A fractional Insulator protected by topological order

Martin Greiter

material contained in:

V. Schnells, R. Thomale, MG: Fractional Insulator protected by topological order, manuscript in preparation.

MG, V. Schnells, R. Thomale: Method to identify parent Hamiltonians for trial states, PRB 98, 081113 (2018).

Introduction

- usual view: Chern insulator = integer QHE without a magnetic field.

  alternatively: Chern insulator = integer QHE on a commensurate lattice

  In particular, the Chern insulator proposed by Haldane is a lattice model with a magnetic field of one Dirac quanta per hexagon.

  → new (fractional) phases can be identified either on the lattice as Chern insulators or in the continuum as quantum Hall states.

- Topological insulator (2D TI) = IQHE + $\overline{\text{IQHE}}$, protected by T

- Fractional topological insulator (FTI) = FQHE + $\overline{\text{FQHE}}$, protected by T

  No real innovation, since the topological order (the fractionalization) and the symmetry protection do not entangle!

- A state which respects T and entangles the "layers" has never been found.

Fractional insulator: ground state wave function

$$\psi_{m_\uparrow,m_\downarrow,n}[z;\bar{w}] = \prod_{i=1}^{N_\uparrow} e^{-\frac{1}{4}|z_i|^2} \prod_{k=1}^{N_\downarrow} e^{-\frac{1}{4}|\bar{w}_k|^2}$$

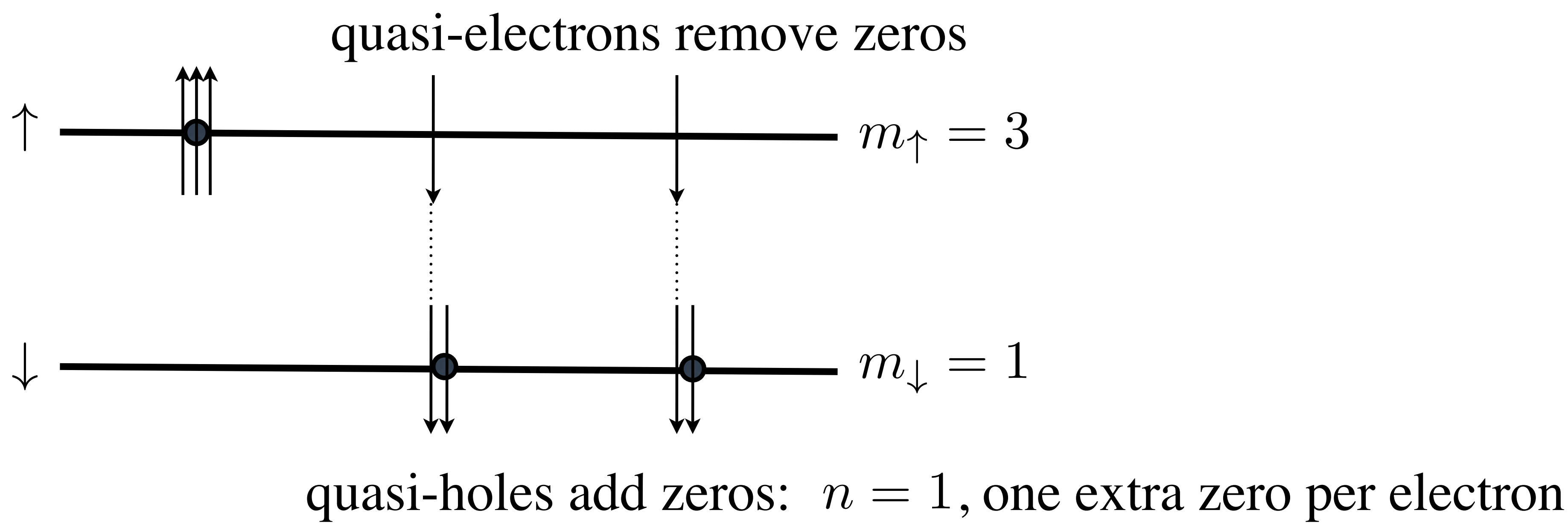
$$\cdot \prod_{k<l}^{N_\downarrow} (\bar{w}_k - \bar{w}_l)^{m_\downarrow} \prod_{k=1}^{N_\downarrow} \prod_{i=1}^{N_\uparrow} (\bar{w}_k - 2\partial_{z_i})^n \prod_{i<j}^{N_\uparrow} (z_i - z_j)^{m_\uparrow}$$

filling factors for each spin: $\nu_\uparrow = \dfrac{1}{m_\uparrow - n}, \quad \nu_\downarrow = \dfrac{1}{m_\downarrow + n}$

$(m_\uparrow, m_\downarrow, n) = (3,1,1): \quad \nu_\uparrow = \nu_\downarrow = \frac{1}{2}$

The 311-state has topological order which breaks time reversal symmetry T.
It is also an SPT (symmetry protected) phase, with T as protecting symmetry!

Illustration of the 331 state:



Finally, the # of zeros in each "layer" is 2 per electron, $\nu_\uparrow = \nu_\downarrow = \frac{1}{2}$.

Chiral edge modes are T conjugates of each other, i.e. protected by T!

Effective field theory of the 331 state:

$$\mathcal{L} = -\frac{1}{4\pi}\varepsilon^{\mu\nu\rho}\boldsymbol{a}_\mu^\top K \partial_\nu \boldsymbol{a}_\rho - \frac{1}{2\pi}\varepsilon^{\mu\nu\rho}(\boldsymbol{a}_\mu^\top \boldsymbol{q})\partial_\nu A_\rho + (\boldsymbol{a}_\mu^\top \boldsymbol{l}) J^\mu_{\mathrm{QP}}$$

specified by $K = \begin{pmatrix} 3 & -1 \\ -1 & -1 \end{pmatrix}, \quad \boldsymbol{q} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \boldsymbol{l} = \begin{pmatrix} l_\uparrow \\ l_\downarrow \end{pmatrix}.$

calculate $\sigma_{\mathrm{xy}} = \frac{e^2}{h}\boldsymbol{q}^\top K^{-1}\boldsymbol{q}, \quad Q_{\mathrm{QP}} = \boldsymbol{q}^\top K^{-1}\boldsymbol{l}, \quad \theta = \pi \boldsymbol{l}^\top K^{-1}\boldsymbol{l}:$

$\sigma_{\mathrm{xy}} = 0, \quad Q_\uparrow = Q_\downarrow = \frac{1}{2}, \quad \theta_\uparrow = \frac{\pi}{4}$ while $\theta_\downarrow = -\frac{3\pi}{4}$.

The T violation is only manifest in non-local quantities, like the QP statistics.

Therefore, the edge states are still topologically protected by T.

General method to find a parent Hamiltonian

Ask whether $|\psi_0\rangle$ is the ground state of Hamiltonian with L terms, $H = \sum_{i=1}^{L} a_i H_i$ and determine the coefficients a_i .

Must be an eigenstate, write $(H + a_0)\,|\psi_0\rangle = 0$ where $a_0 = -E_0$

$$\text{with } H_0 \equiv 1: \quad 0 = \sum_{i=0}^{L} a_i H_i\,|\psi_0\rangle = \sum_{i=0}^{L} a_i\,|\psi^i\rangle = \sum_{i=0}^{L} a_i\,\langle\psi^j|\psi^i\rangle \;\; \forall j$$

We get one solution for each zero eigenvalue of $M_{ji} = \langle\psi^j|\psi^i\rangle$.

Usually one zero eigenvalue corresponds to a conserved quantities, e.g. total angular momentum for a quantum Hall state on the sphere. If there is an exact parent Hamiltonian involving only the terms H_i , however, it will correspond to another

If there is no exact parent Hamiltonian, there is often one eigenvalue which is much smaller than all the others, maybe by 1000 times. The associated eigenvector then usually corresponds to a highly optimized, approximate parent Hamiltonian.

Spinor coordinates and the 311 state on the sphere

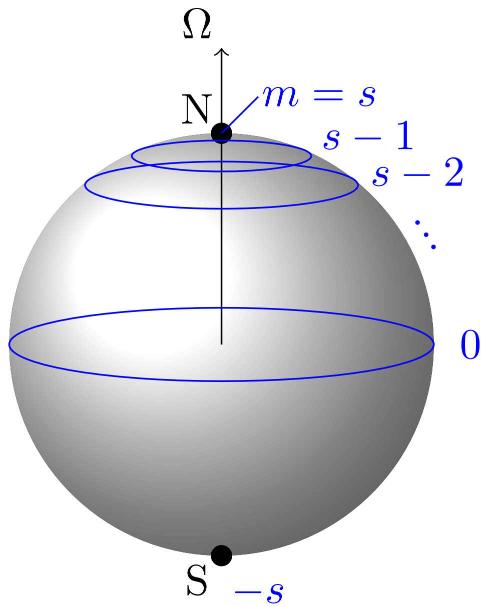
↑-layer

$$u = \cos\left(\frac{\theta}{2}\right) e^{\mathrm{i}\frac{\varphi}{2}},$$

$$v = \sin\left(\frac{\theta}{2}\right) e^{-\mathrm{i}\frac{\varphi}{2}}$$

$$\mathbf{L} = \frac{1}{2}(u,v)\boldsymbol{\sigma}\begin{pmatrix}\partial_u \\ \partial_v\end{pmatrix}$$

$$\psi^{s}_{m,0}(u,v) = u^{s+m}v^{s-m}$$



↓-layer

$$\bar{a} = \cos\left(\frac{\theta}{2}\right) e^{-\mathrm{i}\frac{\varphi}{2}}$$

$$\bar{b} = \sin\left(\frac{\theta}{2}\right) e^{\mathrm{i}\frac{\varphi}{2}}$$

$$\mathbf{L}_{\downarrow} = -\frac{1}{2}(\bar{a},\bar{b})\boldsymbol{\sigma}^{\top}\begin{pmatrix}\partial_{\bar{a}} \\ \partial_{\bar{b}}\end{pmatrix}$$

$$\psi^{s'}_{m',0}(\bar{a},\bar{b}) = \bar{a}^{s'-m'}\bar{b}^{s'+m'}$$

$$\Psi_{m_\uparrow,m_\downarrow,n}[u,v,\bar{a},\bar{b}] =$$

$$\frac{1}{\mathcal{N}}\prod_{k<l}^{N_\downarrow}(\bar{a}_k\bar{b}_l - \bar{a}_l\bar{b}_k)^{m_\downarrow}\prod_{k}^{N_\downarrow}\prod_{i}^{N_\uparrow}\left(\bar{b}_k\frac{\partial}{\partial u_i} - \bar{a}_k\frac{\partial}{\partial v_i}\right)^n\prod_{i<j}^{N_\uparrow}(u_iv_j - u_jv_i)^{m_\uparrow}$$

Landau level pseudopotentials on the sphere

two particle states with relative angular momentum $l = 2s - j$ in the LLL:

$$\{\mathbf{\Omega}(\alpha,\beta)\cdot(\mathbf{L}_1+\mathbf{L}_2)\}\psi^{s,j}_{(\alpha,\beta),0}[u,v] = j\psi^{s,j}_{(\alpha,\beta),0}[u,v]$$

$$\mathbf{\Omega}(\alpha,\beta) = (\alpha,\beta)\boldsymbol{\sigma}\begin{pmatrix}\bar\alpha\\ \bar\beta\end{pmatrix}$$

$$(\mathbf{L}_1+\mathbf{L}_2)^2\psi^{s,j}_{(\alpha,\beta),0}[u,v] = j(j+1)\psi^{s,j}_{(\alpha,\beta),0}[u,v]$$

solution: $$\psi^{s,j}_{(\alpha,\beta),0}[u,v] = (u_1v_2 - u_2v_1)^{2s-j}\prod_{i=1,2}(\bar\alpha u_i + \bar\beta v_i)^j$$

Pseudopotentials specify interactions projected onto relative angular momentum states:

$$V^{2s}_l = \frac{\langle\psi^{s,j}_{(\alpha,\beta),0}|\,V(\mathbf{\Omega}(u_1,v_1)\cdot\mathbf{\Omega}(u_2,v_2))\,|\psi^{s,j}_{(\alpha,\beta),0}\rangle}{\langle\psi^{s,j}_{(\alpha,\beta),0}|\psi^{s,j}_{(\alpha,\beta),0}\rangle}$$

Interactions in terms of two-body pseudopotentials

two body interaction in a basis of L_z eigenstates:

$$H = \sum_{m_1=-s}^{s} \sum_{m_2=-s}^{s} \sum_{m_3=-s}^{s} \sum_{m_4=-s}^{s} c^\dagger_{m_1} c^\dagger_{m_2} c_{m_4} c_{m_3} \quad \delta_{m_1+m_2,m_3+m_4}$$

$$\cdot \sum_{l=0}^{2s} \langle s, m_1; s, m_2 | 2s-l, m_1+m_2 \rangle \, V_l^{2s} \, \langle 2s-l, m_3+m_4 | s, m_3; s, m_4 \rangle$$

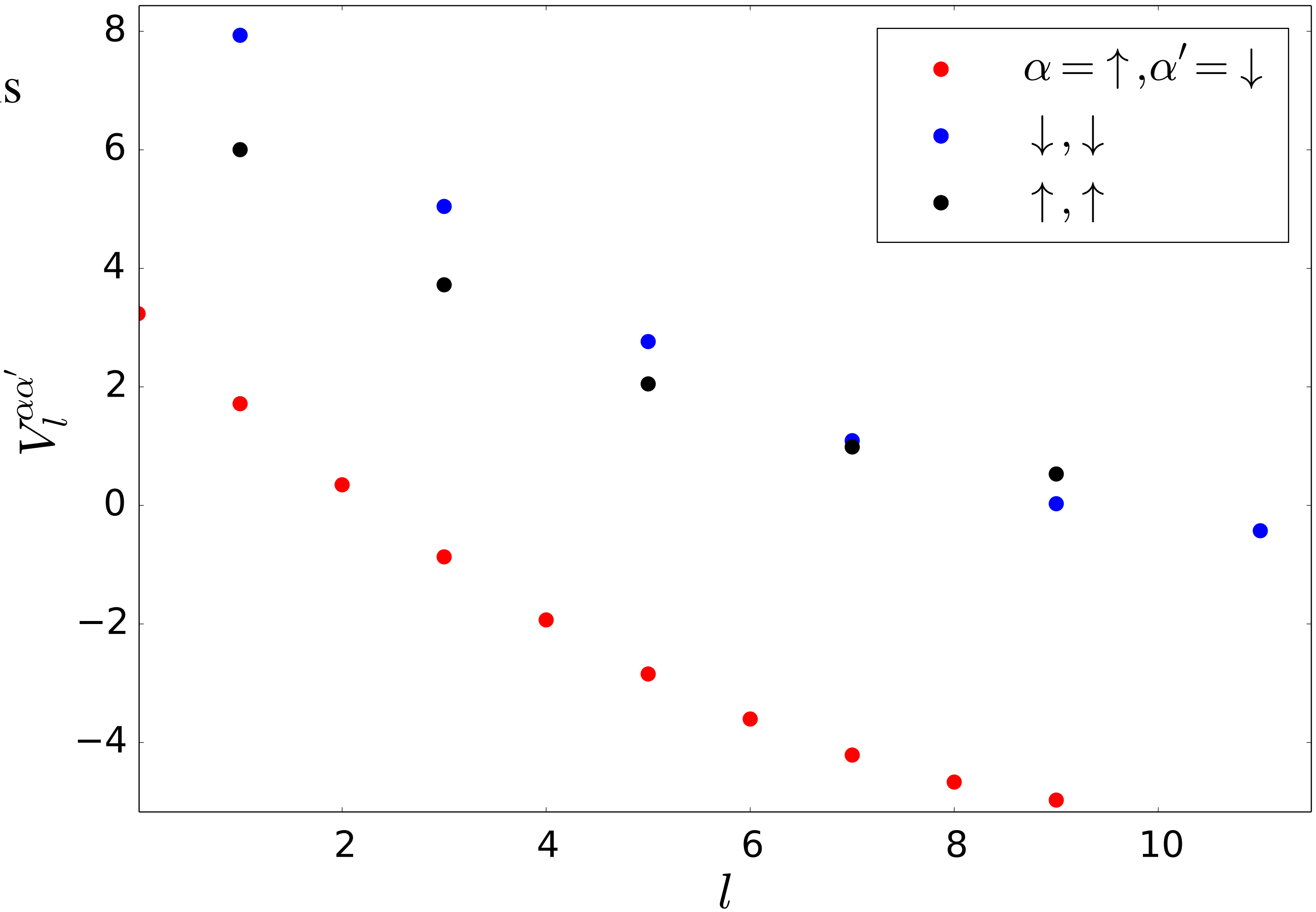
$$=: \sum_{l}^{2s} V_l^{2s} H^{(l)}$$

for the 311 state with $\uparrow$ and $\downarrow$ spin layers:

$$H = \sum_{l=1}^{2s'-1}{}' V_l^{2s'} H^{(\downarrow\downarrow,l)} + \sum_{l=1}^{2s-1}{}' V_l^{2s} H^{(\uparrow\uparrow,l)} + \sum_{l=0}^{2s} V_l^{s+s'} H^{(\uparrow\downarrow,l)}$$

An approximate parent Hamiltonian for 311 state on the sphere

pseudopotentials



$N_\uparrow = N_\downarrow = 6$

$\uparrow$ -sphere: $2s = 9$

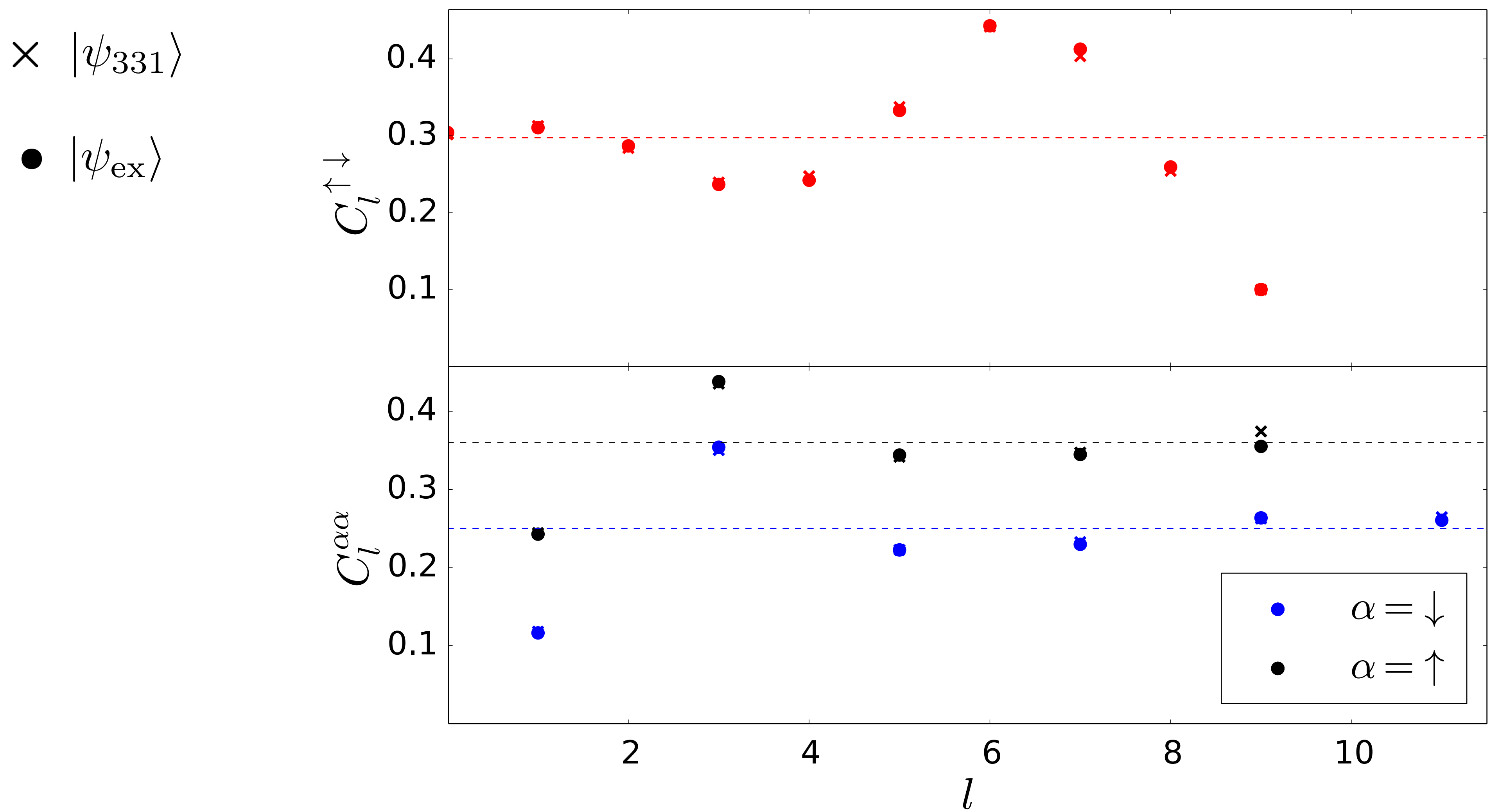
$\downarrow$ -sphere: $2s = 11$

$\langle \psi_{\text{ex}} | \psi_{311} \rangle = 0.82$

$\dim \mathcal{H} \approx 10000$ in the $S^z_{\text{tot}} = 0$ subspace

relative angular momenta

Relative angular momentum resolved correlators

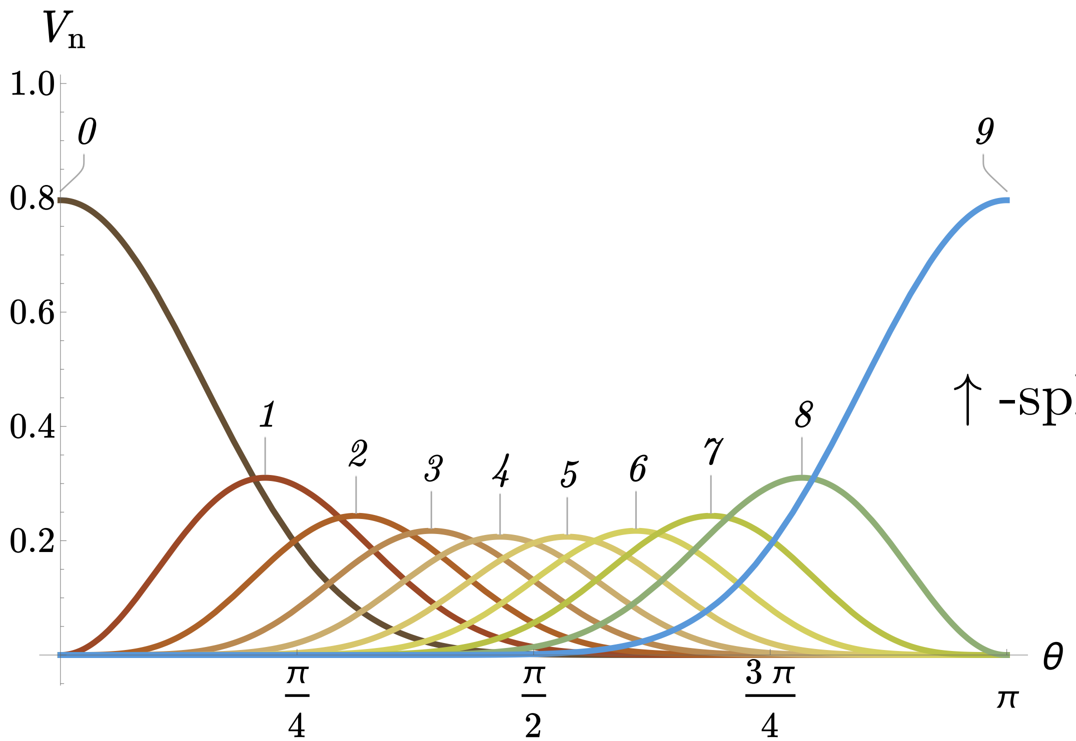


relative angular momenta

Real space basis for the pseudopotentials

$$V_n(\theta) = \mathcal{N}_n \left(\cos\frac{\theta}{2}\right)^{2(2s-n)} \left(\sin\frac{\theta}{2}\right)^{2n}, \quad n = 0, 1, \ldots, 2s$$

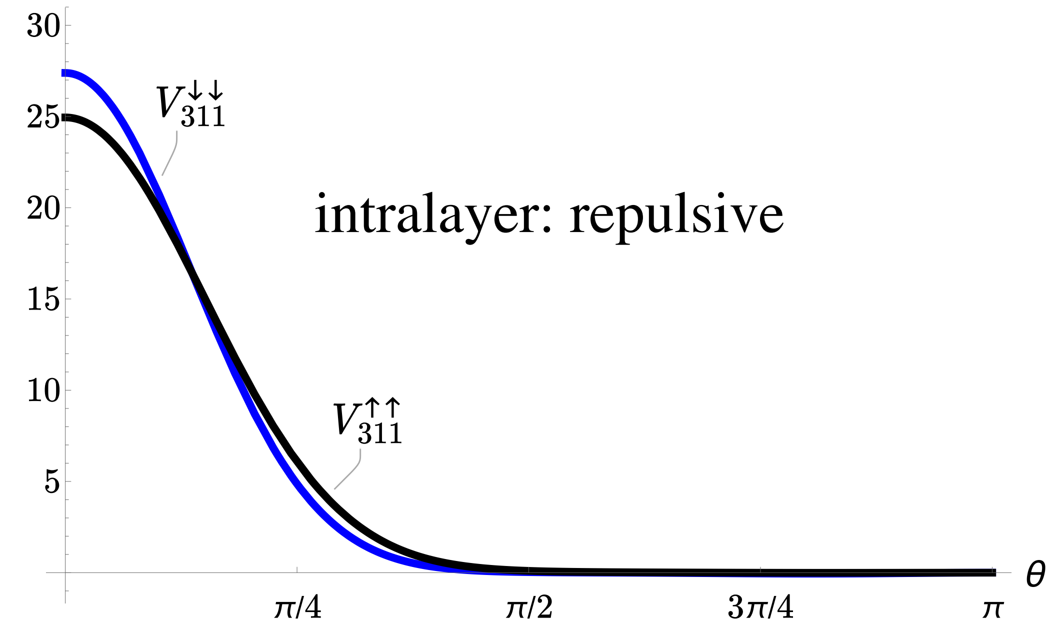
$$\mathcal{N}_n = \int_0^\pi d\theta \; 2\pi \sin\theta \; V_n(\theta)$$



$\uparrow$ -sphere: $2s = 9$

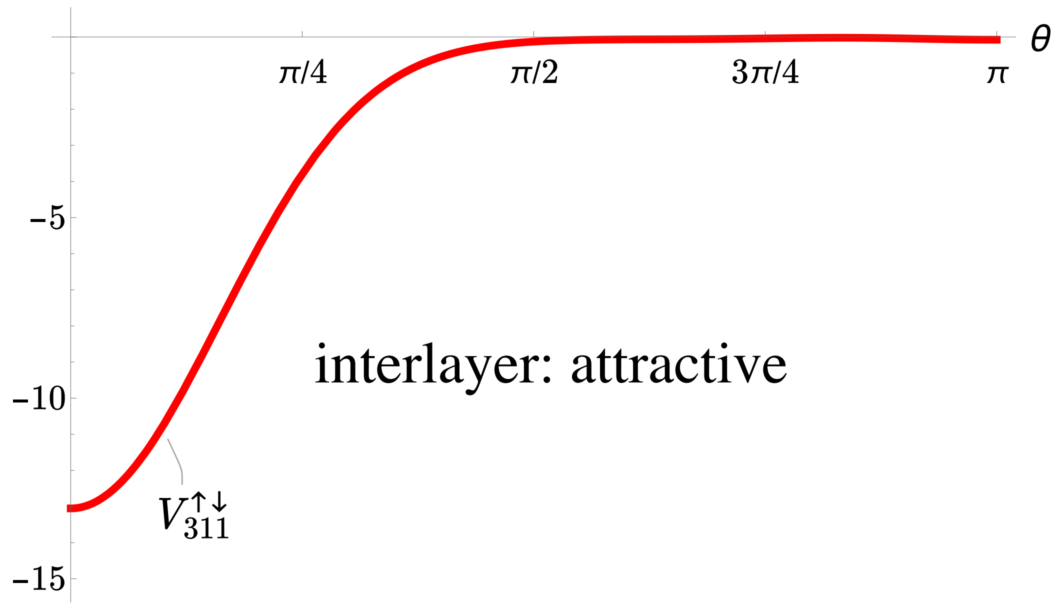
since there are as many basis potentials as there are pseudopotentials $V_l^{\alpha\alpha'}$, we can uniquely convert our pseudopotentials into a the real space potential spanned by these basis potentials.

The parent Hamiltonian in real space



Note that for the edge states to be protected by T the Hamiltonian must be T-invariant, i.e. $V_{311}^{\uparrow\uparrow} = V_{311}^{\downarrow\downarrow}$

For a finite system, however, we break T as $2s_\uparrow = 2s_\downarrow - 2$



Conclusion

Fractional Insulator protected by topological order

The 311-state has topological order which breaks time reversal symmetry T.
It is also an SPT (symmetry protected) phase, with T as protecting symmetry!

It can be stabilized with local potentials common in quantum Hall systems:
repulsive in the layers and attractive between them