Understanding coherent structure emergence in homogeneously forced turbulence by means of the statistical state dynamics

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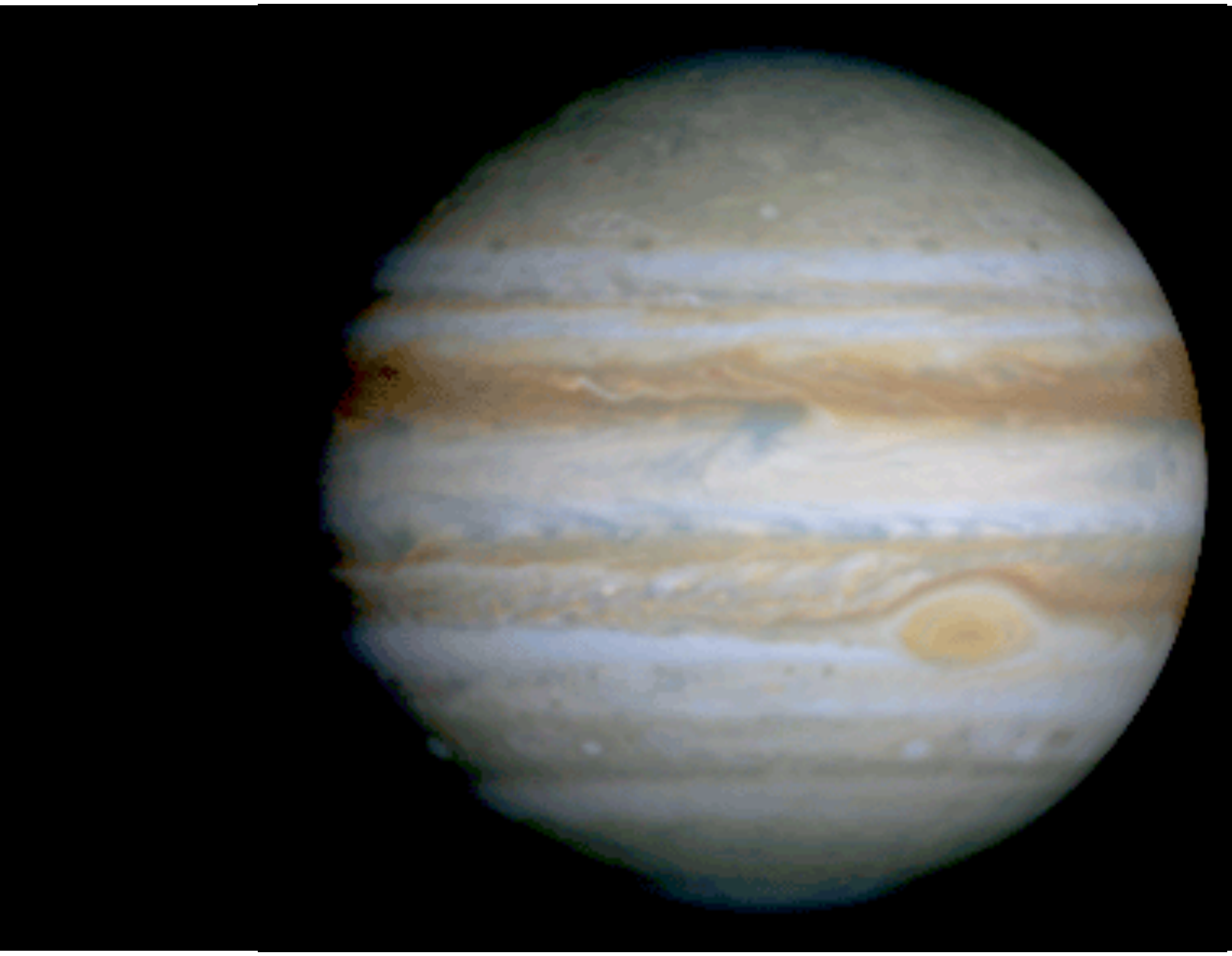
KITP
11 Jan. 2017

Planetary turbulence

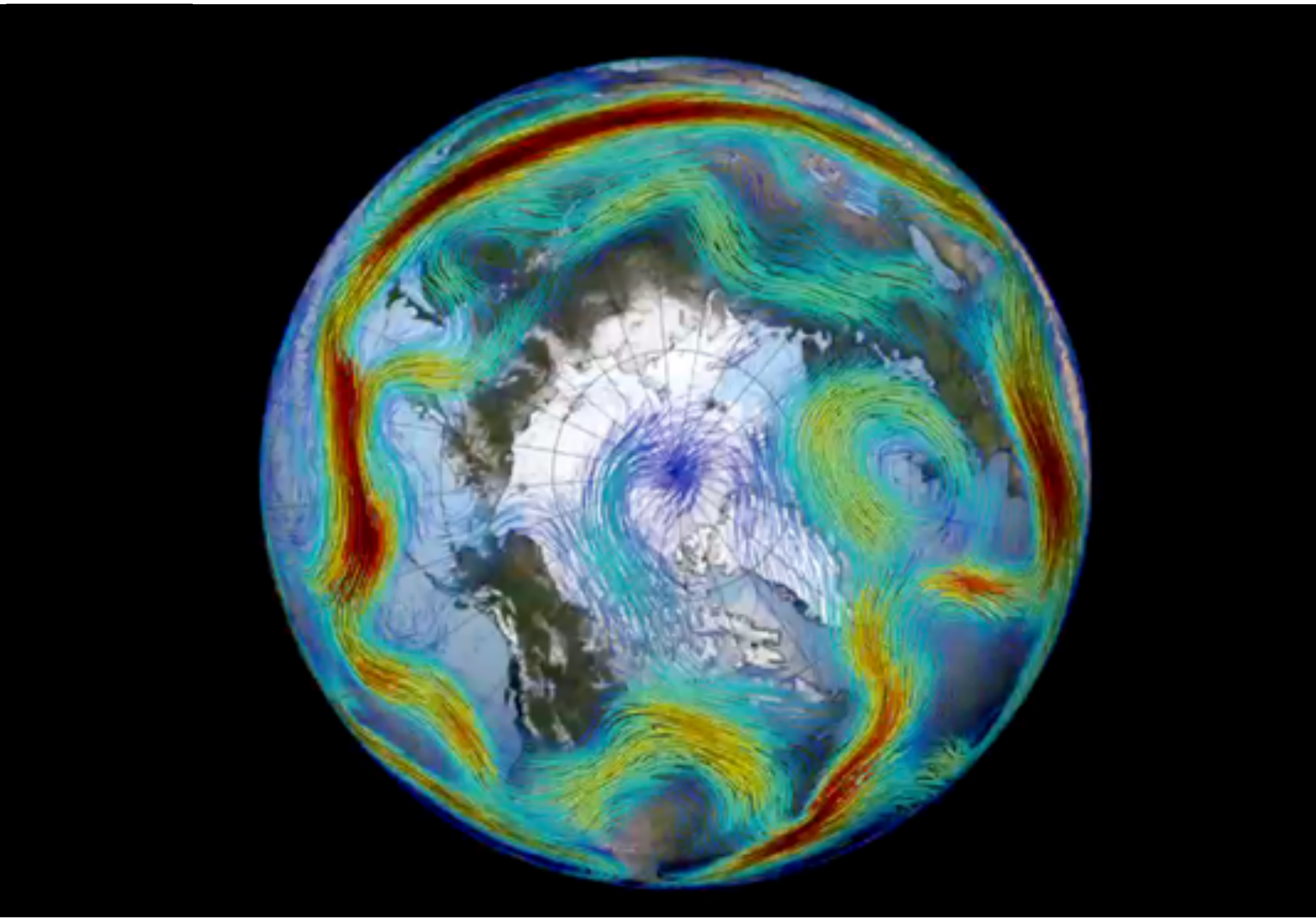
most of the energy of the flow is in large-scale coherent jets and vortices of specific form

not at the largest allowed scale (as inverse cascade might imply)
arrest of the cascade by jets



banded Jovian jets

NASA/Cassini Jupiter Images



polar front jet

NASA/Goddard Space Flight Center

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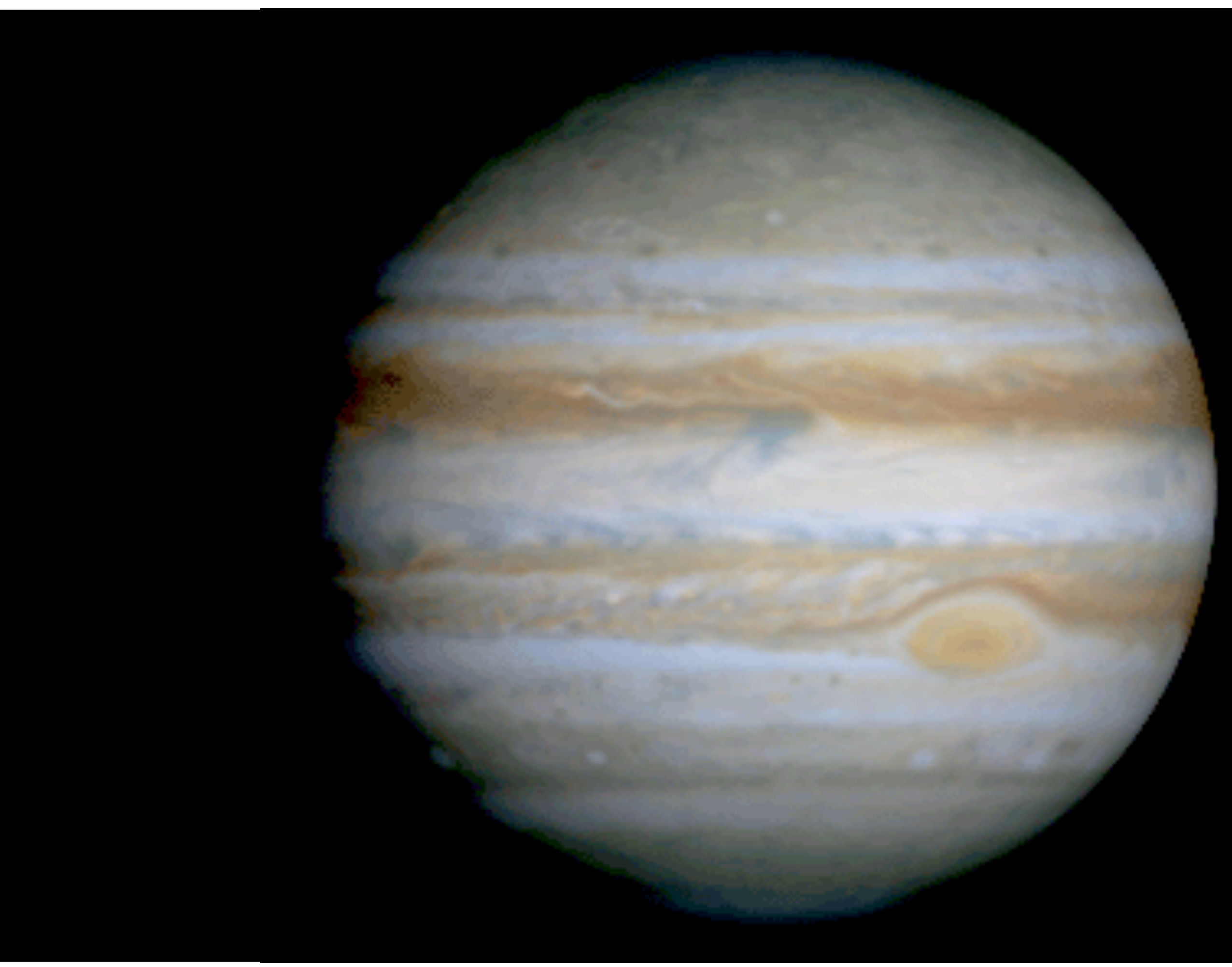

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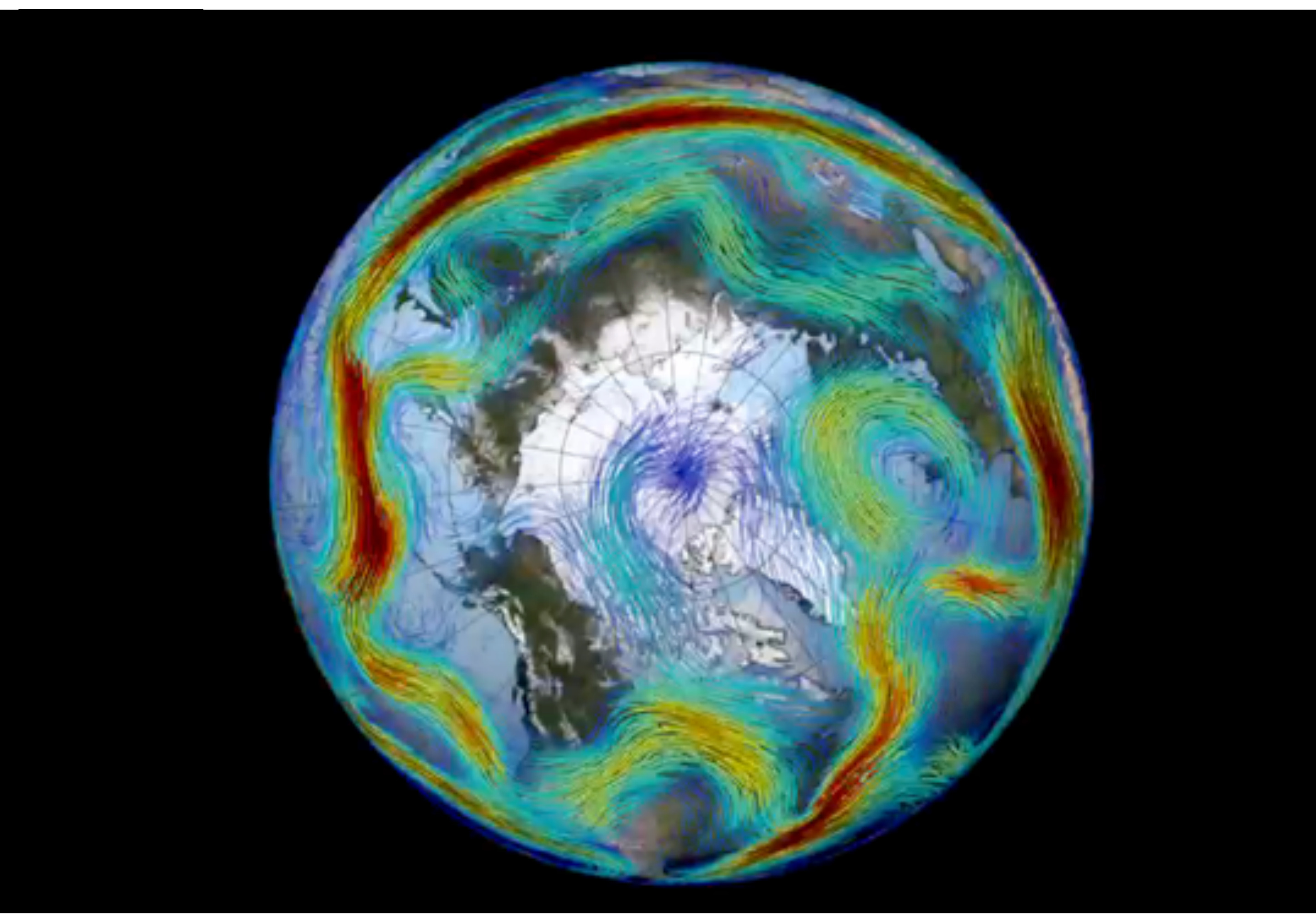

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Boundary layer turbulence

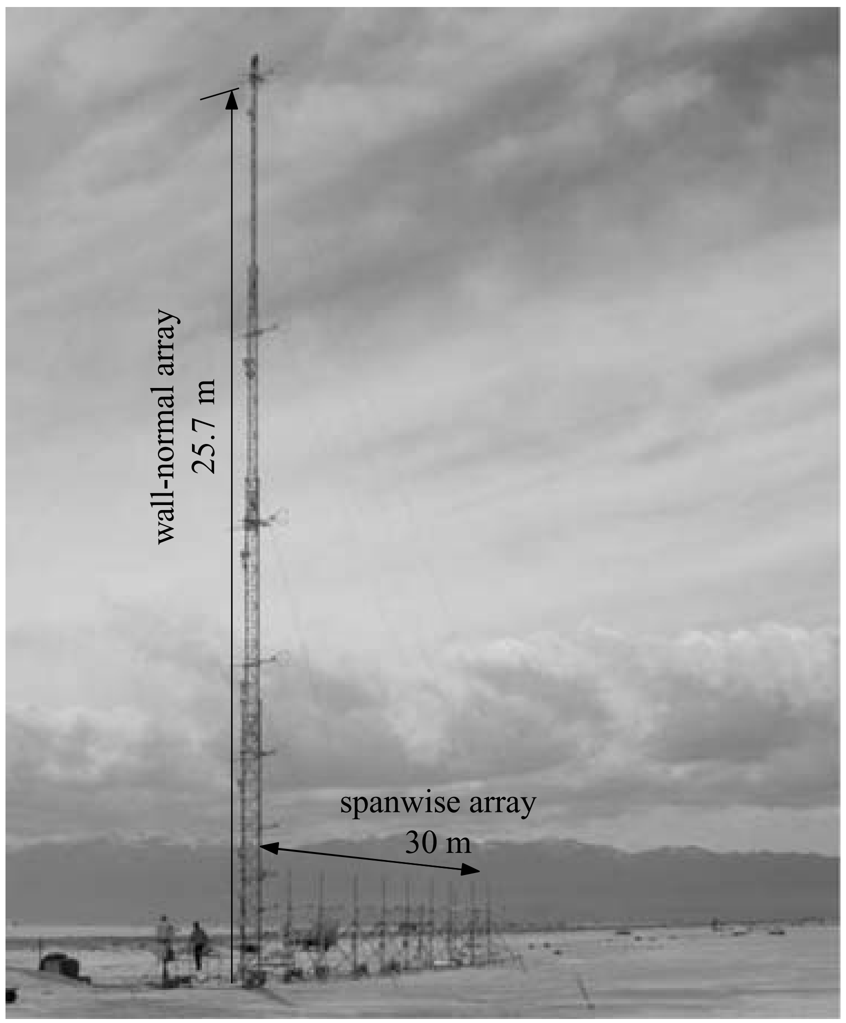
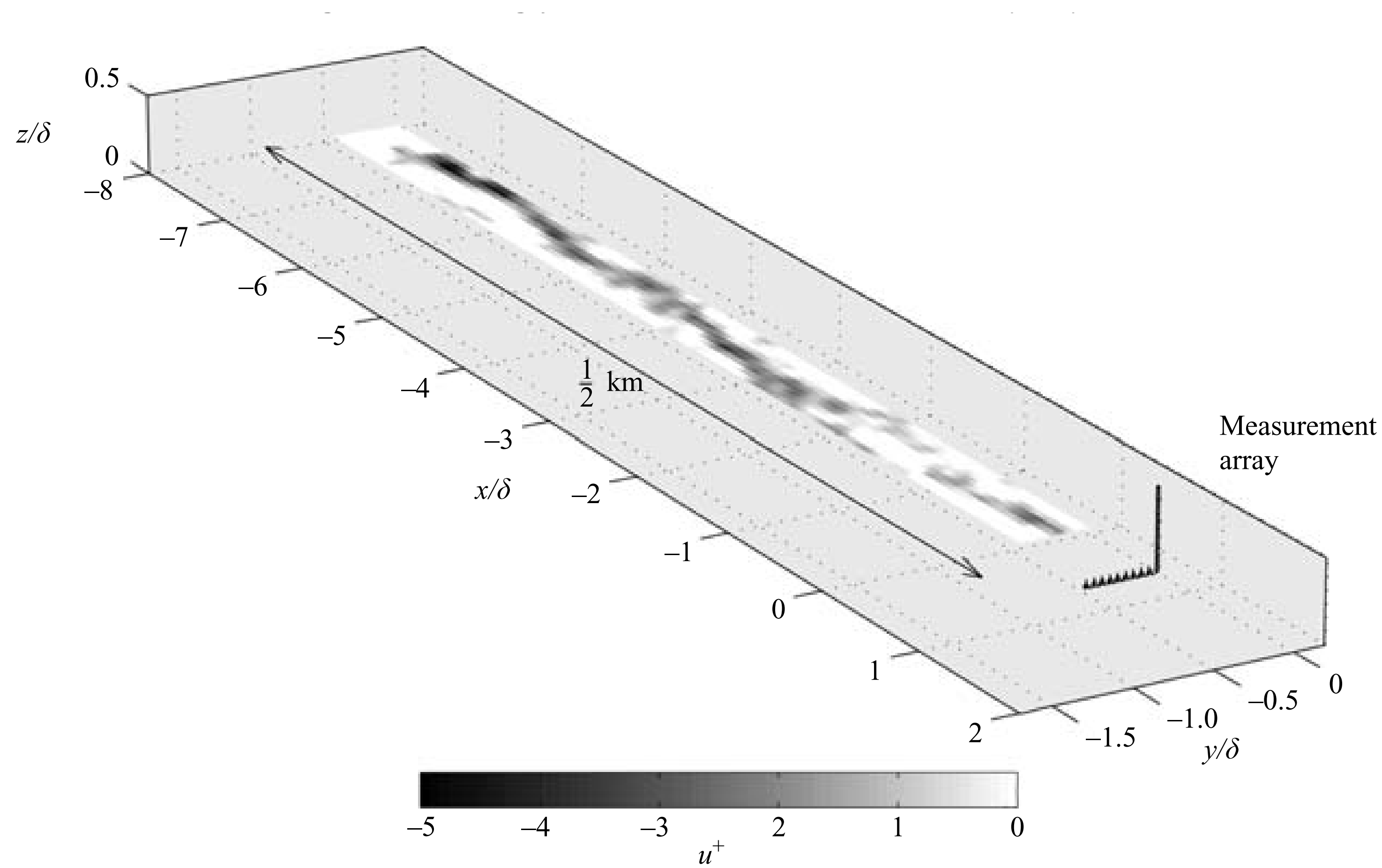
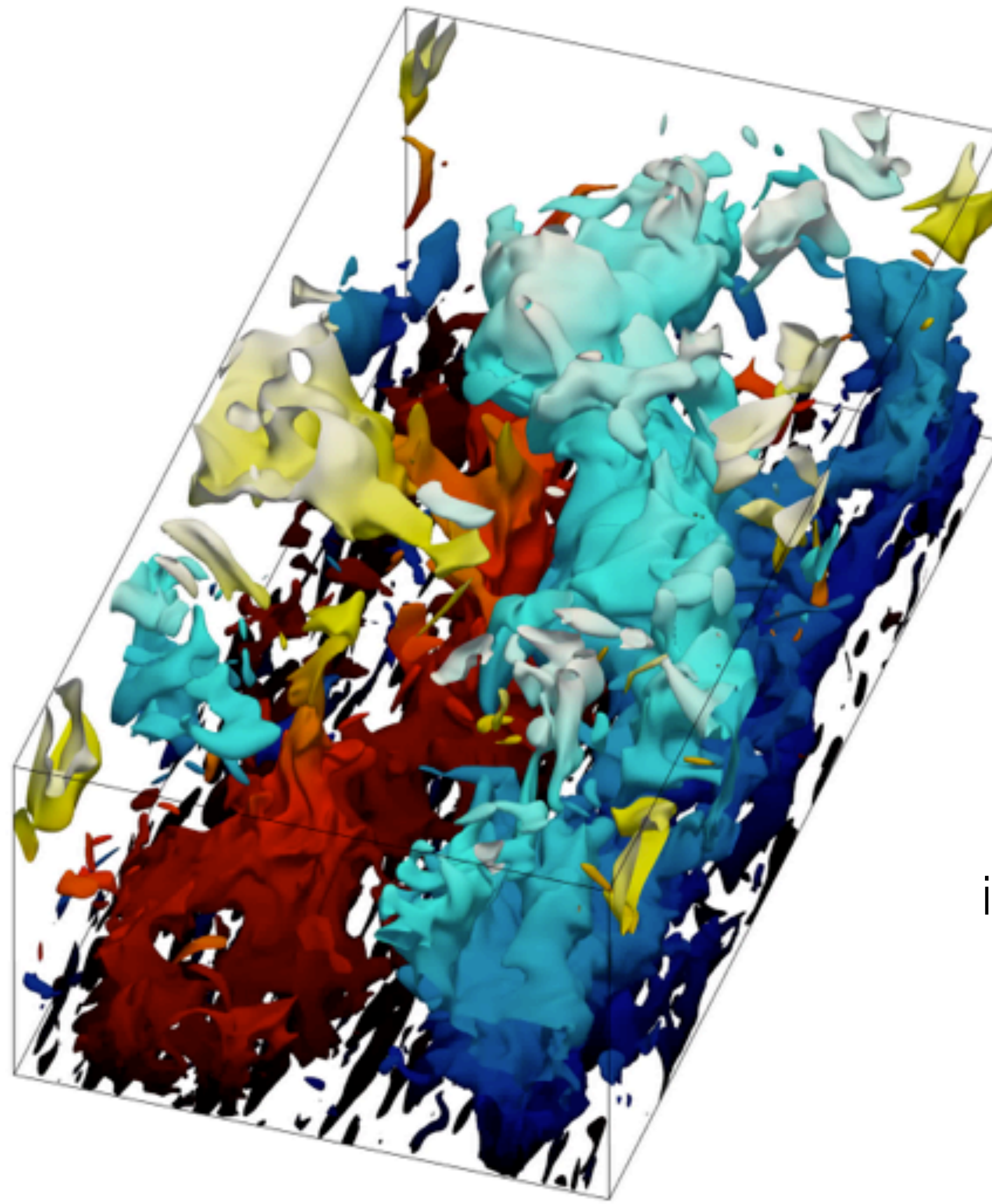


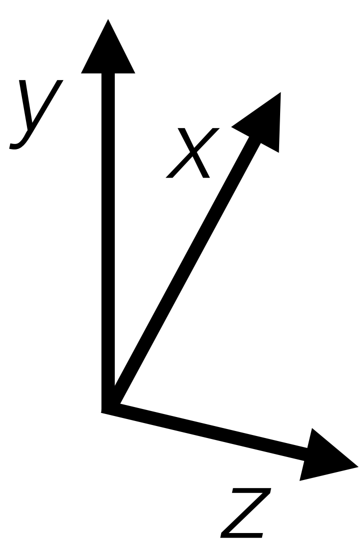
FIGURE 12. View of the measurement array installed at the SLTEST site.



Hutchins & Marusic 2007

Wall-bounded turbulence

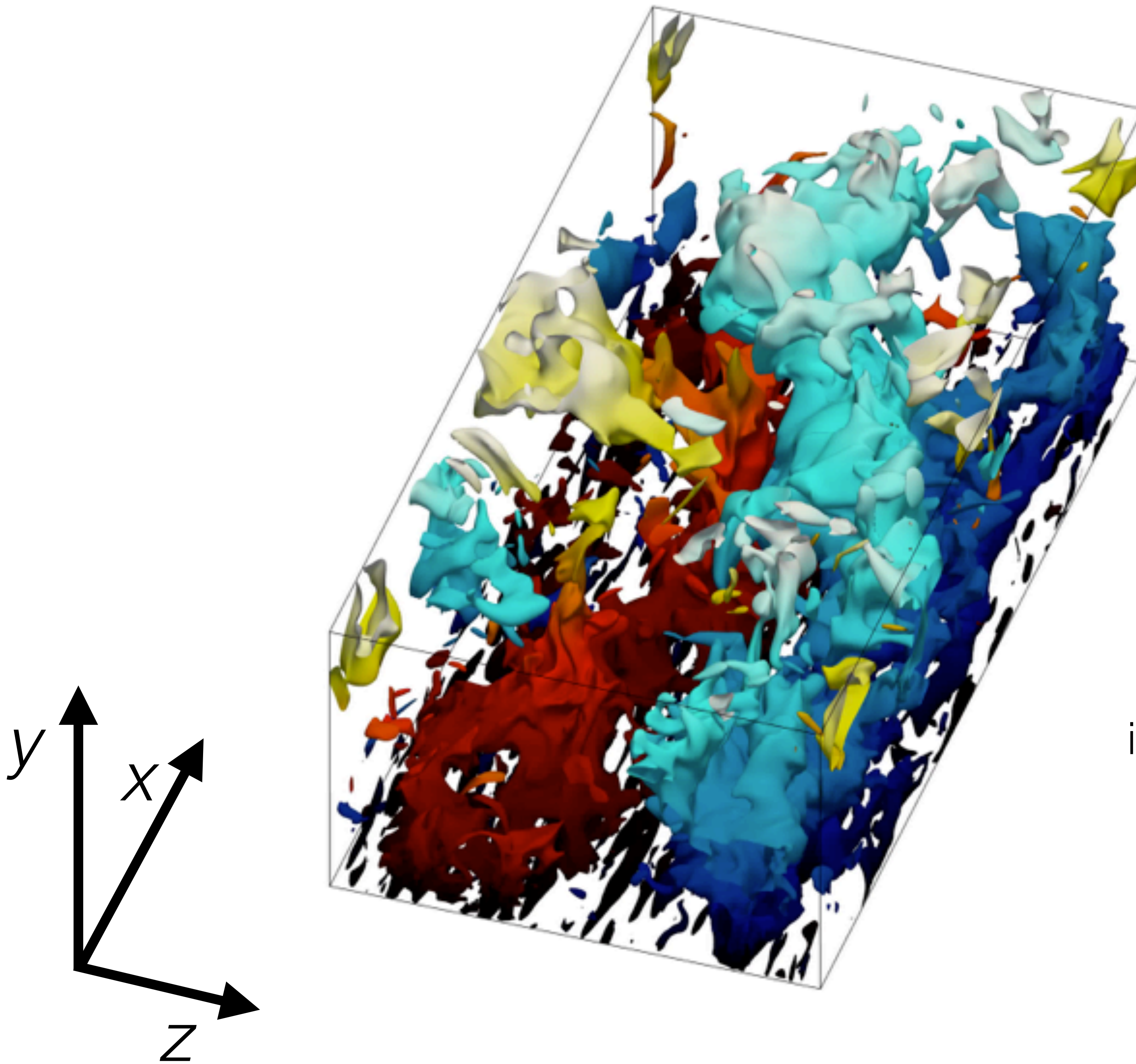




high and low speed
streak isocontours
in Poiseuille turbulence
at $Re_\tau = 950$

Credit: A Lozano-Durán

Wall-bounded turbulence



high and low speed
streak isocontours
in Poiseuille turbulence
at $Re_\tau = 950$

Credit: A Lozano-Durán

The problem to be addressed:

Understand how these *specific* structures arise
and how are they maintained

Claims

I. The underlying dynamics of structure formation lies in the interaction of turbulent eddies with mean flows

II. Often, structure formation has analytic expression *only* in the Statistical State Dynamics (SSD/DSS)
(the dynamics that govern the statistics of the flow rather than the dynamics governing single flow realizations)

III. Because of (**I**) a second-order closure of the SSD is adequate

Statistical State Dynamics (SSD)

1. split the flow variables into: $\langle$mean$\rangle$ + eddy $'$

$$\boldsymbol{u}(\boldsymbol{x},t) \;=\; \langle \boldsymbol{u}(\boldsymbol{x},t)\rangle \;+\; \boldsymbol{u}'(\boldsymbol{x},t)$$

2. form the hierarchy of same-time statistical moments/cumulants

$$\underbrace{\langle \boldsymbol{u}(\boldsymbol{x}_a,t)\rangle}_{=C^{(1)}_a}\;,\quad \underbrace{\langle \boldsymbol{u}'(\boldsymbol{x}_a,t)\boldsymbol{u}'(\boldsymbol{x}_b,t)\rangle}_{=C^{(2)}_{ab}}\;,\quad \underbrace{\langle \boldsymbol{u}'(\boldsymbol{x}_a,t)\boldsymbol{u}'(\boldsymbol{x}_b,t)\boldsymbol{u}'(\boldsymbol{x}_c,t)\rangle}_{=C^{(3)}_{abc}}\;,\quad \dots$$

3. find how each one of the moments/cumulants evolve

$$\partial_t C^{(1)}_a = \mathcal{F}_1\left(C^{(1)}_a\;,\;C^{(2)}_{ab}\right)$$

$$\partial_t C^{(2)}_{ab} = \mathcal{F}_2\left(C^{(1)}_a\;,\;C^{(2)}_{ab}\;,\;C^{(3)}_{abc}\right)$$

$$\partial_t C^{(3)}_{abc} = \mathcal{F}_3\left(C^{(1)}_a\;,\;C^{(2)}_{ab}\;,\;C^{(3)}_{abc}\;,\;C^{(4)}_{abcd}\right)\quad ,\ \text{etc ...}$$

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4. S3T/CE2: closure at second-order

Remarks on SSD — What is novel here?

$$\partial_t C_a^{(1)} = \mathcal{F}_1 \left(C_a^{(1)} \ , \ C_{ab}^{(2)} \right)$$

$$\partial_t C_{ab}^{(2)} = \mathcal{F}_2 \left(C_a^{(1)} \ , \ C_{ab}^{(2)} \ , \ C_{abc}^{(3)} \right)$$

$$\partial_t C_{abc}^{(3)} = \mathcal{F}_3 \left(C_a^{(1)} \ , \ C_{ab}^{(2)} \ , \ C_{abc}^{(3)} \ , \ C_{abcd}^{(4)} \right) \quad , \text{etc} \ ...$$

Remarks on SSD — What is novel here?

Usually (inspired by homogeneous isotropic turbulence) people took $\langle \boldsymbol{u}(\boldsymbol{x},t) \rangle = 0$

$$\partial_t C^{(1)}_{a} = \mathcal{F}_1 \left(C^{(1)}_{a} \; , \; C^{(2)}_{ab} \right)$$

but this is fundamental for structure formation (claim (**I**))

$$\partial_t C^{(2)}_{ab} = \mathcal{F}_2 \left(C^{(1)}_{a} \; , \; C^{(2)}_{ab} \; , \; C^{(3)}_{abc} \right)$$

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Remarks on SSD — What is novel here?

Usually (inspired by homogeneous isotropic turbulence) people took $\langle \boldsymbol{u}(\boldsymbol{x},t)\rangle \;=\; 0$

Main effort/interest was to obtain the equilibrium statistics: $\partial_t = 0$

$$\partial_t C^{(1)}_a = \mathcal{F}_1\left(C^{(1)}_a \;,\; C^{(2)}_{ab}\right)$$

but this is fundamental for structure formation (claim (**I**))

$$0 \;= \mathcal{F}_2\left(C^{(1)}_a \;,\; C^{(2)}_{ab} \;,\; C^{(3)}_{abc}\right)$$

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By studying the *dynamics* of the statistics new phenomena arise that are either not present or are obscured in single flow realizations

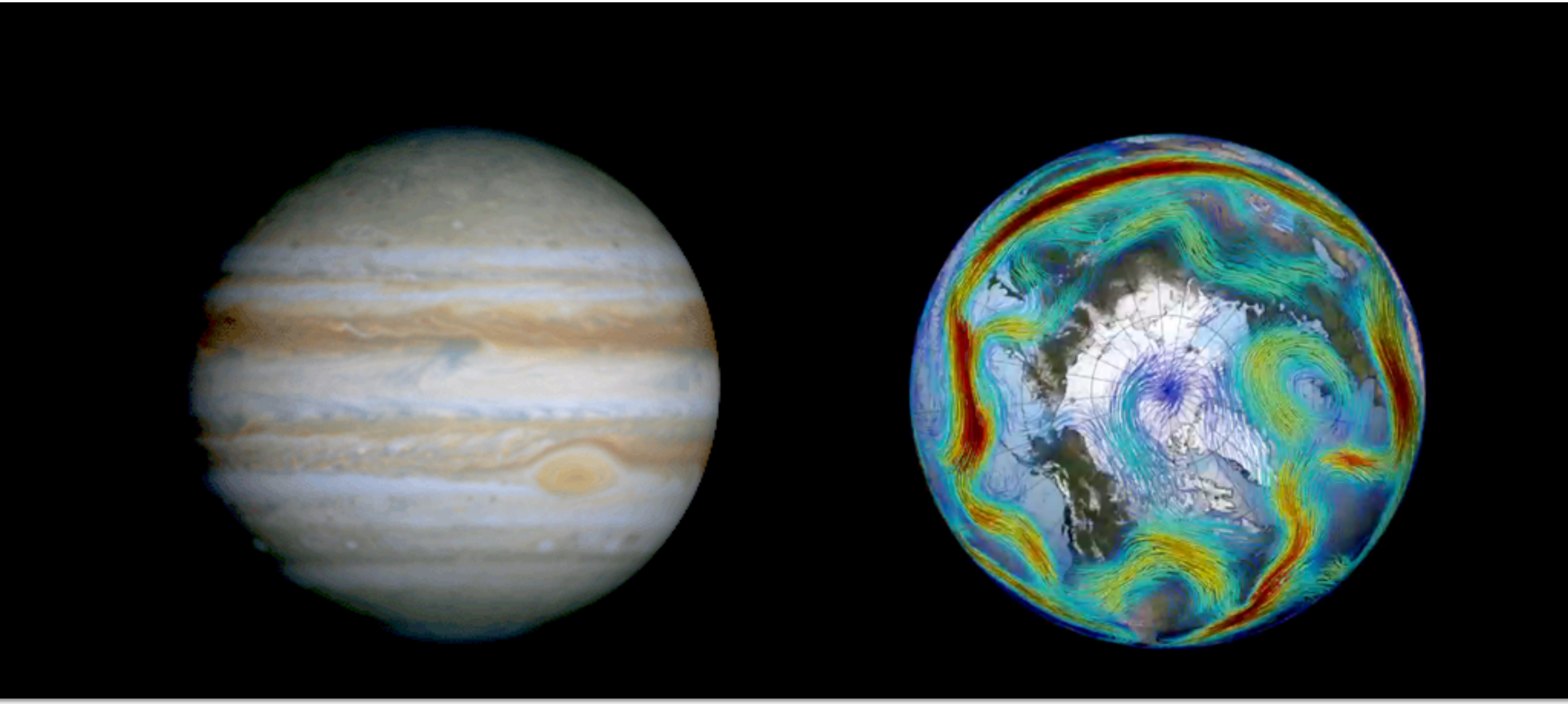
I will show that within the framework of SSD we understand:

A. Jet/large-scale wave emerge in planetary turbulence
as an instability of the SSD
(this shows that SSD capture the mechanism)

B. Roll/streak structures
in pre-transitional free-stream Couette turbulence
emerge as an instability of the SSD

A.

Jet/Large-scale wave emergence in planetary turbulence



Farrell & Ioannou 2003, 2007; Srinivasan & Young 2012; Tobias & Marston 2013; NCC, Farrell & Ioannou 2014, 2016
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Ait-Chaalal, Schneider, Meyer, Marston 2016

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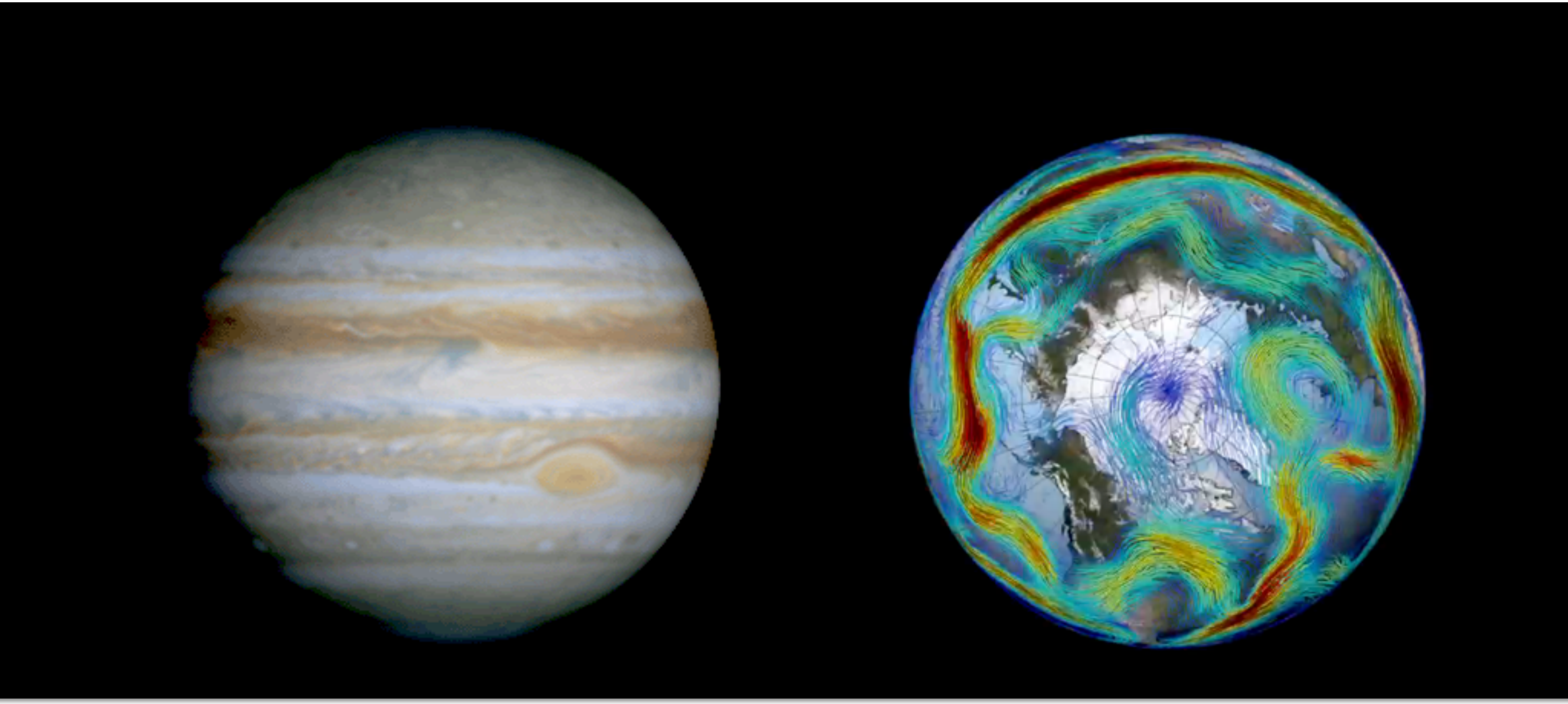

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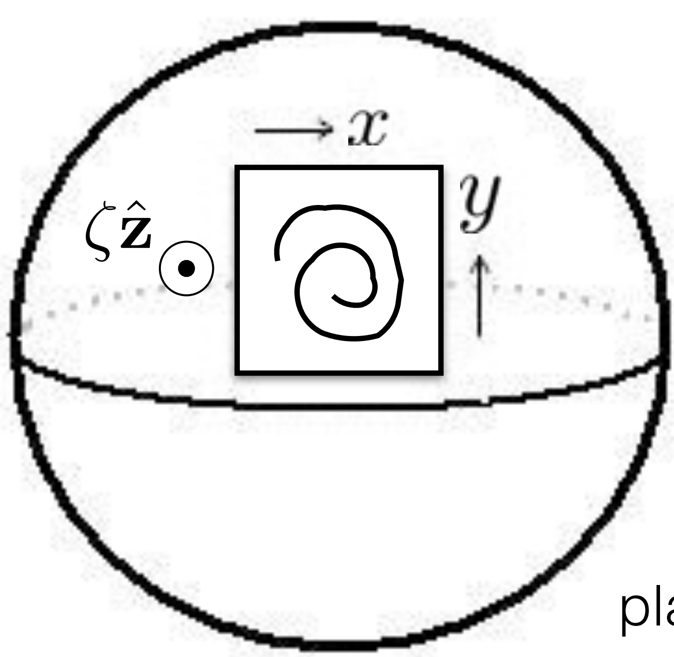
barotropic vorticity equation on a β-plane

$$\partial_t \zeta + \boldsymbol{u} \cdot \boldsymbol{\nabla}(\zeta + \beta y) = -r\zeta + \sqrt{\varepsilon}\xi$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

$$\boldsymbol{u} = (u, v) = (-\partial_y \psi, \partial_x \psi)$$

$$\zeta = (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \hat{\mathbf{z}} = \Delta\psi$$



β: gradient of planetary vorticity

barotropic vorticity equation on a β-plane

$$\partial_t \zeta + \boldsymbol{u} \cdot \boldsymbol{\nabla}(\zeta + \beta y) = -r\zeta + \sqrt{\varepsilon}\xi$$

linear dissipation at rate *r*

stochastic forcing

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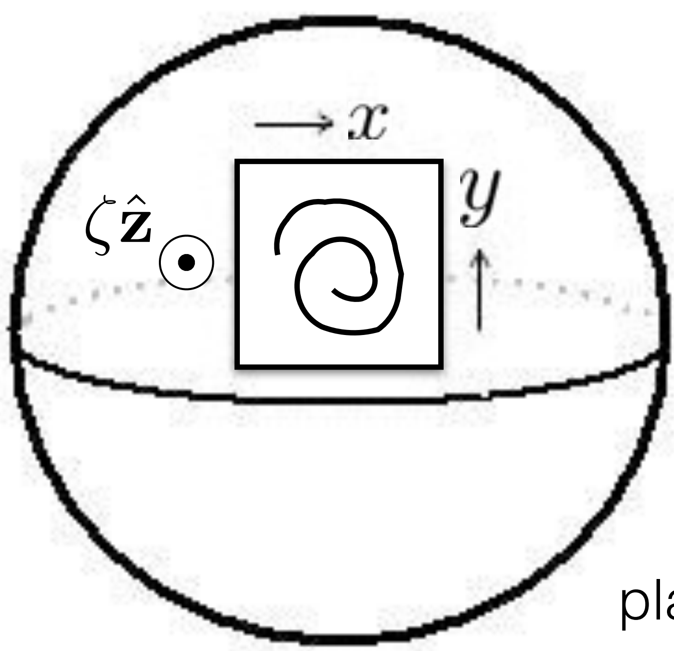
zero mean
white in time
&
statistically homogeneous

$$\langle \xi(\boldsymbol{x}_a, t_a)\xi(\boldsymbol{x}_b, t_b)\rangle = Q(\boldsymbol{x}_a - \boldsymbol{x}_b)\,\delta(t_a - t_b)$$



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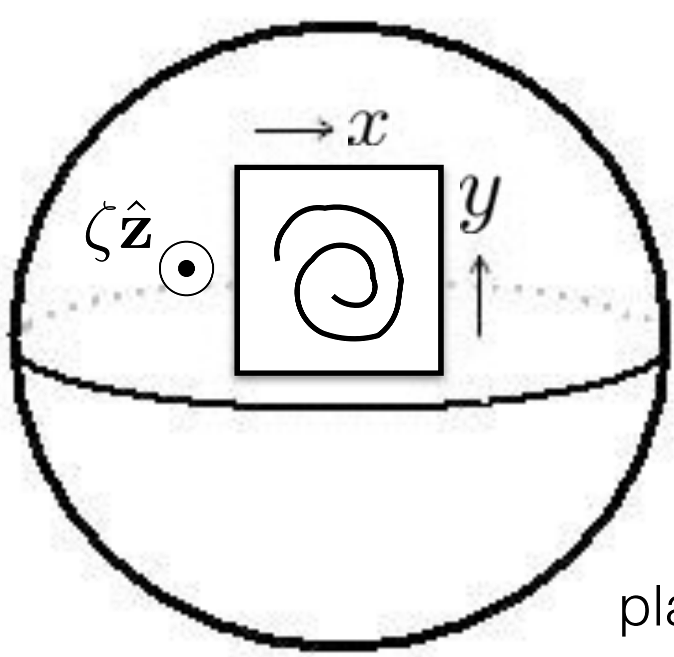
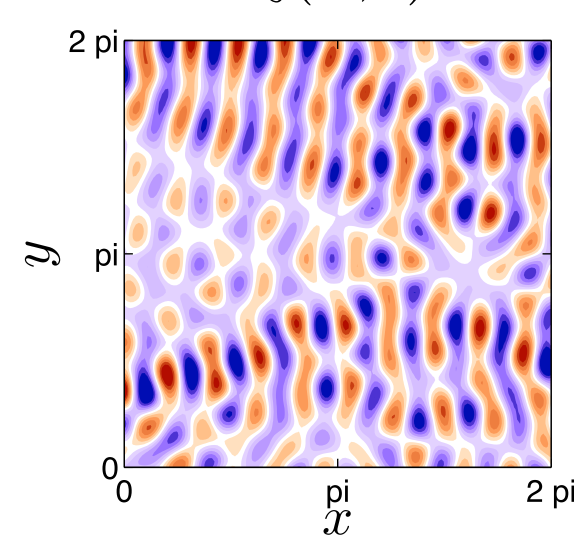


β: gradient of planetary vorticity

$\xi(\boldsymbol{x}, t)$



anisotropic Earth-like forcing modeling energy injected to the barotropic mode by baroclinic instability

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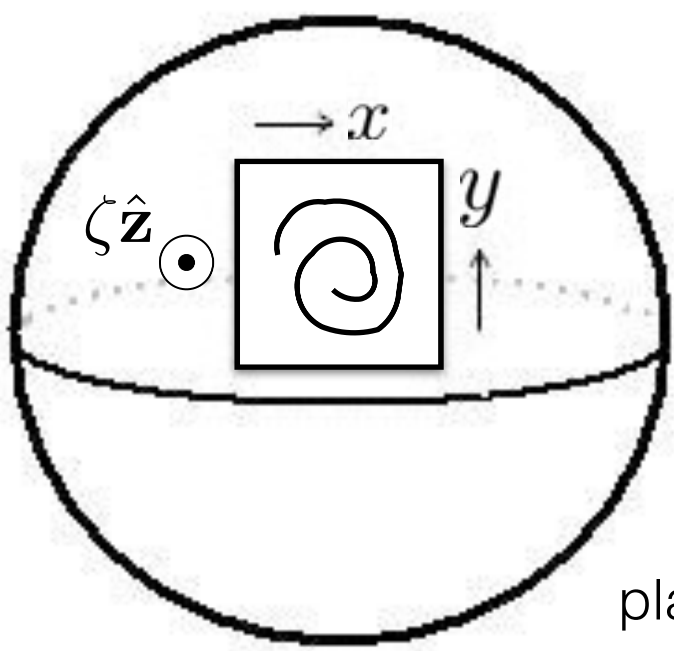


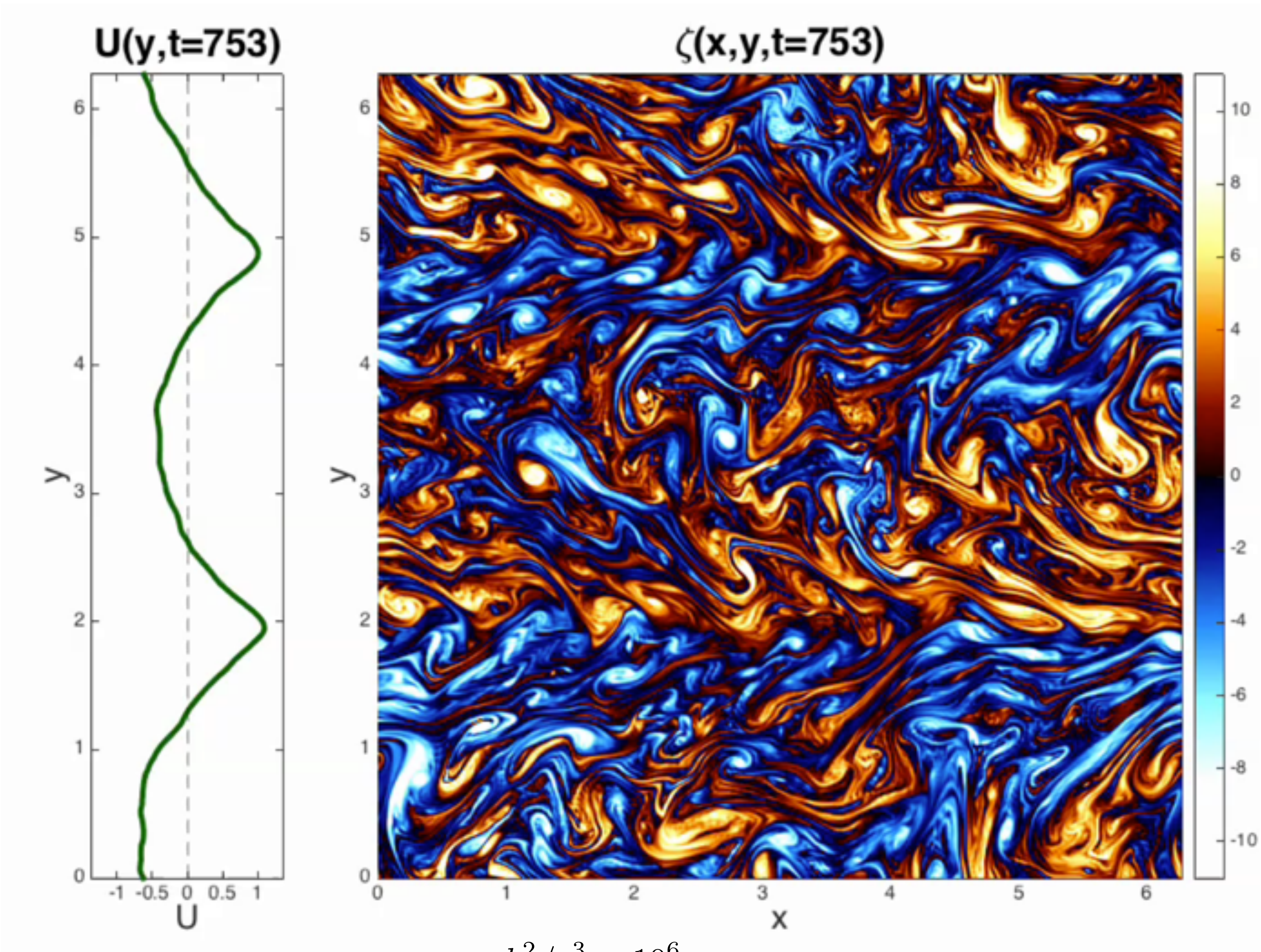
β: gradient of planetary vorticity

two non-dimensional parameters

$\varepsilon k_f^2 / r^3$

$\beta / (k_f r)$

barotropic β-plane turbulence exhibits large-scale structure formation



$$\varepsilon k_f^2 / r^3 = 10^6$$

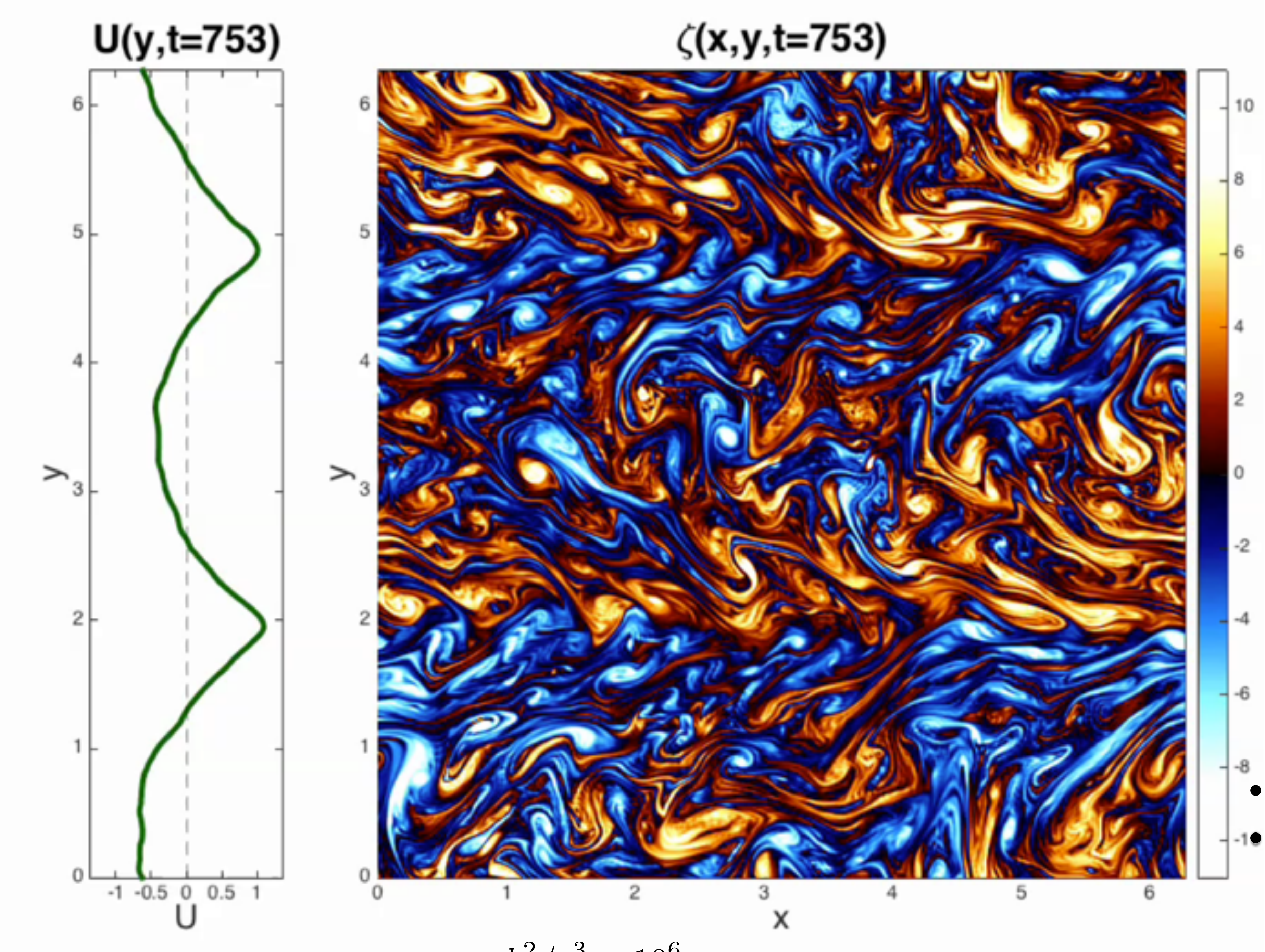
$$\beta / (k_f r) = 67$$

statistically homogeneous forcing

(no inhomogeneity is imposed by the forcing)

any random flow inhomogeneities organize the turbulence in a manner so that they are reinforced

barotropic β-plane turbulence exhibits large-scale structure formation



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http://www.youtube.com/watch?v=ZtzW25NooDk

statistically homogeneous forcing

(no inhomogeneity is imposed by the forcing)

any random flow inhomogeneities organize the turbulence in a manner so that they are reinforced

we observe:

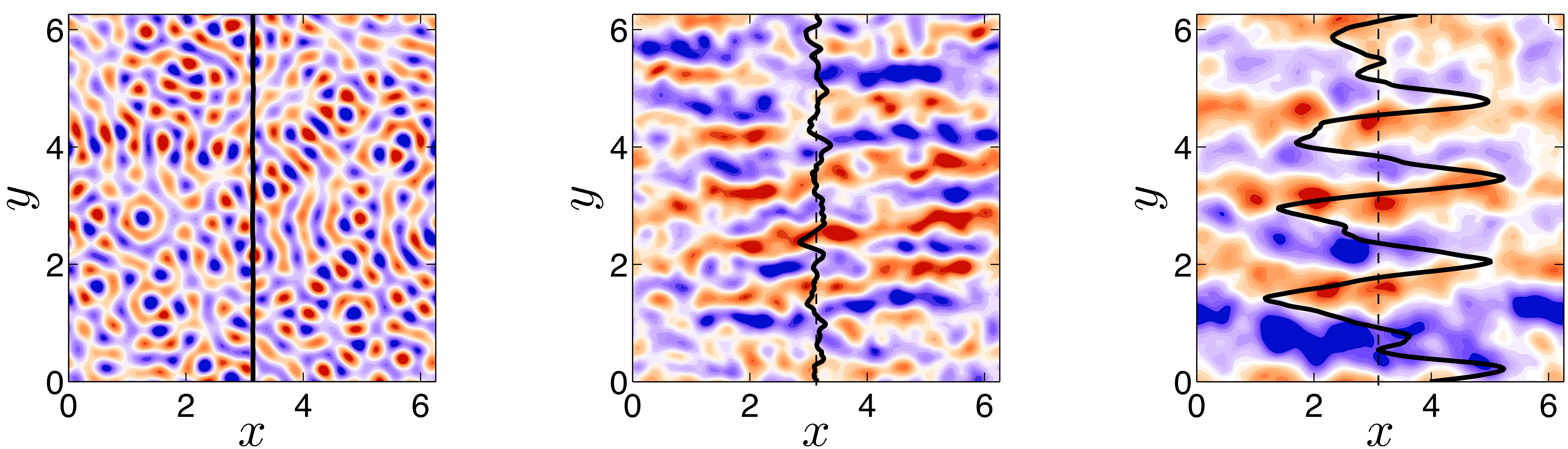
- jet emerge
- jets appear to change much slower compared to the eddies
- jet have a particular structure

various β-plane turbulence flows at statistically steady state:

homogeneous — traveling waves — zonal jets

$$\beta/(k_f r) = 67$$

$\varepsilon k_f^2/r^3 = 10^2$ | 5×10^3 | 5×10^4



this suggests that there is some kind of transition as ε is increased

[snapshots of the streamfunction $\psi(\mathbf{x},t)$ with instantaneous zonal mean zonal flow $U(y,t)$]

S3T closure of SSD

take the ⟨mean⟩ as a zonal mean

under the ergodic assumption that

⟨mean⟩ = ensemble average over forcing realizations

$$Z(\mathbf{x},t) = \langle \zeta(\mathbf{x},t) \rangle \quad , \qquad C_{ab}^{(2)} = \langle \zeta'(\mathbf{x}_a,t) \zeta'(\mathbf{x}_b,t) \rangle$$

1st cumulant

2nd cumulant

S3T closure of SSD

$$\partial_t Z + \mathbf{U}\cdot\boldsymbol{\nabla}(Z+\beta y) = \mathcal{R}(C^{(2)}_{ab}) - rZ$$

$$\partial_t C^{(2)}_{ab} = \left[\mathcal{A}(\mathbf{U})_a + \mathcal{A}(\mathbf{U})_b\right] C^{(2)}_{ab} + \varepsilon Q_{ab}$$

with

$$\mathbf{U} \stackrel{\text{def}}{=} (-\partial_y, \partial_x)\Delta^{-1} Z$$

$$C^{(2)}_{ab} \stackrel{\text{def}}{=} \langle \zeta'(\mathbf{x}_a, t)\zeta'(\mathbf{x}_b, t)\rangle$$

$$Q_{ab} \stackrel{\text{def}}{=} Q(\mathbf{x}_a - \mathbf{x}_b) \longrightarrow$$ the spatial covariance of the statistically homogeneous stochastic forcing

$$\mathcal{R}(C^{(2)}_{ab}) \stackrel{\text{def}}{=} -\langle \boldsymbol{u}'\cdot\boldsymbol{\nabla}\zeta'\rangle = \boldsymbol{\nabla}\cdot\left[\frac{\hat{\mathbf{z}}}{2}\times\left(\boldsymbol{\nabla}_a\Delta_a^{-1} + \boldsymbol{\nabla}_b\Delta_b^{-1}\right) C^{(2)}_{ab}\right]_{a=b}$$

(the Reynolds stresses are given as a linear function of *C*)

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neglect of third cumulant
is *equivalent* with
neglect of the eddy—eddy term in eddy equation in the EOM
($\longrightarrow$ PainInNeck-term Tobias was talking about on Monday)

Note: The dynamics of the 1st & 2nd cumulants is necessarily quasi-linear (Herring 1963)

S3T closure of SSD

$$\partial_t Z + \mathbf{U}\cdot\boldsymbol{\nabla}(Z+\beta y) = \mathcal{R}(C^{(2)}_{ab}) - rZ$$

$$\partial_t C^{(2)}_{ab} = \left[\mathcal{A}(\mathbf{U})_a + \mathcal{A}(\mathbf{U})_b\right] C^{(2)}_{ab} + \varepsilon Q_{ab}$$

The S3T system

- nonlinear
- autonomous, deterministic (central limit theorem)
- admits fixed point solutions $\Big(\mathbf{U}^e(\mathbf{x}), C^e(\mathbf{x}_a,\mathbf{x}_b)\Big)$
- associated perturbation equations used to determine stability of these fixed points

S3T equilibria for homogeneous forcing

$$\mathbf{U}^e = 0 \quad , \quad C^e(\mathbf{x}_a - \mathbf{x}_b) = \frac{\varepsilon\, Q}{2r}$$

(for any ε, β and homogeneous Q)

zero mean flow + non-zero second-order eddy statistics

$$\mathbf{U}^e(\mathbf{x}) = \Big(U^e(y), 0\Big) \quad , \quad C^e(x_a - x_b, y_a, y_b)$$

zonal jet mean flow + non-zero zonally homogeneous 2nd-order eddy statistics

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zonal jet mean flow + non-zero zonally homogeneous 2nd-order eddy statistics

Perturbations about these equilibria are governed by:

we linearized about a turbulent state!

$$\partial_t \delta Z = \mathcal{A}(\mathbf{U}^e)\, \delta Z + \mathcal{R}(\delta C)$$

$$\partial_t \delta C_{ab} = [\mathcal{A}_a(\mathbf{U}^e) + \mathcal{A}_b(\mathbf{U}^e)]\, \delta C_{ab} + (\delta\mathcal{A}_a + \delta\mathcal{A}_b) C^e_{ab}$$

$$\delta\mathcal{A} \overset{\text{def}}{=} \mathcal{A}(\mathbf{U}^e + \delta\mathbf{U}) - \mathcal{A}(\mathbf{U}^e)$$

eigenanalysis of this system determines the stability of $\Big(\mathbf{U}^e(\mathbf{x}), C^e(\mathbf{x}_a, \mathbf{x}_b)\Big)$

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zonal jet mean flow + non-zero zonally homogeneous 2nd-order eddy statistics

Perturbations about these equilibria are governed by:

hydrodynamic stability

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eigenanalysis of this system determines the stability of $\Big(\mathbf{U}^e(\mathbf{x}), C^e(\mathbf{x}_a, \mathbf{x}_b)\Big)$

Consider the homogeneous turbulent equilibrium:

$$\mathbf{U}^e = 0 \quad , \quad C^e(\mathbf{x}_a - \mathbf{x}_b) = \frac{\varepsilon\, Q}{2r}$$

(for any ε, β and homogeneous Q)

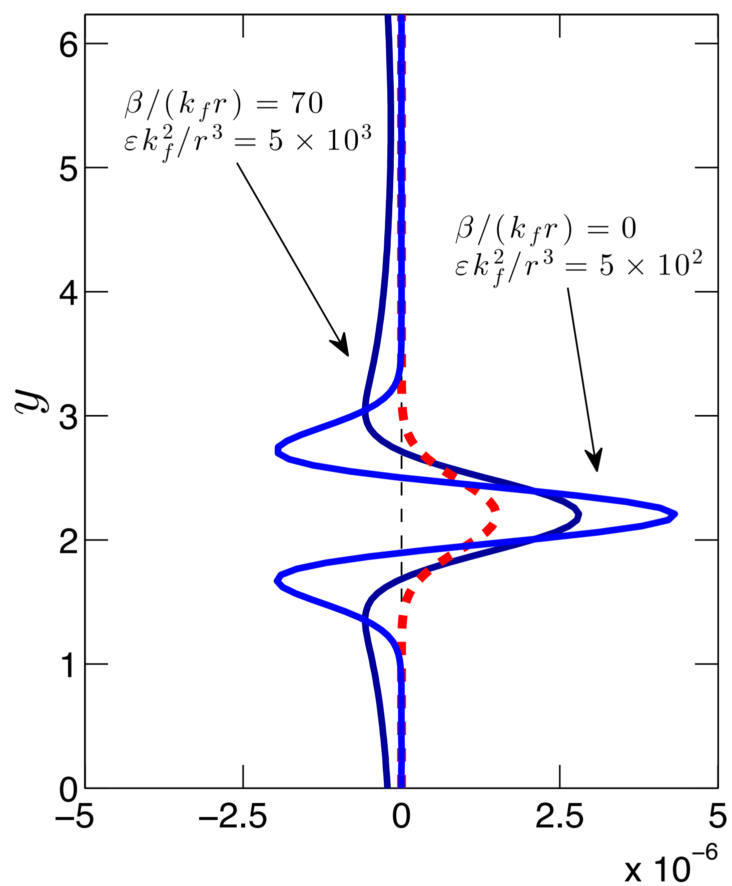
zero mean flow + non-zero second-order eddy statistics

How does the state with *no mean flow* becomes unstable?

proof of concept

An infinitesimal mean flow *δU* distorts the turbulence in a manner so as to produce Reynolds stresses R(*δC*) that reinforce the *δU* itself

$$\partial_t \delta Z = \mathcal{A}(\ 0\ )\, \delta Z + \mathcal{R}(\delta C)$$



$(--)\ \delta U\ \ ,\ \ (—)\ \mathcal{R}(\delta C)$

proof of concept

An infinitesimal mean flow δU distorts the turbulence in a manner so as to produce Reynolds stresses R(δC) that reinforce the δU itself

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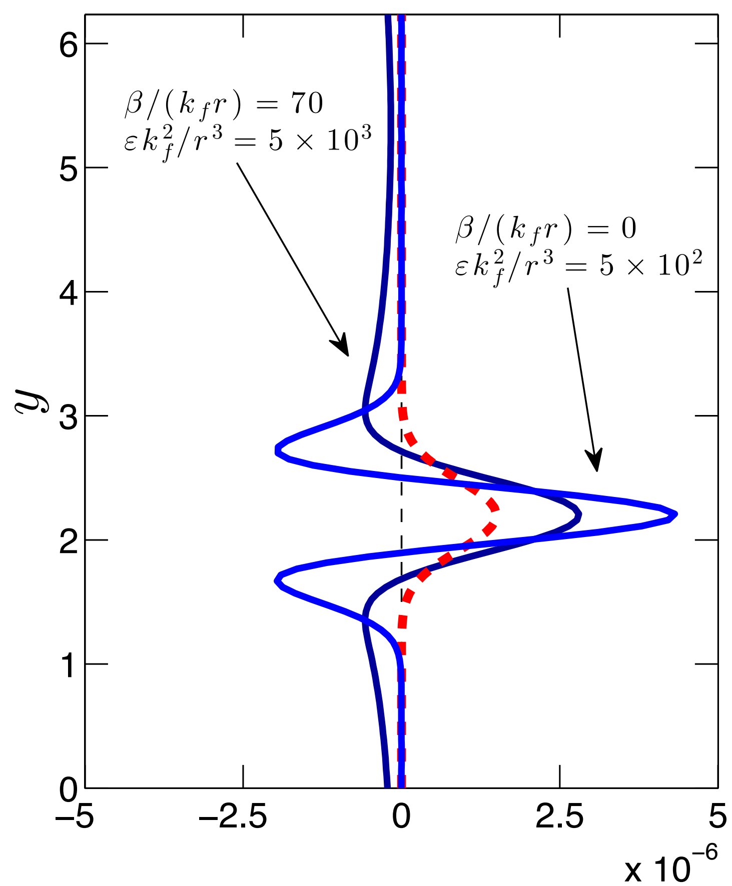
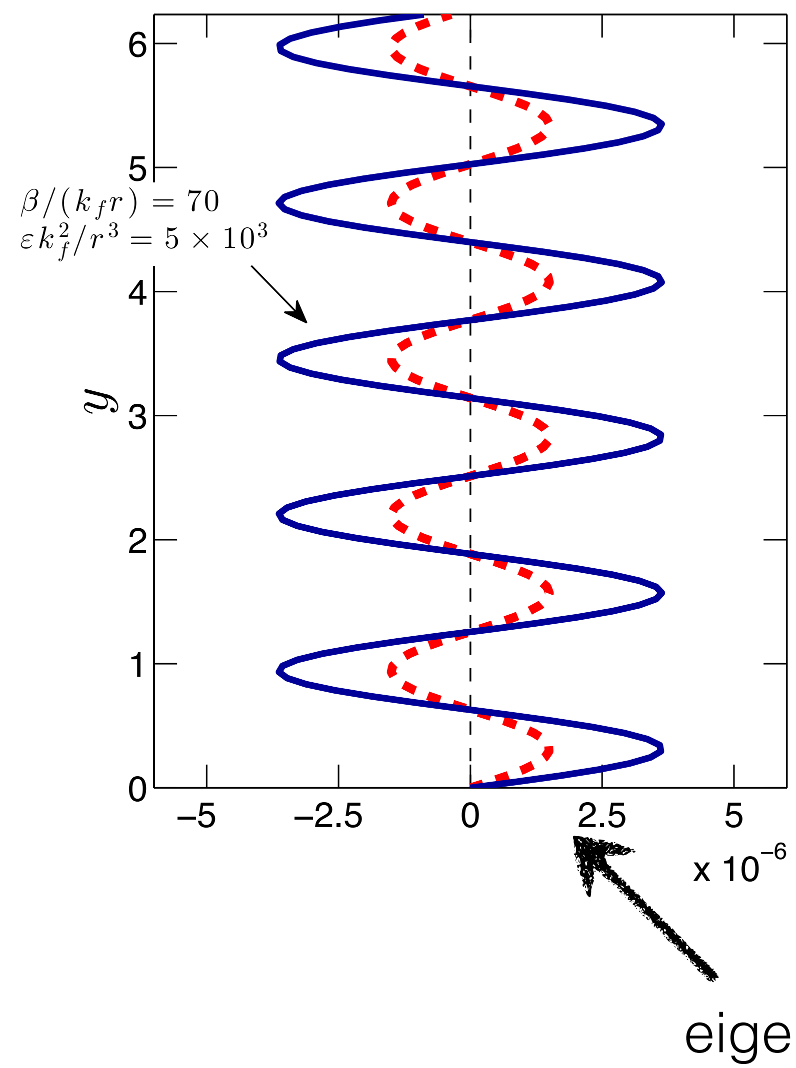
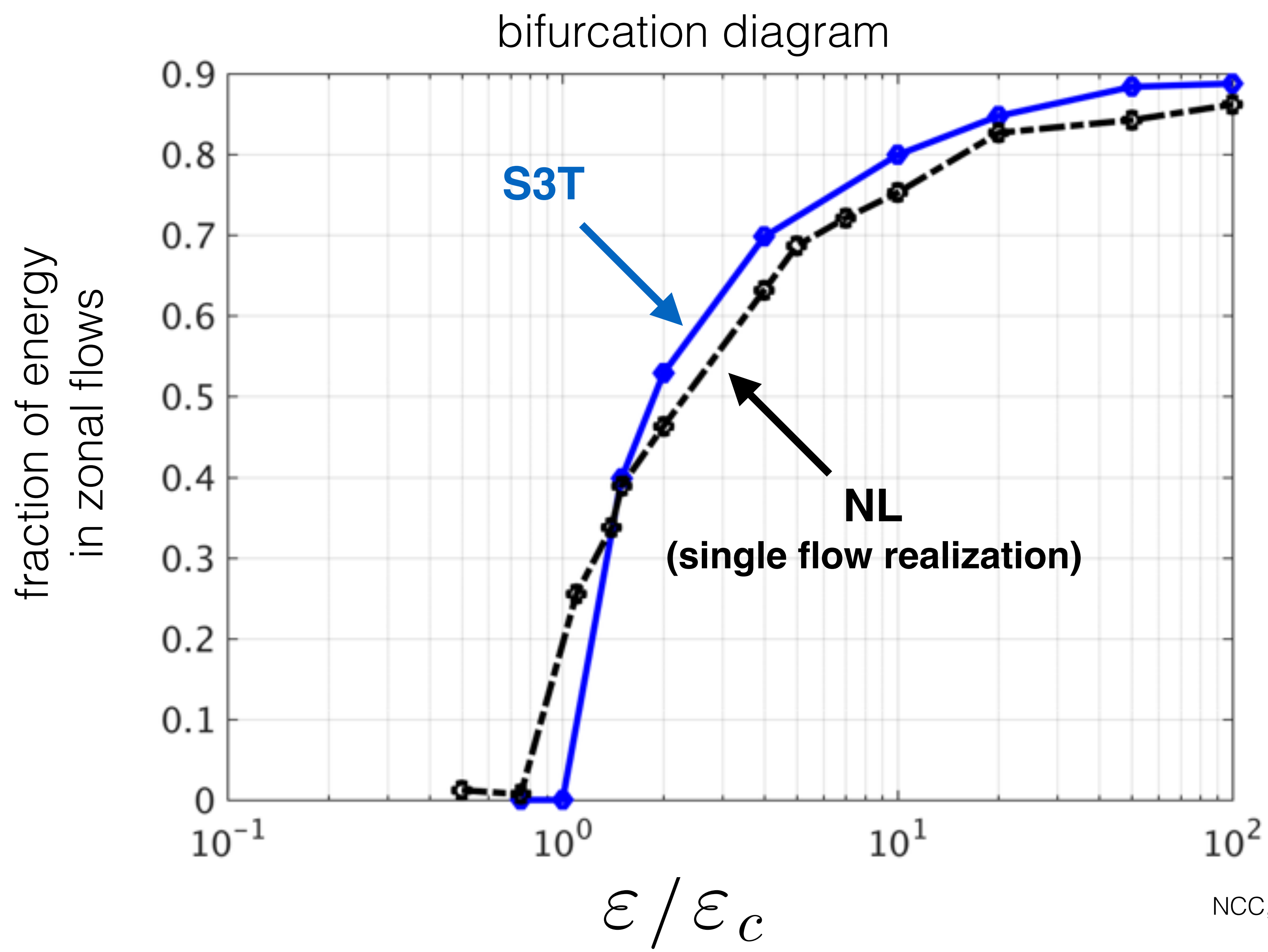


$$(--)\ \delta U \quad , \quad (—)\ \mathcal{R}(\delta C)$$



Farrell & Ioannou 2007
Srinivasan & Young 2012
NCC, Farrell & Ioannou 2014
Bakas, NCC & Ioannou 2015

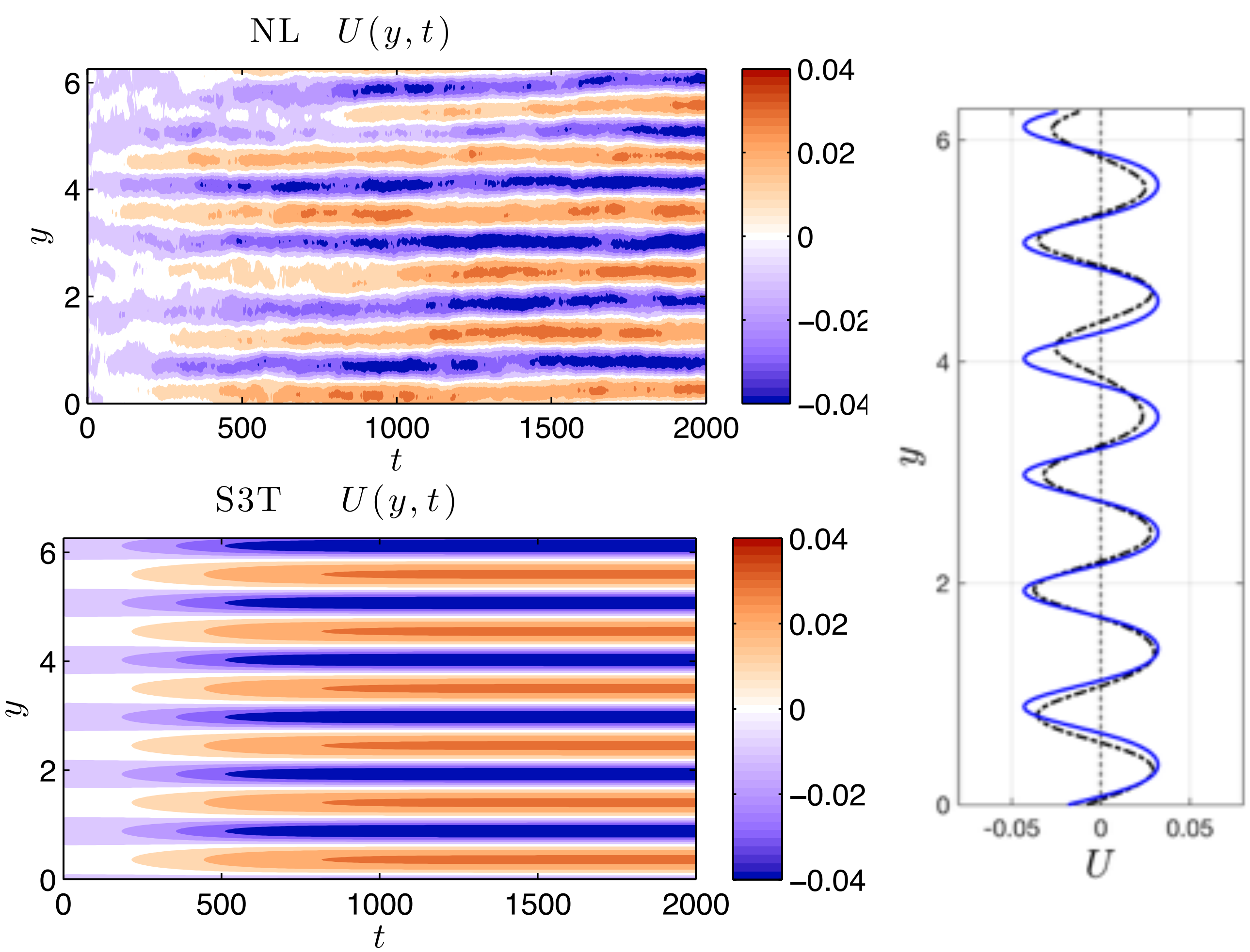
Verification of S3T predictions for the jet formation bifurcation



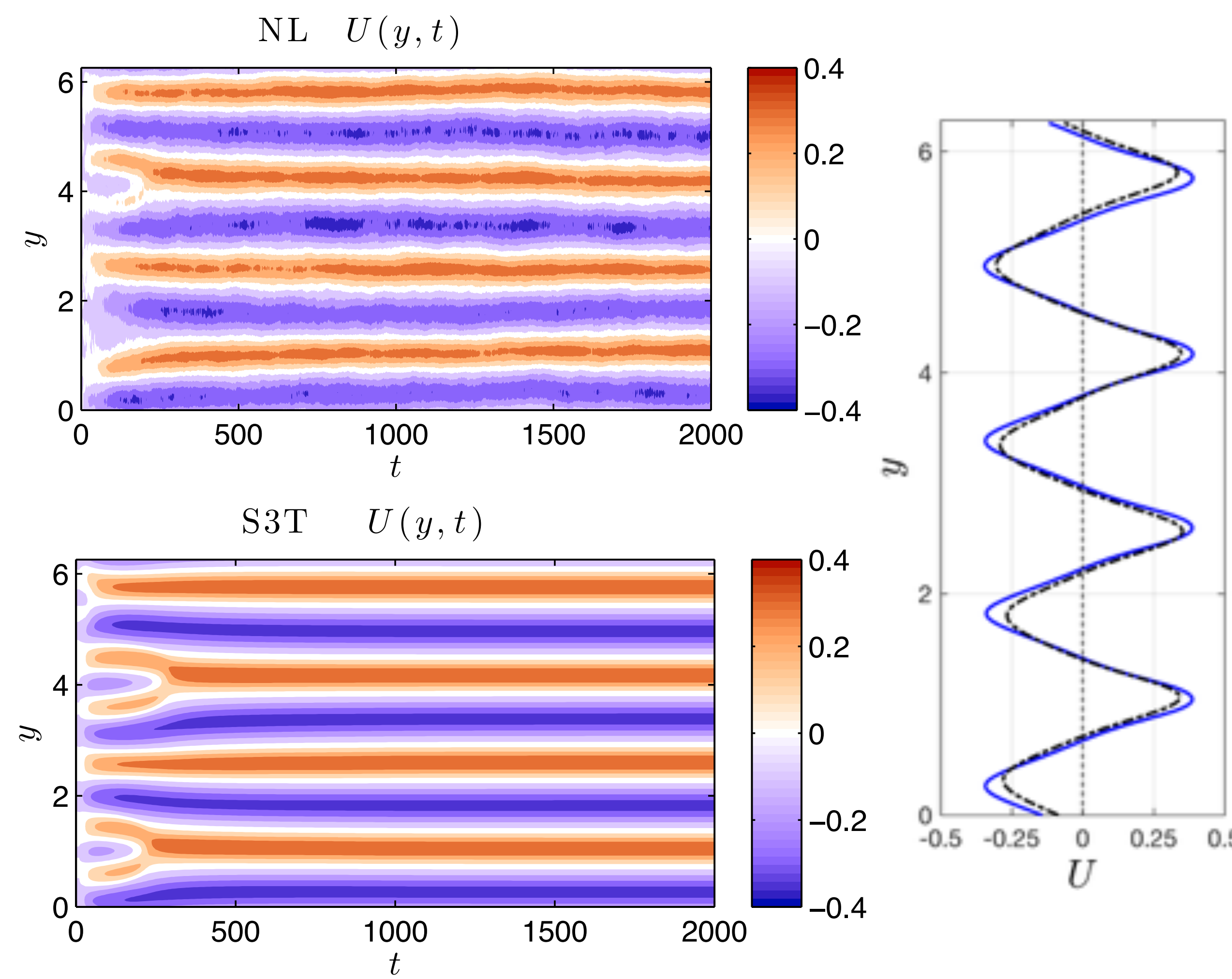
NCC, Farrell & Ioannou 2014

Verification of the S3T predictions for the structure of the finite amplitude jet equilibria

$\varepsilon/\varepsilon_c$ = 1.5



$\varepsilon/\varepsilon_c$ = 20

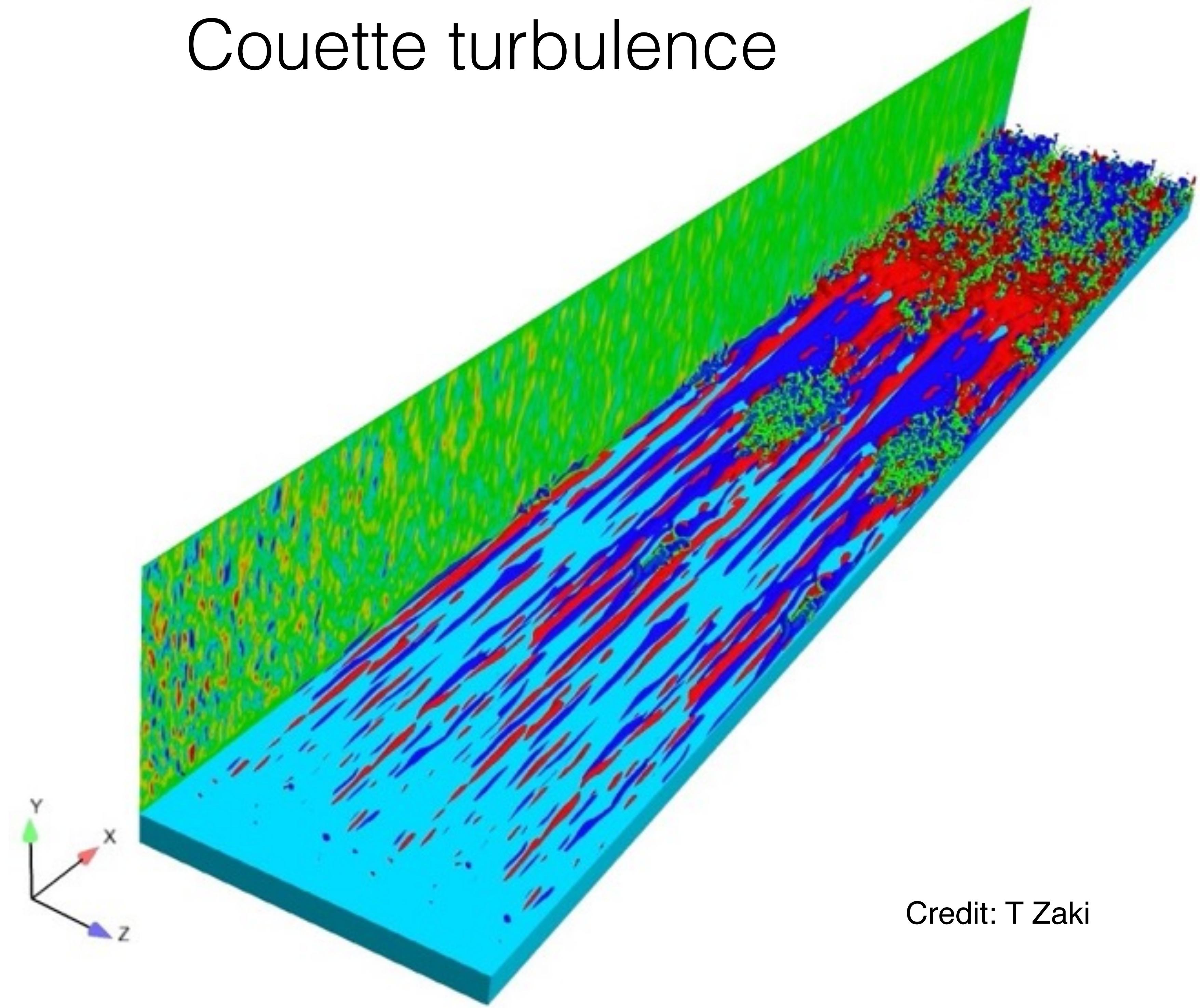


NCC, Farrell & Ioannou 2014

S3T instabilities grow and reach finite amplitude to produce new inhomogeneous S3T equilibria

B.

Roll/streak formation in pre-transitional free-stream Couette turbulence



Credit: T Zaki

Farrell, Ioannou & Nikolaidis (2016) Instability of the roll/streak structure induced by free-stream turbulence in pre-transitional Couette flow, *Phys. Rev. Fluids.* (sub judice, arXiv:1607.05018)

roll/streak formation in free-stream Couette turbulence

flow = streamwise mean + perturbations

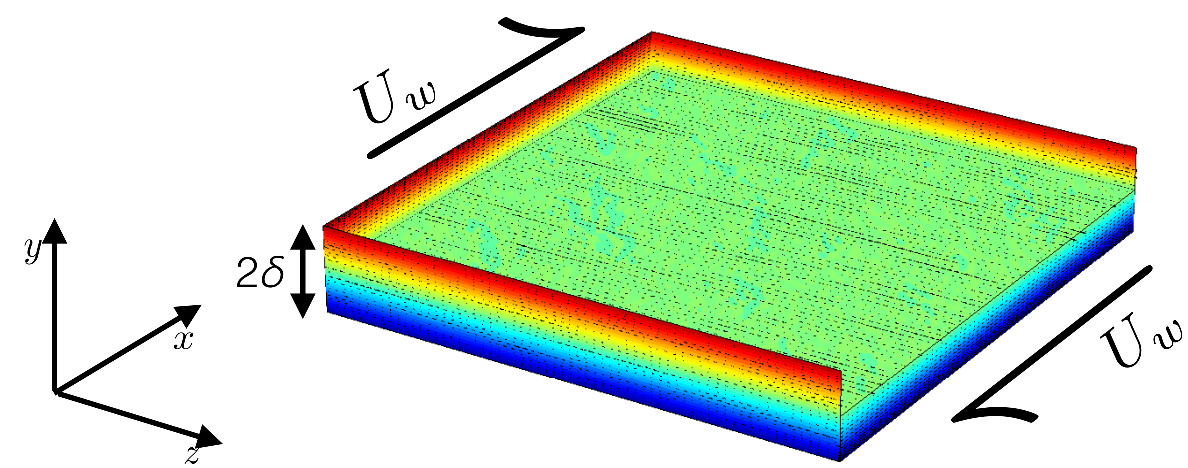
$$\boldsymbol{u} = \mathbf{U} + \boldsymbol{u}'$$

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \boldsymbol{\nabla} \mathbf{U} + \boldsymbol{\nabla} P - \frac{1}{Re} \Delta \mathbf{U} = -\langle \boldsymbol{u}' \cdot \boldsymbol{\nabla} \boldsymbol{u}' \rangle$$

$$\partial_t \boldsymbol{u}' + \mathbf{U} \cdot \boldsymbol{\nabla} \boldsymbol{u}' + \boldsymbol{u}' \cdot \boldsymbol{\nabla} \mathbf{U} + \boldsymbol{\nabla} p' - \frac{1}{Re} \Delta \boldsymbol{u}' = -\left(\boldsymbol{u}' \cdot \boldsymbol{\nabla} \boldsymbol{u}' - \langle \boldsymbol{u}' \cdot \boldsymbol{\nabla} \boldsymbol{u}' \rangle \right) + \sqrt{\varepsilon} \boldsymbol{\xi}$$

$$\boldsymbol{\nabla} \cdot \mathbf{U} = \boldsymbol{\nabla} \cdot \boldsymbol{u}' = \boldsymbol{\nabla} \cdot \boldsymbol{\xi} = 0$$

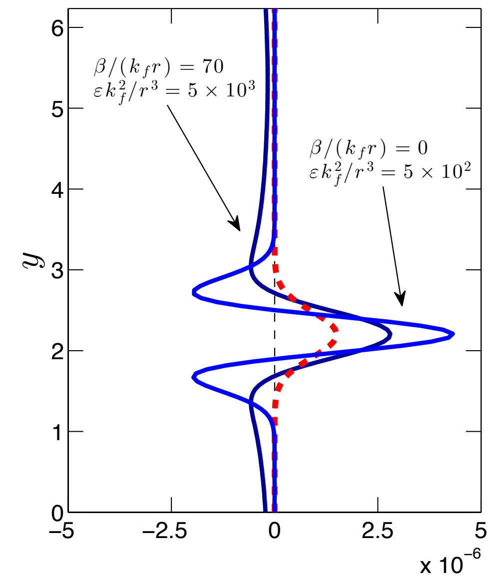
$$Re = \frac{U_w \delta}{\nu}$$



Credit: V Thomas

proof of concept

2D problem

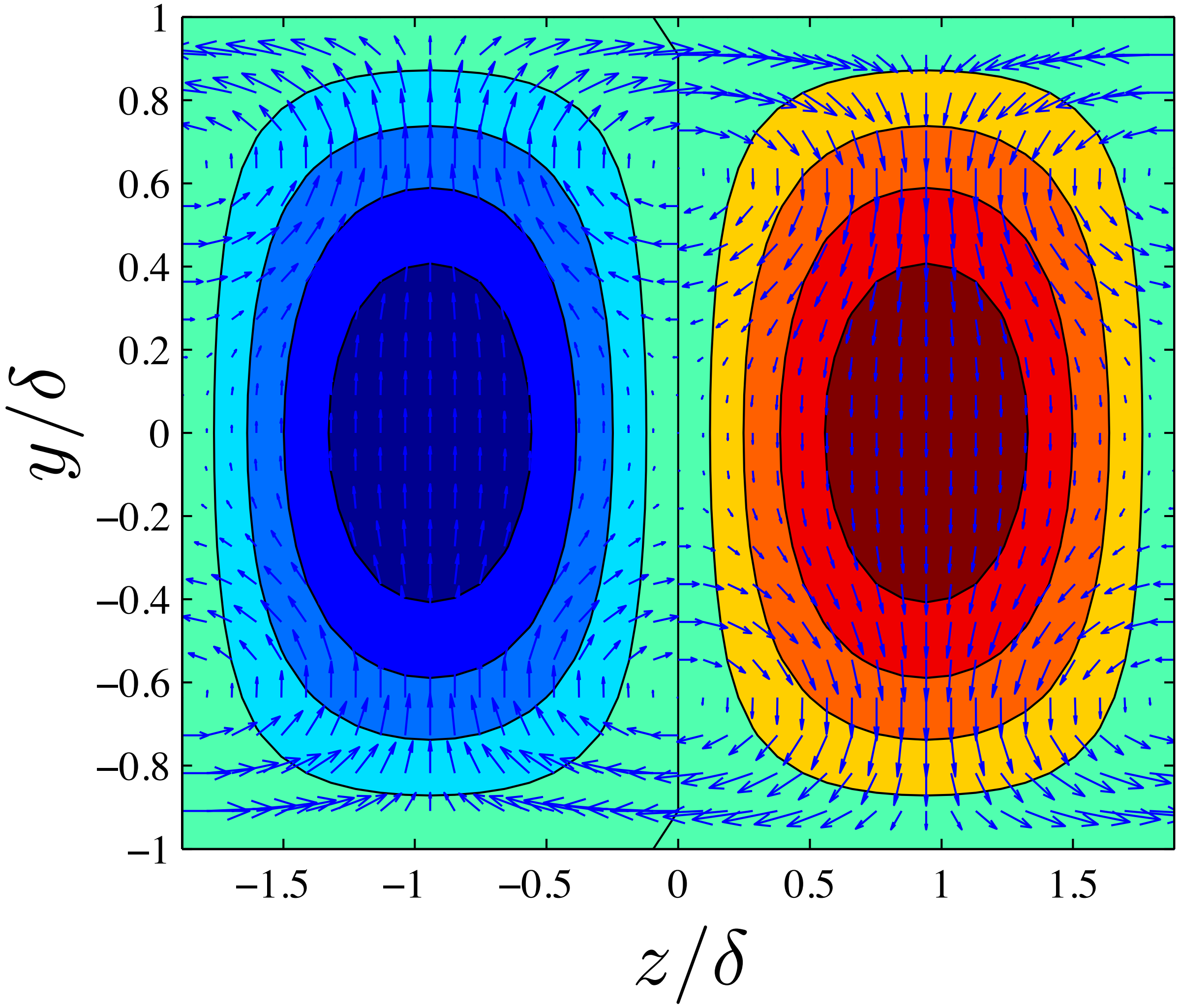


$$(--)\ \delta U \quad , \quad (—)\ \mathcal{R}(\delta C)$$

Analogously, in the 3D problem
infinitesimal mean flows organize the turbulent Reynolds stresses
so as to reinforce the very same mean flow

proof of concept

1. Perturb a shear flow by an infinitesimal streak in the presence of turbulence
2. Calculate the response of the turbulence and the Reynolds stresses the are produce.



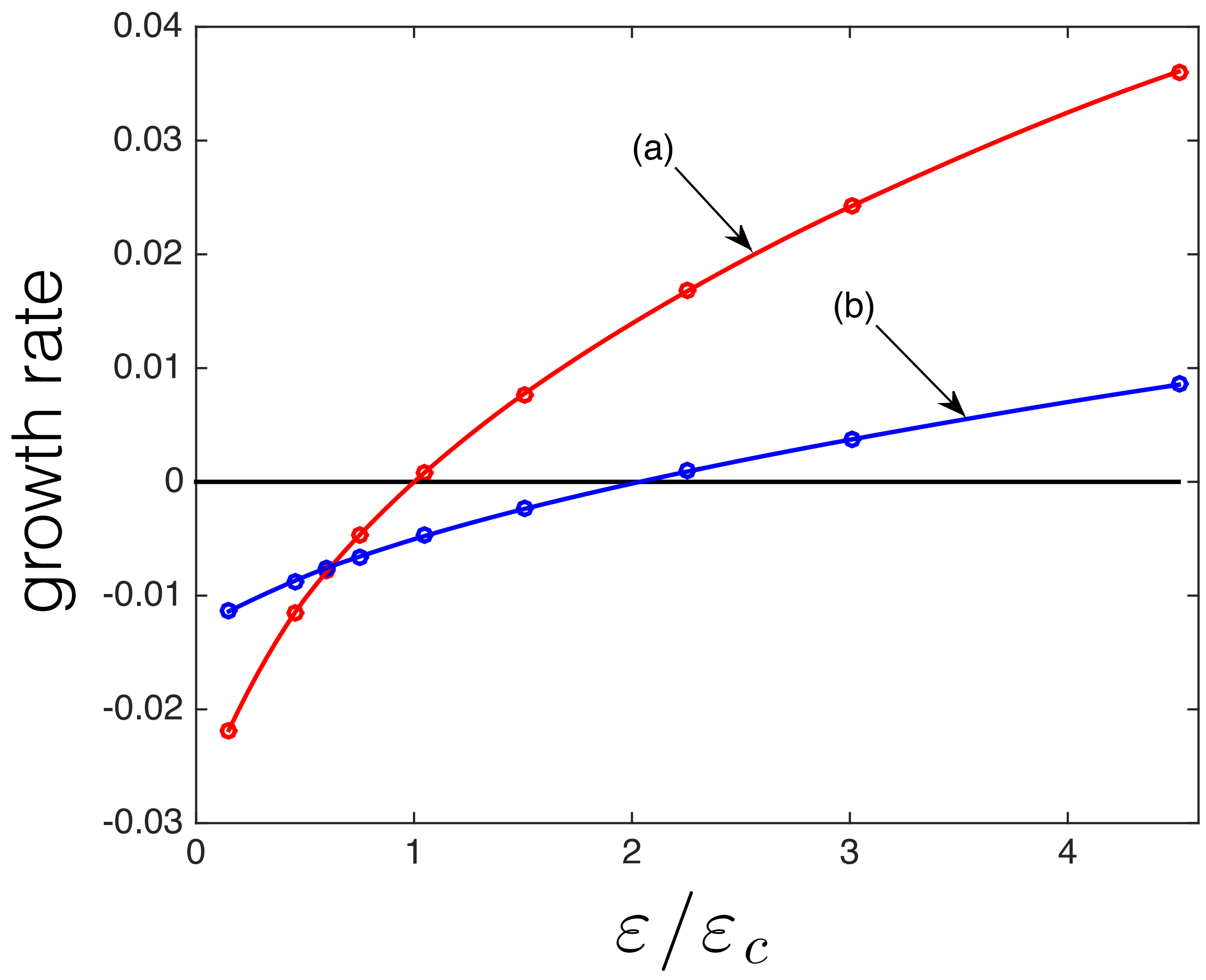
Farrell & Ioannou 2012

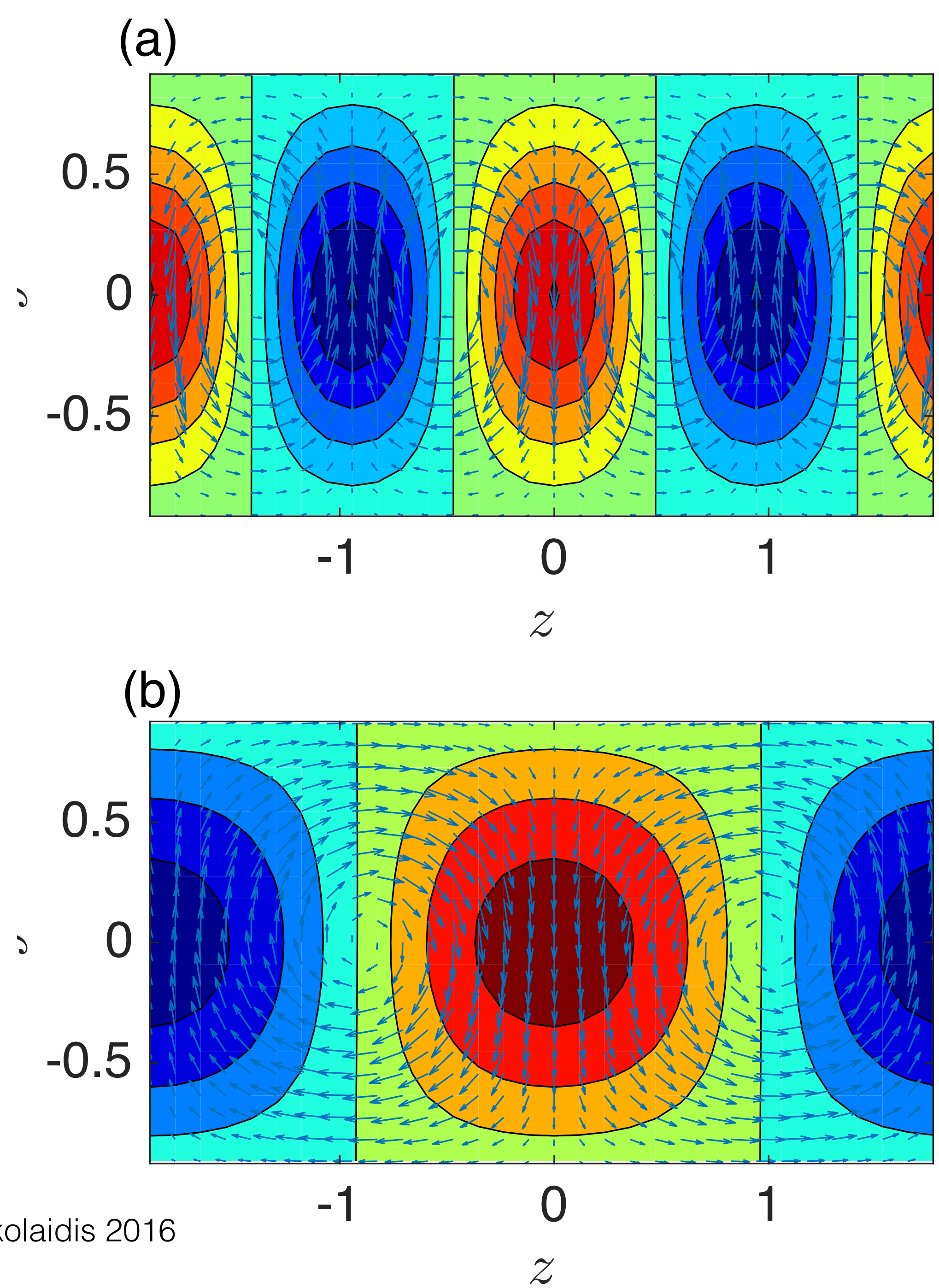
minimal channel
Re=400

it turns out that the stresses force a roll (V, W)
exactly such as to amplify the streak

Interpretation: turbulent Reynolds stresses are organized by the streak to force a roll circulation configured to amplify the streak

eigenvalues/eigenmodes of the least stable S3T roll/streak modes



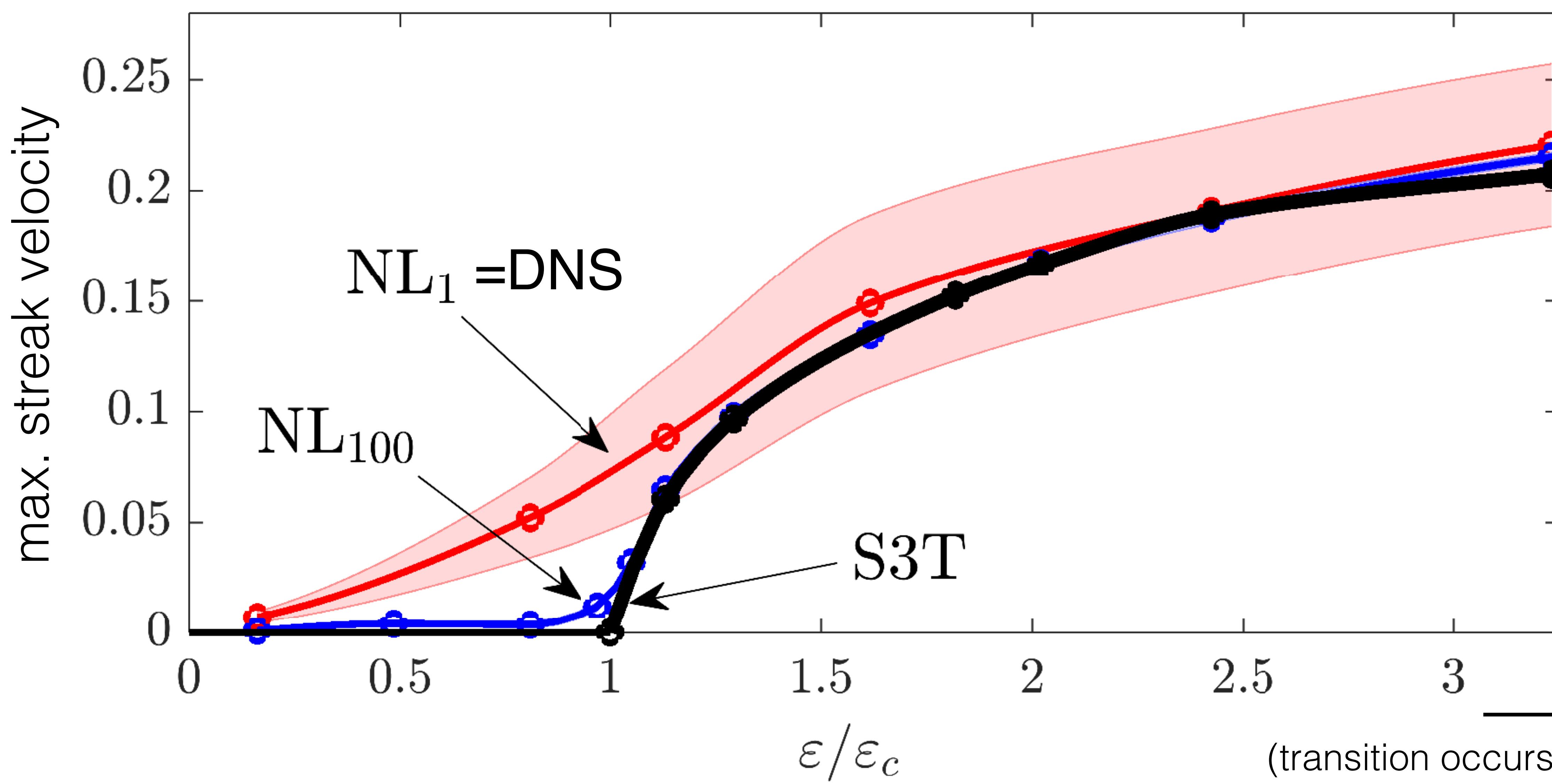


Farrell, Ioannou & Nikolaidis 2016

minimal channel: $L_x = 1.75\pi$, $L_z = 1.2\pi$, $Re = 400$, stochastic excitation at $k_x = 2\pi/L_x$

ε_C sustains turbulence with energy 0.14% of the Couette flow energy.

bifurcation structure

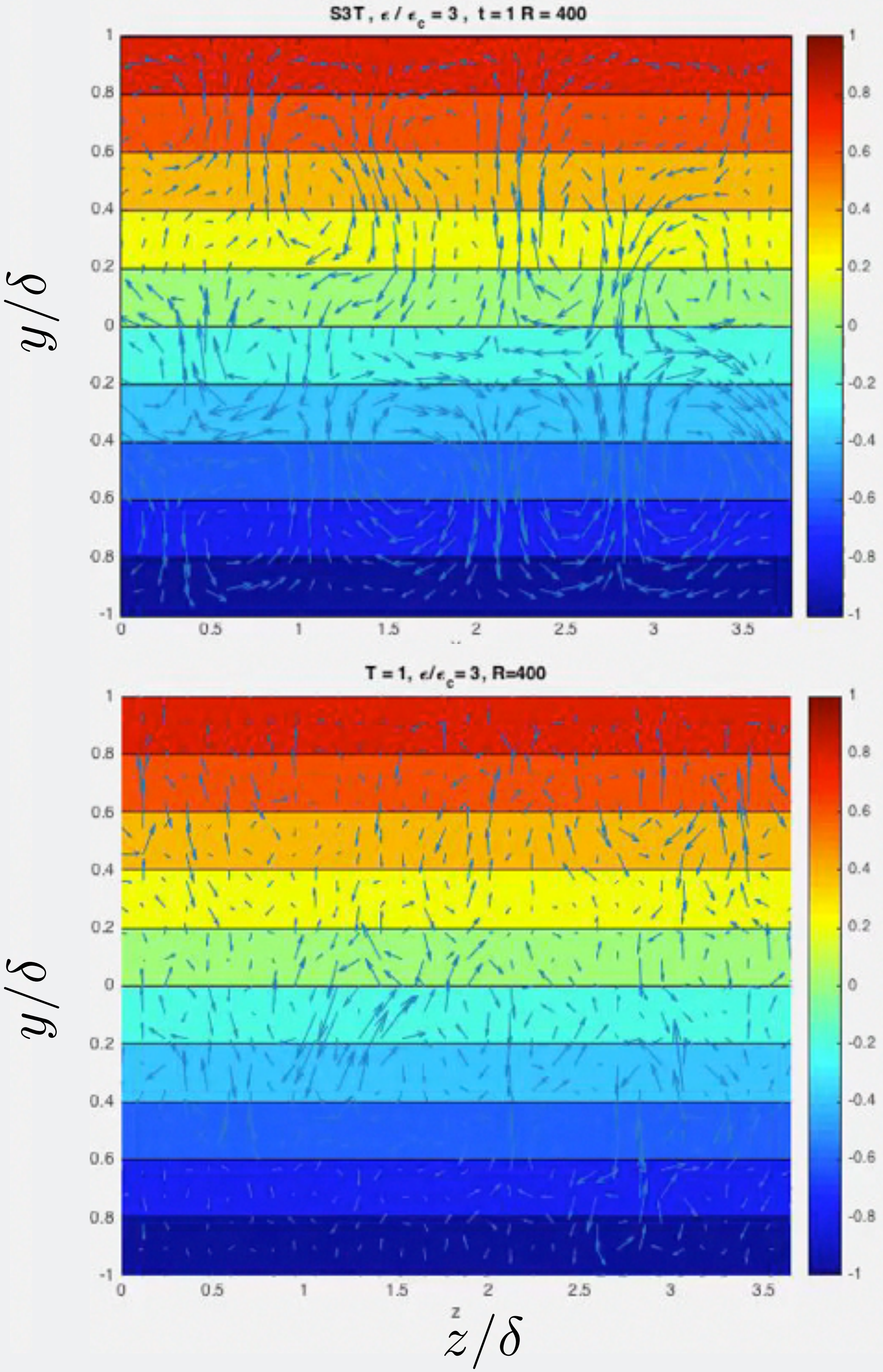


(transition occurs at ~6)

Farrell, Ioannou & Nikolaidis 2016

minimal channel
Re=400

S3T



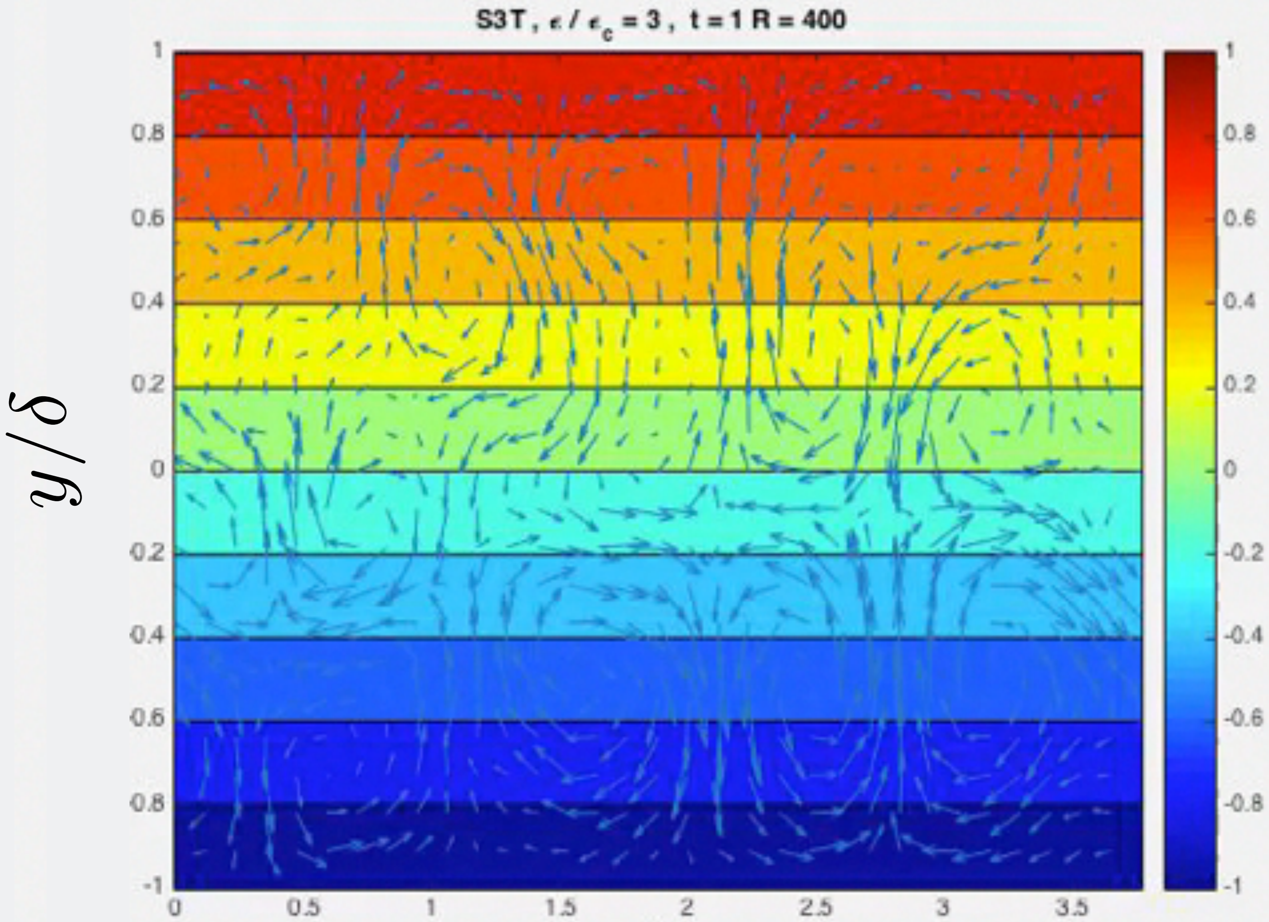
DNS

Farrell, Ioannou & Nikolaidis 2016

$\varepsilon/\varepsilon_c = 3$

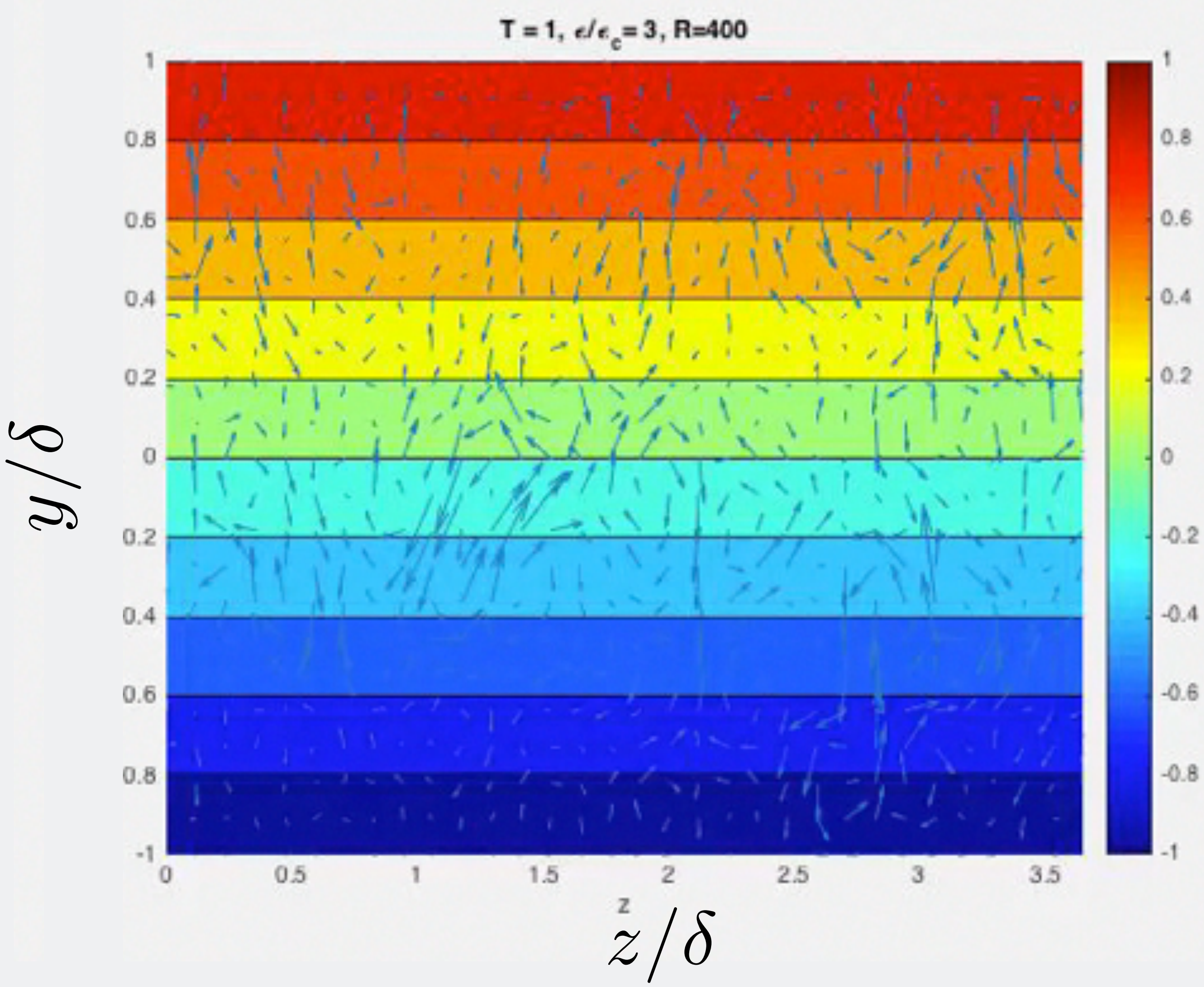
(pre-transitional)

S3T



Farrell, Ioannou & Nikolaidis 2016

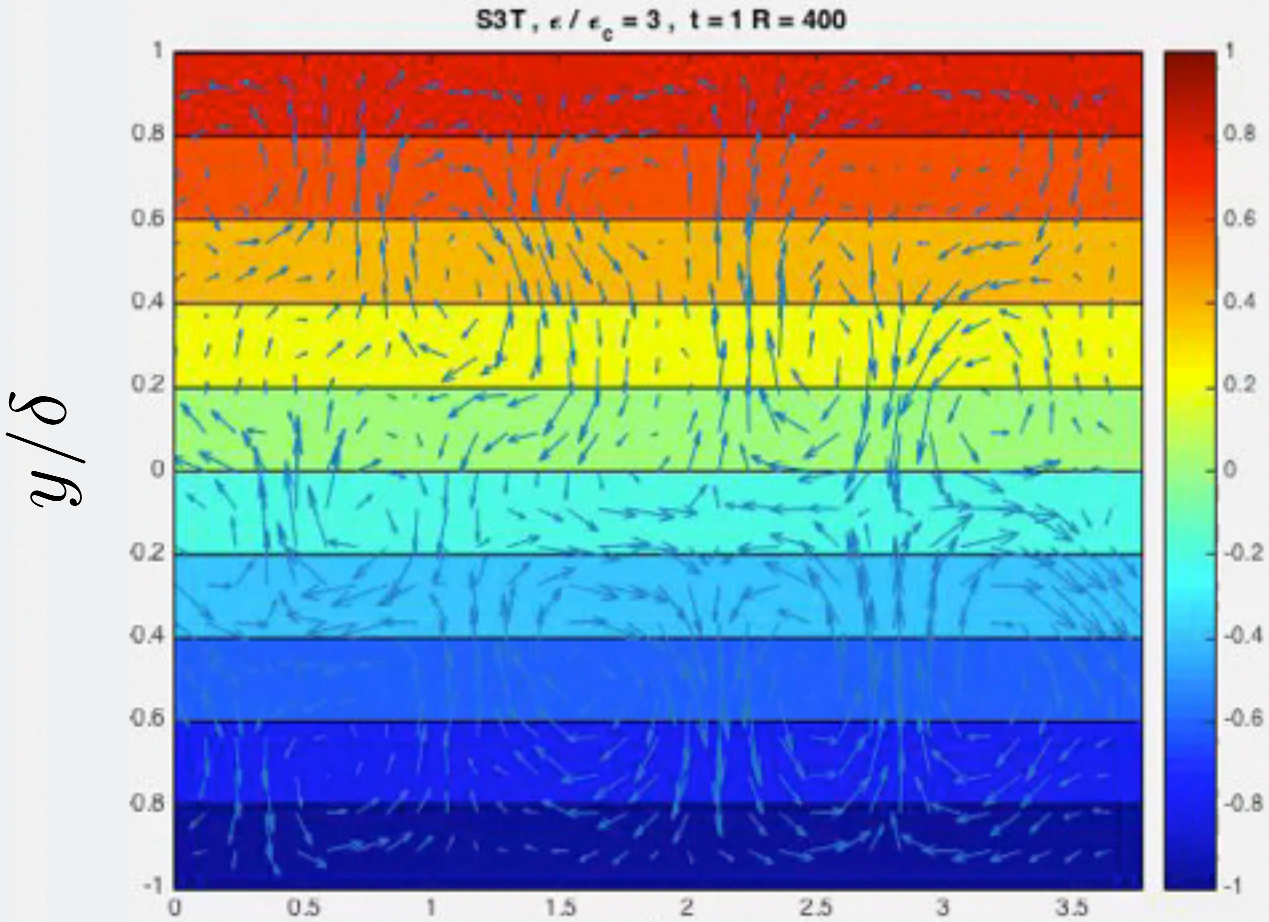
DNS



$\varepsilon/\varepsilon_c = 3$

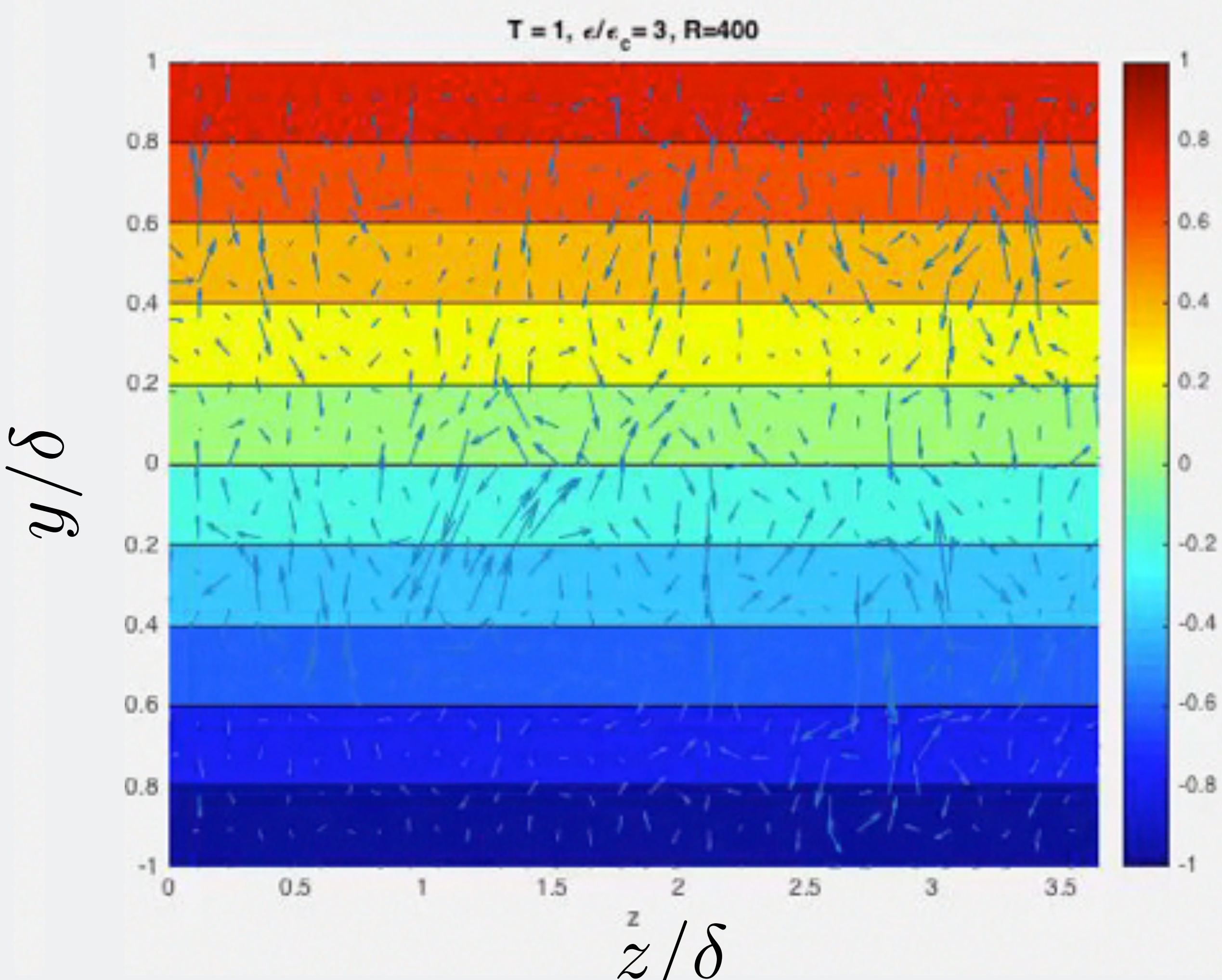
(pre-transitional)

S3T



Farrell, Ioannou & Nikolaidis 2016

DNS



$\varepsilon/\varepsilon_c = 3$
(pre-transitional)

Conclusions

- S3T generalizes the hydrodynamic stability of Rayleigh and allow us to study the stability of turbulent flows
- The emergence of coherent structures in a variety of flow settings is (analytically) predicted as an instability of the turbulent state
- S3T also predicts the final inhomogeneous turbulent state at which the system bifurcates to after the homogeneous state becomes unstable
- This is a first tool that enables us to determine the tipping points of the climate (climate = statistical turbulent equilibrium state)

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thanks