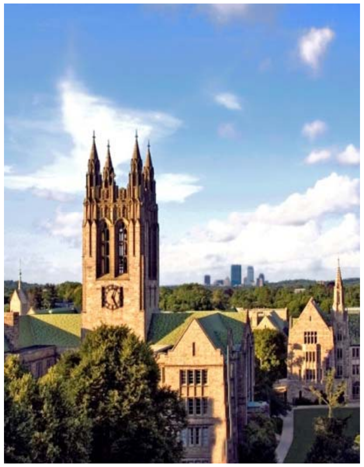
Symmetric Tensor Networks and Topological Phases

Ying Ran (Boston College)



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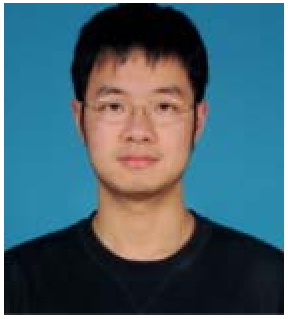

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- **References:**
  - arXiv: 1505.03171, S. Jiang, Y. Ran
  - arXiv: 1509.04358, P. Kim, H. Lee, S. Jiang, B. Ware, C. Jian, M. Zaletel, J. Han, Y. Ran
  - arXiv: 1610.02024, S. Jiang, P. Kim, J. Han, Y. Ran
  - arXiv: 1611.07652, S. Jiang, Y. Ran

Motivations

- We focus on bosonic topological (SET or SPT) phases, which require strong interactions to realize.

(1) Conceptual issues:

-- Classification problems

(SPT phases with spatial symmetries.)

(2) "Practical" issues: How to realize them?

-- Physical intuitions/guiding principles?

(Are there criteria like the band-inversion picture in topological insulators?)

-- Numerical methods suitable for searching for these topological phases in models?

(How to write down generic variational wavefunctions?)

Main result

Based on tensor-network formulation, we develop a machinery to:

(1) systematically (but partially) classify topological phases

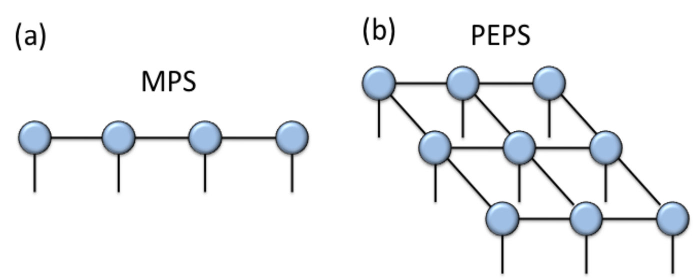
(2) construct generic variational wavefunctions for these phases

(onsite and spatial) symmetries of the system → (Partially) classification of topological phases and construction of generic wavefunctions for each class

- This machinery answers:

How many classes of symmetric tensor-network wavefunctions that cannot be smoothly deformed into each other under certain assumptions?

1D-MPS, 2D-PEPS, and 3D generalizations



figures from R. Orus, Annals Phys. (2014)

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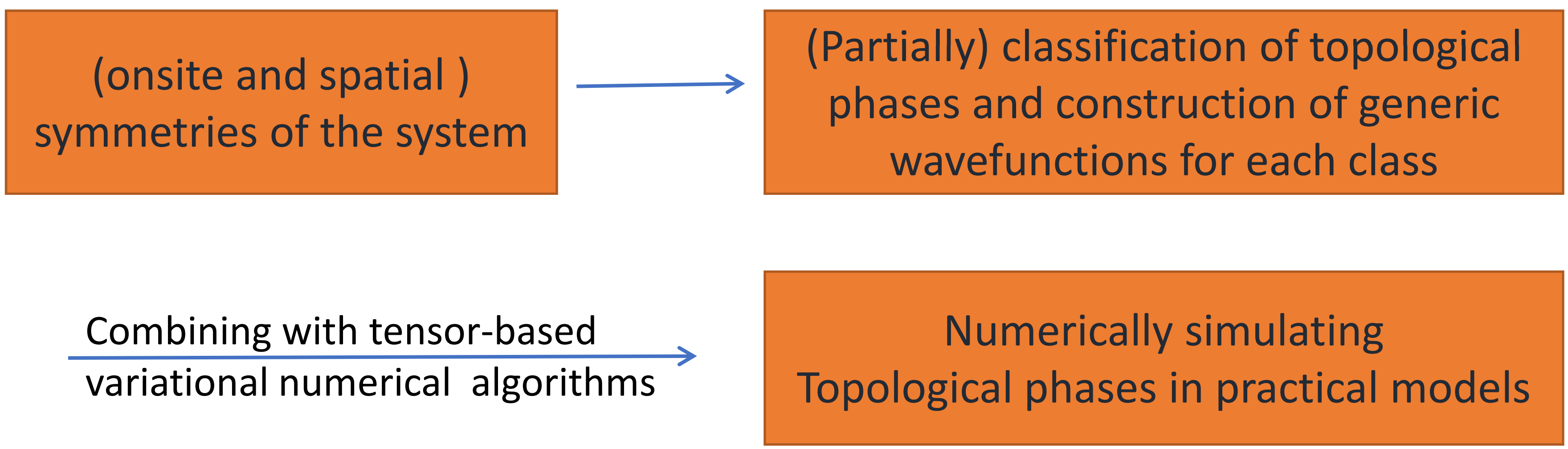
(2) construct generic variational wavefunctions for these phases



Plan: Three applications of this machinery

(1) Classification and simulation of competing spin liquids in the spin-1/2 Heisenberg model on the kagome lattice

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(2) Classification of bosonic cohomological SPT: $H^{d+1}(SG, U(1))$

- SG: on-site and lattice symmetries (onsite (Chen, Liu, Gu, Wen…), lattice (Chen, Hermele, Fu, Qi, Furusaki, Cheng…))
- T and P(mirror) should be treated as "anti-unitary"
- Generic tensor wavefunctions for every class (if SG is discrete)

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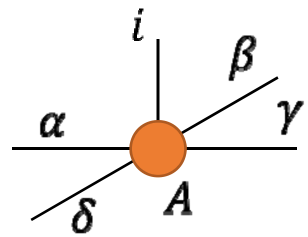
(3) A by-product: a general connection between "conventional" SET phases and SPT phases in 2D. For example:

- Toric code + Z_2 Ising symmetry = $\{I, g\}$ with $[g(e)]^2 = -1, [g(m)]^2 = 1$
- Condense m with $g(m) = 1$ → trivial SPT
- Condense m with $g(m) = -1$ → nontrivial Z_2 SPT

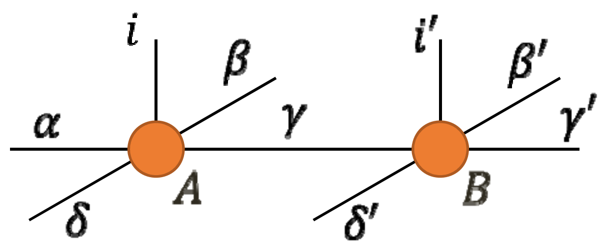
What are tensor networks?

tensor



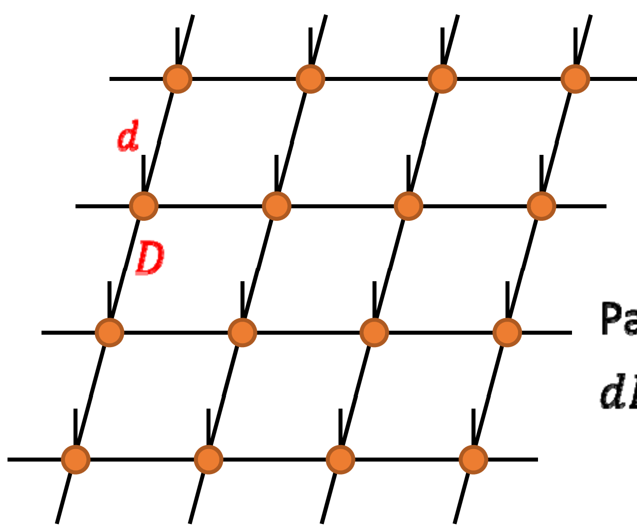
$$= A^{i}_{\alpha\beta\gamma\delta} \sim \sum A^{i}_{\alpha\beta\gamma\delta} |i\rangle \otimes |\alpha\beta\gamma\delta\rangle$$

tensor contraction



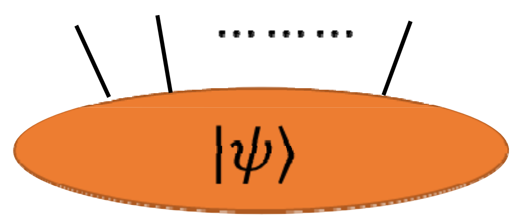
$$= \sum_{\gamma} A^{i}_{\alpha\beta\gamma\delta} \cdot B^{i'}_{\gamma\beta'\gamma'\delta'}$$

PEPS



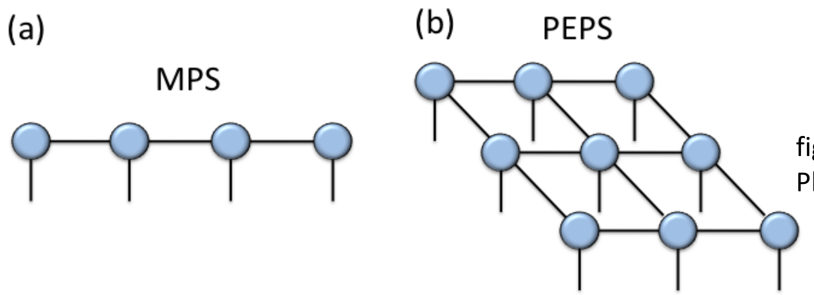
$$|\psi\rangle = \sum_{\{i\}} c_{i_1 i_2 \ldots i_n} |i_1, i_2, \ldots, i_n\rangle$$

Parameters: dD^4/site



Why tensor networks?

1D-MPS, 2D-PEPS, and 3D generalizations …)



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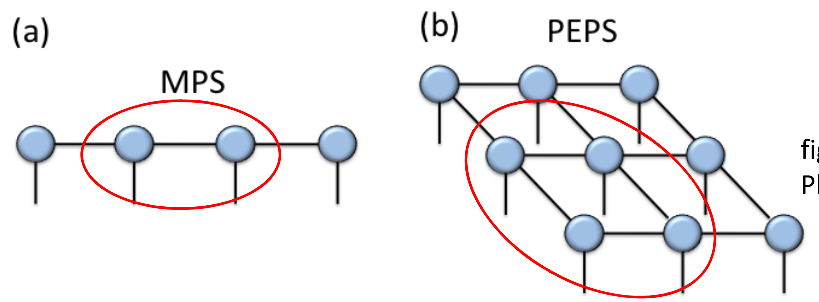
- In the past, indeed the first deep insight and systematic results of SPT phase were obtained by studying symmetry properties of MPS in 1d. (Pollmann, Berg, Turner, Oshikawa, Chen, Gu, Wen…)

- Powerful numerical algorithms
  - 1D MPS: DMRG (S. White …)
  - 2D PEPS: iTEBD, CTM, TRG… (Cirac, Verstraete, Vidal, Gu, Levin, Wen, Xiang …)

Why tensor networks?

1D-MPS, 2D-PEPS, and 3D generalizations …)



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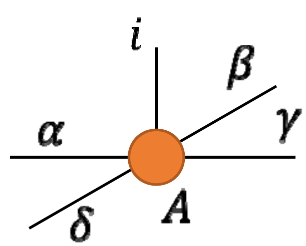
- In particular, the tensor-network formulation is particularly suitable to understand the local symmetry properties of a global wavefunction, which are essential for symmetric topological phases.



$$= A^{i}_{\alpha\beta\gamma\delta} \sim \sum A^{i}_{\alpha\beta\gamma\delta} |i\rangle \otimes |\alpha\beta\gamma\delta\rangle$$

symmetry properties of these local quantum states (in enlarged Hilbert space)
➔ symmetry properties of the global physical state

Plan

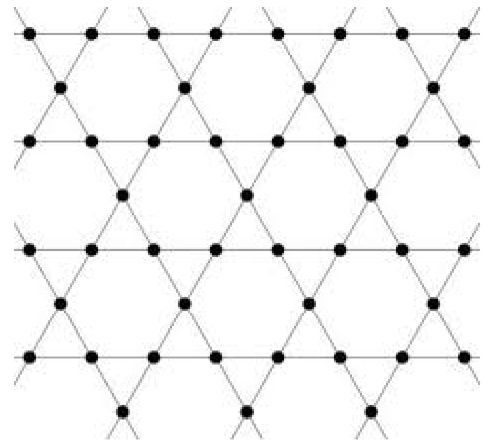
- Spin liquids on the kagome lattice
- SPT phases: $H^{d+1}(SG, U(1))$
- Anyon condensation mechanism:

  "conventional" SET phases → SPT phases

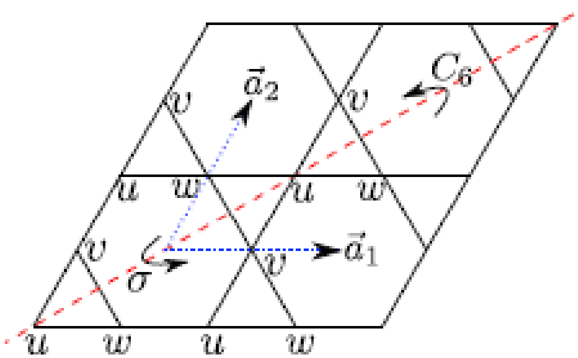
The Heisenberg model on the kagome lattice

- **For systems on the kagome lattice with spin-$\frac{1}{2}$ per site, what is the quantum phase of the following Hamiltonian?**

$$H = J\sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$



Symmetries: on-site spin rotation, time reversal, lattice translation, rotation, reflection



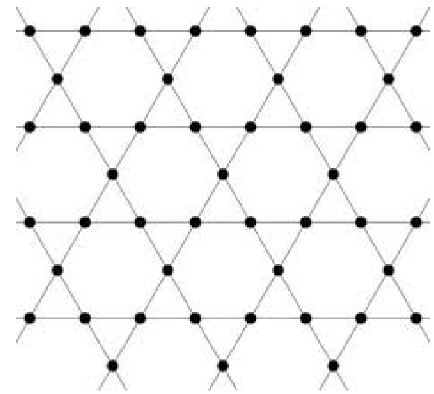
Sachdev, Marston, Senthil, Singh, Evenbly, Vidal, Ran, Hermele, Wen, Lee, Wang, Vishwanath, Iqbal, Becca, Sorella, Poilblanc, White, Huse, Depenbrock, McCulloch, Schollwock, Jiang, Balents, Mei, Xiang, He, Zaletel, Oshikawa, Pollmann… and many more

DMRG → spin liquid phase (SET for both onsite and lattice symmetries)?

Which spin liquid???

Different Z2 Spin liquids in tensor formulation

- **Let me firstly summarize some results:**



Classifying symmetric PEPS describing Z2 Spin liquids

→32 classes (Z_2^5)

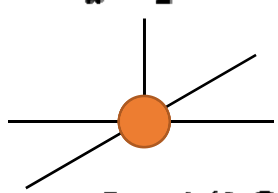
- χ_σ , χ_T → new classes, "weak SPT" index
- η_{12} , η_{c6} , η_σ → symmetry fractionalization of e

(consistent with Schwinger boson results) (Sachdev,Wang,Vishwanath)

For every class, constrained Hilbert space for a local tensor

→ generic wavefunctions

$d = 2$



$D = 6\ (0 \oplus \frac{1}{2} \oplus 1)$

For $D = 6$, can realize **four** classes

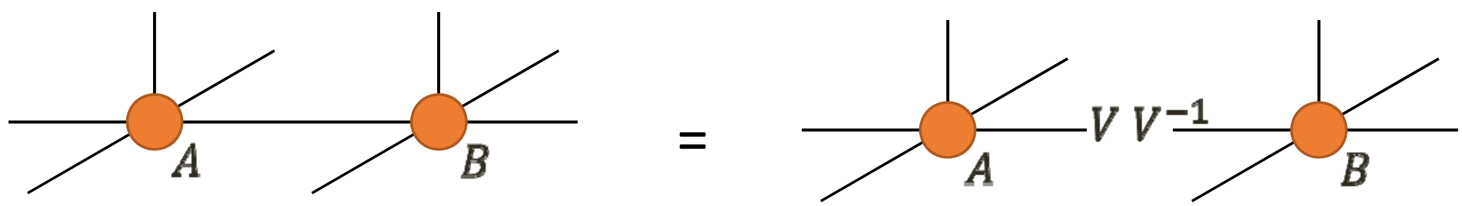
- Unconstrained: $DIM_{tot} = d \cdot D^6 \approx$ **2600**
- Constrained: $DIM_{constraint} =$ **19**

- How to implement global symmetries into tensor-network?

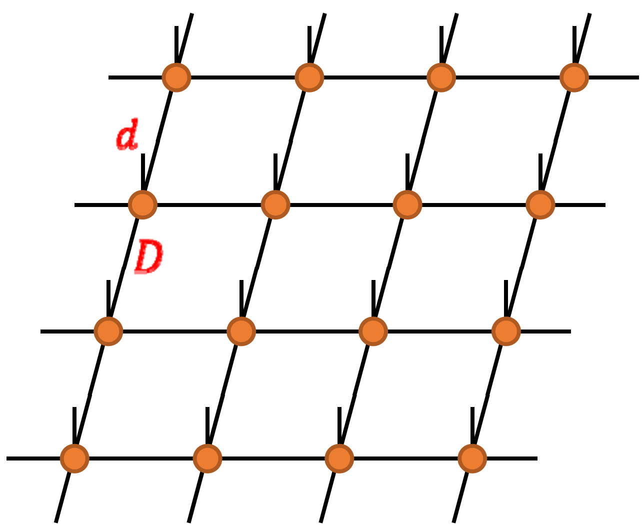
Basic assumption:

Symmetries on physical wavefunctions ~ Gauge transformation on internal legs

Gauge redundancy in a tensor-network



One internal bond ~ $GL(D, \mathbf{C})$ gauge redundancy



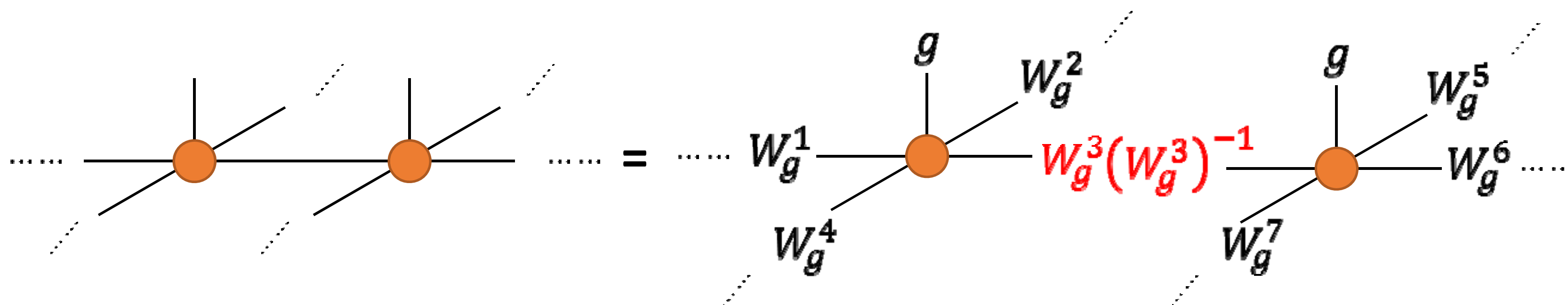
Gauge redundancy

~ $[GL(D, \mathbf{C})]^{N_{bond}}$

Gauge redundancy & symmetry

$$|\psi\rangle = g|\psi\rangle$$

global symmetries on physical wavefunctions ~ gauge transformation on internal legs

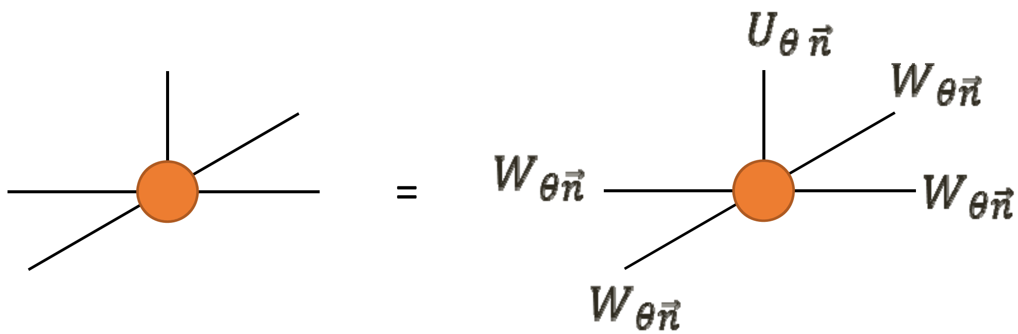


$$T^a = W_g g \circ T^a$$

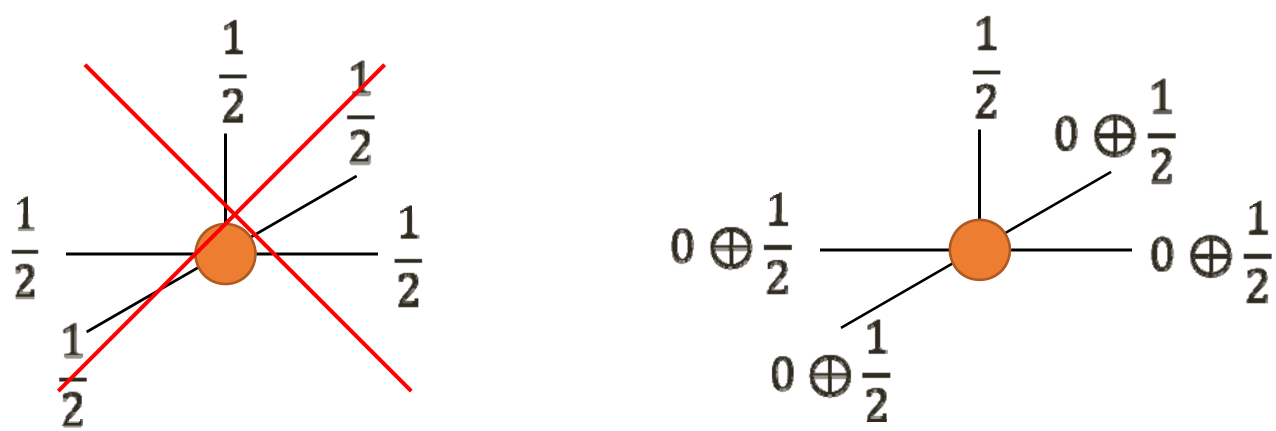
Classify symmetric phases by different symmetry transformation rules of local tensors.

What are consistent conditions for W_g?

Implementing spin rotation symmetry

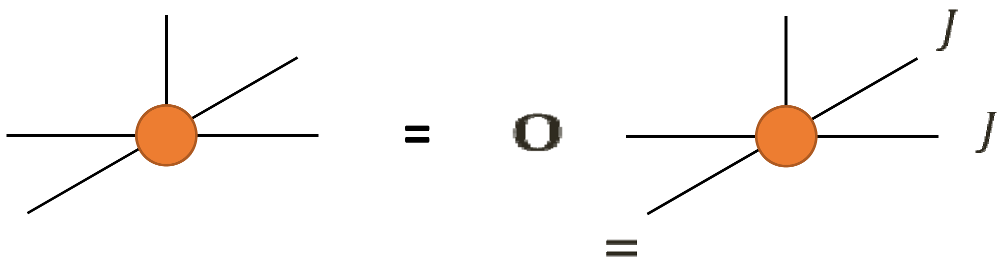


$W_{\theta\vec{n}}$: representation of SU(2) symmetry → Local tensors are spin singlets



Invariant Gauge Group ($\boldsymbol{IGG}$)

2π spin rotation:

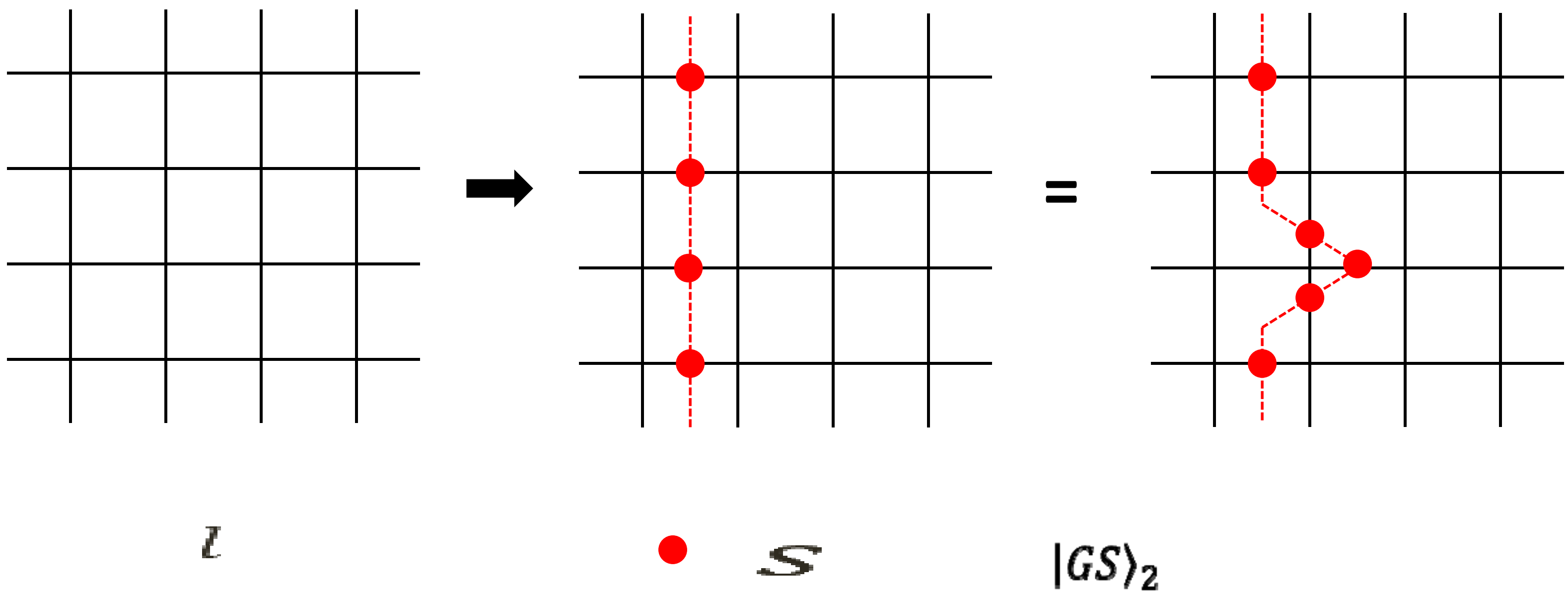


$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{matrix} |0\rangle \\ |\uparrow\rangle \\ |\downarrow\rangle \end{matrix}$$

- $J^2 = I$, Z_2 matrix IGG:
  - a pure gauge transformation
  - leaves a single tensor invariant
  - relate to Z_2 gauge theory (Swingle, Wen, Poilblanc, Schuch, Pérez-García,Cirac ....)
- Minimal required Z_2 IGG in spin-1/2 kagome system ~ no featureless symmetric phase
- There is always "trivial IGG"— leg dependent phase factors

Physical interpretation of $\boldsymbol{IGG}$

- What is the physical meaning of IGG?
- Z_2 IGG ~ Z_2 gauge theory (toric code); J ~ flux line
    - Four-fold GSD on torus



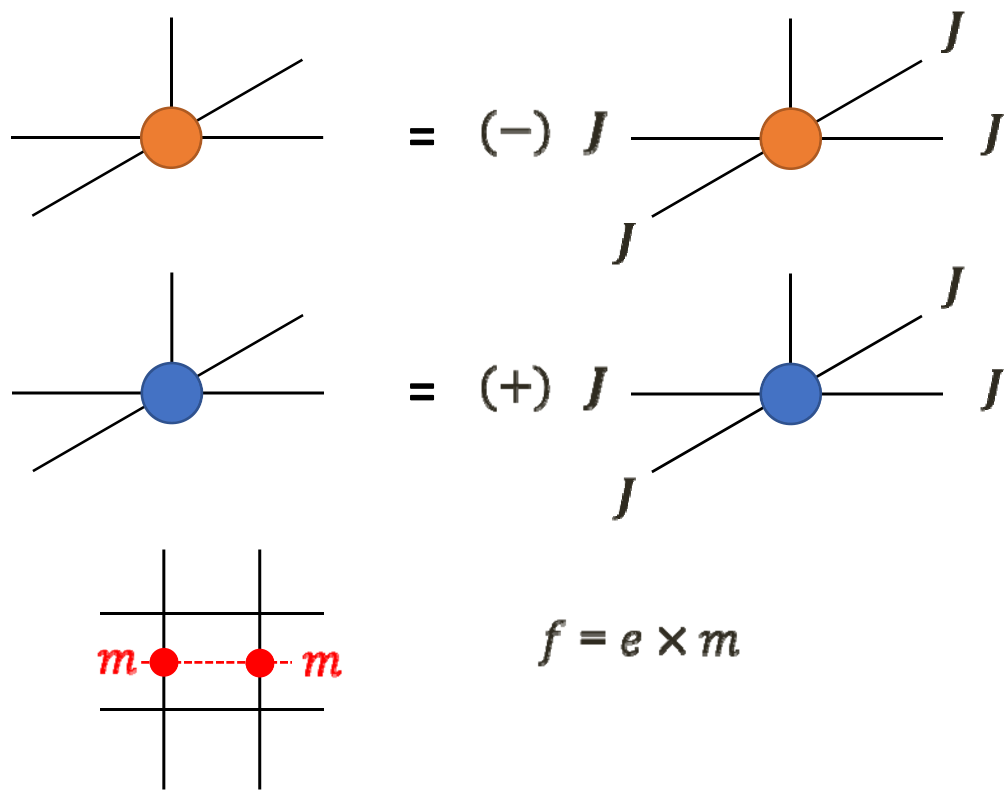
Physical interpretation of $\boldsymbol{IGG}$

- Topological excitations:

1

e (spinon)

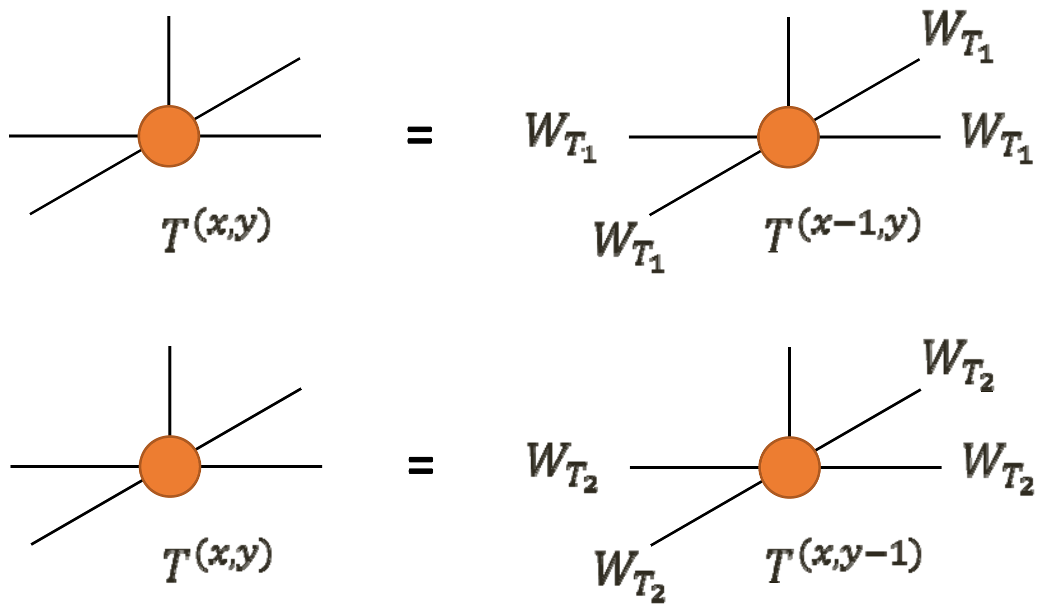
m (vison)



$f = e \times m$

- deconfined → spin liquid
- confined → ordered phase (VBS, magnetic order)

Tensor equations: interplay between symmetries and $\boldsymbol{IGG}$



- translation form a $Z \times Z$ group, defined by $T_1 T_2 = T_2 T_1$

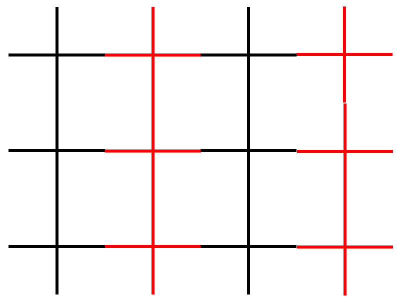
  → $W_{T_1} T_1 W_{T_2} T_2 = \chi \cdot \eta \cdot W_{T_2} T_2 W_{T_1} T_1$

  χ: leg dependent $U(1)$, $\eta = I \; or \; J$

Physical interpretation for tensor equations

$$W_{T_1}T_1W_{T_2}T_2 = \chi \cdot \eta \cdot W_{T_2}T_2W_{T_1}T_1$$

- For translations, χ can always be set to 1 by redefining W
- η label **symmetry fractionalization** of spinon e
  - $\eta = \boldsymbol{I}$ → zero flux spin liquid
  - $\eta = \boldsymbol{J}$ → π flux spin liquid
- Solving equations by fixing gauge
- zero-flux class: $W_{T_1} = W_{T_2} = I$ → tensors translation invariant
- π-flux class: $W_{T_2} = I, W_{T_1}(x,y,i) = \eta^y$ → unit cell of tensors doubled



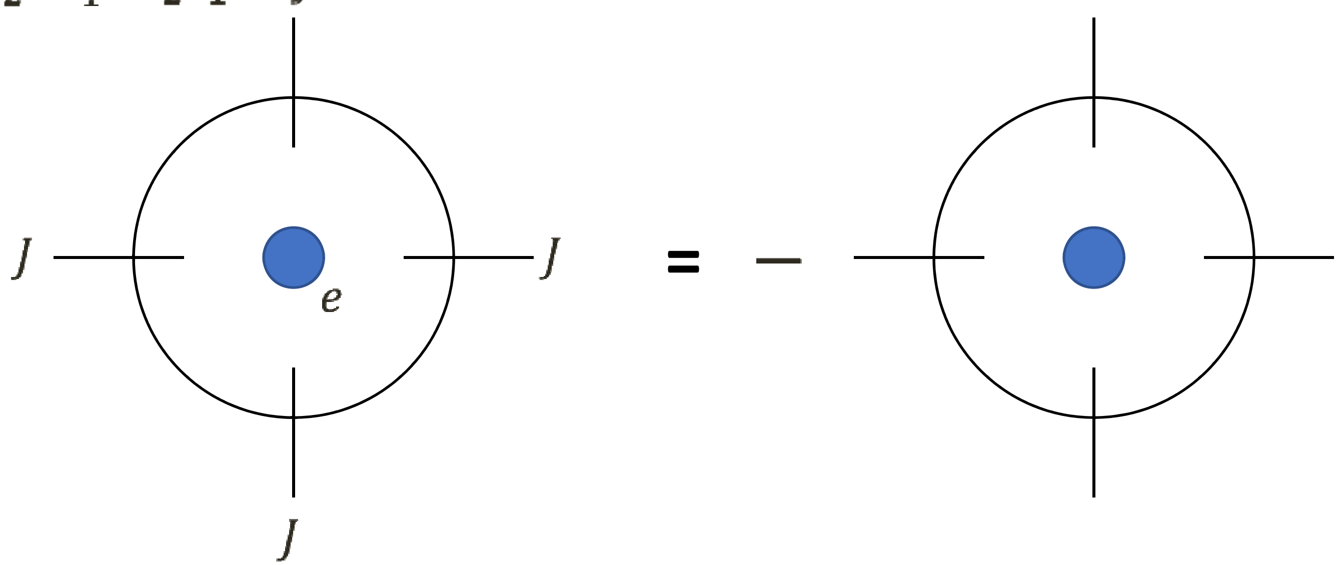
π-flux class

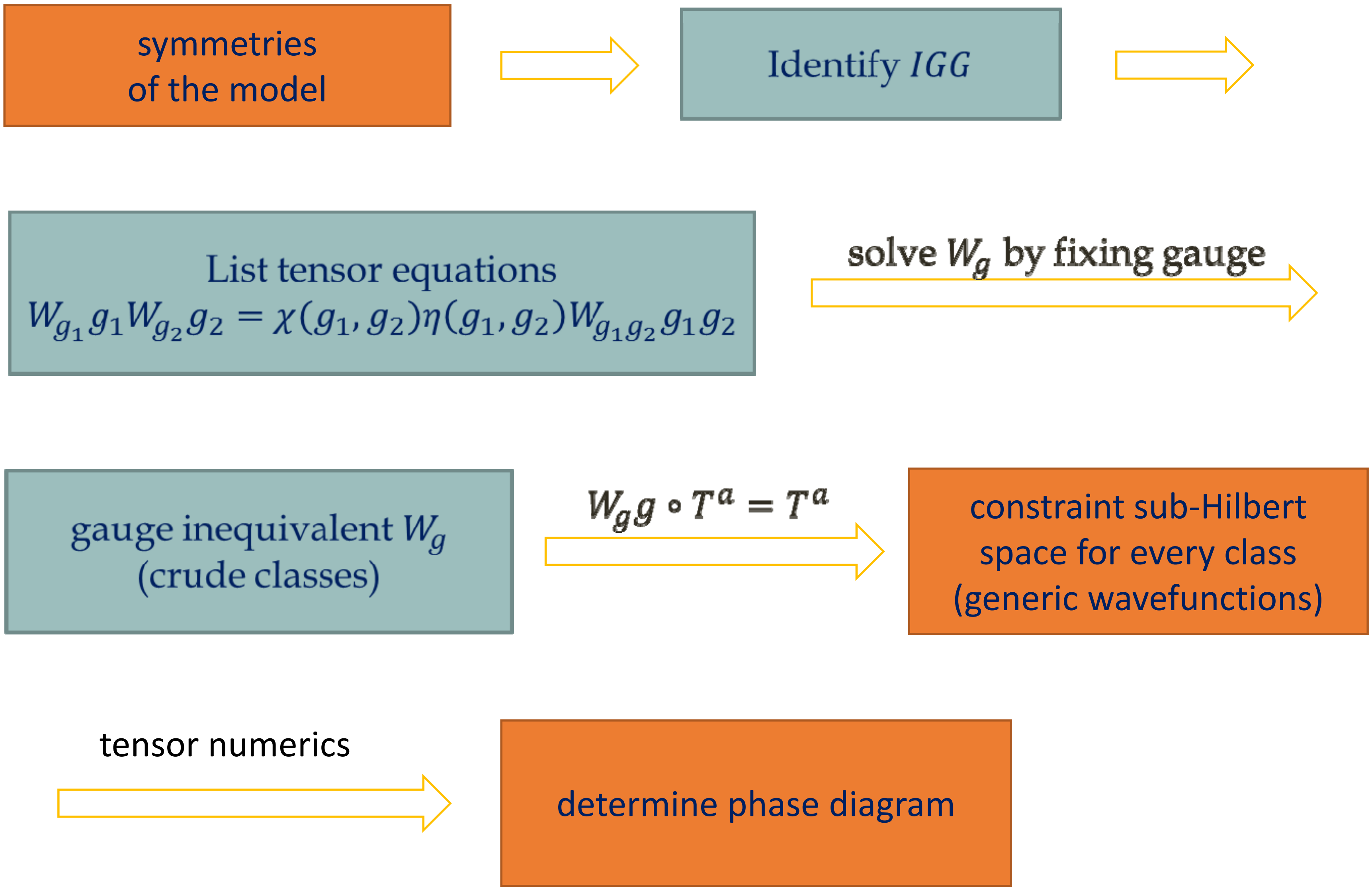
Symmetry fractionalization from tensor equations

$$W_{T_1}T_1W_{T_2}T_2 = \eta \cdot W_{T_2}T_2W_{T_1}T_1$$

- $\eta = I$, trivial SET
- $\eta = J$, e carries fractional "translational" quantum number

$T_2^{-1}T_1^{-1}T_2T_1 \to J$




symmetries
of the model
Identify IGG
List tensor equations
$W_{g_1} g_1 W_{g_2} g_2 = \chi(g_1, g_2)\eta(g_1, g_2) W_{g_1 g_2} g_1 g_2$
solve W_g by fixing gauge
gauge inequivalent W_g
(crude classes)
$W_g g \circ T^a = T^a$
constraint sub-Hilbert
space for every class
(generic wavefunctions)
tensor numerics
determine phase diagram

Kagome Heisenberg model

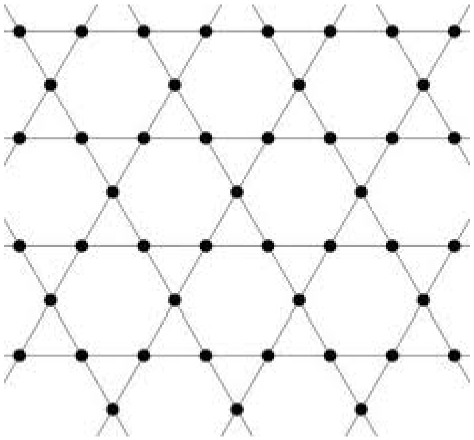
Topological invariants of tensor symmetry transformation rules:

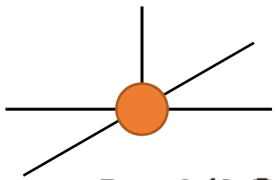
- χ_σ , χ_T

"weak SPT" index, 2D AKLT like physics

- η_{12} , η_{C6} , η_σ

label symmetry fractionalization of spinon-e in the Z_2 QSL member phase.





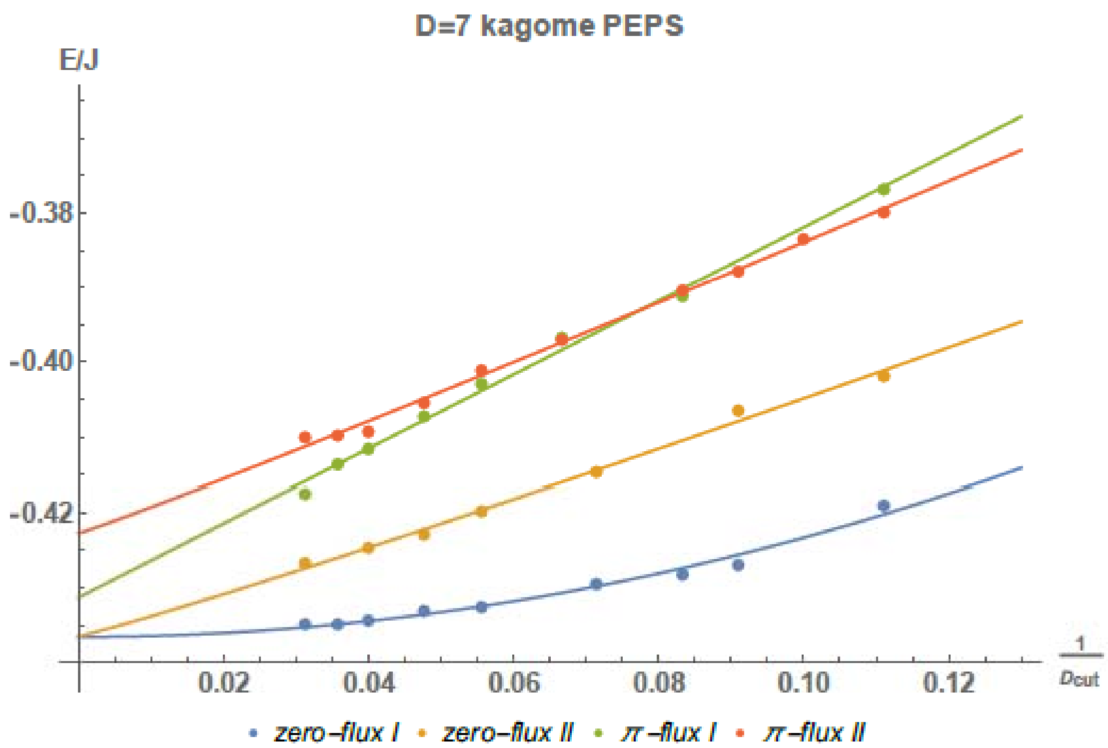
For $D = 6$, can realize **four** classes

- Unconstraint: $DIM_{tot} = d \quad D^6 \approx$ **2600**
- Constraint: $DIM_{constraint} =$ **19**

Symmetric iPEPS algorithm

- Focus on infinite PEPS (iPEPS)
- Optimization
  - Minimize “approximate” energy densities within constrained Hilbert spaces of four promising classes (Simple update method) (Jiang,Xiang…)
- Measurement
  - Measure energy density for the optimized state
  - Tensor RG + variational Monte Carlo (Nave,Levin,Gu,Wen,Xiang,Jiang,Wang, Sandvik, Verstraete,…)

Energy densities for optimal state of four classes



D_{cut} ~ virtual states kept when performing tensor contraction

- For $D = 7, 8 \times 8 \times 3$ lattice size, $E \sim -\mathbf{0.4366(3)}J$, comparable to (slightly higher than) DMRG report.
- Two zero-flux classes have nearly degenerate energy

Which class?

Competing spin liquids?

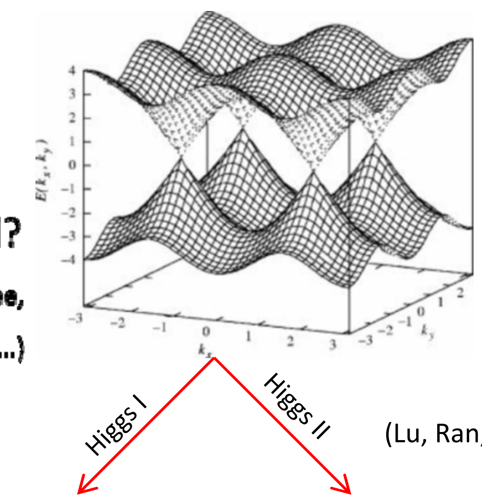
$$E(\text{Zero-Flux I}) \approx E(\text{Zero-Flux 2})$$

Sachdev's Q1=Q2 state   Q1=-Q2 state

- Two possibilities:
  1. Their energy densities are different. But our numerics is not accurate enough to distinguish them.
  2. They indeed share degenerate energy. Any physical reason?

$U(1)$ Dirac spin liquid?

(Hastings, Ran, Hermele, Lee, Wen, Iqbal, Becca, Poilblanc...)



(Lu, Ran, Lee)

zero-flux I   zero-flux II

Future work:

1. Long-range behavior?
2. Excitation spectrum?
3. More advanced optimization method

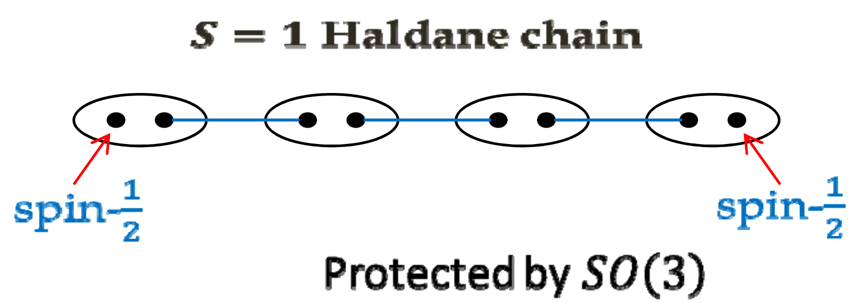
Plan

- Spin liquids on the kagome lattice
- SPT phases: $H^{d+1}(SG, U(1))$
- Anyon condensation mechanism:

  "conventional" SET phases → SPT phases

Bosonic cohomological SPT phases

- In our framework, SPT: $H^{d+1}(SG, U(1))$ (not complete)
  - SG: **on-site** and **lattice symmetries** (onsite (Chen, Liu, Gu, Wen), lattice (Chen, Hermele, Fu, Qi, Furusaki,Cheng...))
  - T and P(mirror) ~ "anti-unitary"
- Example:



  - 1d, $H^2(Z_2^T, U(1)) = H^2(Z_2^P, U(1)) = Z_2$, the "Haldane phase"
  - 2d, P & T → $H^3\left(Z_2^T \times Z_2^P, U(1)\right) = Z_2^2$
- Generic wavefunctions (constrained tensor Hilbert space) for every class
- A by-product: a general connection between SET and SPT phases in 2D.

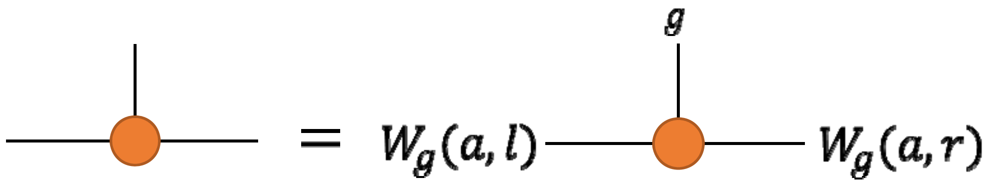
SPT phases: $H^{d+1}(SG, U(1))$ (d=1)

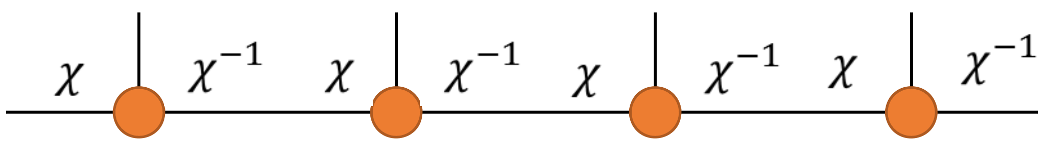
Warm-up: d=1 (Pollmann, Berg, Turner, Oshikawa, Chen, Gu, Wen…)

The main purpose here is to demonstrate the anti-unitary action of mirror reflection in the tensor formulation.

Just like the spin liquid example, let us firstly identify IGG, then find the consistency equations for the tensor symmetry transformation rules.

SPT phases: $H^{d+1}(SG, U(1))$ (d=1)

Symmetry Condition: 

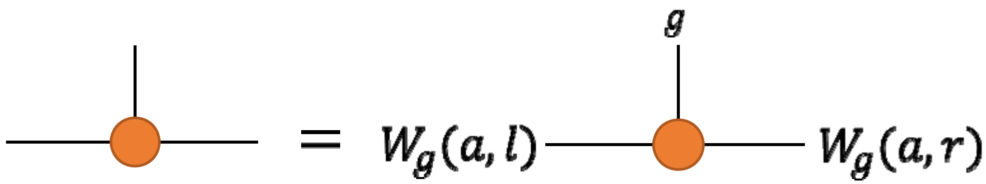
IGG: 

a single U(1) phase variable over the whole lattice.

(On an infinite lattice, we require $W_g \cdot g$ ($\forall g \in SG$) as well as ***IGG*** to send all local tensors back to themselves without extra U(1) phase)

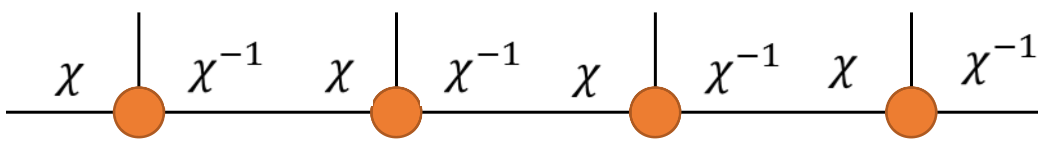
SPT phases: $H^{d+1}(SG, U(1))$ (d=1)

Symmetry Condition:

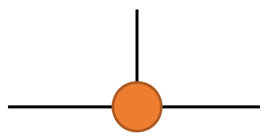
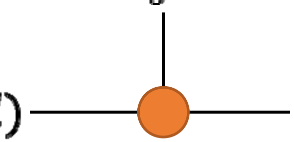


$$\Rightarrow W_{g_1}g_1W_{g_2}g_2 = \chi(g_1,g_2)W_{g_1g_2}g_1g_2,\ \chi(g_1,g_2) \in IGG$$

IGG:



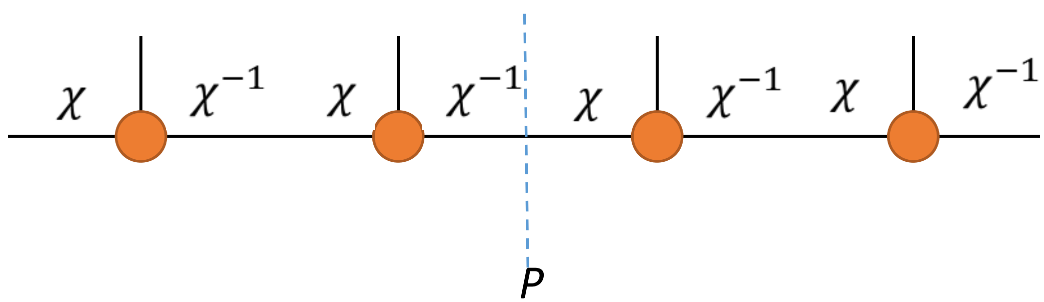
SPT phases: $H^{d+1}(SG, U(1))$ (d=1)

Symmetry Condition:  $= W_g(a,l)$  $W_g(a,r)$

$\Longrightarrow W_{g_1} g_1 W_{g_2} g_2 = \chi(g_1, g_2) W_{g_1 g_2} g_1 g_2$, $\chi(g_1, g_2) \in IGG$

and $\chi(g_1, g_2)\, \chi(g_1 g_2, g_3) = {}^{g_1}\chi(g_2, g_3) \chi(g_1, g_2 g_3)$. $\chi(g_1, g_2) \in H^2(SG, U(1))$

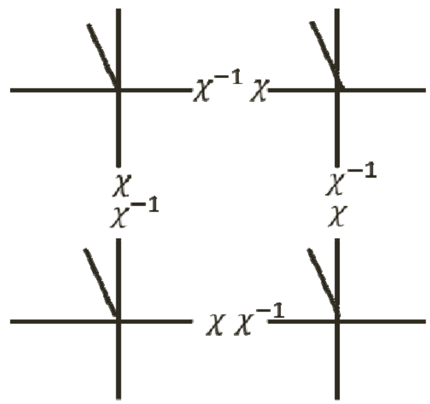
IGG:



Just as time-reversal, mirror reflection send $\chi \to \chi^{-1}$,
so they should be treated as anti-unitary

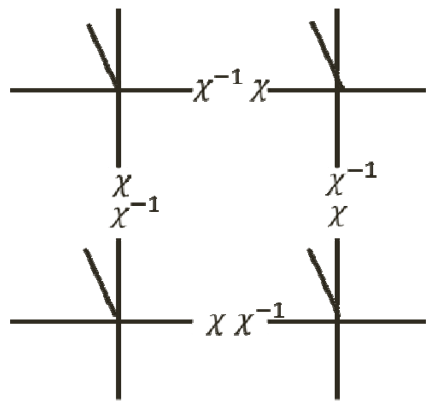
SPT phases: $H^{d+1}(SG, U(1))$ (d=2)

Plaquette IGG:



For every plaquette, there is
at least a phase variable $\in IGG$.

SPT phases: $H^{d+1}(SG, U(1))$ (d=2)



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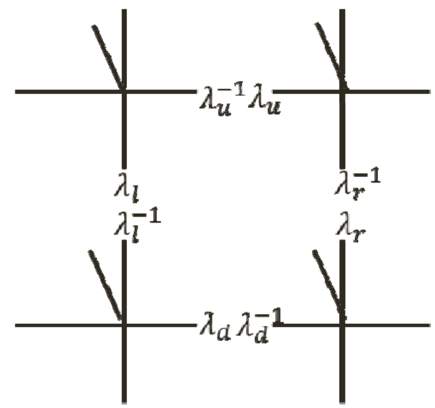
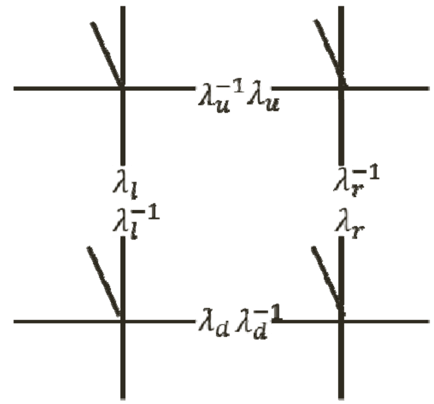


For every plaquette, there is at least a phase variable $\in IGG$.

It turns out that, to describe strong SPT phases, plaquette-IGG need to contain nontrivial matrix transformations.

SPT phases: $H^{d+1}(SG, U(1))$ (d=2)

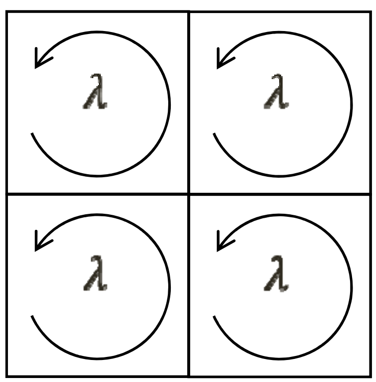
Plaquette IGG:



In order to describe cohomological SPT phases, we assume:

global matrix IGG decomposition:

$$J = \prod_p \lambda_p$$



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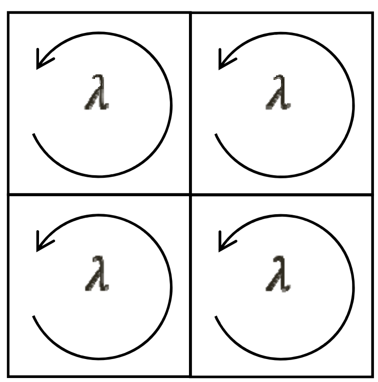
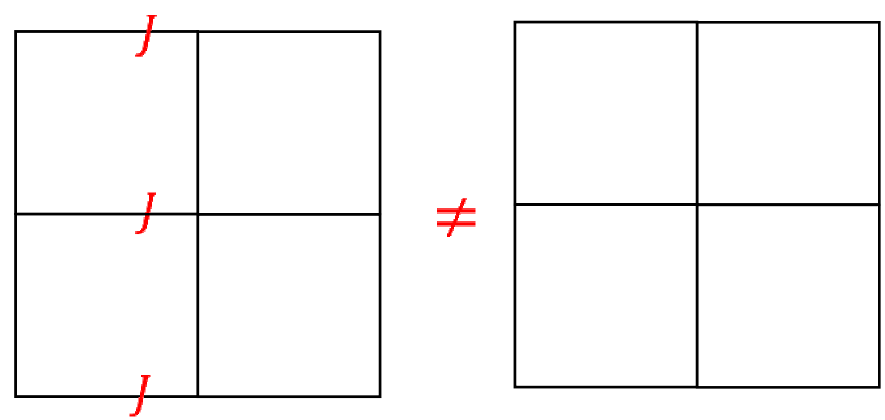


Physical interpretation:

If J cannot be decomposed, then the tensor-network would be topologically ordered.



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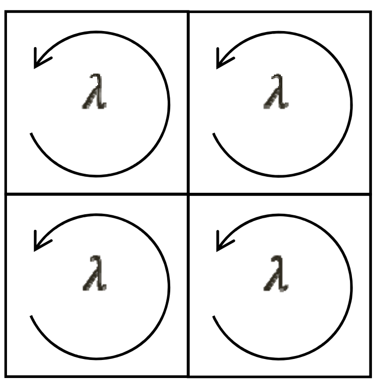
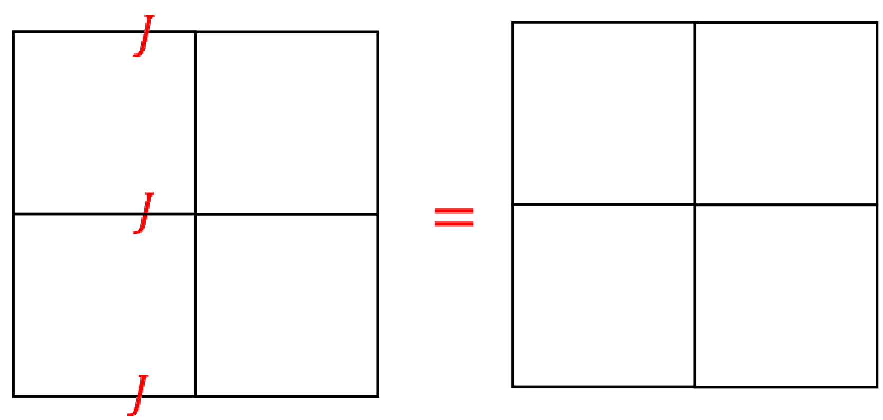


Physical interpretation:

If J can be decomposed, then:



i.e. the would-be topological order is confined by the J-string condensation.

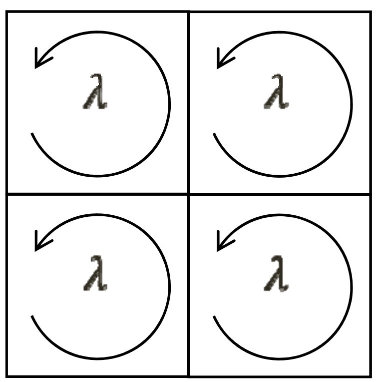
SPT phases: $H^{d+1}(SG, U(1))$ (d=2)

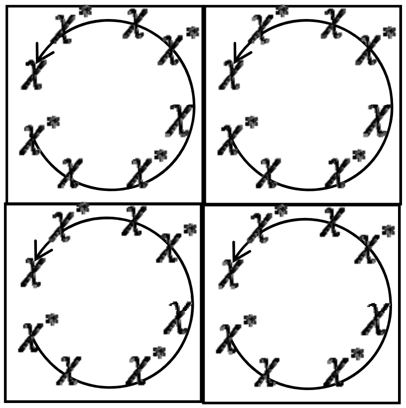
In order to describe cohomological SPT phases, we assume:

global matrix IGG **decomposition:**

$J = \prod_p \lambda_p = \prod_p \chi \cdot \lambda_p$

This decomposition has an overall U(1) phase ambiguity:





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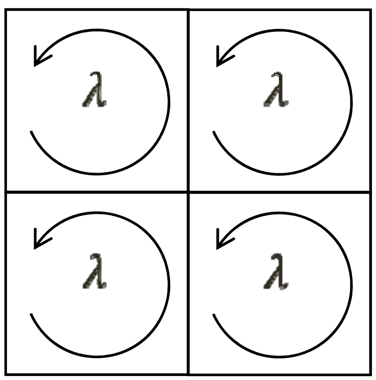

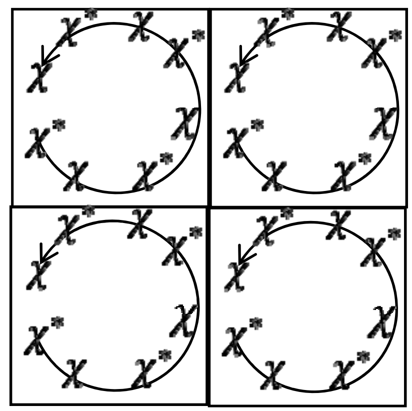

It turns out that the χ ambiguity in this decomposition directly leads to topological invariants of tensor symmetry transformation rules: 3-cocycles

SPT phases: $H^{d+1}(SG, U(1))$ (d=2)

$$W_{g_1} g_1 W_{g_2} g_2 = \eta(g_1, g_2) W_{g_1 g_2} g_1 g_2$$

$$\eta(g_1, g_2)\eta(g_1 g_2, g_3) = {}^{W_{g_1} g_1}\eta(g_2, g_3)\eta\,(g_1, g_2 g_3)$$

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"Global" IGG $\eta(g_1, g_2)$ can be decomposed to plaquette IGG:

$$\eta = \prod_p \lambda_p$$

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$$\eta = \prod_p \lambda_p$$

$$\lambda_p(g_1, g_2)\lambda_p(g_1 g_2, g_3) =$$

3-cocycle →

$$\chi(g_1, g_2, g_3)\ {}^{W_{g_1} g_1}\lambda_p(g_2, g_3)\lambda_p(g_1, g_2 g_3)$$

SPT phases: $H^{d+1}(SG, U(1))$ (d=2)

$$W_{g_1} g_1 W_{g_2} g_2 = \eta(g_1, g_2) W_{g_1 g_2} g_1 g_2$$

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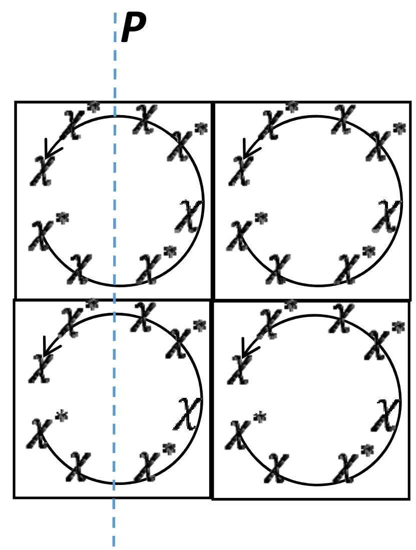
3-cocycle

$$\chi(g_1, g_2, g_3)\, {}^{W_{g_1} g_1}\lambda_p(g_2, g_3)\lambda_p(g_1, g_2 g_3)$$



Just like time-reversal,
mirror reflections send $\chi \rightarrow \chi^{-1}$ (anti-unitary),

while rotations/translations send $\chi \rightarrow \chi$ (unitary)

SPT phases: $H^{d+1}\left(SG, U(1)\right)$ (d=2)

Examples:

- $H^3\left(Z_2, U(1)\right) = Z_2$

  Constructing the nontrivial SPT with D=4 PEPS (square lattice)

  Constrained tensor sub-Hilbert space→ 15 variational parameters

- $H^3(Z_2^T \times Z_2^P, U(1)) = Z_2 \times Z_2$

  Constructing three nontrivial SPT with D=6 PEPS (square lattice)

Constrained sub-Hilbert spaces→ 79/79/87 variational parameters

SPT phases: $H^{d+1}(SG, U(1))$ (d=3)

- Similar construction can be generalized to 3D. (need cubic IGG& plaquette IGG).

- After complicated algebra, one can show that the topological invariants of tensor symmetry transformation rules are given by four cocycles.

- And time-reversal and mirror reflections should be treated as anti-unitary.

Plan

- Spin liquids on the kagome lattice
- SPT phases: $H^{d+1}(SG, U(1))$
- Anyon condensation mechanism:

  "conventional" SET phases → SPT phases

A by-product: anyon condensation – from SET to SPT

An example:

Toric code —condense m particle→ Trivial phase

A by-product: anyon condensation – from SET to SPT

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Toric code with global Z_2 symmetry $\{I,g\}$ —condense m particle→ ???

A by-product: anyon condensation – from SET to SPT

An example:

Toric code —condense m particle→ Trivial phase

Toric code with global Z_2 symmetry $\{I,g\}$ —condense m particle→ ???

- SET: $[g(e)]^2 = -1, [g(m)]^2 = 1$

This is a rather conventional SET without gapless edge states

A by-product: anyon condensation – from SET to SPT

An example:

Toric code —condense m particle→ Trivial phase

Toric code with global Z_2 symmetry $\{I, g\}$ —condense m particle→ ???

- SET: $[g(e)]^2 = -1, [g(m)]^2 = 1$
  - Condense m with $g(m) = 1$ → trivial Ising paramagnet
  - Condense m with $g(m) = -1$ → nontrivial Ising SPT

Schematic phase diagram

A by-product: anyon condensation – from SET to SPT

An example:

Toric code —condense m particle→ Trivial phase

Toric code with global Z_2 symmetry $\{I,g\}$ —condense m particle→ ???

- SET: $[g(e)]^2 = -1, [g(m)]^2 = 1$
  - Condense m with $g(m) = 1$ → trivial Ising paramagnet
  - Condense m with $g(m) = -1$ → nontrivial Ising SPT

Subtlety in the definition of $g(m)$:
Precisely, these quantum numbers are measured by g-defect featuring trivial symmetry fractionalizations

anyon condensation after gauging the symmetry

- Why the phase we obtained is nontrivial SPT? One could justify by gauging the Z_2 symmetry $\{1, g\}$. (Levin, Gu)

	Z_2 SPT??	double semion??
"Parent" phase	SET with e carry fractional quantum number	Z_4 gauge theory
g ($g \times g = m$)	Z_2 symmetry defect	Z_4 gauge flux
m ($m \times m = 1$)	Z_2 gauge flux	double-Z_4 gauge flux
$\Omega_g(*) = -1$	Z_2 symmetry charge	double-Z_4 charge
Condensing object	m with $\Omega_g(m) = -1$	double-Z_4 charge & double-Z_4 flux

anyon condensation after gauging the symmetry

The gauged theory describes Z_4 gauge theory
→ double-semion theory via condensing $2e2m$

		X		X
	X		X	
		X		X
	X		X	

.

anyon condensation after gauging the symmetry

The gauged theory describes Z_4 gauge theory
→ double-semion theory via condensing $2e2m$

		X		X
	X		X	
		X		X
	X		X	

Gauging picture of time reversal or spatial symmetries is unclear beyond TN formulation.

But it turns out these nontrivial SPT can also be obtained from anyon condensation.

e.g.: toric code with $Z_2^T \times Z_2^P$
e fractionalized in different ways
(Z_2^T: time-reversal, Z_2^P: mirror)

→ condense m with $T \cdot P(m) = -1$ →

four types of SPT phases
$(H^3(Z_2^T \times Z_2^P, U(1)) = Z_2 \times Z_2)$

General criteria for anyon condensation

an SET phase —Condense certain fluxes→ an SPT phase

- Gauge group: $Z_{N_1} \times Z_{N_2} \times \cdots$ & symmetry group: SG
- e-particles feature nontrivial symmetry fractionalization
- m-particles have trivial fractionalization, but can carry usual quantum numbers

$\Omega_{g_1} \cdot \Omega_{g_2} = \lambda(g_1, g_2) \cdot \Omega_{g_1 g_2}$ Ω_g ~ symmetry defect, λ ~ certain m-particle

(Barkeshli, Bonderson, Cheng, Wang, Hermele, Chen,Fidkowski…)

General criteria for anyon condensation

Condense certain fluxes

an SET phase ⟶ an SPT phase

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- Condensing m-particles without breaking symmetry, which requires:

  1. Condensed m's carry 1D symmetry irrep: $\chi_m(g)$
  2. $\chi_m(g) \cdot \chi_{m'}(g) = \chi_{mm'}(g)$

- After condensing those m's, we get an SPT phase

$$\omega(g_1, g_2, g_3) \equiv \chi_{\lambda(g_2, g_3)}(g_1), \qquad [\omega] \in H^3(SG, U(1))$$

Possible model realizations?

- Quantum dimer models on non-bipartite lattice can host Z_2 toric code topological order. (Rokhsar,Kivelson,Sondhi,Moessner)

- These models can be mapped to hard-core boson models (or XXZ models), with a U(1) symmetry, and spinons carry half-charge. (Balents, Fisher, Girvin, Isakov, Kim...)

- Tuning parameters, these models (e.g.: kagome 1/3-filled hard-core boson model) can go from the Z_2 spin liquid into Valence Bond Solid phases (VBS) via vison condensation. (Pollmann et.al.)

- One could add interactions breaking the U(1) down to Ising. If the the condensed vison is Ising odd, then, the resulting VBS phase is a nontrivial Ising-SPT phase. (SPT-VBS)

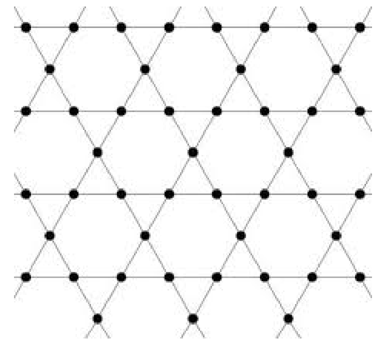
(In tensor-based algorithms, the quantum number carried by low energy vison near condensation can be measured.)

Summary

Input: symmetries of the model

↓ **Running Machinery**

Output: SET/SPT classes and generic wavefunctions for every classes



32 gapped Z_2 spin liquid

Nearly degenerate energy densities for two classes

SPT partially classified by $H^{d+1}[SG, U(1)]$

SG: on-site & spatial symmetries

T and P antiunitary

SET $\xrightarrow{\text{anyon condensation}}$ SPT

Discussion/future directions

- Previously in 2D PEPS, MPO-invariance was used to characterize onsite SPT in PEPS(Williamson et.al.), connection with our formulation?
- Classifying and simulating fermion phases?
- Combining state of the art numerical techniques with this analytical construction (Vanderstraeten, Verstraete, Corboz…)
  - More accurate energy density, correlators, …
  - Excitation spectrum?
- Possible realization of SPT?
  - Numerical simulation for SPT tensor wavefunctions
  - Condensing visons carrying nontrivial quantum number in spin liquid phases → SPT-VBS phase?

Thank You!

Motivations

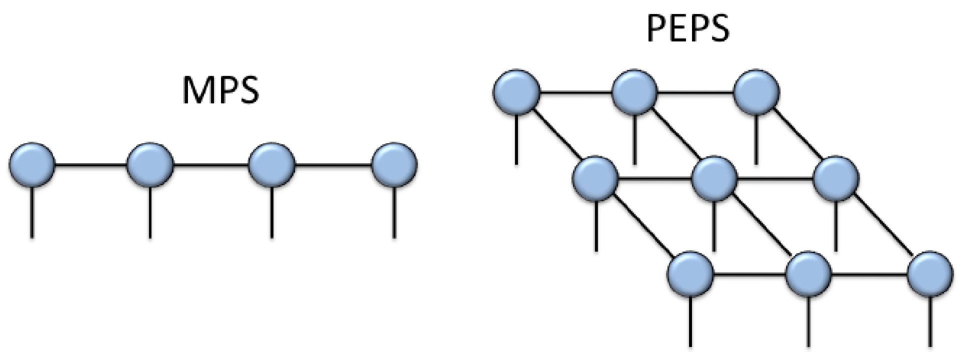
Motivations

s

Motivations

- How to represent generic SPT states using tensor-network wavefunctions (particularly in 2 and higher spatial dimensions) ?

In this talk: tensor-network=MPS(1d), PEPS(2d), and their 3d analog (Cirac, Verstraeta, Vidal…)

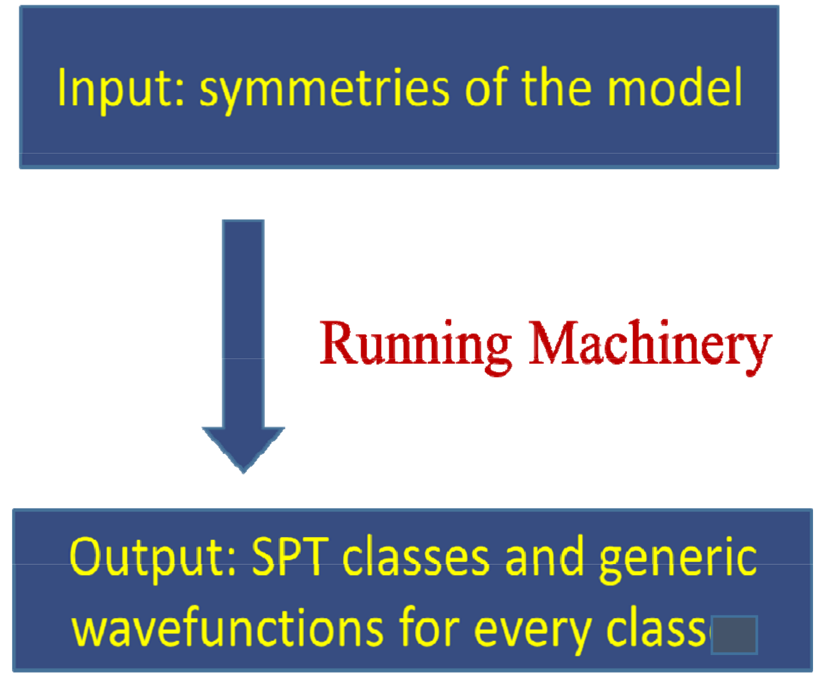


(figures from Orus, Annals Phys. (2014))

But how to systematically understand higher dimensional SPT using tensor-network formulation?

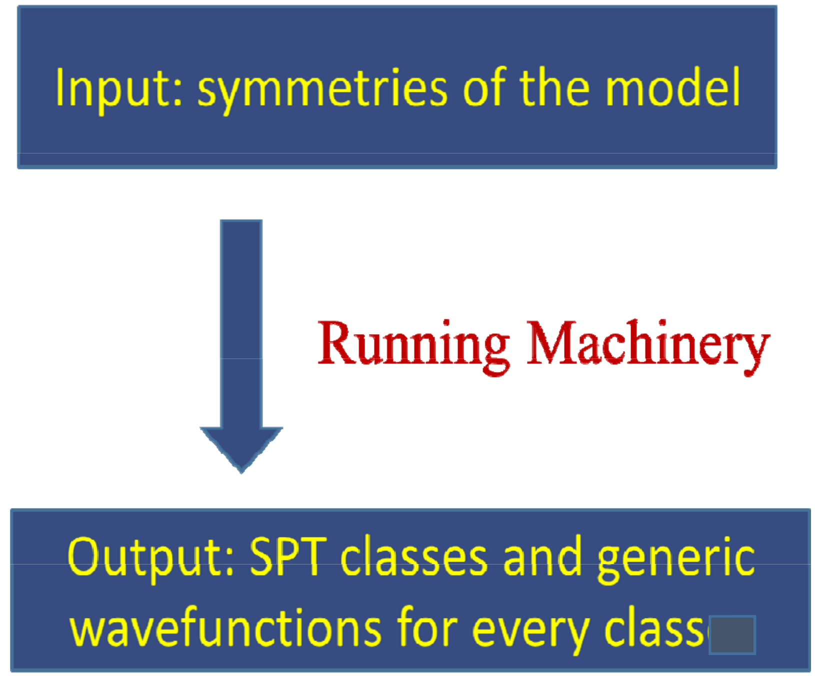
Main results:

- We focus on bosonic cohomological SPT.
- We have identified a general machinery to classify/construct generic SPT states using tensor-network.



(Finite bond-dimension tensor-network construction works for all discrete symmetries, and continuous symmetries in some cases)

Main results:



- We focus on bosonic cohomological SPT.

- We have identified a general machinery to classify/construct generic SPT states using tensor-network.

- In our construction, SPT are classified by $H^{d+1}(SG, U(1))$, where *SG* is the full symmetry group including both onsite and space-group. Both *time-reversal and mirror reflections should be treated as anti-unitary operations.*

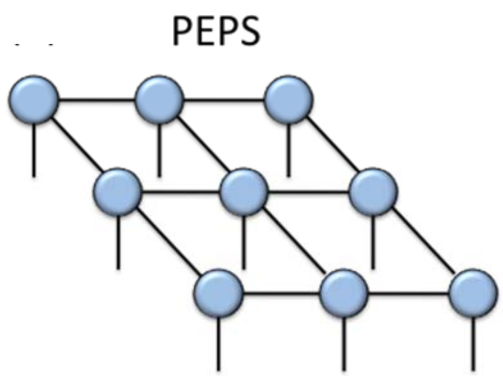
Previously $H^{d+1}(SG, U(1))$ classification is obtained for onsite *SG* (Chen,Liu,Gu,Wen…)

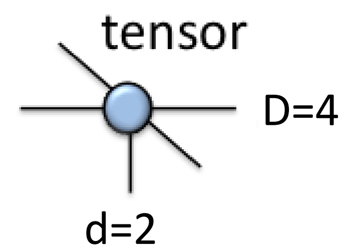
many new phases can be constructed:

e.g. in 2d, inversion symmetry (180° spatial rotation) → Z_2 classification

A simple example

- PEPS representation of an Ising system on a square lattice with bond-dimension D=4:





In the absence of symmetry:
each local tensor lives in a local Hilbert space whose dimension=$d \cdot D^4$=512

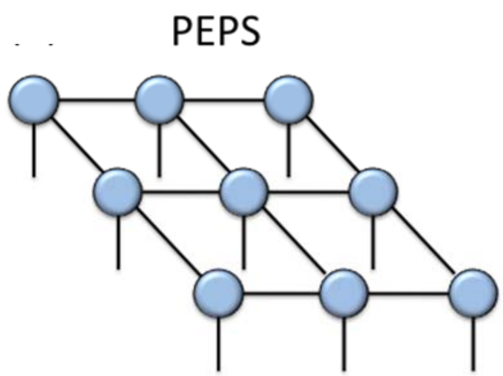
a trial wavefunction with a lot of variational parameters:

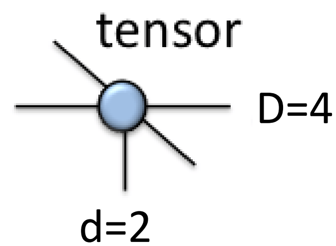
# of parameters=512-1(normalization)=511

The price to pay is that one may not be able to sharply distinguish different quantum phases.

A simple example

- PEPS representation of an Ising system on a square lattice with bond-dimension D=4:





In the absence of symmetry:
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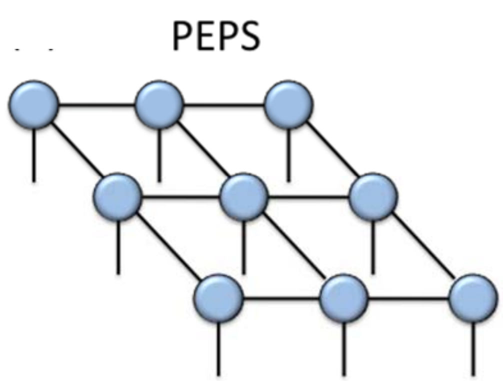
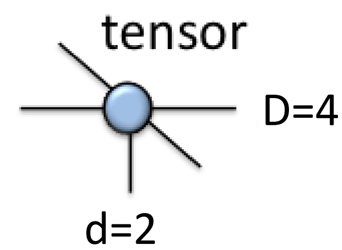
- For onsite Ising symmetry: $H^3\big(SG = Z_2, U(1)\big) = Z_2$. There is one nontrivial SPT. (Chen,Liu,Gu,Wen,Levin...)

A simple example

- PEPS representation of an Ising system on a square lattice with bond-dimension D=4:





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- For onsite Ising symmetry: $H^3\left(SG = Z_2, U(1)\right) = Z_2$. There is one nontrivial SPT.
- To represent this SPT by PEPS with D=4, it turns out each local tensor lives in a sub-Hilbert space whose dimension=16.

--- a generic SPT trial wavefunction with 15 variational parameters

Plan:

Instead of keep going on tensor-network wavefunctions,

let me talk about a *by-product* of our main results, which is also quite general and can be formulated in more conventional languages.

A by-product of main results

- a general connection between SET (topological ordered states with symmetry) and SPT phases via anyon condensation.

Basic idea:

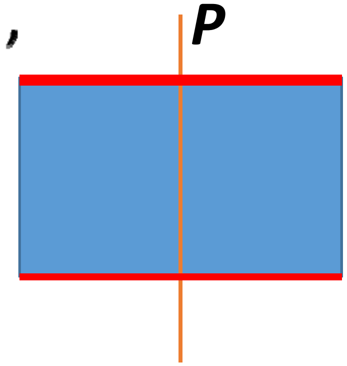
Starting from a rather conventional gauge theory (i.e., an SET phase like a Z_2 spin liquid), if one condenses certain bosonic anyons to confine the gauge field, under "certain conditions", the resulting phase is necessarily an SPT phase.

We will soon provide a general criterion about the "certain conditions". But let me show you some examples first.

Examples: vison condensation in a Z_2 spin liquid

- Let us consider a 2d spin system with $SG = Z_2^P \times Z_2^T$, where $Z_2^P = \{I, P\}$, $Z_2^T = \{I, T\}$.

  P is a mirror reflection, T is the time-reversal.



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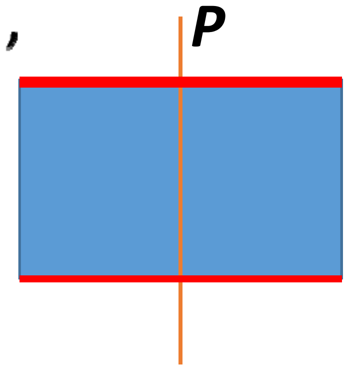

Starting from a Z_2 spin liquid phase-A, in which :

the bosonic spinon e has nontrivial symmetry fractionalization:

$$T(e)^2 = -1, P(e)^2 = +1, T \cdot P(e) = P \cdot T(e)$$

But the bosonic vison m has no symmetry fractionalization. (m still can carry usual quantum numbers.)

Examples: vison condensation in a Z_2 spin liquid

- Let us consider a 2d spin system with $SG = Z_2^P \times Z_2^T$, where $Z_2^P = \{I, P\}$, $Z_2^T = \{I, T\}$.

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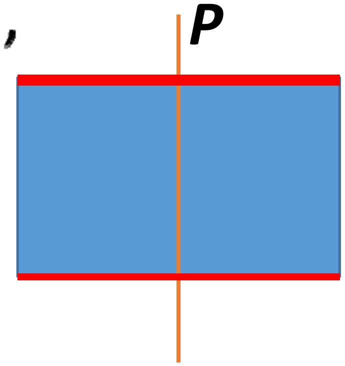

Starting from a Z_2 spin liquid phase-A, in which :

the bosonic spinon e has nontrivial symmetry fractionalization:

$$T(e)^2 = -1, P(e)^2 = +1, T \cdot P(e) = P \cdot T(e)$$

But the bosonic vison m has no symmetry fractionalization. (m still can carry usual quantum numbers.)

This spin liquid is rather usual, there is no stable gapless edge states.

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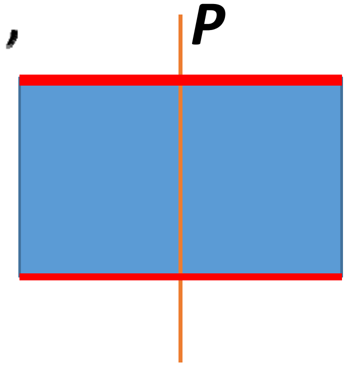

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But the bosonic vison m has no symmetry fractionalization.

Claim:

If condensing the $T \cdot P$ odd vison m to confine the Z_2 gauge,

then the resulting phase is an SPT phase (a bosonic topological crystalline insulator) with gapless edge states on the P symmetric edges. --- let me call it as SPT phase-A

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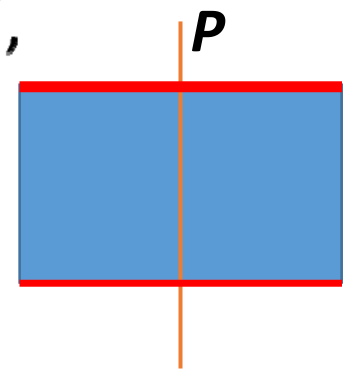

Summary:

Z2 Spin liquid	Spinon *e* nontrivial symmetry fractionalization
Spin-liquid-A	$T(e)^2 = -1$

Condensing

$T \cdot P$ odd vison m

Resulting Phase
SPT-A

visons are assumed to have trivial symm. fractionalization,
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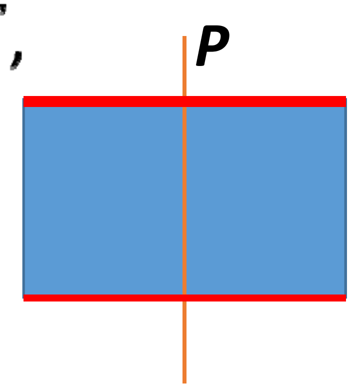

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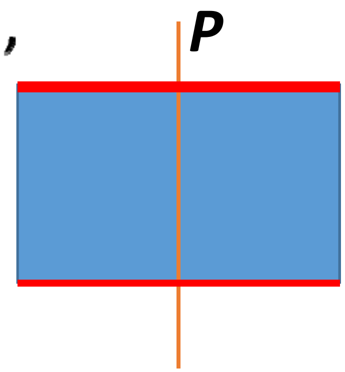

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SPT-A
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$$H^3(SG, U(1)) = Z_2^2$$

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And we know how to write down generic tensor-network wavefunc for each of them

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visons are assumed to have trivial symm. fractionalization, but could carry usual quantum numbers.

Possible realizations?

- In spin-1/2 systems, $T(e)^2 = -1$ in Z_2 spin liquids basically comes for free.

- Condensing vison ➔ VBS (valence bond solids)

  Breaking translational symmetry, a symmetry that we care but do not really care.

SPT-VBS phases?

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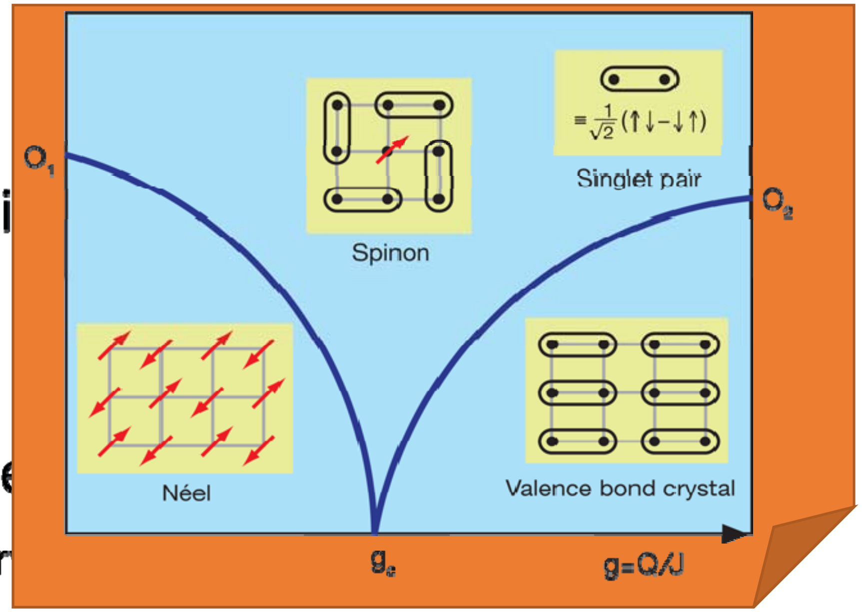

(Figure from Singh, Physics, 2010)

SPT-VBS phases?

One may say that it is still highly nontrivial to have Z_2 spin liquids to begin with, but what we really care is the confined VBS phase.

VBS are quite common in spin models, e.g. deconfined criticality.

(Senthil, Balents, Sachdev, Vishwanath, Fisher, Sandvik…)

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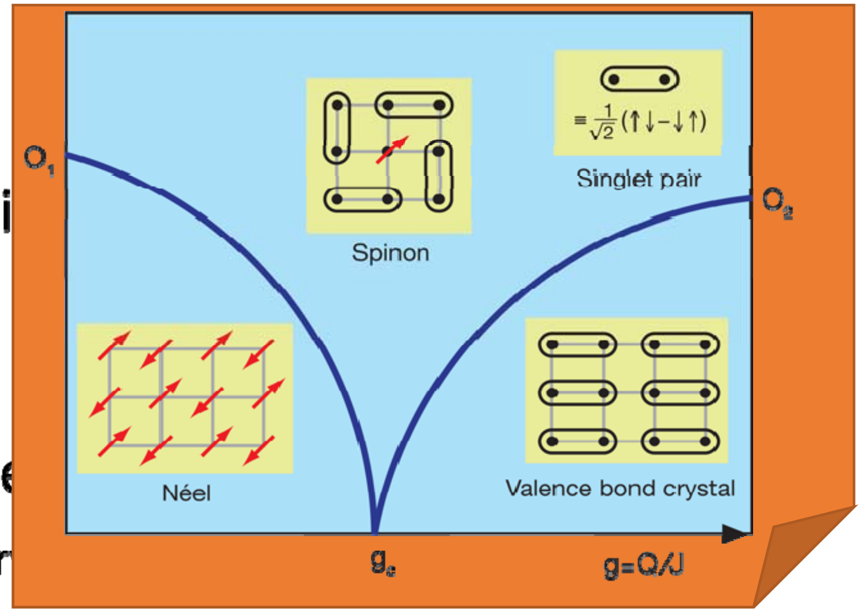

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SPT-VBS phases?

Consider the easy-plane case of deconfined-criticality.

The question is: what is the quantum number carried by the condensed vortices?

This has to be determined by numerical simulations of models.

(e.g., in the J-Q model, Sandvik...)

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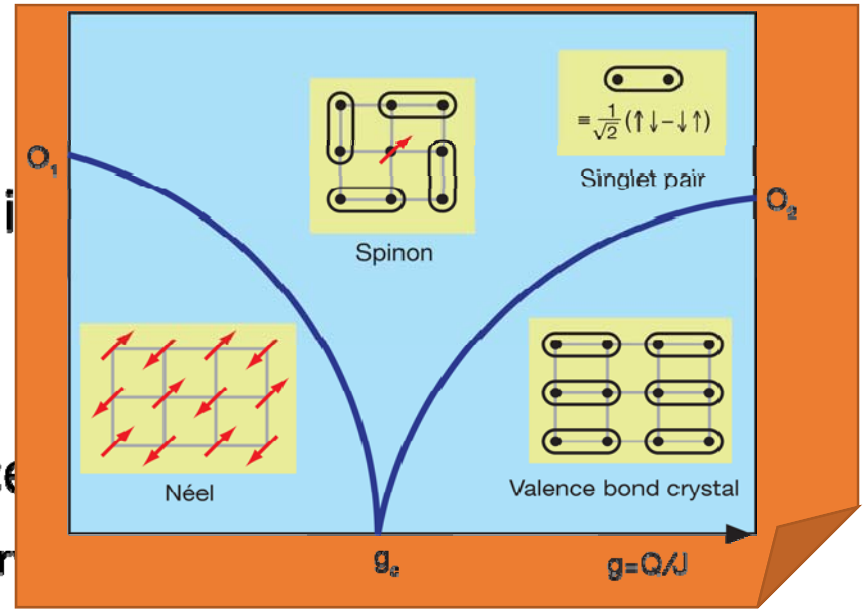

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SPT-VBS phases?

Consider the easy-plane case of deconfined-criticality.

The question is: what is the quantum number carried by the condensed vortices?

It is in fact nontrivial to numerically measure quantum numbers of a single vison or a vortex.

If I am allowed to use tensor-network wavefunctions, I have an algorithm to do the job.

A general Criterion to obtain SPT from SET in 2+1d

(1) Consider an SET phase described by "usual" discrete Abelian gauge theory (e.g. $Z_N, Z_N \times Z_M$) in the presence of symmetry group *SG*.

We have a bunch of bosonic gauge charges (*e*-particles), and gauge fluxes (*m*-particles).

"usual" means: e.g., for Z_2, only toric-code-like but not double-semion-like.

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$$\Omega_{g_1} \cdot \Omega_{g_2} = \lambda(g_1, g_2)\Omega_{g_1 g_2}, \quad \lambda(g_1, g_2) \text{ is an } m\text{-particle}$$

Ω_g is symmetry action on anyons. (M. Barkeshli, P. Bonderson, M. Cheng, Z. Wang, L. Fidkowski, N. H. Lindner, A. Kitaev, X. Chen, F. J. Burnell, A. Vishwanath, L. Fidkowski, G. Y. Cho, J. C. Y. Teo, and S. Ryu ....)

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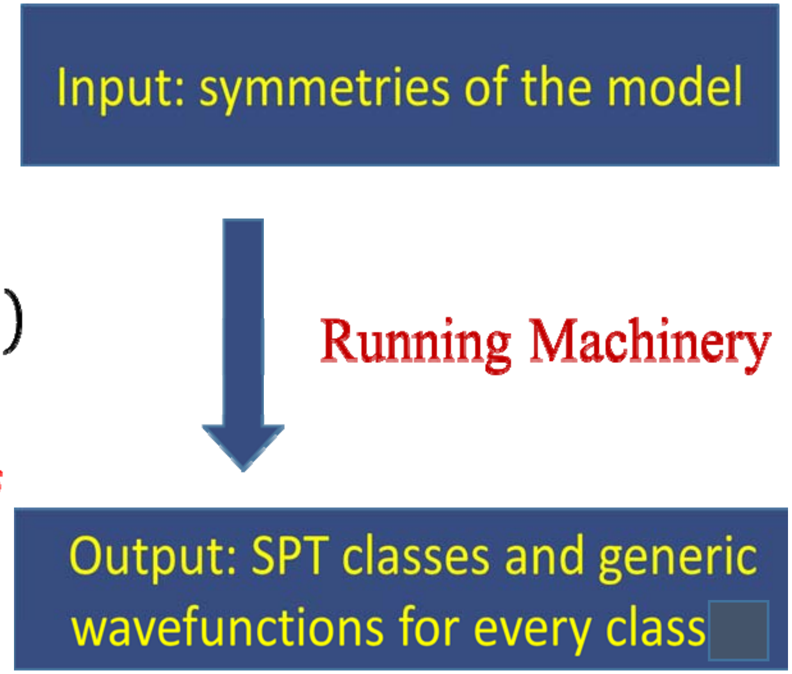
(4) The resulting phase is an SPT phase characterized by the 3-cocycle:

$$\omega(g_1, g_2, g_3) \equiv \chi_{\lambda(g_2, g_3)}(g_1) \in U(1)$$

Summary

- **Classification/construction of bosonic cohomological SPT using tensor: $H^{d+1}(SG, U(1))$**

SG **is the full symmetry group including both onsite and space-group.** ***Time-reversal and mirror reflections should be treated as anti-unitary operations.***



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↓ Running Machinery

Output: SPT classes and generic wavefunctions for every class

- A by-product: A general Criterion to obtain SPT from SET via anyon condensation:

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gauge-flux quantum number: $\chi_m(g)$

⇒ Condensing gauge-fluxes while preserving symmetry

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Thank you!

By-products of main results

- Lattice translational symmetry ($Z^d \subset SG$) leads to “weak indices”.

e.g.: recently we showed that there are 4 featureless Mott insulators at half-filling on the honeycomb lattice. Now we understand it is due to two Z_2 weak indices in $H^{d+1}(SG, U(1))$.