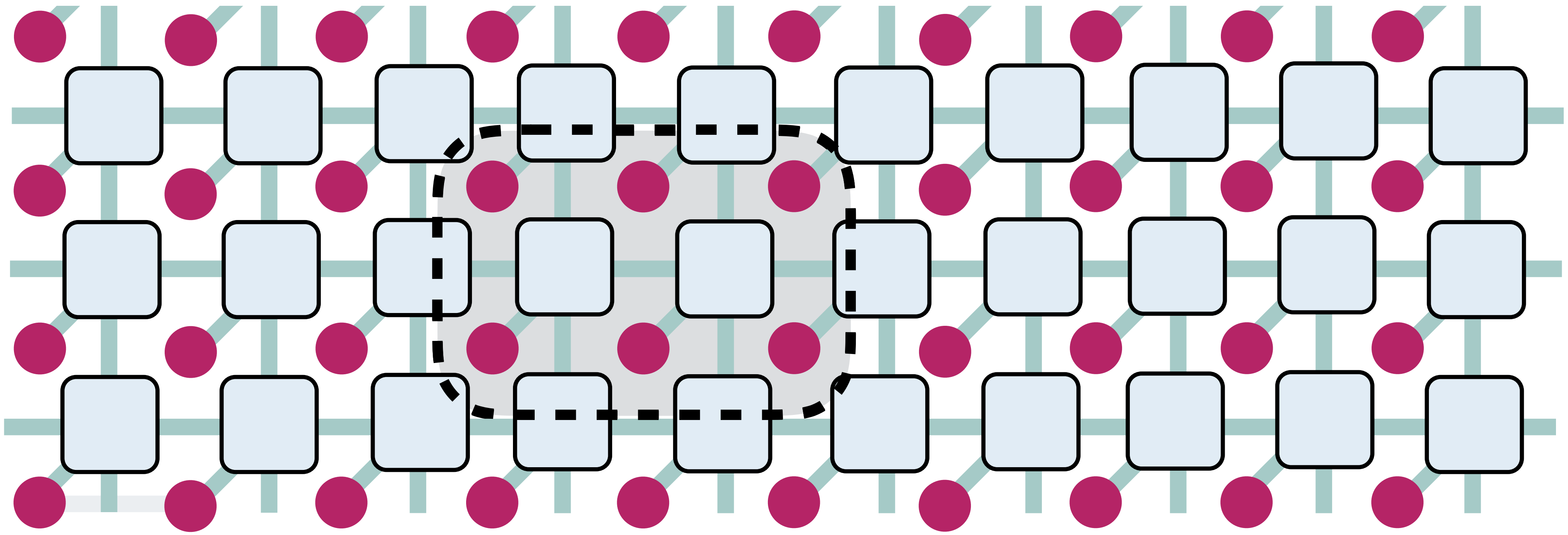
- **Theorem:** • All states that satisfy area laws (for Renyi entropies $\alpha < 1$) have efficient approximation in $\|.\|_1$-norm ($\text{poly}(n, 1/\epsilon)$)

Verstraete, Cirac, Murg, Adv Phys 57, 143 (2008)
Eisert, Cramer, Plenio, Rev Mod Phys 82, 277 (2010)



Matrix product states parametrise physical corner

Two surprising observations



- Similar intuition holds true, but…

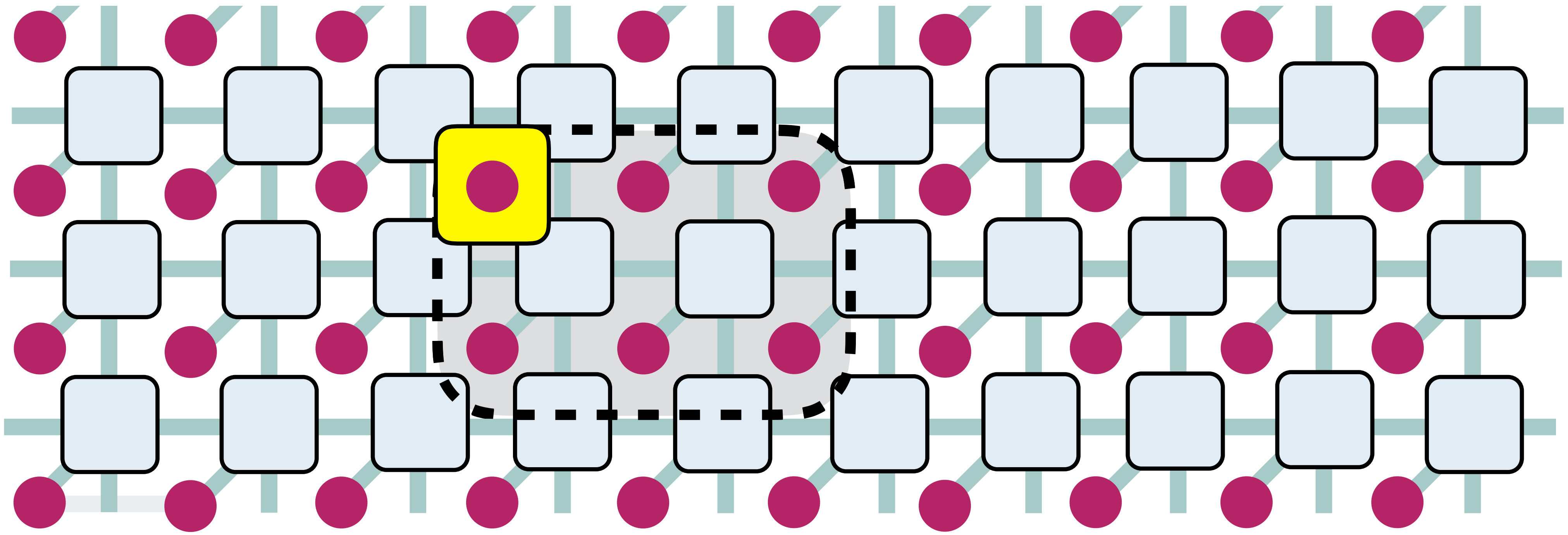
> - **Theorem:** There are states that satisfy **all** (Renyi entropy) **area laws**, yet cannot be efficiently approximated by any tensor network state

Ge, Eisert, New J Phys 18, 083026 (2016)

- Good reasons to believe that they capture low energy physics

Ge, Molnar, Cirac, Phys Rev Lett 116, 080502 (2016)

Two surprising observations



- Another cute twist

- **Theorem:** PEPS contraction is #P-complete

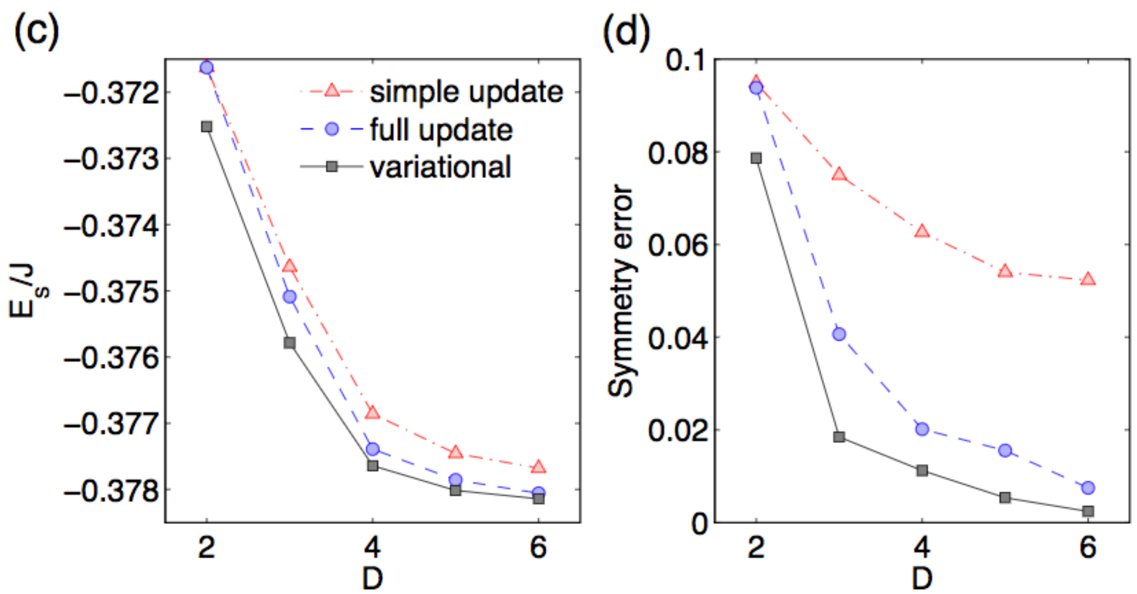
Schuch, Wolf, Verstraete, Cirac, Phys Rev Lett 98, 140506 (2007)

- Cannot efficiently compute expectation values in worst case complexity!

- **Theorem:** PEPS that approximate gapped ground states well [in the sense that they are $\|\cdot\|_1$ norm close and have uniformly gapped parent], can be contracted in quasi-polynomial time

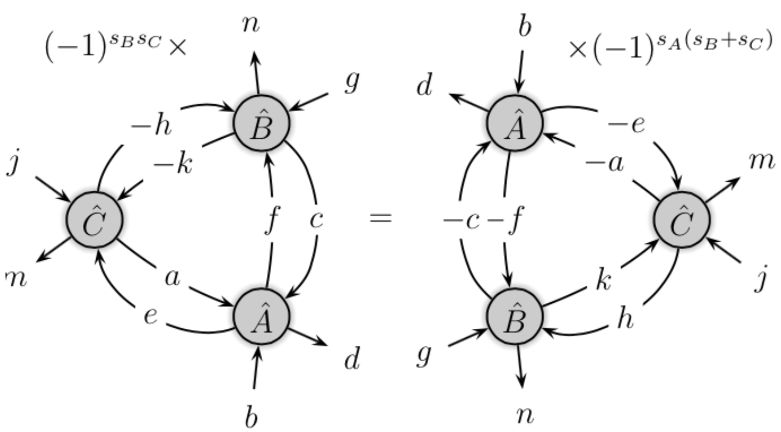
Schwarz, Buerschaper, Eisert, arXiv:1606.06301

- Finite PEPS and iPEPS: Excellent numerical performance



Shustry-Sutherland model

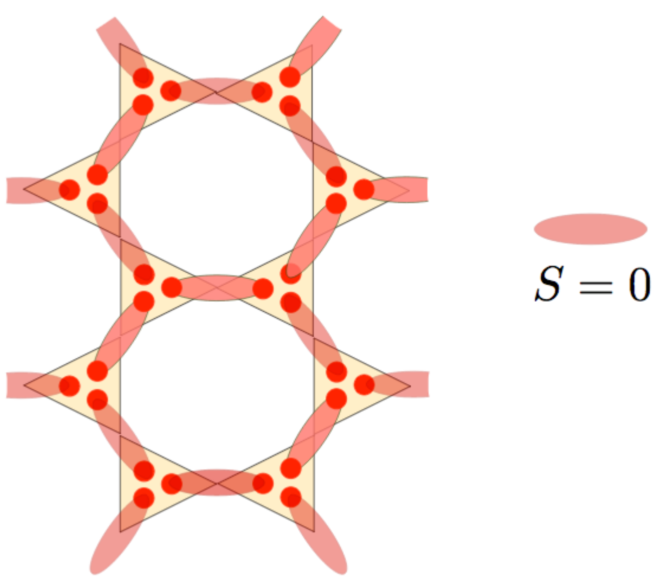
Corboz, arXiv:1605.03006



Fermionic tensor networks

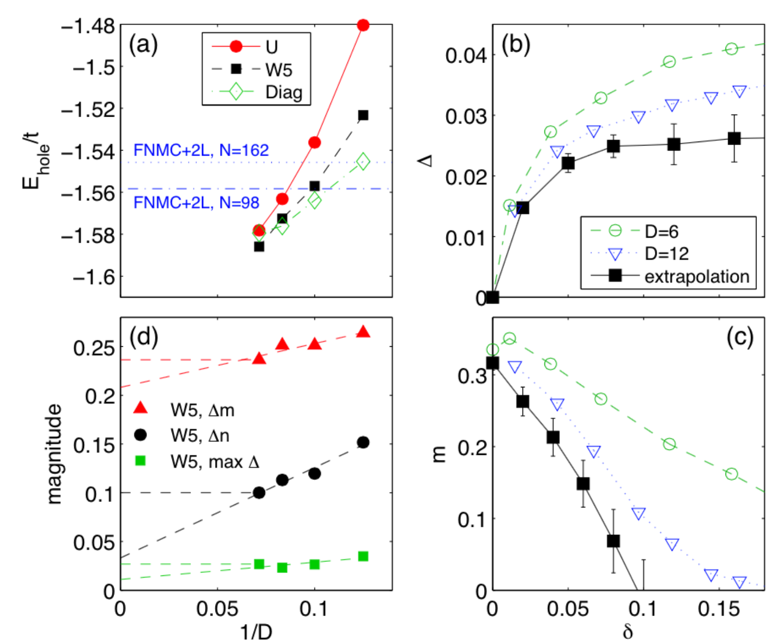
Pineda, Barthel, Eisert, Phys Rev A 81, 050303 (2010)

Corboz, Evenbly, Verstraete, Vidal, Phys Rev A 81, 010303 (2010)



Spin-3/2 AKLT spin liquids

Lavoie at al, Nature Phys 6, 850 (2010)



Ground state energies in the t-J model

Corboz, arXiv:1605.03006

Orus, Ann Phys 349, 17 (2014)

Topological order in PEPS

A new type of order

- Definition of topological order

  - **Degeneracy** of the Hamiltonian constant and depends on **topology**
  - Excitations behave like quasi-particles having **anyonic statistics**
  - All GS are **locally indistinguishable** (no local order parameter)
  - To map between them, one needs a **non-local operator**

- How can it be captured in PEPS governed by single tensor?

- Parent Hamiltonian

- **Theorem:** All MPS and PEPS have frustration-free parent Hamiltonians
$$H = \sum_j h_j \;,\; h_j|\psi\rangle = 0$$

- **Injective PEPS:** PEPS projection has left inverse
- Any action achievable on the virtual indices by acting on the physical spins

- But they are unique ground states of their parents

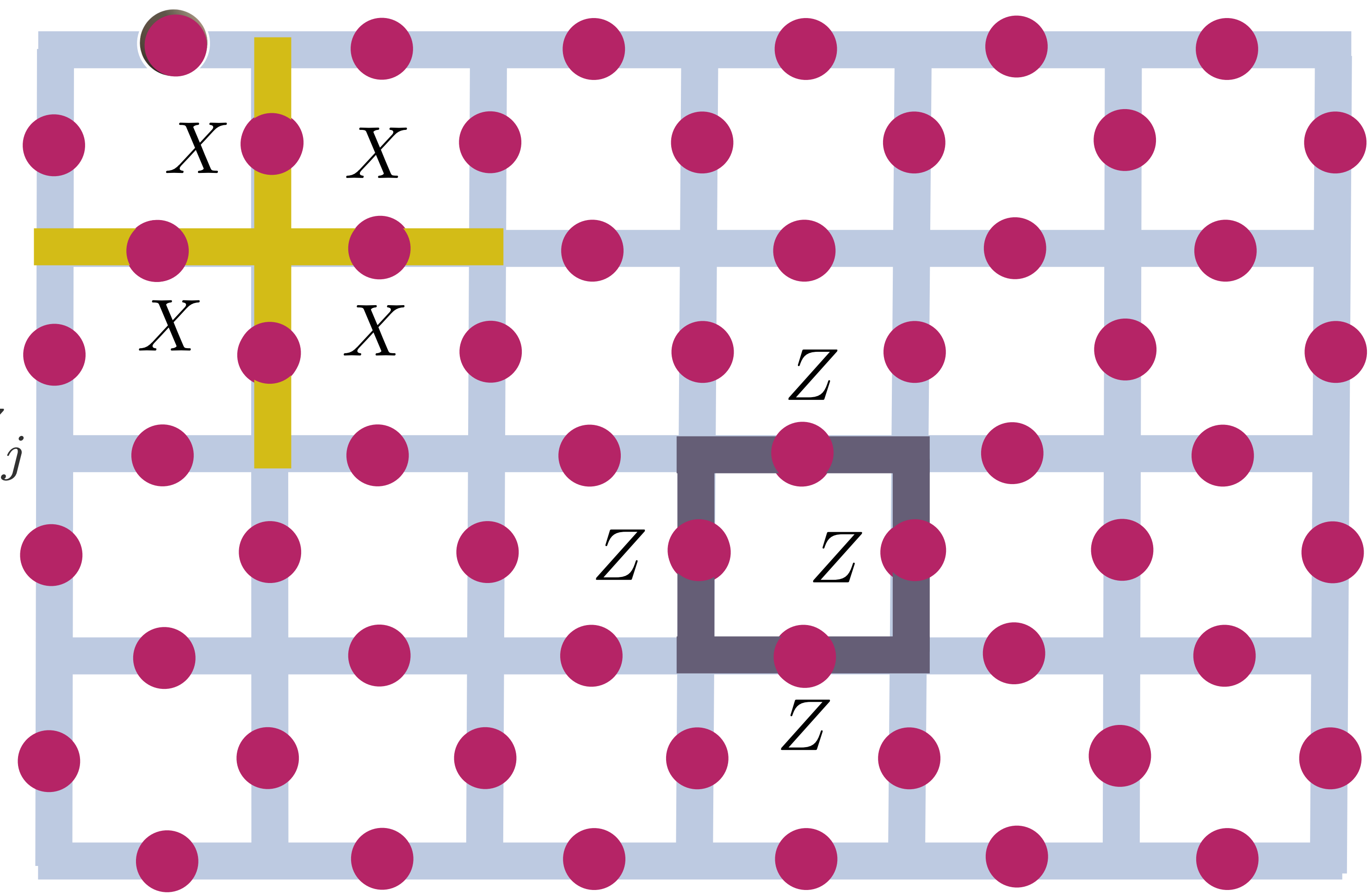
- Capture systems like **toric code**

  - Star operators $\prod_{j\in+} X_j$

    (flux at plaquette)

  - Plaquette operators $\prod_{j\in\Box} Z_j$

    (charge at vertex)



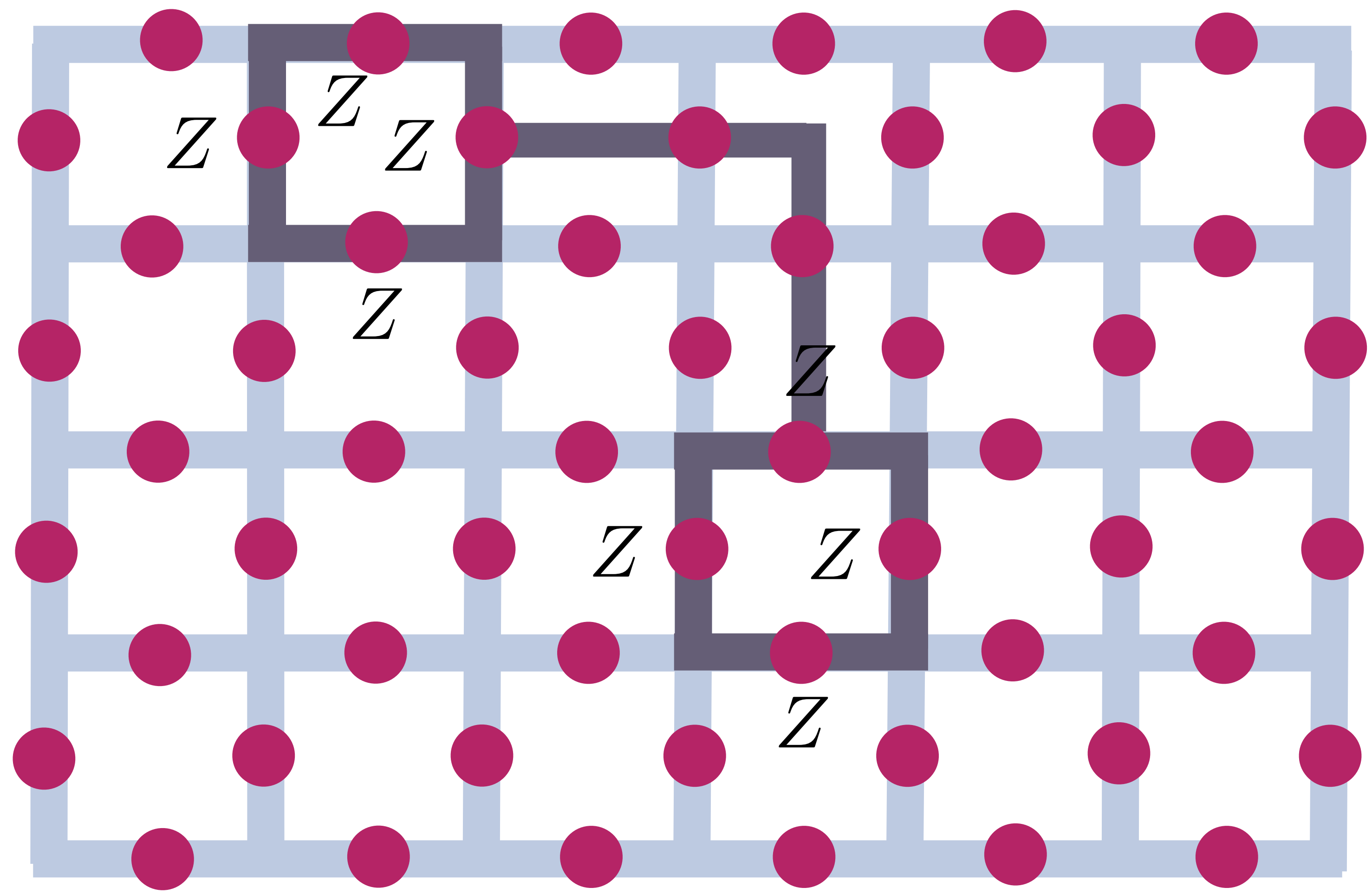
  - Frustration-free parent Hamiltonian (stars and plaquettes act trivially on GS)

$$H = -J\sum_k \left[\prod_{j\in\Box_k} Z_j + \prod_{j\in+_k} X_j\right]$$

Kitaev, quant-ph/9707021
Dennis, Kitaev, Landahl, Preskill, J Math Phys 43, 4452 (2002)

Toric code

- Capture systems like **toric code**
  - Define **string operators**

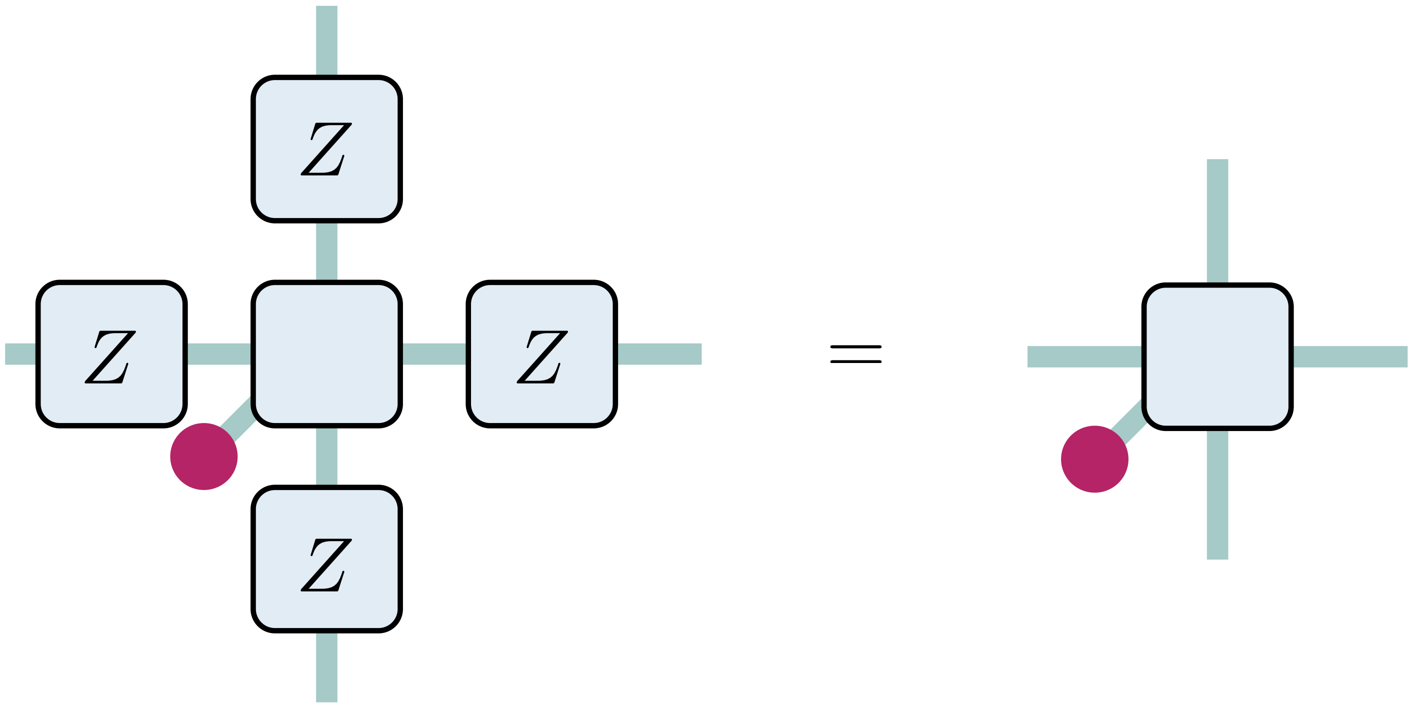


  - Ground state formed by closed loop configurations
  - Shows $\mathbb{Z}_2$-topological order
  - Excitations behave like quasi-particles with **anyonic statistics** (e - anyons on vertices, m- anyons on plaquettes)

Kitaev, quant-ph/9707021
Dennis, Kitaev, Landahl, Preskill, J Math Phys 43, 4452 (2002)

Gauge symmetry

- Let G be any finite group, e.g., $G = \mathbb{Z}_2 = \{1, Z\}$



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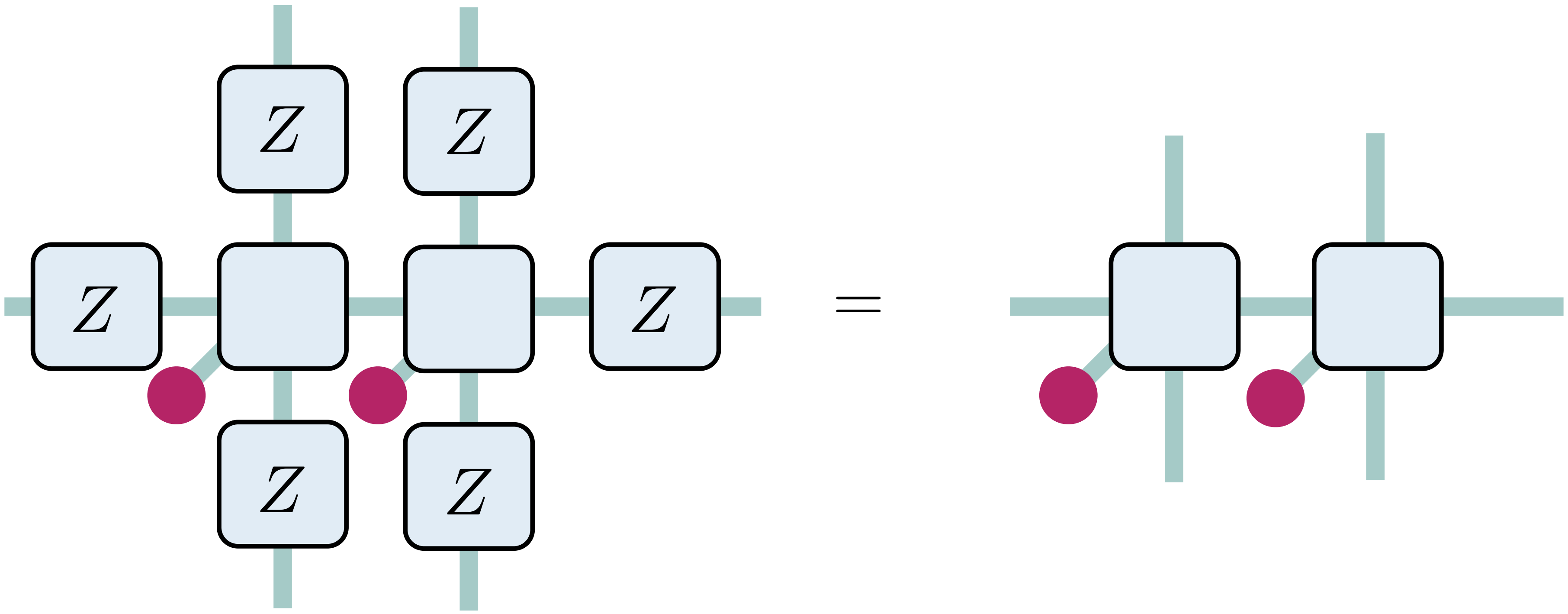

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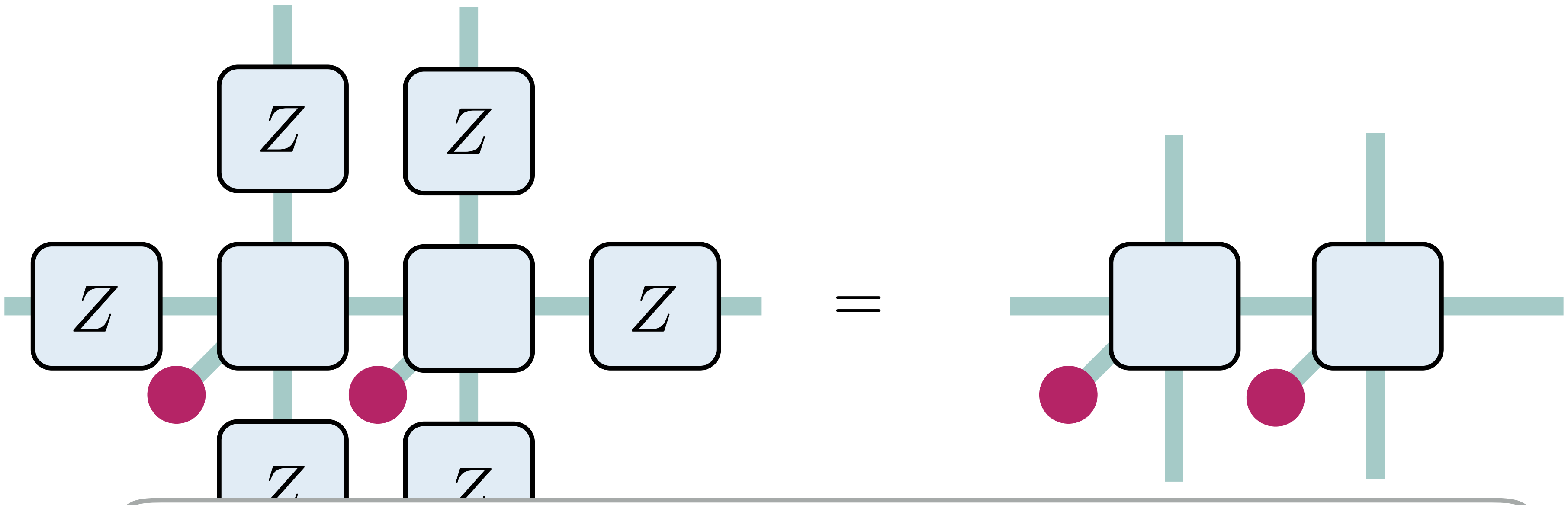
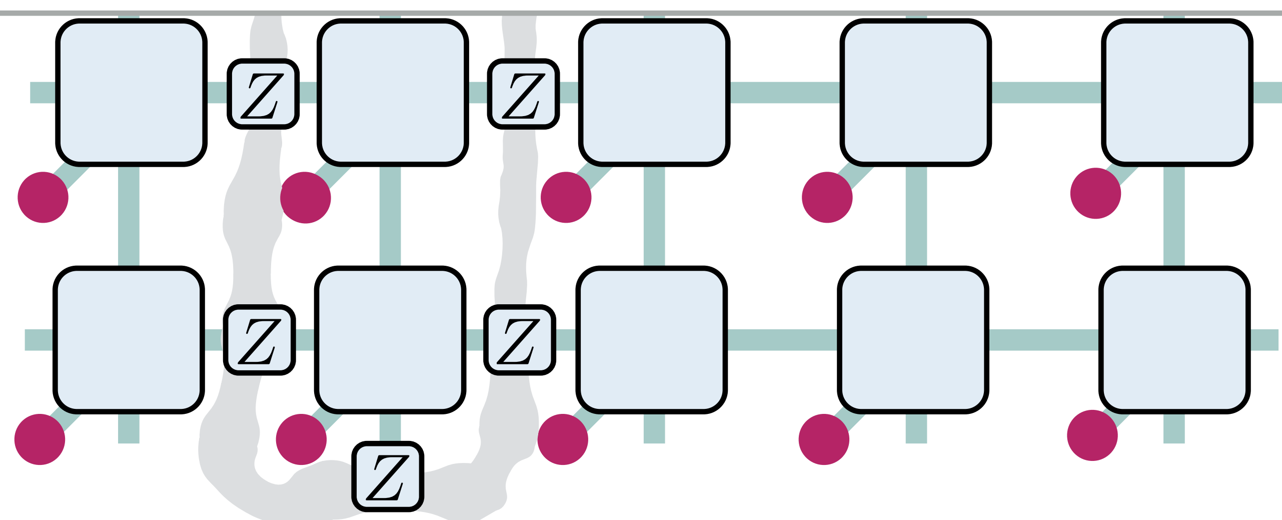
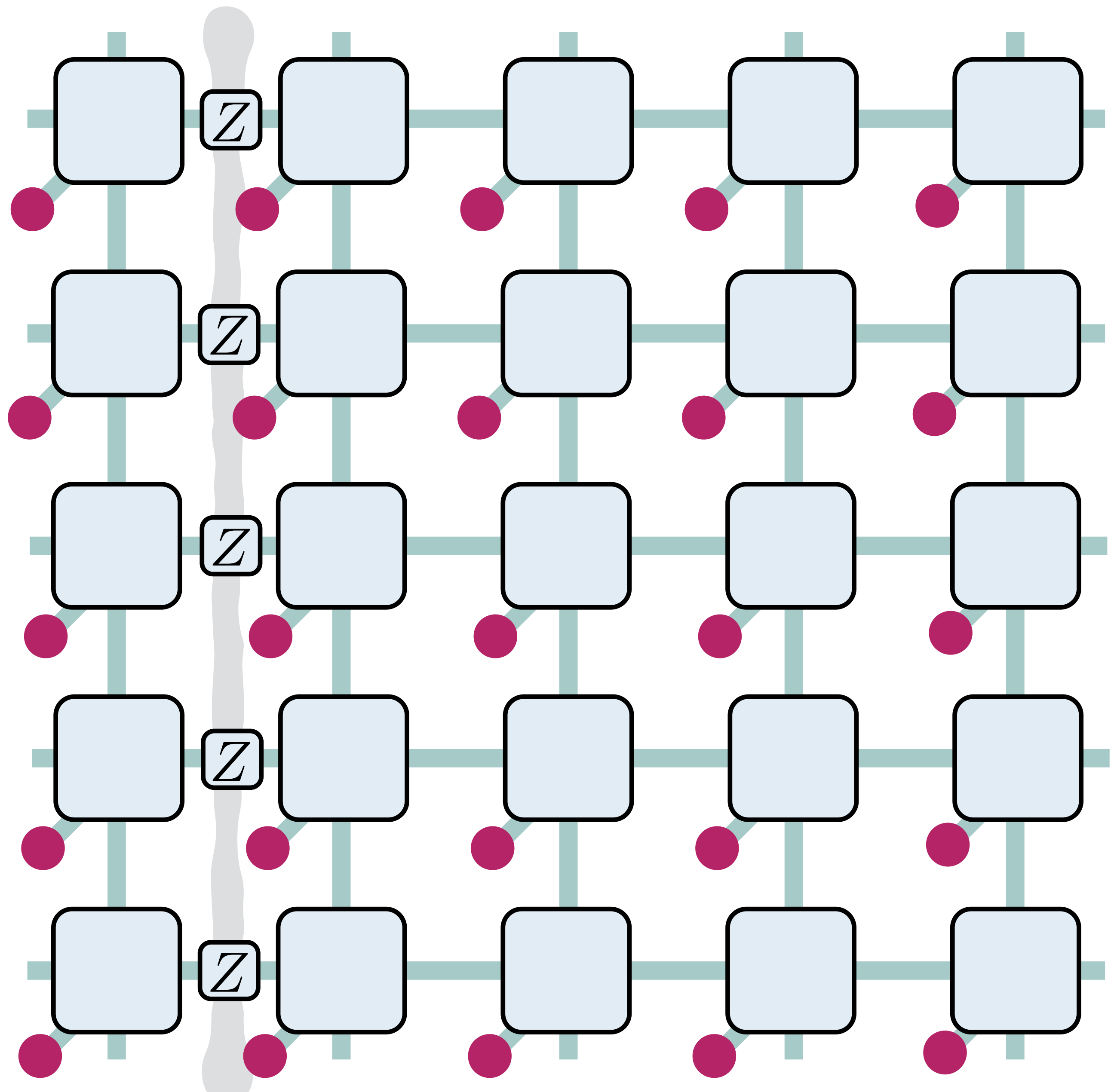


- How about loops that are non-contractible?



- They can be arbitrarily deformed, but do not vanish



- Gives new ground states of parent Hamiltonian

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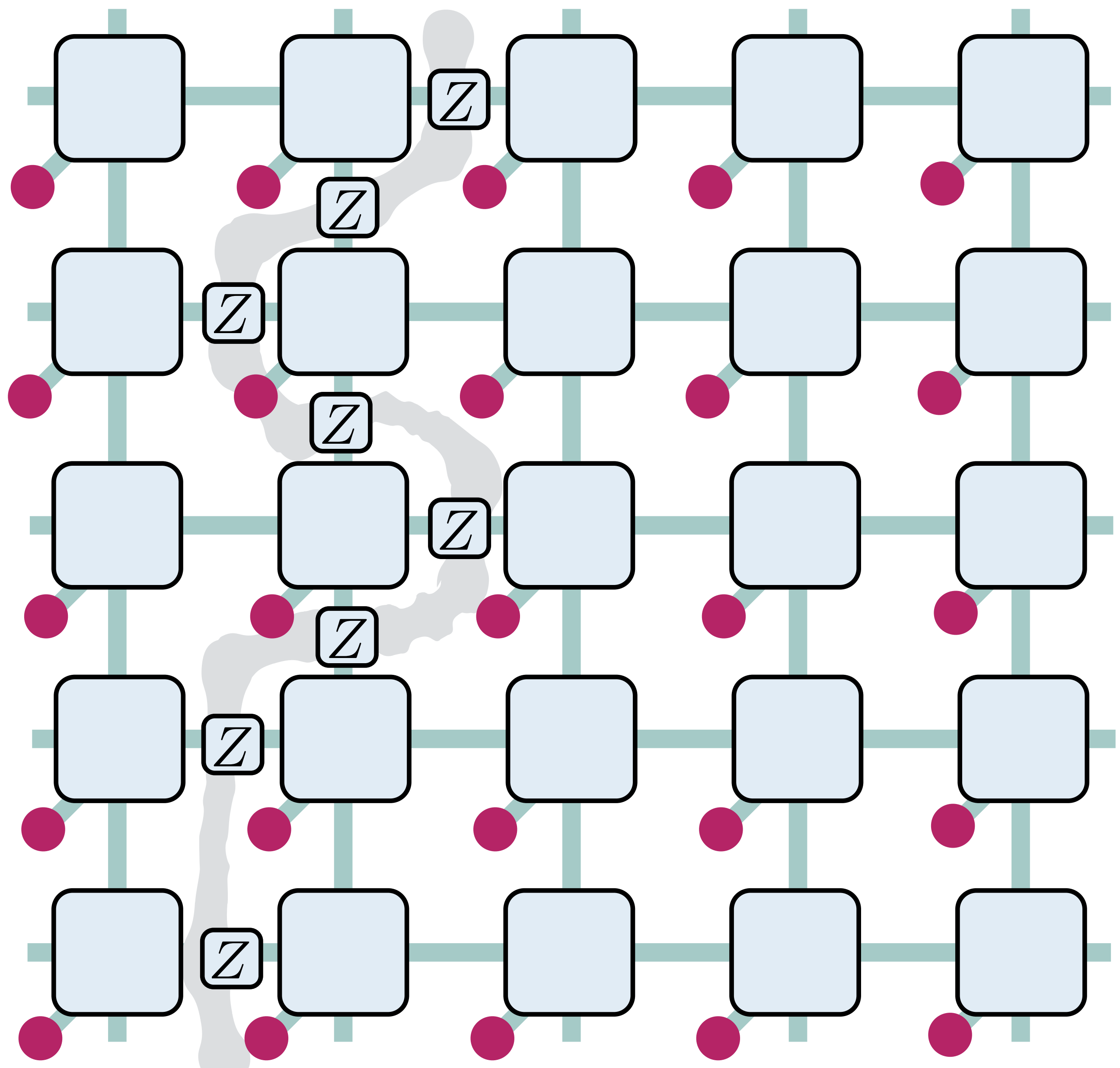
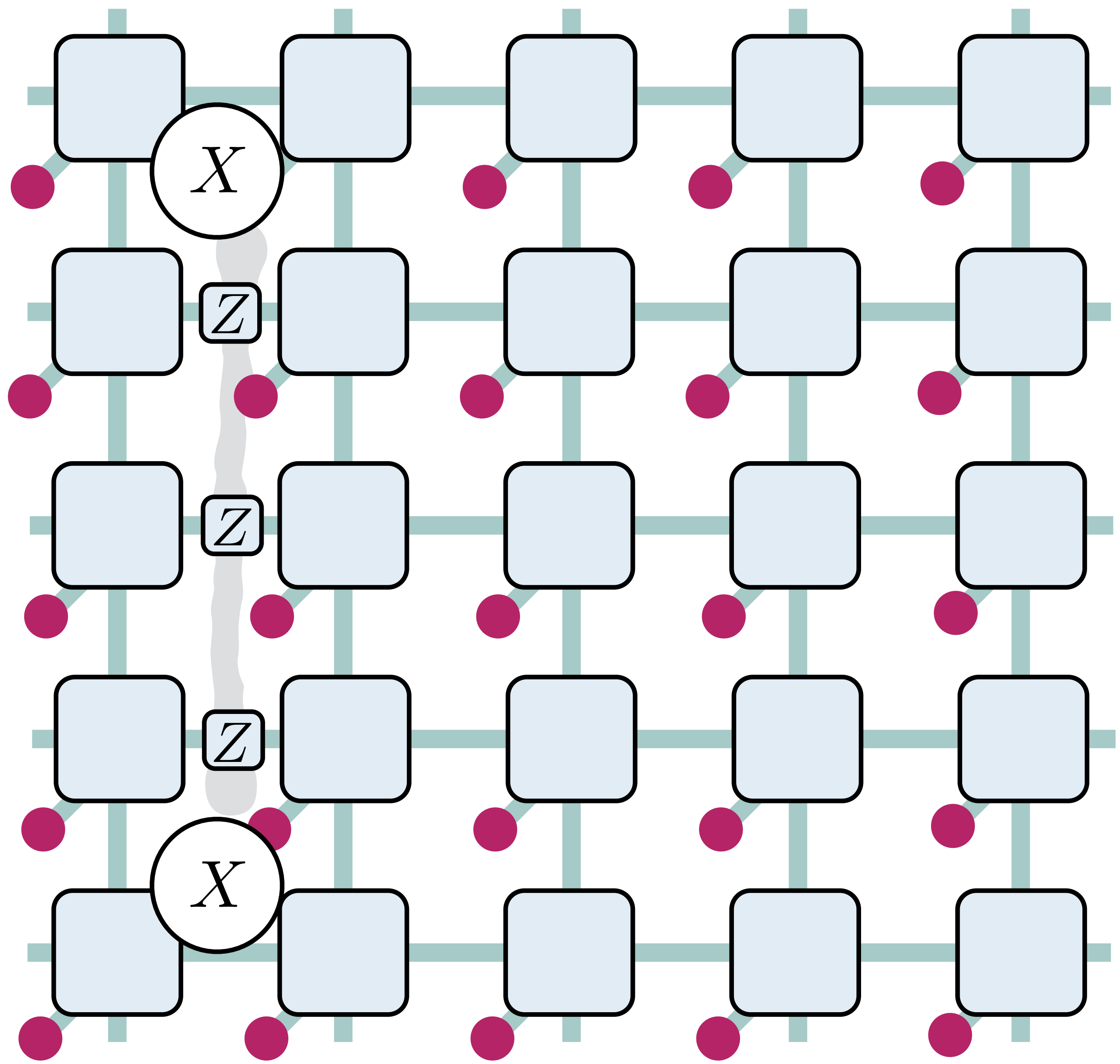


- Gives new ground states of parent Hamiltonian

- Open strings can be deformed, except from end points (quasi-particles)



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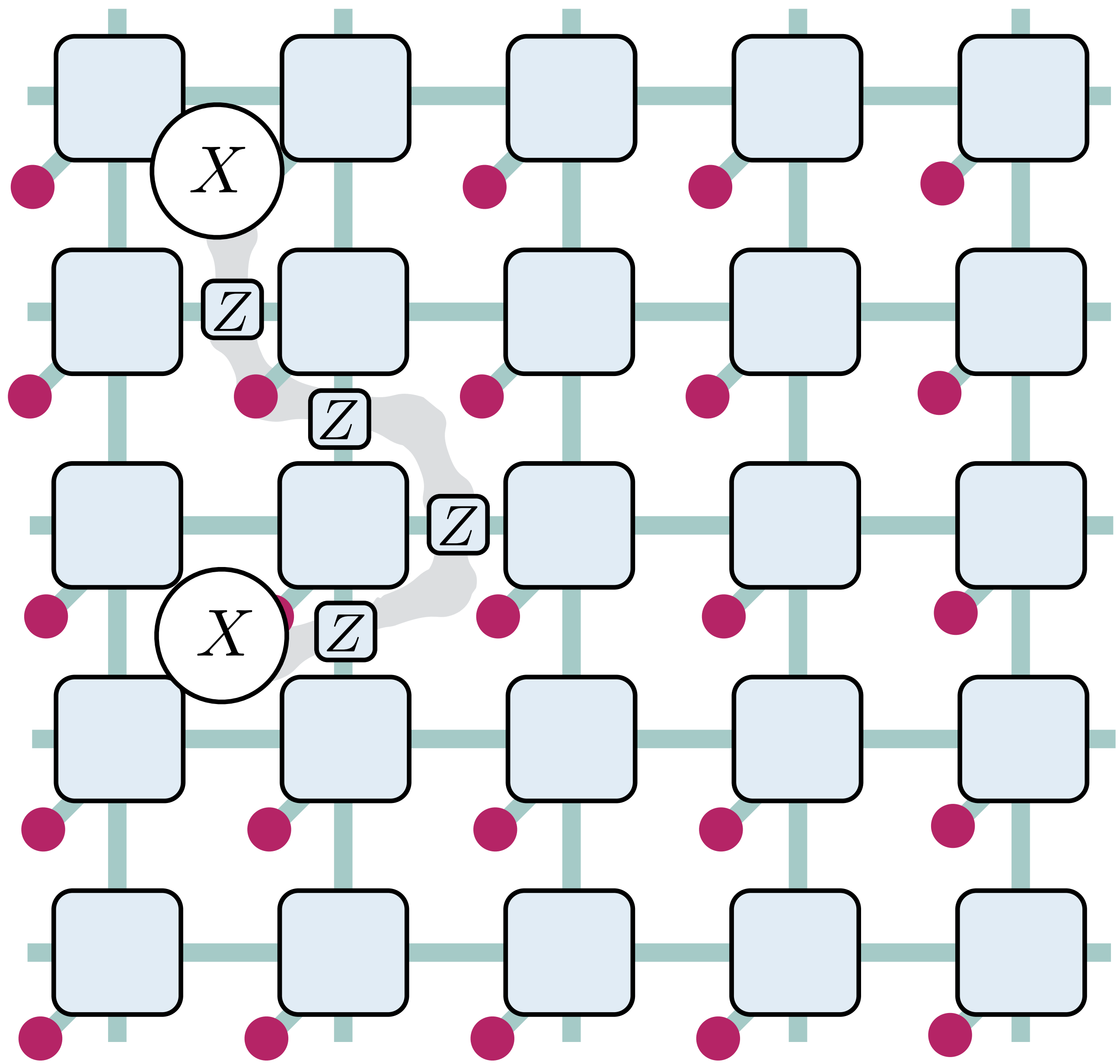


G-injective and G-isometric PEPS

- G**-injective PEPS:** Symmetry group G is acting on virtual indices and PEPS tensors are left-invariant on the G-invariant subspace
- G**-isometric PEPS:** All PEPS tensors are isometries

- It is possible to unitarily transform between any two states in ground space by acting on two stripes wrapping around the torus
- …, the states in the GS cannot be distinguished by local operations
- … the **entanglement entropy** of any topologically trivial region is

$$S(\rho_A) = \log|G||\partial A| - \log|G|$$

- Here $-\log|G|$ is the topological correction to the area law

Schuch, Cirac, Perez-Garcia, Ann Phys 325, 2153 (2010)

Topological order

- We recover topological order

- **Degeneracy** of the Hamiltonian constant and depends on **topology** ✓
- All GS are **locally indistinguishable** (no local order parameter) ✓
- To map between them, you need a **non-local operator** ✓
- Excitations behave like quasi-particles with **anyonic statistics** ✓

A complete picture?

- Good enough to capture **toric code, quantum double models** etc

  Kitaev Ann Phys 303, 2 (2003)

- Take $G = S_3$, suitable for universal **topological quantum computation**

- Not capturing **string net models**

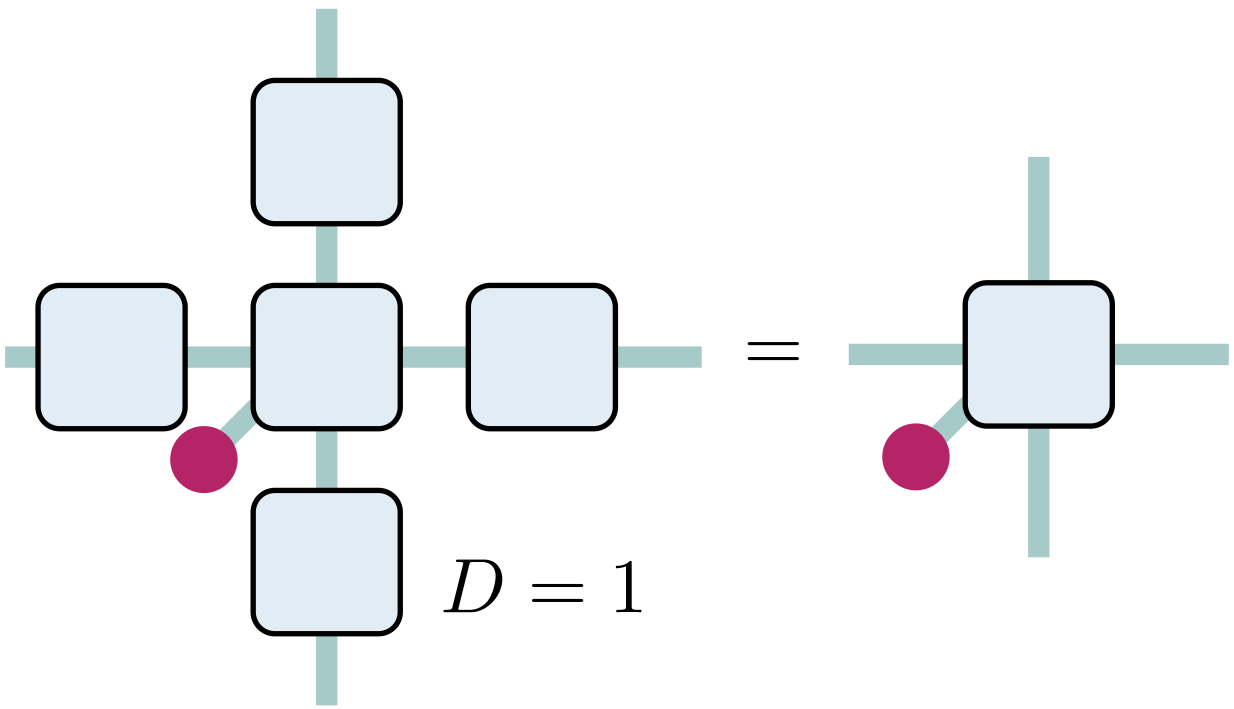
  Levin, Wen, Phys Rev B 71, 045110 (2005)

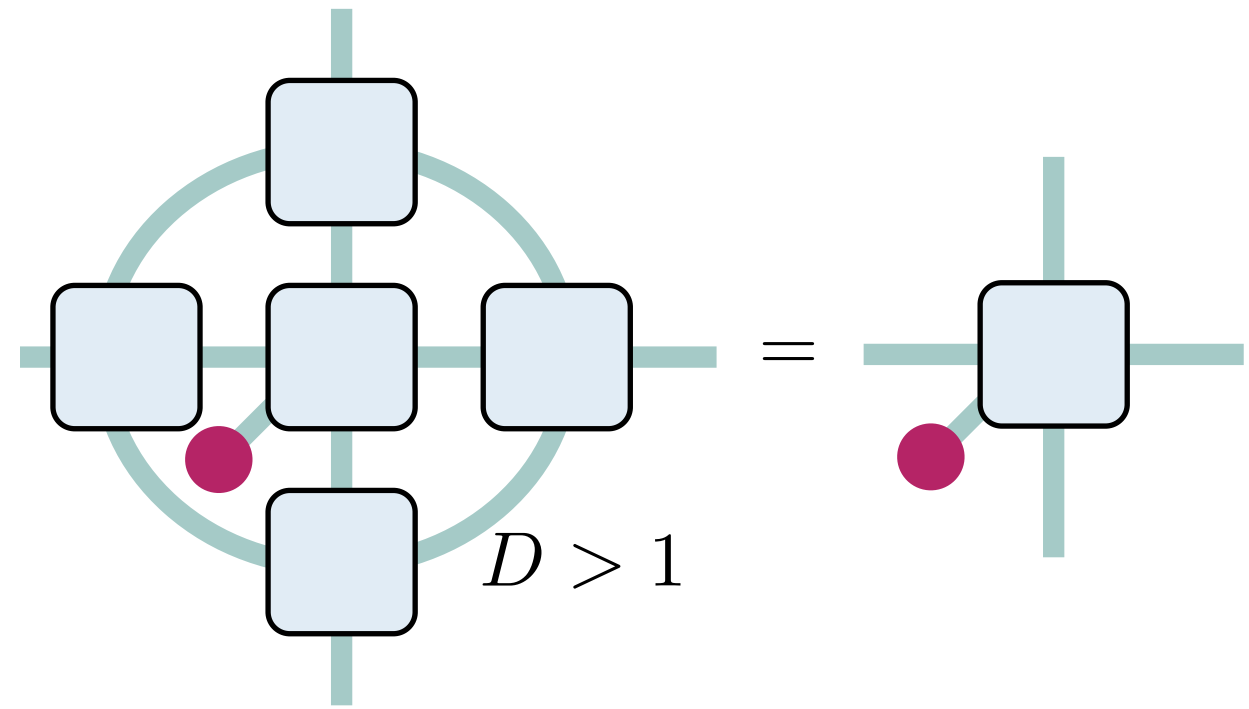
> - Can a complete understanding of topological order be achieved in terms of PEPS?

MPO-injective PEPS

Beyond G-injective PEPS

- Virtual symmetries

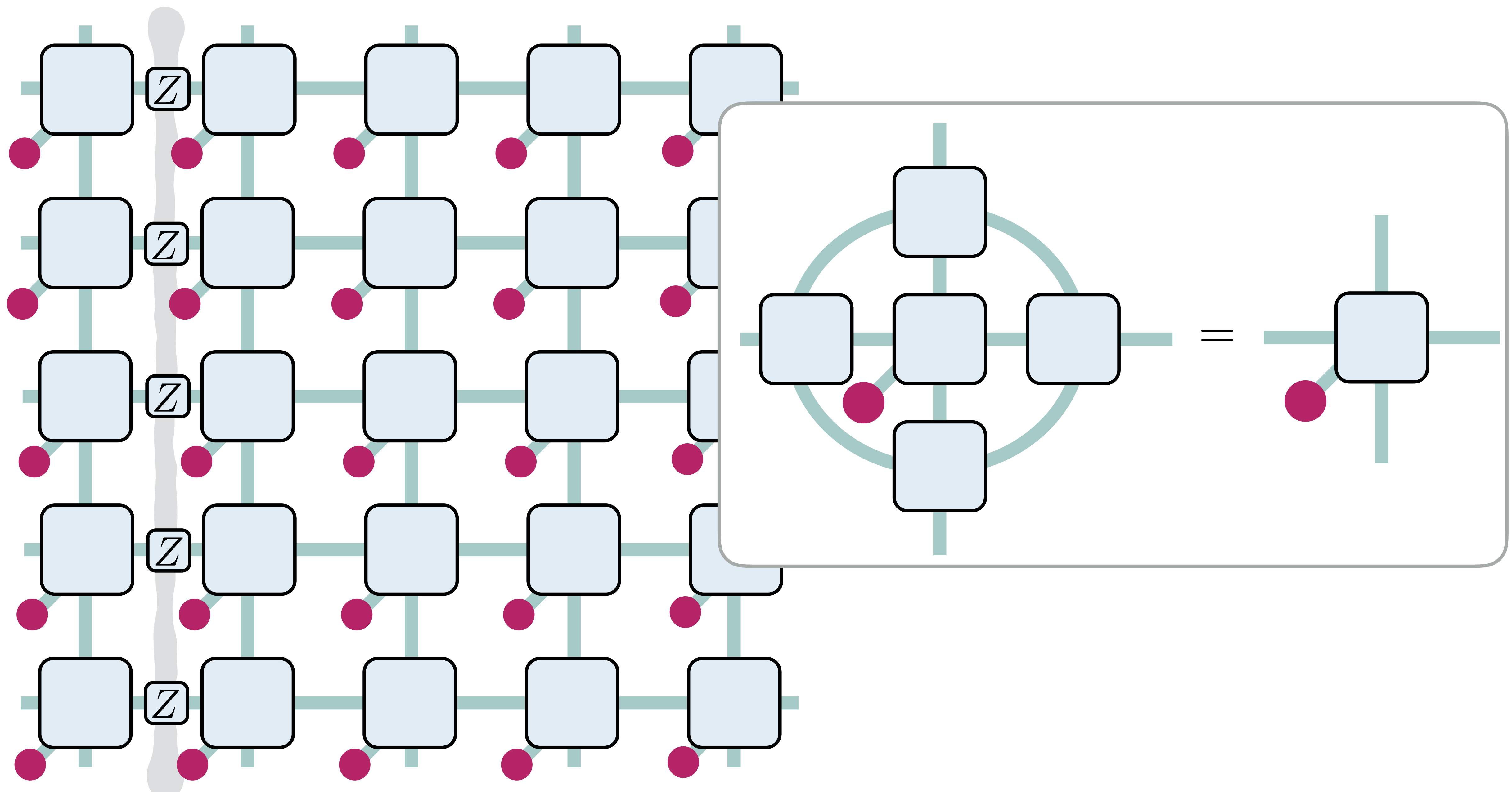




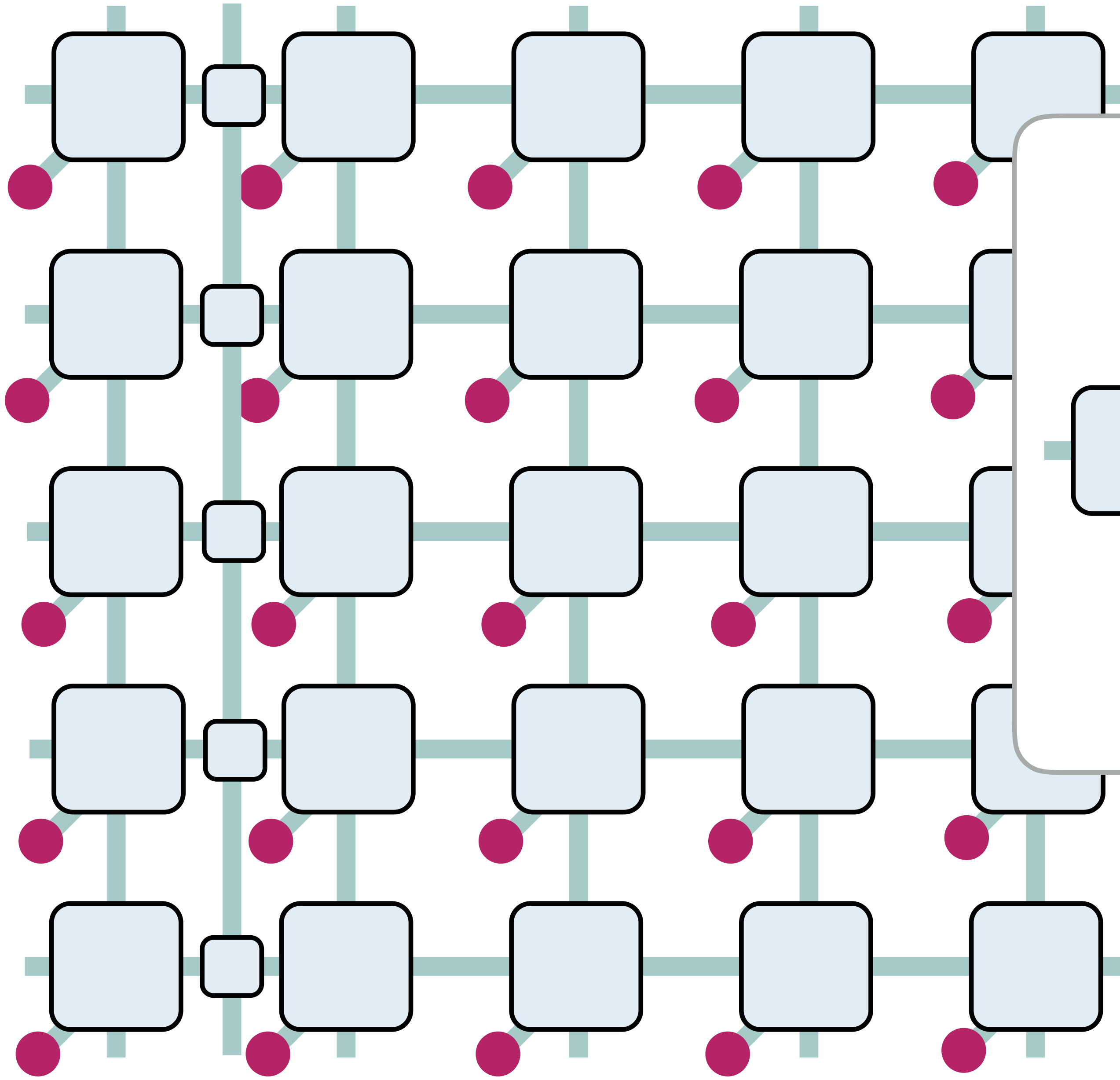
- G **-symmetry**
- Matrices
- Groups

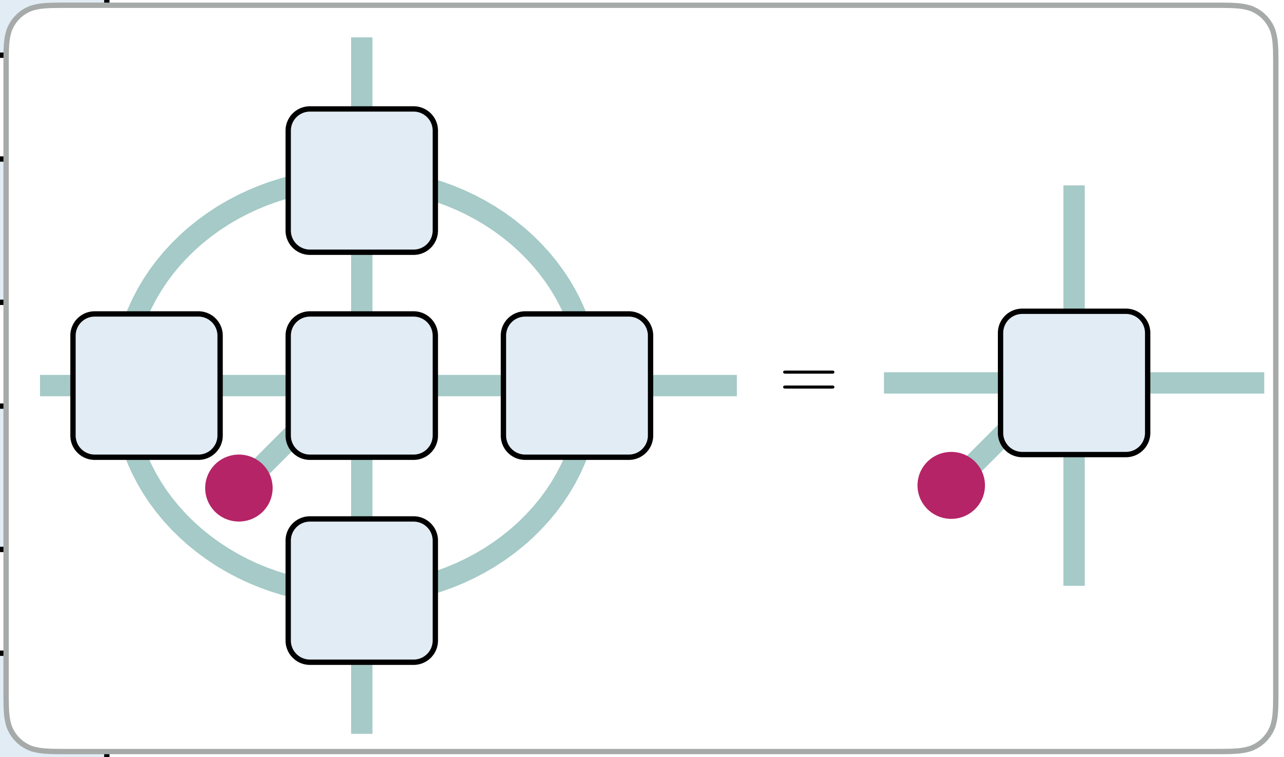
- **MPO-symmetry**
- Matrix-product operator
- Twisted groups and more

Beyond G-injective PEPS



Beyond G-injective PEPS





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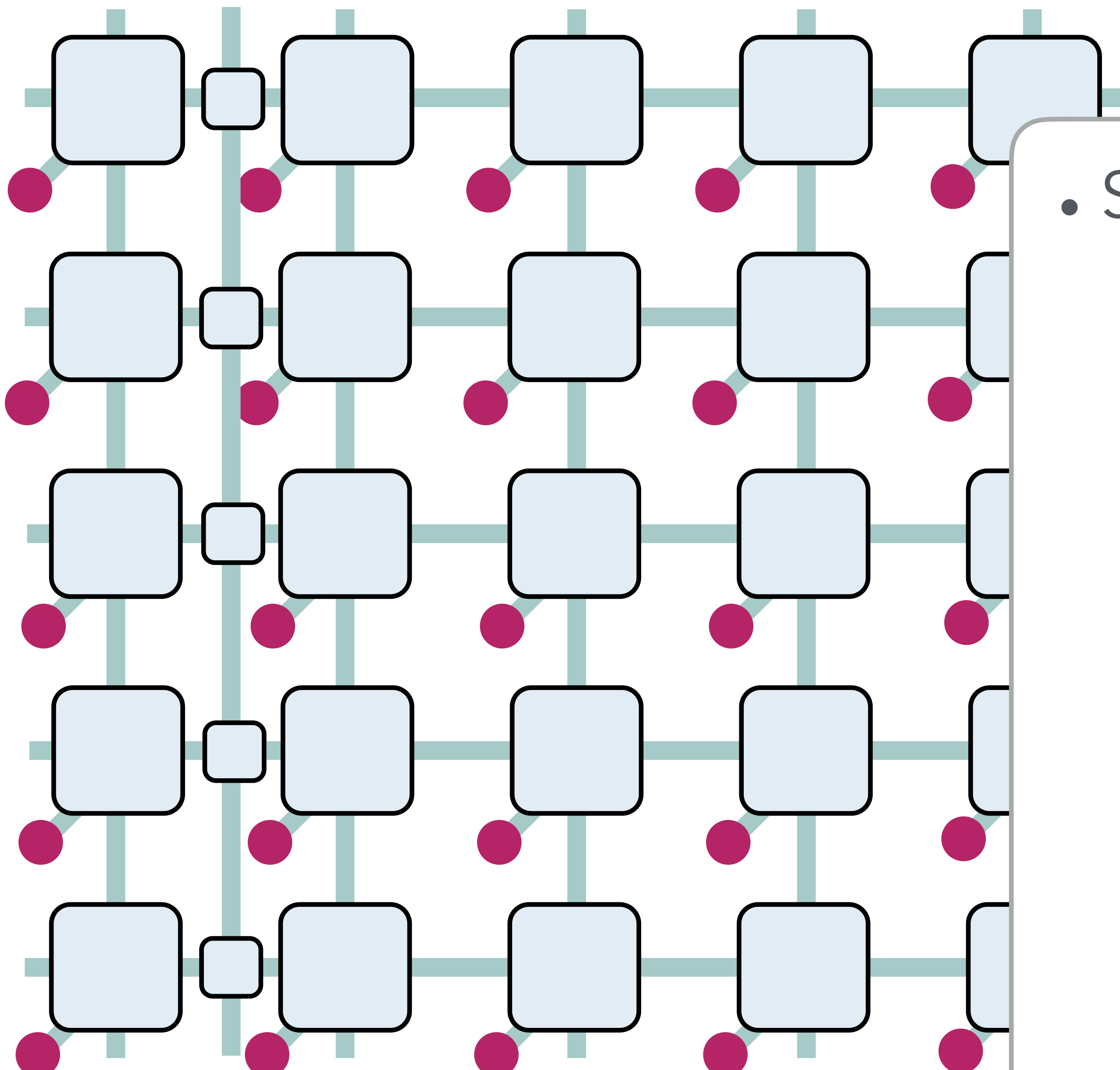
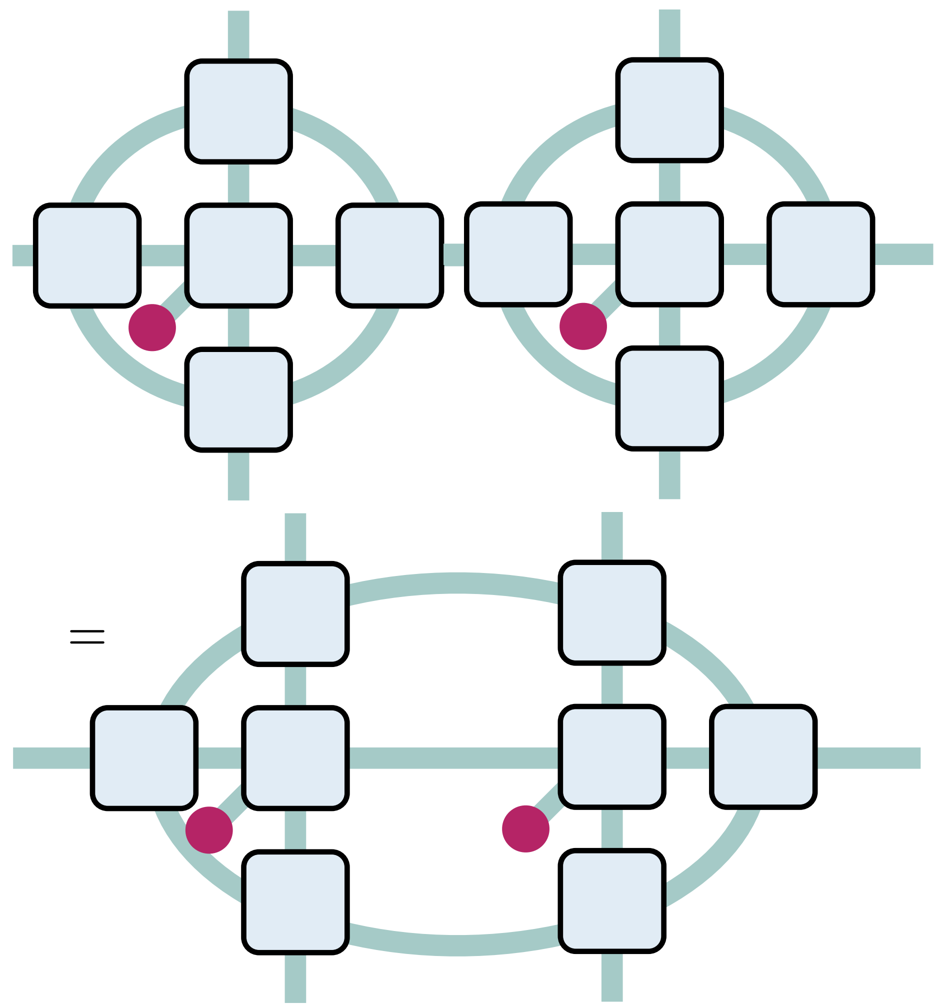


- Stable under concatenation



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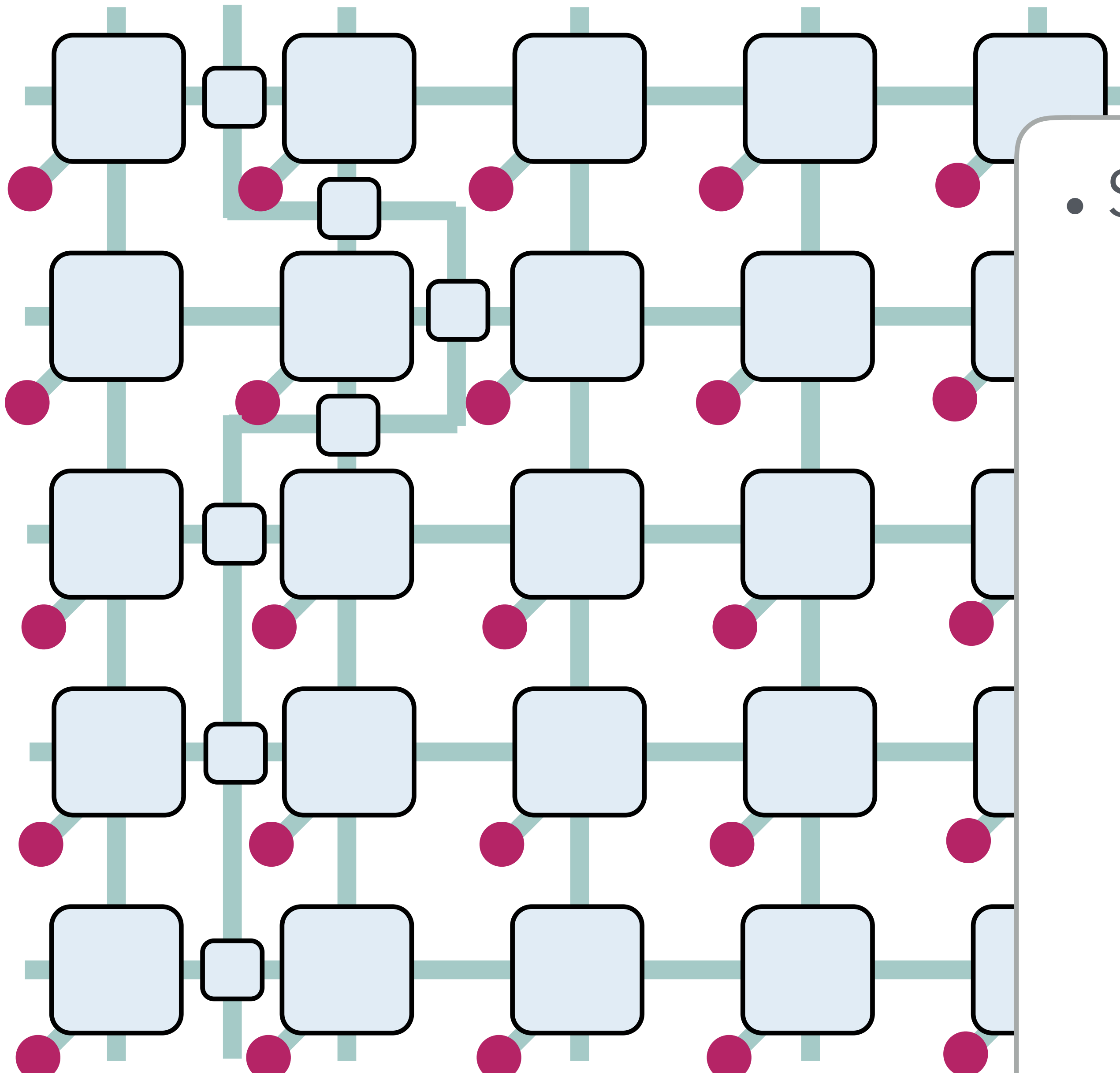

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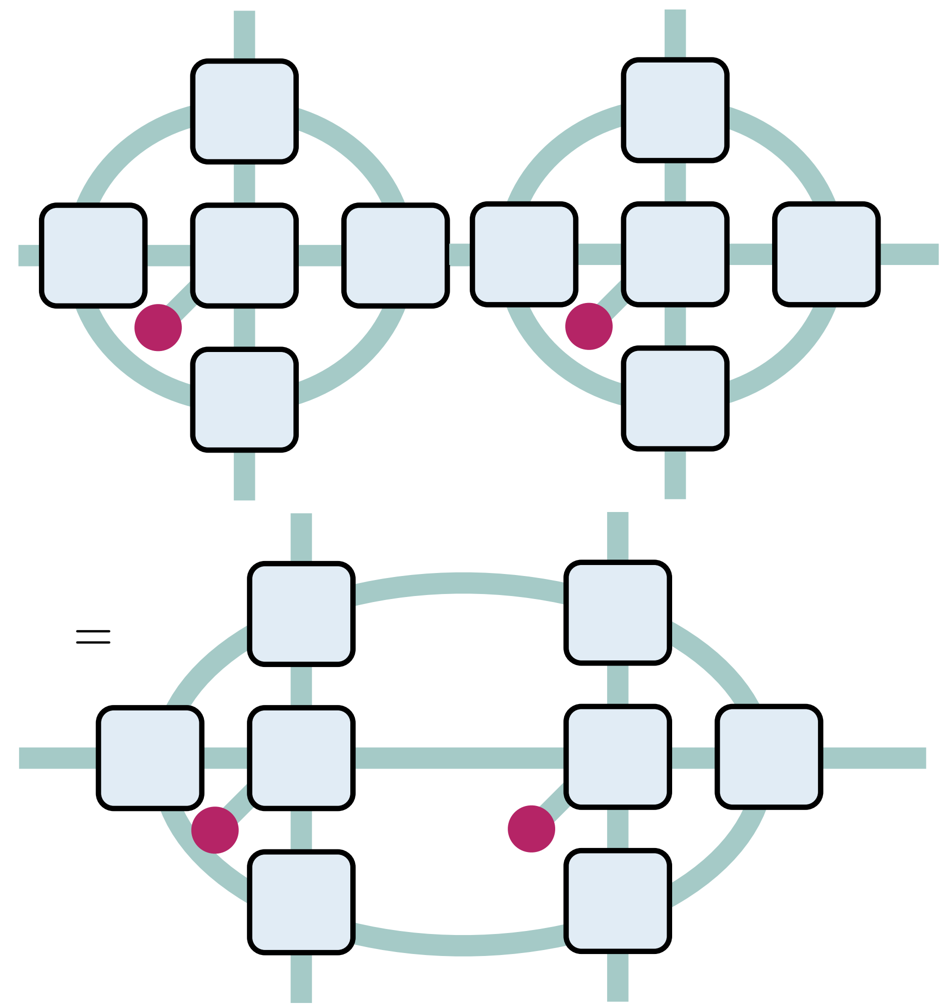
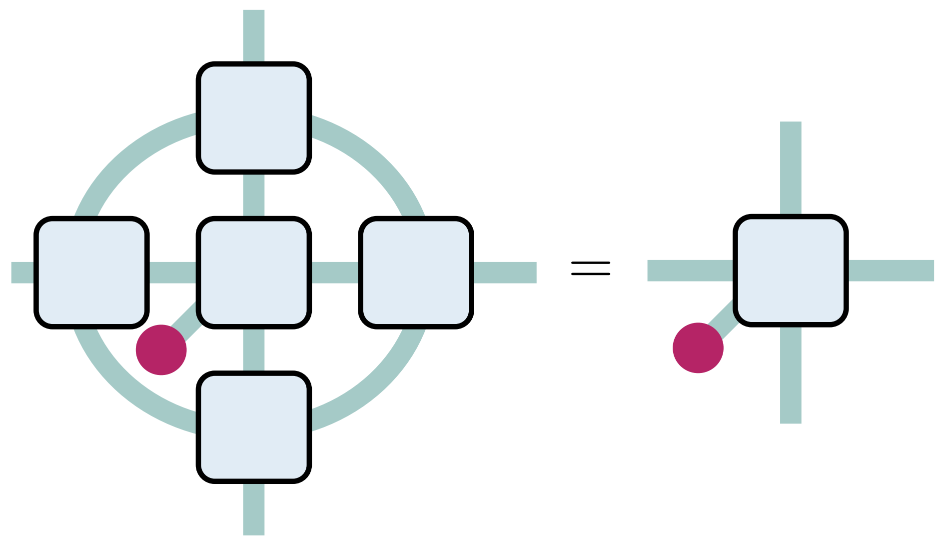


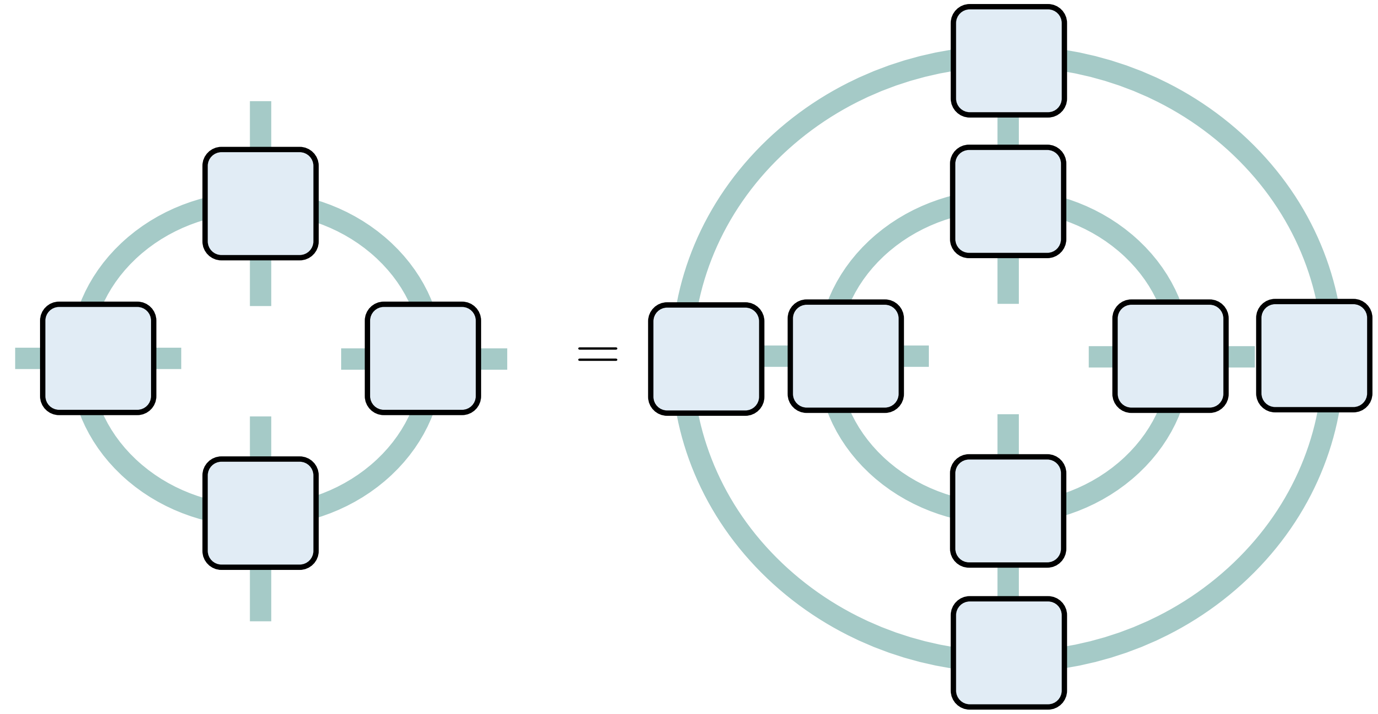
Axioms of MPO-injectivity

- MPO symmetry



Axioms of MPO-injectivity

- MPO symmetry
- MPO projector



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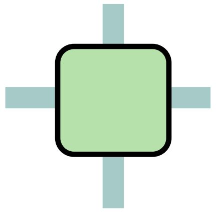
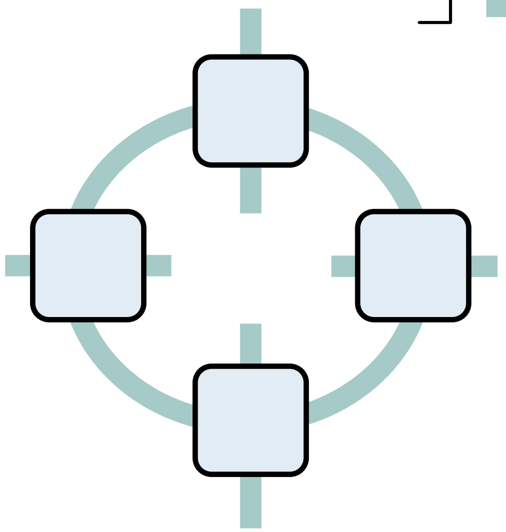
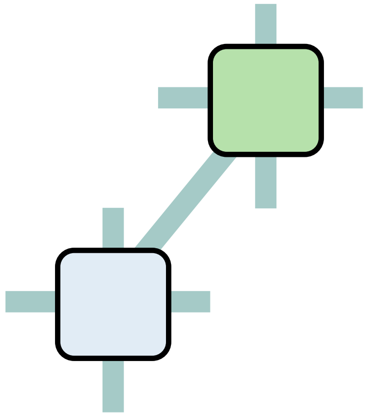
$\exists$  :

 $=$ 

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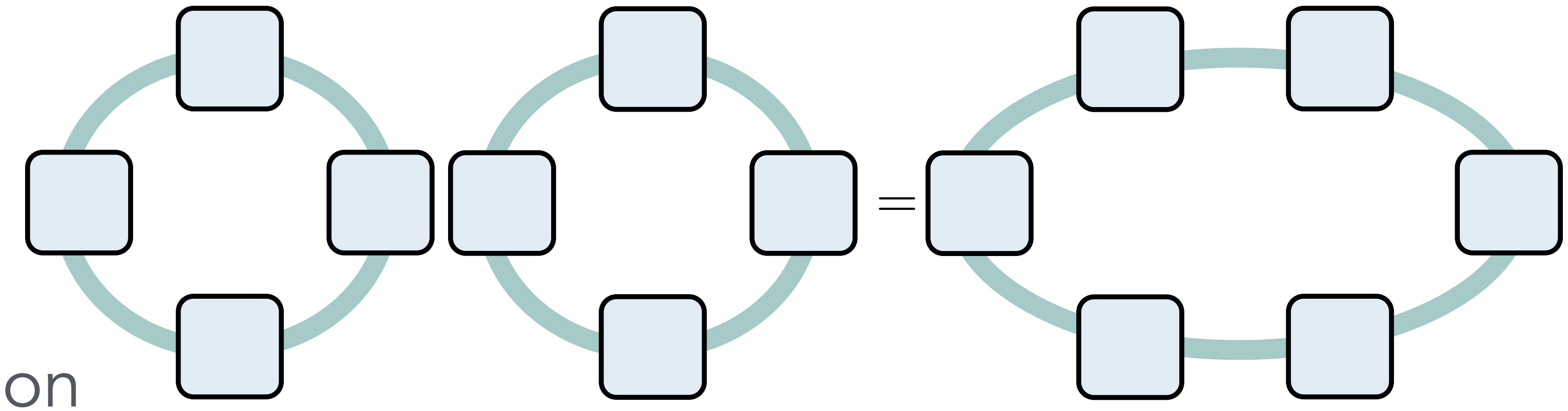

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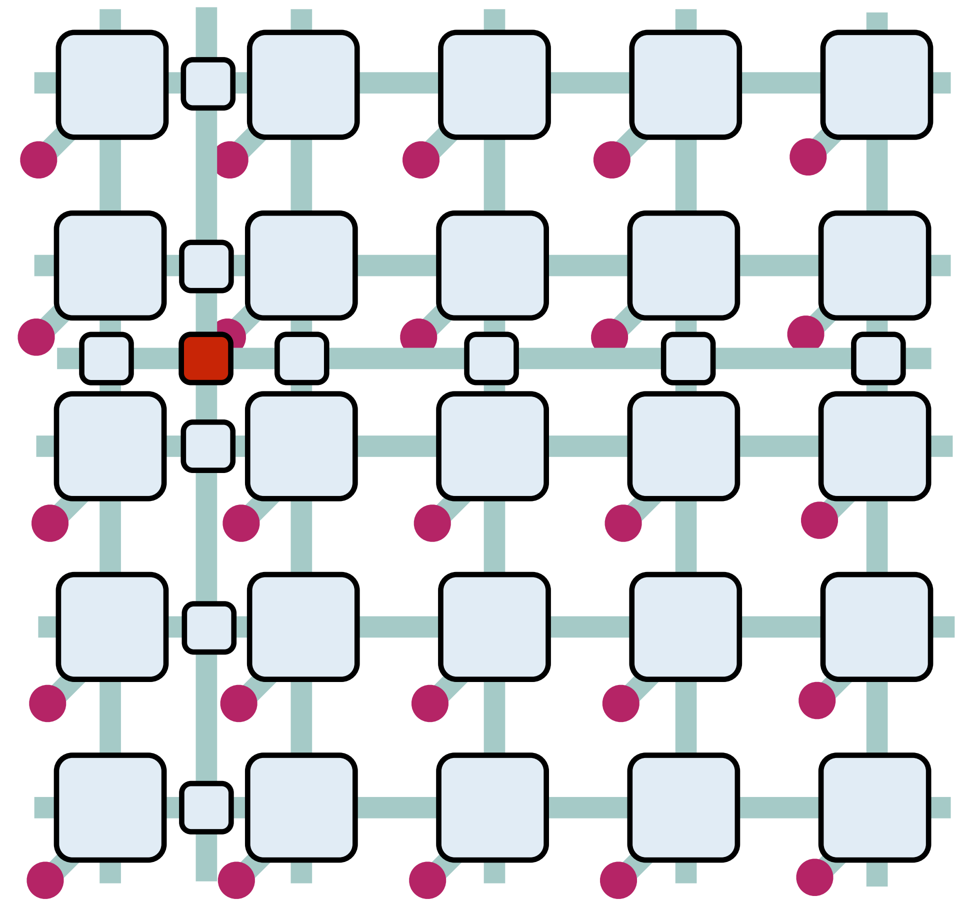

- **Can compute:**
  - Topological correction to area law

$$S(\rho_A) = c|\partial A| - \gamma$$

  - Ground state space

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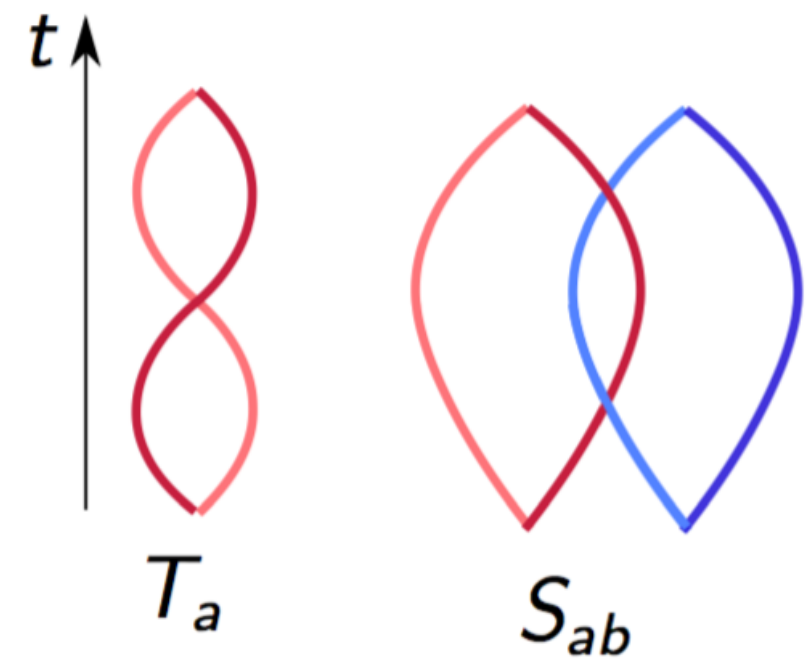

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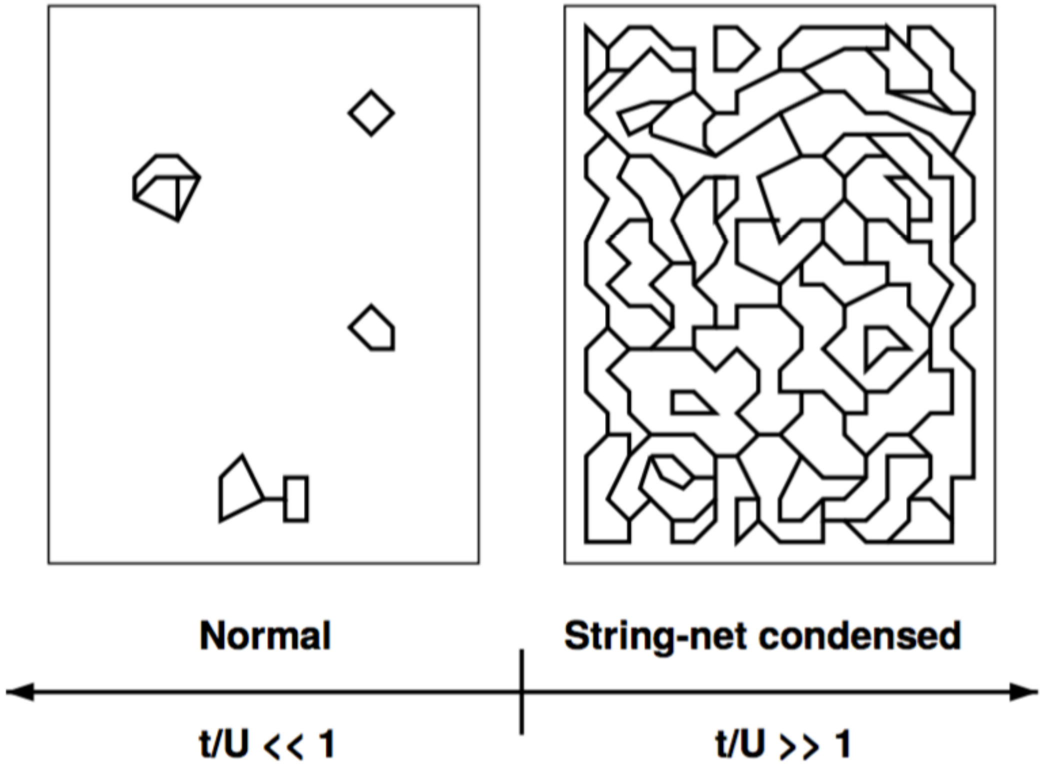


- Can compute:
  - Topological correction to area law

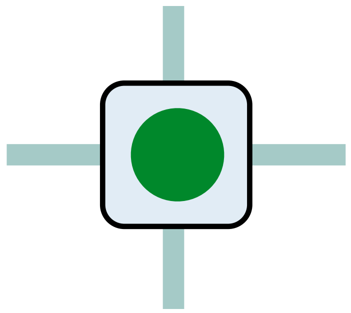
$$S(\rho_A) = c|\partial A| - \gamma$$

  - Ground state space
  - Anyonic statistics: S and T matrices
  - Captures Levin-Wen string net models

Levin, Wen, Phys Rev B 71, 045110 (2005)
Gu, Levin, Swingle, Wen, Phys Rev B 79, 085118 (2009).

Towards tensor networks for fermionic systems

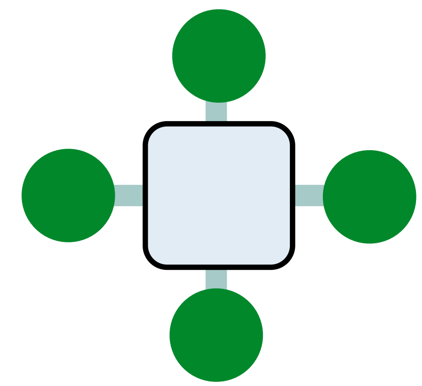
- **Tensors with physical fermions**
  - Book-keeping of the order (manual)

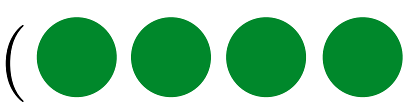


Wille, Buerschaper, Eisert, arXiv:1609.02574

Axioms of MPO-injectivity

- **Tensors with physical fermions**
  - Book-keeping of the order (manual)
- **Add virtual fermions**
  - Book-keeping of the order (in-built)
      - Fermionic entangled pairs
      - Grassmann numbers

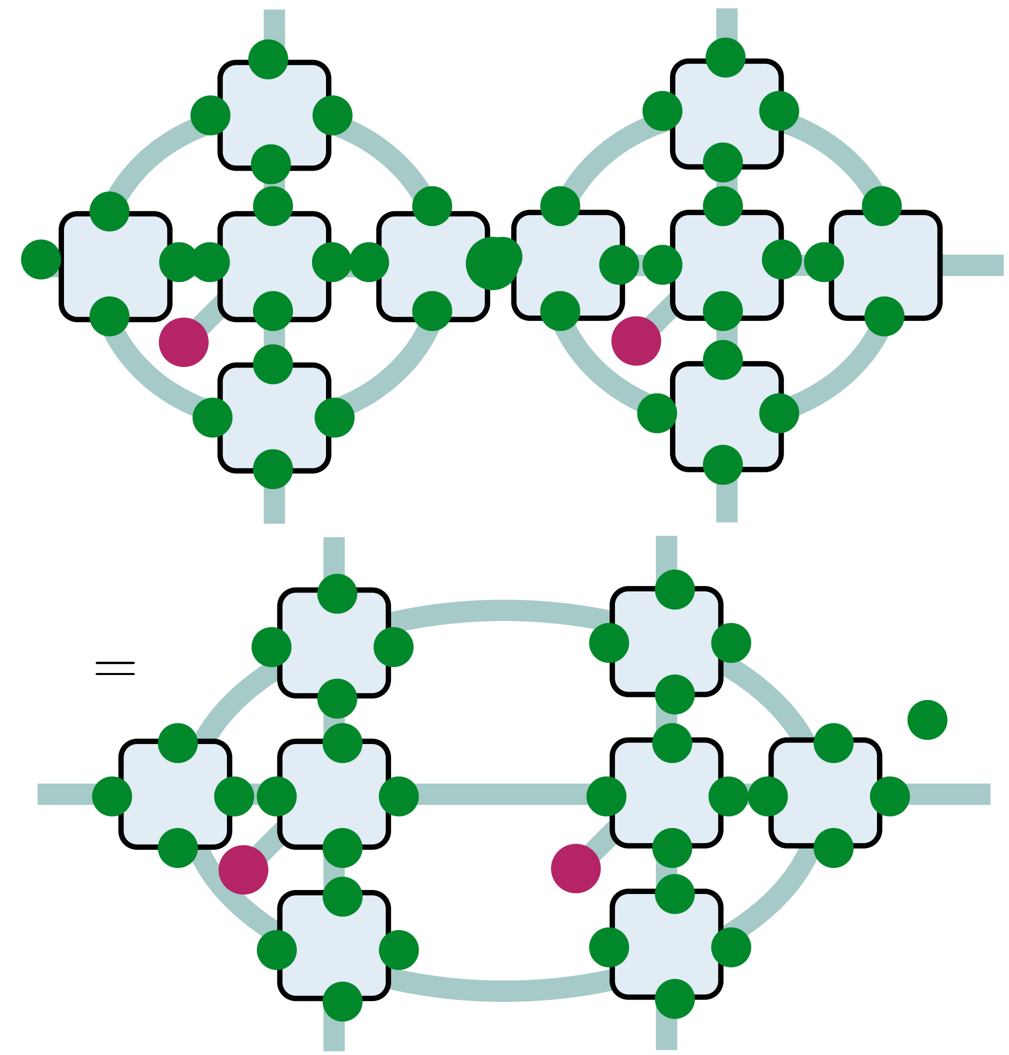


#() even

Wille, Buerschaper, Eisert, arXiv:1609.02574
Williamson, Bultinck, Haegeman, Verstraete, arXiv:1609.02897

Fermionic MPOs?

- **Fermionic MPOs**
- Axioms take analogous form
- Graded algebraic structure
- Axioms fulfillable?



Wille, Buerschaper, Eisert, arXiv:1609.02574

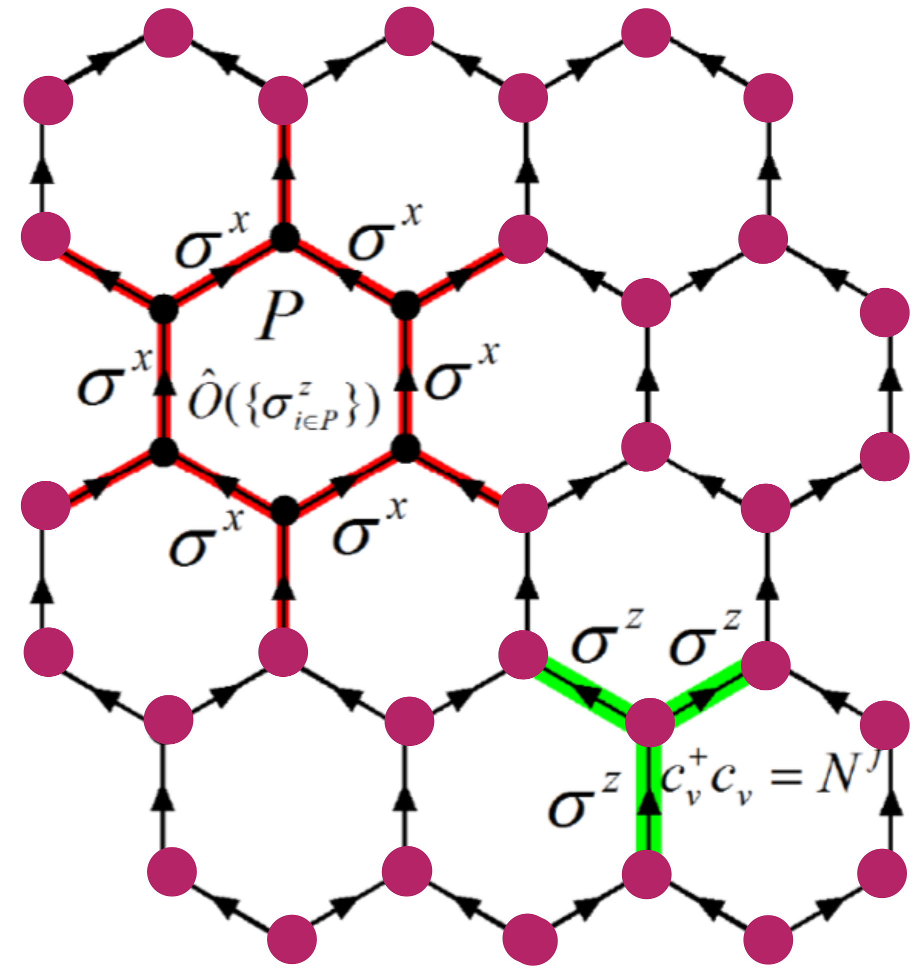
Fermionic toric code

- **Edges:** Spin 1/2
- **Vertices:** Fermions

$$H = \sum_v Q_v + \sum_p Q_p$$

$$Q_v = \frac{1}{2}(1 + \prod_{i\in v} \sigma_i^Z) F_v$$

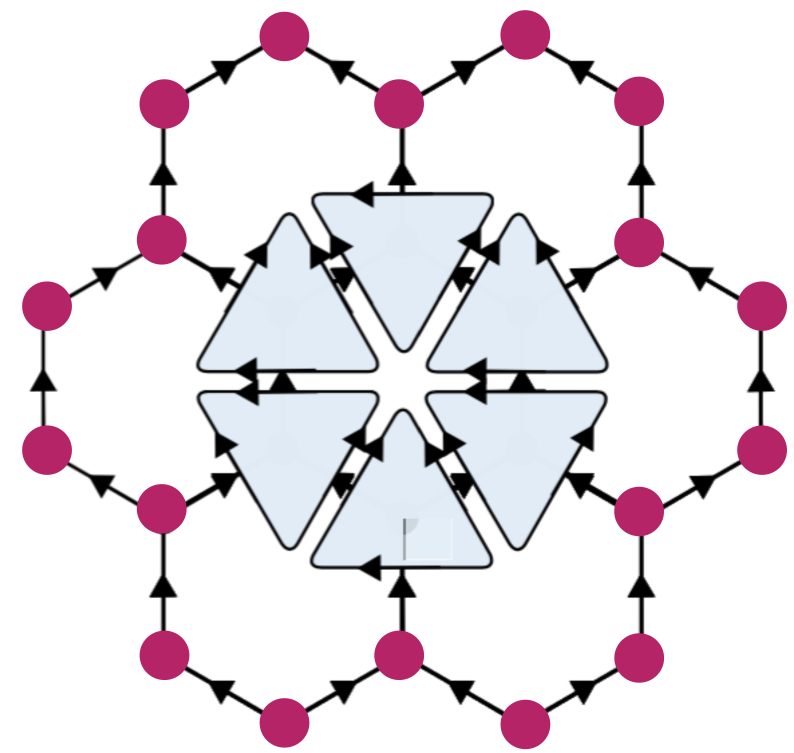
$$Q_p = \frac{1}{2}(1 + \prod_{i\in p} \sigma_i^X) F_p$$



Gu, Wang, Wen, Phys Rev B 90, 085140 (2014)

Fermionic MPO-injectivity

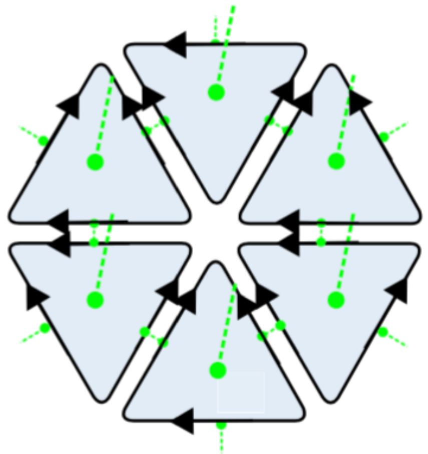
- Dual lattice
- Grassmann numbers



Wille, Buerschaper, Eisert, arXiv:1609.02574

Fermionic MPO-injectivity

- Dual lattice
- Grassmann numbers

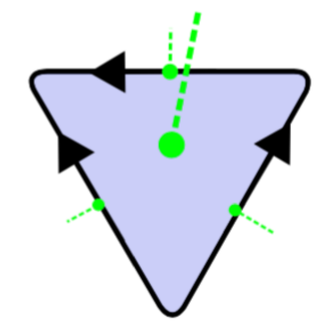


$$A = \sum A^{p_1 p_2 p_3 v_1 v_2 v_3}_{p f_1 f_2 f_3} \; \theta^p \theta^{f_1} \bar{\theta}^{f_2} \bar{\theta}^{f_3} |p_1, p_2, p_3\rangle\langle v_1, v_2, v_3|$$

Wille, Buerschaper, Eisert, arXiv:1609.02574

Fermionic MPO-injectivity

- Dual lattice
- Grassmann numbers
- Virtual symmetries with branching structure*



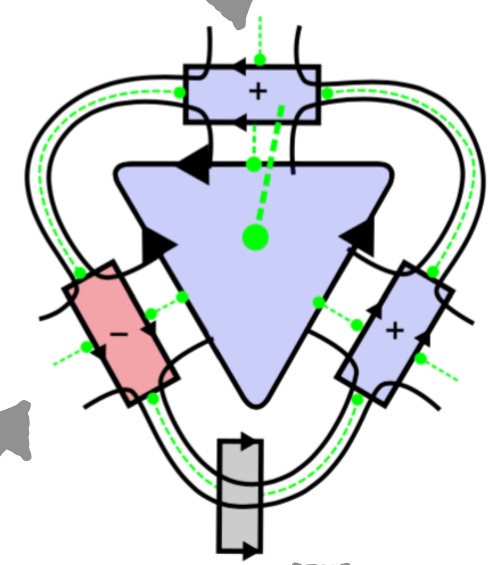
*Edges of PEPS tensor are oriented such that no cyclic orientation arises

Wille, Buerschaper, Eisert, arXiv:1609.02574

Fermionic MPO-injectivity

- Dual lattice
- Grassmann numbers
- Virtual symmetries with branching structure

T_+ at edges parallel to MPO direction



T_- at edges anti-parallel to MPO direction

Purely bosonic Y to ensure concatenation

- **Theorem:** Construction satisfies axioms

- Can compute properties, e.g., ground state degeneracy
- Interesting physical models?

Wille, Buerschaper, Eisert, arXiv:1609.02574

Twisted fermionic double models

- **Twisted fermionic quantum doubles** (instances of fermionic string nets)

  - Graded group cohomology: Triple (G, s, ω)
    - **Group** G, defining bosonic degrees of freedom
    - **2-cocycle** $\mathcal{H}^2(G, \mathbb{Z}_2)$, governing coupling

      $$s(a,b) + s(ab,c) + s(a,bc) + s(b,c) = 0$$

    - **Graded 3-cocycle** $\mathcal{H}^3_f(G, U(1), s)$

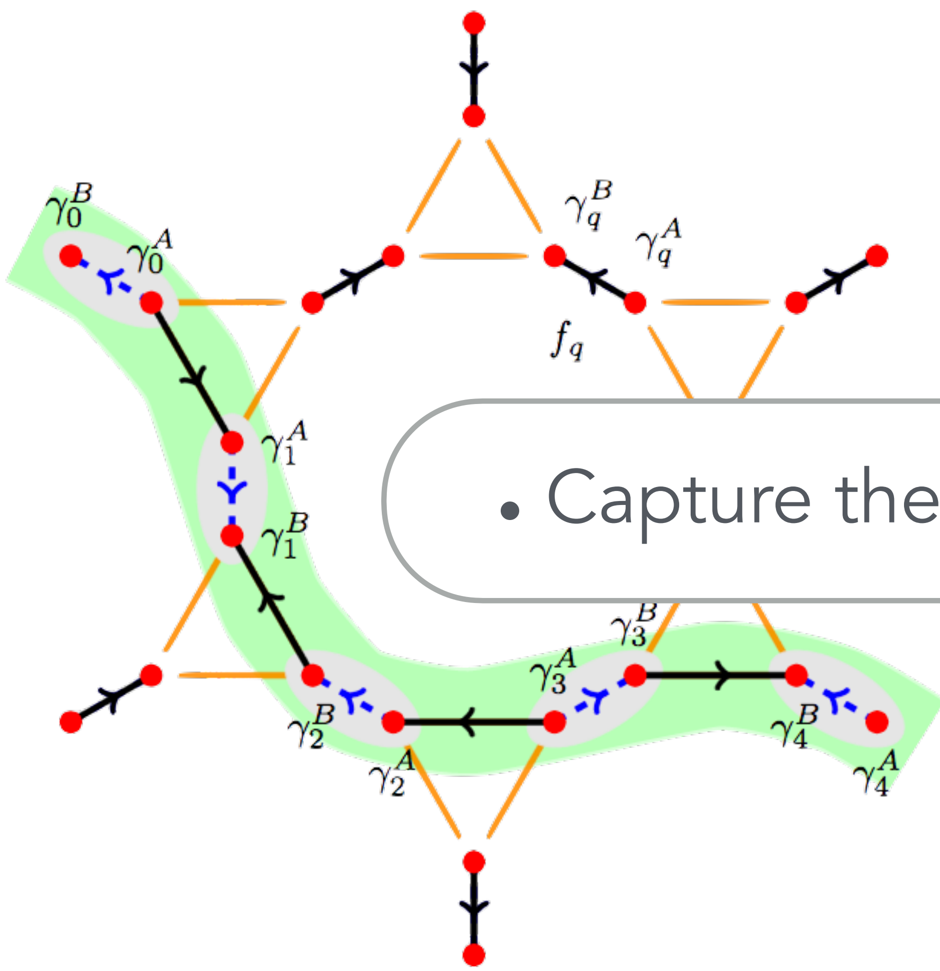
      $$\omega(a,b,c)\omega(a,bc,d)\omega(b,c,d) = (-1)^{s(a,b)s(c,d)}\omega(ab,c,d)\omega(a,b,cd)$$

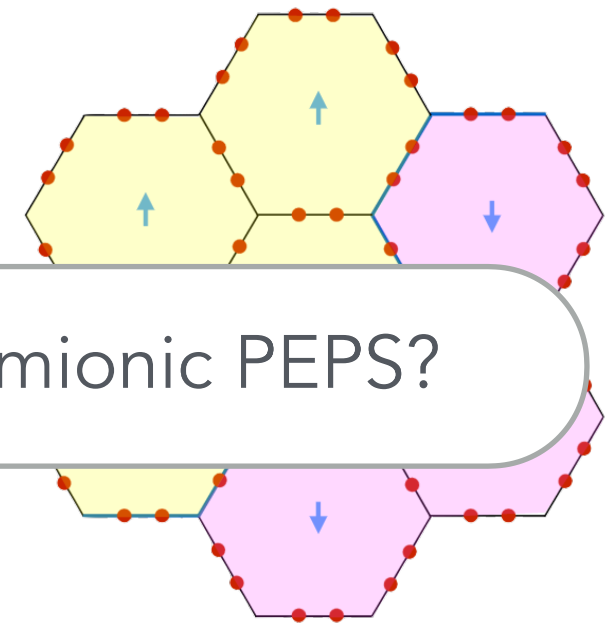
- Can all be shown to satisfy framework (tedious)
- **Fermionic toric code:** Simplest triple
  - $G = \mathbb{Z}_2$
  - $s(1,1) = 1, s = 0$ otherwise

Wille, Buerschaper, Eisert, arXiv:1609.02574
Bultinck, Williamson, Haegeman, Verstraete, arXiv:1610.07849

- Consistent framework of topological PEPS for fermionic systems

Wille, Buerschaper, Eisert, arXiv:1609.02574
Williamson, Bultinck, Haegeman, Verstraete, arXiv:1609.0289





- Capture them as fermionic PEPS?

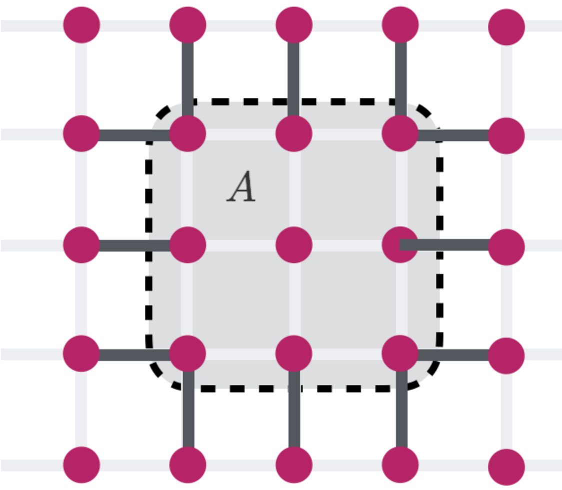
Ising anyons in frustration-free Majorana dimer models

Ware, Son, Cheng, Mishmash, Alicea, Bauer, arXiv:1605.06125

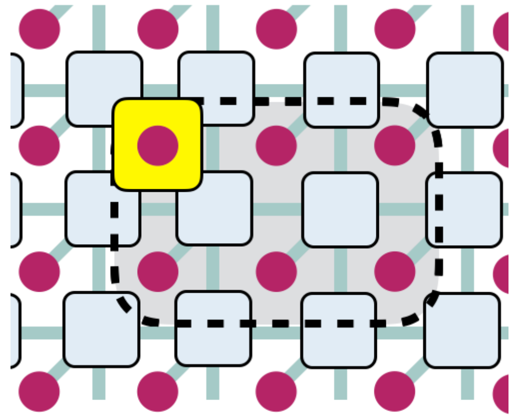
Discrete spin structures and commuting projector models for 2d fermionic symmetry protected topological phases

Tarantino, Fidowski, Phys Rev B 94, 115115 (2016)

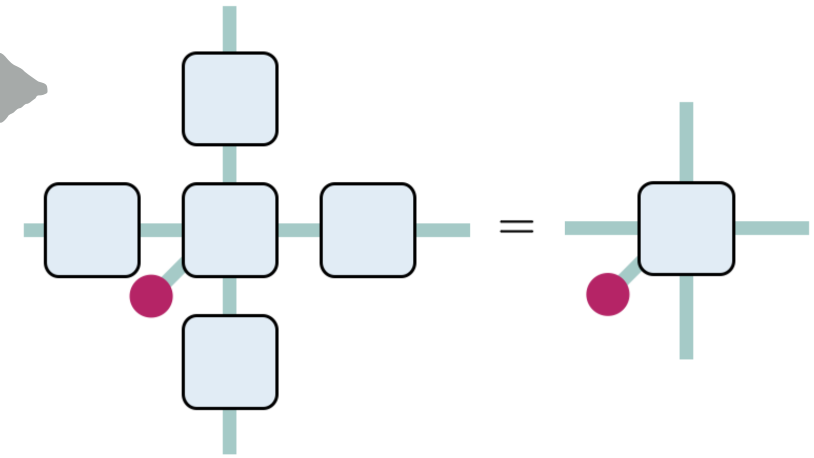
Summary



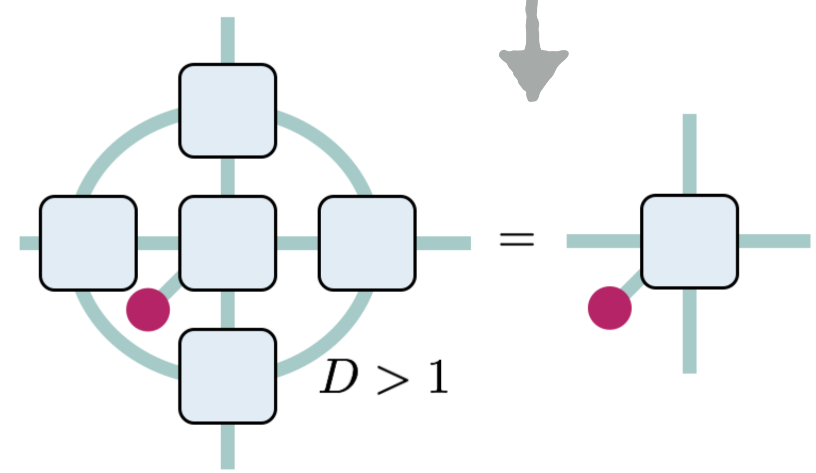
Area laws and tensor networks



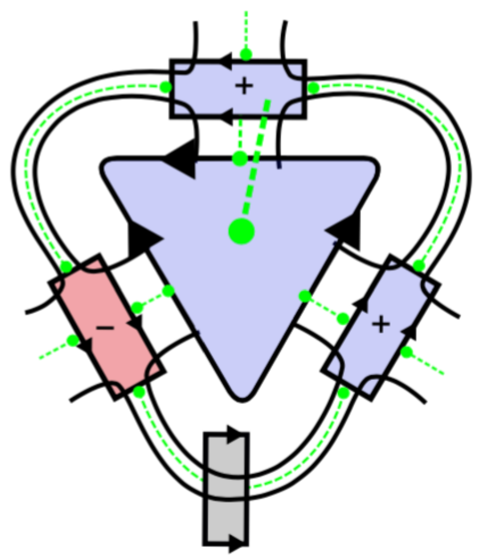
Contraction of PEPS



G-injective PEPS



MPO-injective PEPS



Fermionic topological PEPS and a crime story

Thanks for your attention