Spatial localization… or what is beyond the minimal flow units

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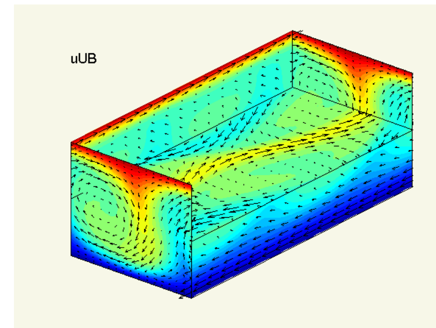
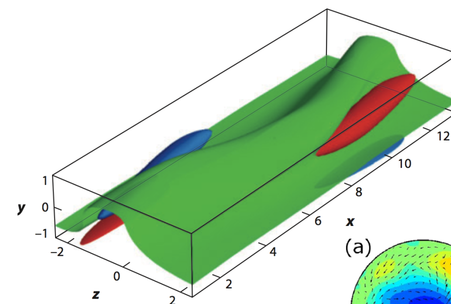
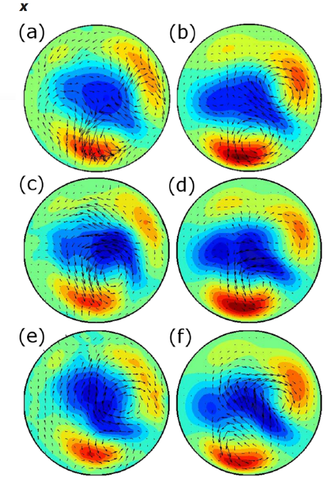
Exact coherent structures

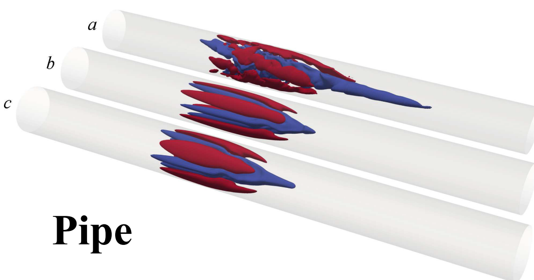
- Plane Couette flow
  - Nagata (1990)
  - Kawahara & Kida (2001)
  - Viswanath (2007)
  - Cvitanovic & Gibson (2010)
- Plane Poiseuille flow
  - Waleffe (2001)
  - Itano & Toh (2001)
  - Mellibovsky & Meseguer (2015)
- Pipe flow
  - Faisst & Eckhardt (2003)
  - Duguet, Pringle, & Kerswell (2008)
  - Willis, Cvitanovic, & Avila (2013)





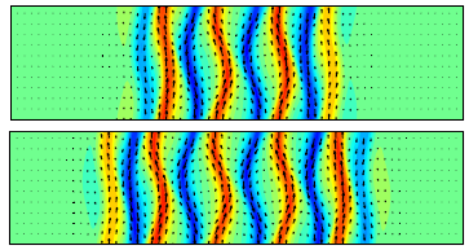


Spatial localization



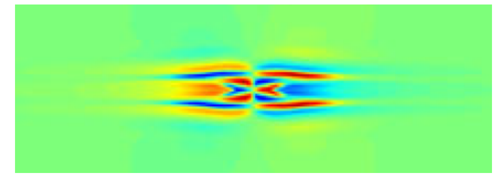
Pipe

Avila *et al.* (2013)

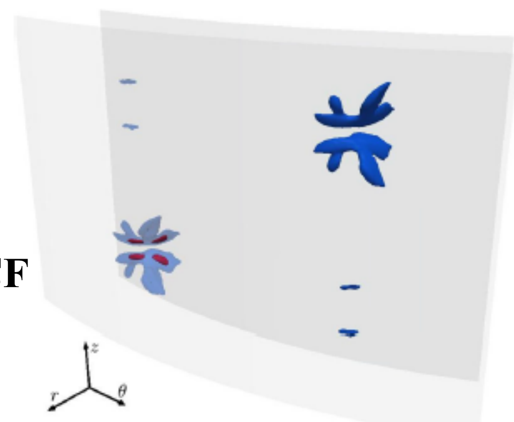


Schneider *et al.* (2013)

PCF

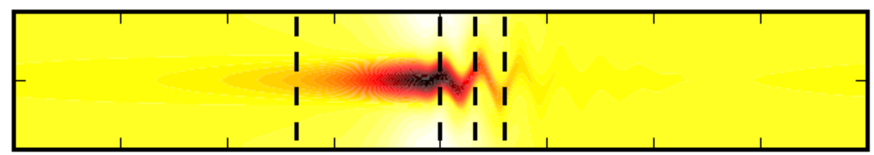


Brand and Gibson (2014)

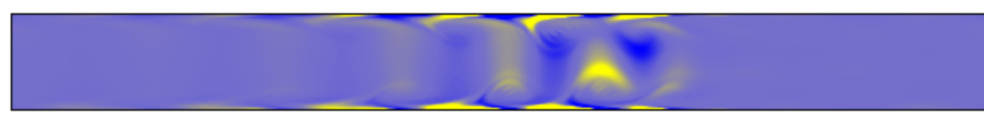


TCF

Deguchi *et al.* (2014)

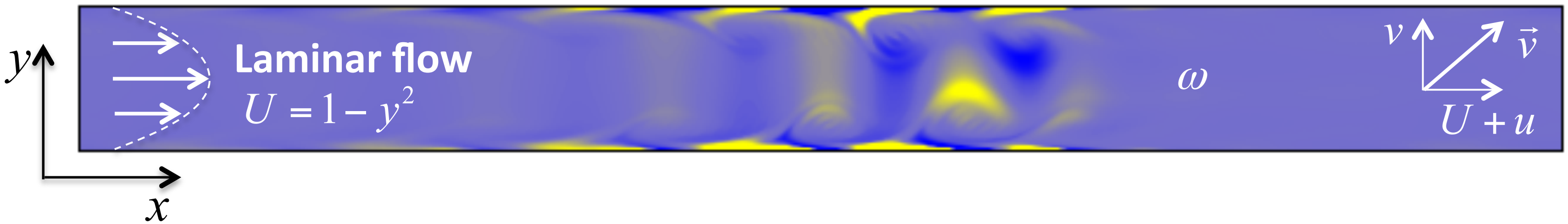


3D PPF Zammert, Eckhardt (2014)



2D PPF Mellibovsky, Meseguer (2015)

Modulated Tollmien-Schlichting wave



Orr-Sommerfeld equation: $(\partial_t + U\partial_x - Re^{-1}\nabla^2)\nabla^2 v = U''\partial_x v$

Squires equation: $(\partial_t + U\partial_x - Re^{-1}\nabla^2)\eta = -U''\partial_z v$

Stream function: $u = \partial_y \psi, \quad v = -\partial_x \psi$

$$\psi(x,y,t) = \varphi(y)e^{i\alpha x}e^{\lambda t}, \quad x \to \pm\infty$$

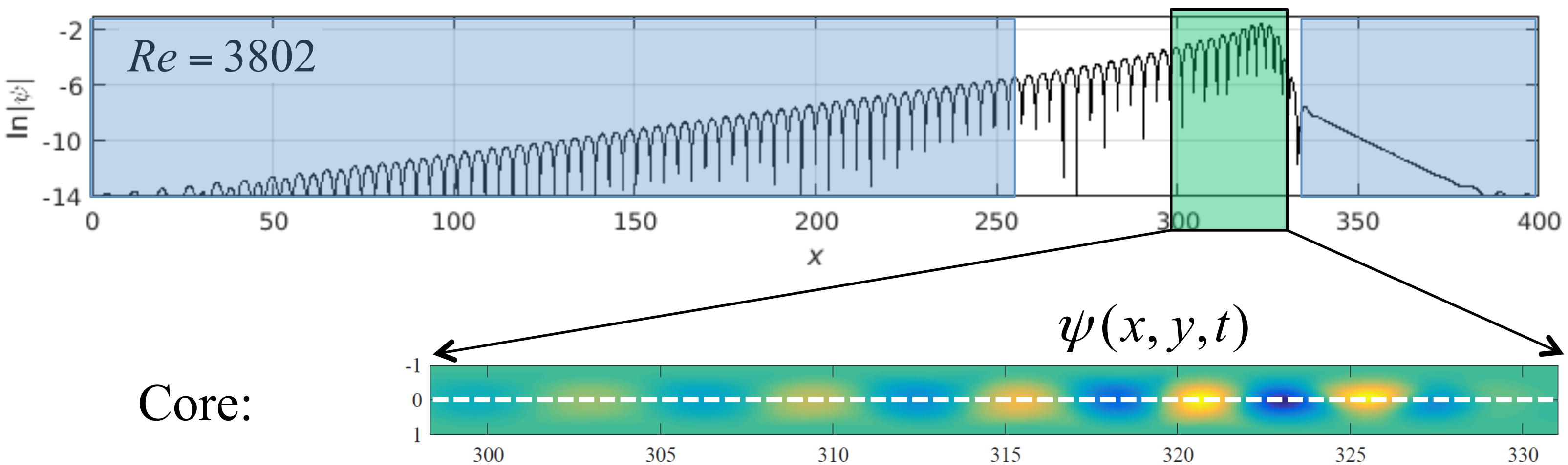
Boundary value problem:

$$\lambda(\partial_y^2 - \alpha^2)\varphi = \left[Re^{-1}(\partial_y^2 - \alpha^2)^2 + i\alpha(U'' - U(\partial_y^2 - \alpha^2)) \right]\varphi$$

$$\varphi(\pm 1) = \varphi'(\pm 1) = 0$$

Asymptotics

Numerically computed solution: (Channelflow + Newton-Krylov)

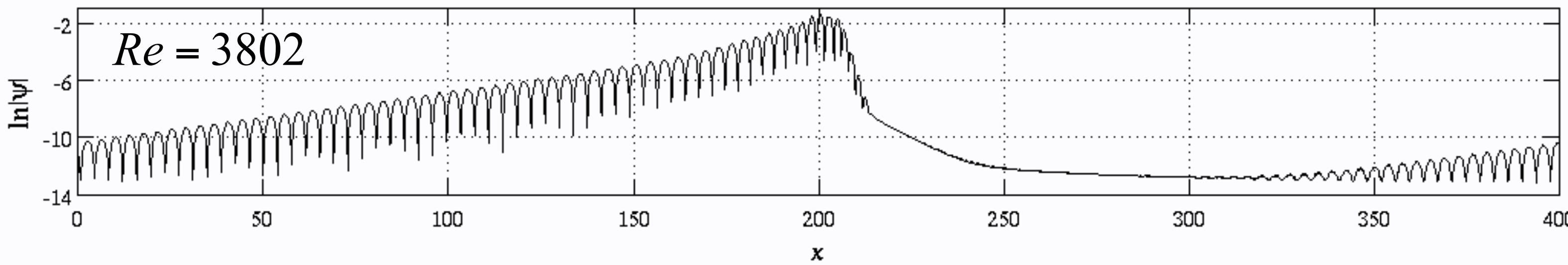


Core:

Asymptotics: $\psi(x,y,t) = \varphi(y)e^{i\alpha x}e^{\lambda t}$

$\alpha = q + is$ $\quad$ $\lambda = \sigma + i\omega$

Relative periodic orbit (RPO)



Asymptotics in a stationary frame:

$$\psi(x,y,t) = \varphi(y)e^{iqx-sx}e^{\sigma t+i\omega t}$$

In a moving reference frame:

$$\xi = x - ct \quad \rightarrow \quad \psi(\xi,y,t) = \varphi(y)e^{iq\xi-s\xi}e^{i(\omega+qc)t}e^{(\sigma-sc)t}$$

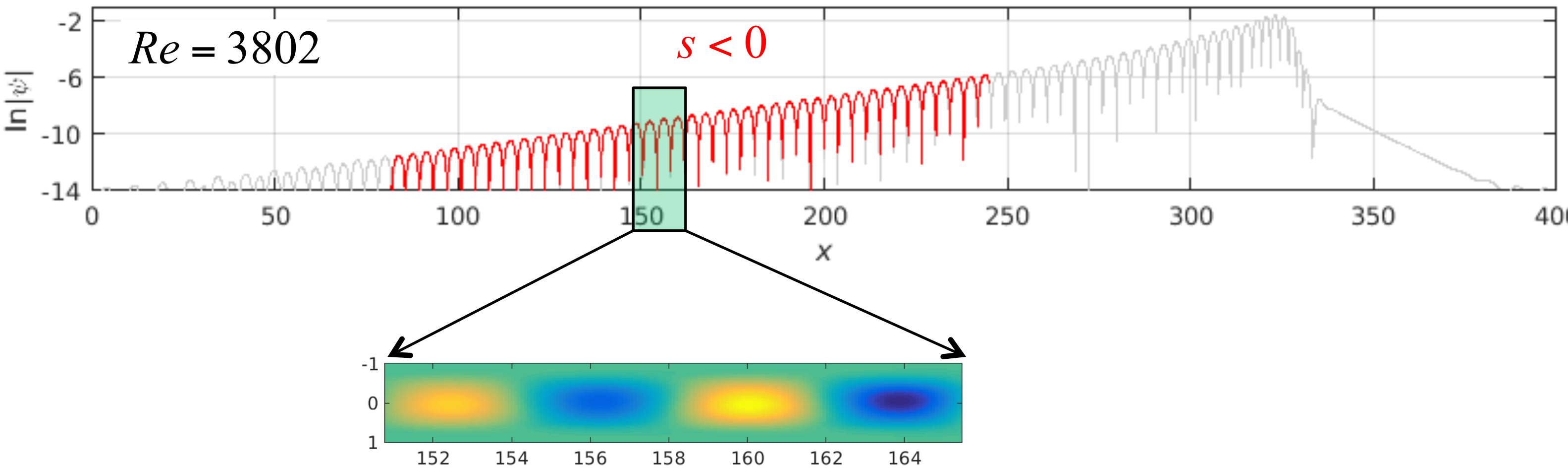
Should be marginally stable:

$$\sigma(q,s) - sc = 0$$

Should have the same period as RPO:

$$\omega(q,s) + qc = \frac{2\pi}{T}n, \quad n = 0,1,\cdots$$

Trailing (upstream) tail

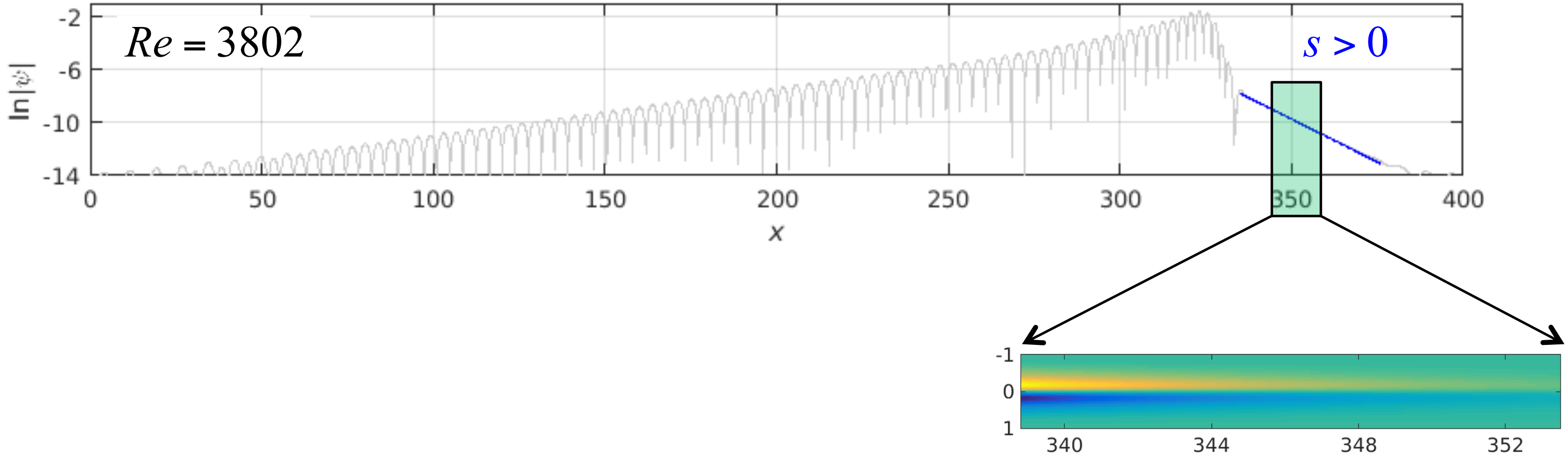


$$\psi(\xi, y, t) = \varphi_{tail}(y) e^{iq\xi - s\xi} e^{i\omega t}$$

Orr-Sommerfeld: $q = 0.826, \quad s = -0.0354, \quad n = 1$

Numerics: $q = 0.826, \quad s = -0.0355$

Leading (downstream) tail

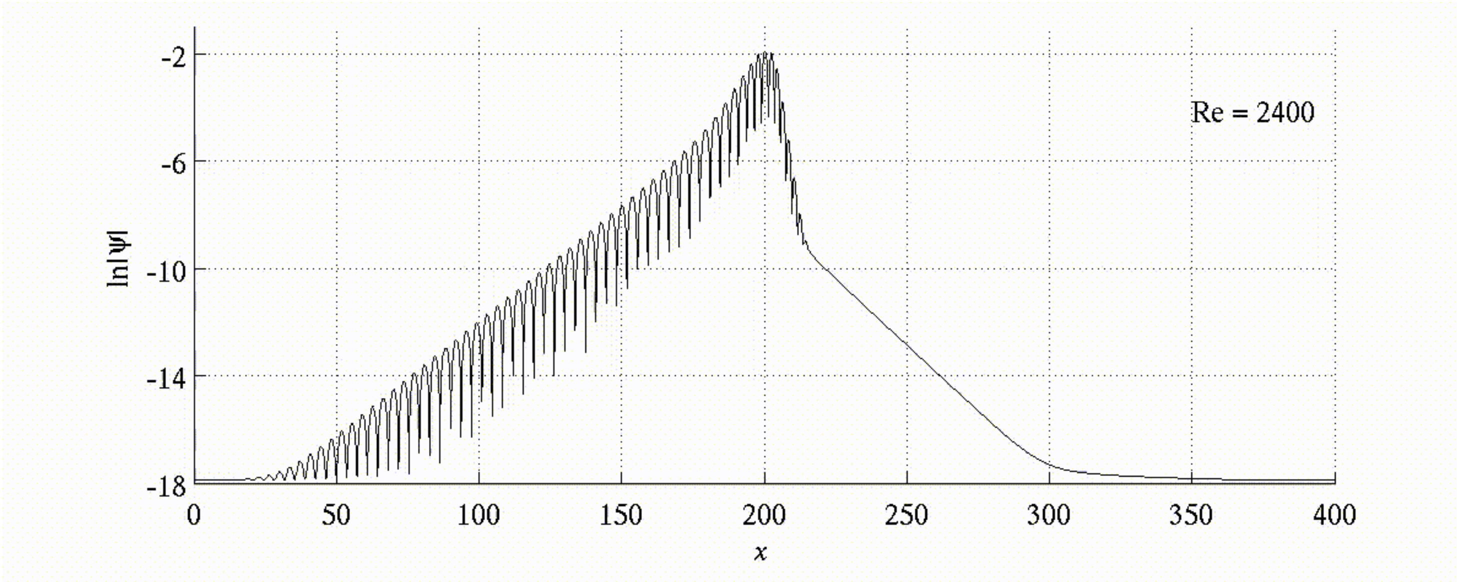


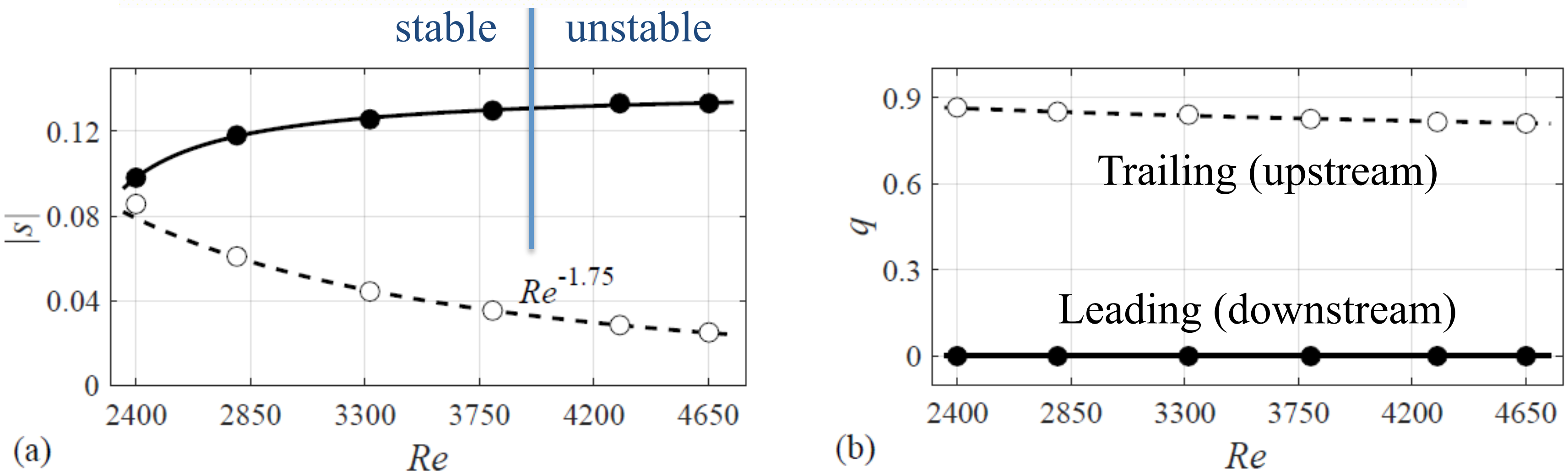
$$\psi(\xi, y, t) = \varphi_{head}(y) e^{-s\xi}$$

Orr-Sommerfeld: $q = 0, \quad s = 0.130, \quad n = 0$

Numerics: $q = 0, \quad s = 0.129$

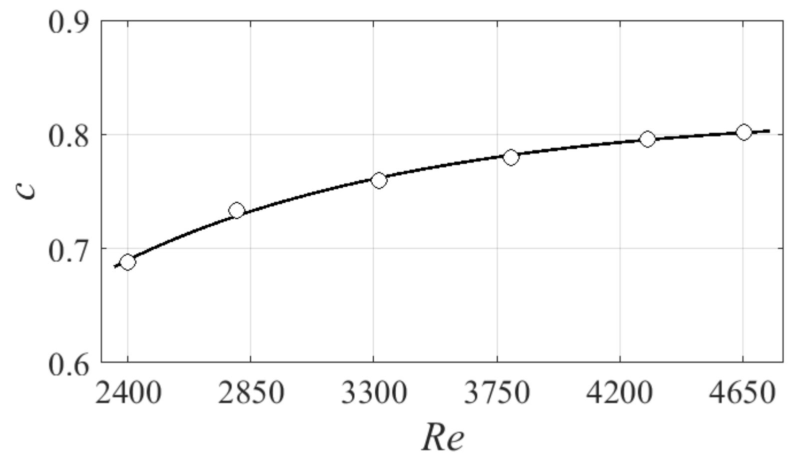
Re-dependence





Group speed

Numerical solution:



Linear front propagation theory:

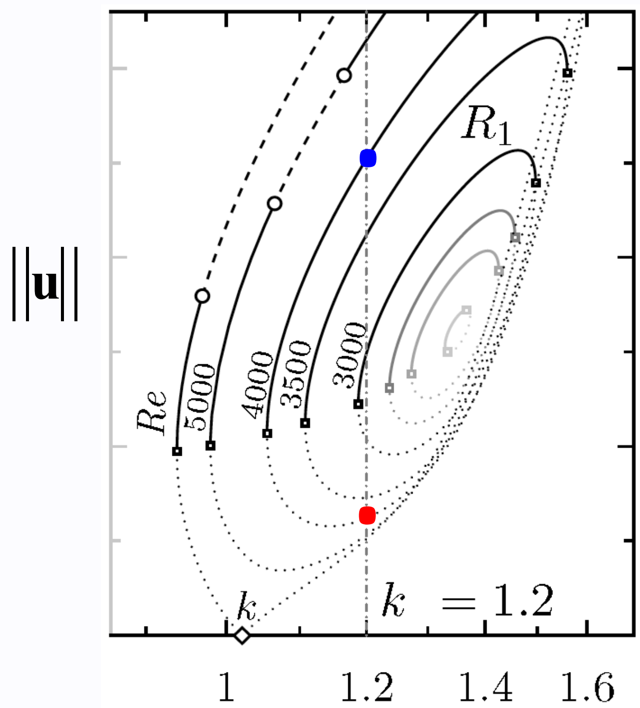
(pulled front)

$$c = i\frac{\partial\lambda}{\partial\alpha} = \frac{\mathrm{Re}(\lambda)}{\mathrm{Im}(\alpha)}$$

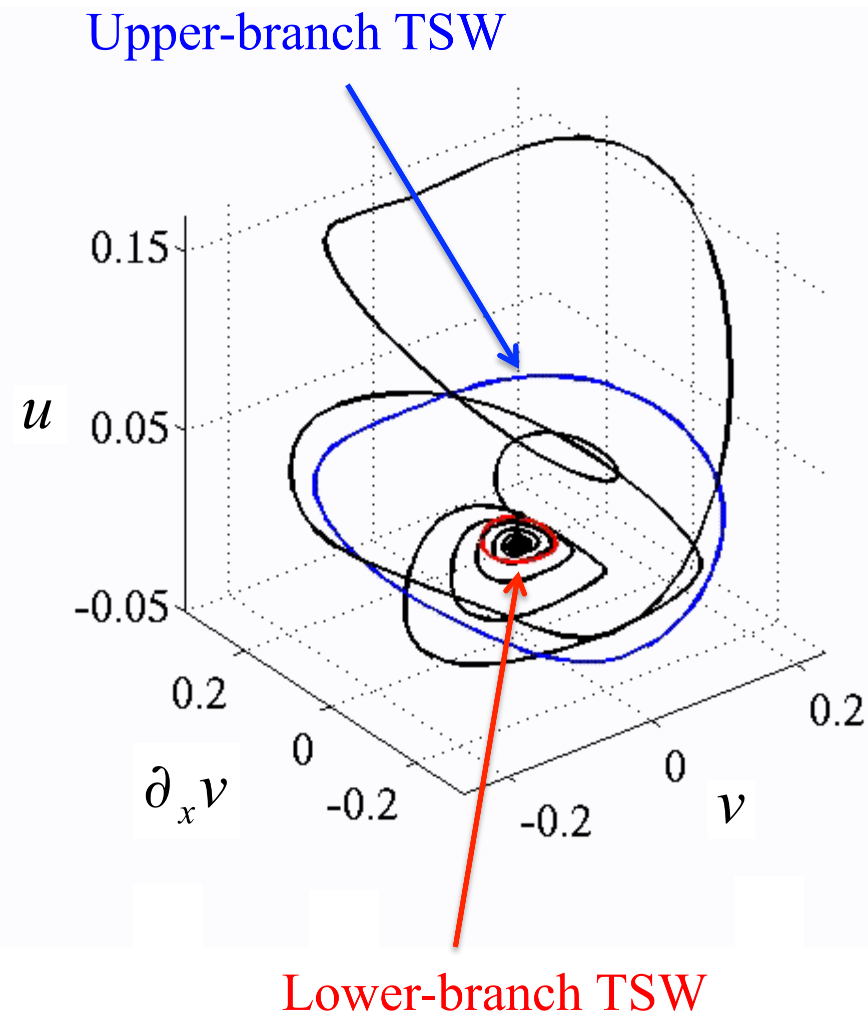
$$\Rightarrow \quad \frac{\sigma}{s} = \frac{\partial\sigma}{\partial s} \quad (\dagger)$$

No solutions of (†) $\Rightarrow$ Speed selected by nonlinear mechanism (pushed front)

Relation to Tollmien-Schlichting waves

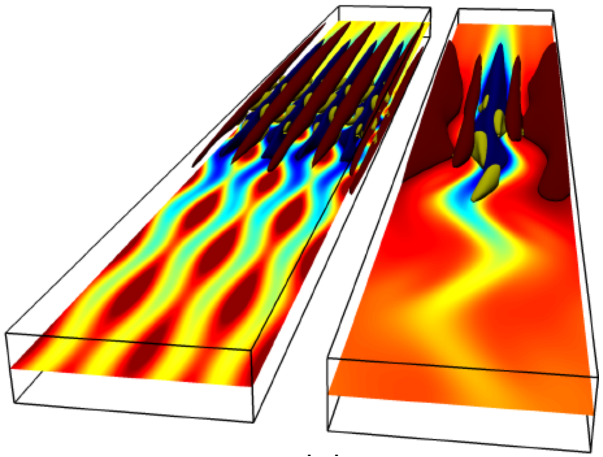


Mellibovsky & Meseguer (2015)



Alternative approach

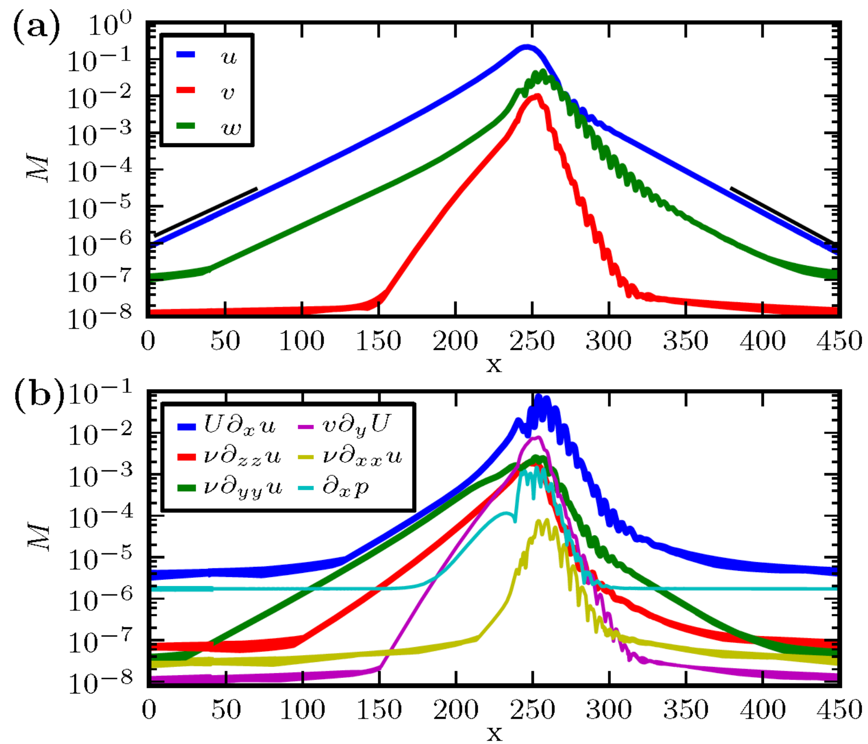
Linearized NSE: $\partial_t \mathbf{u} + U\partial_x \mathbf{u} + v\partial_y U\hat{\mathbf{x}} = -\nabla p + Re^{-1}\nabla^2 \mathbf{u}$



Zammert & Eckhardt (2016)

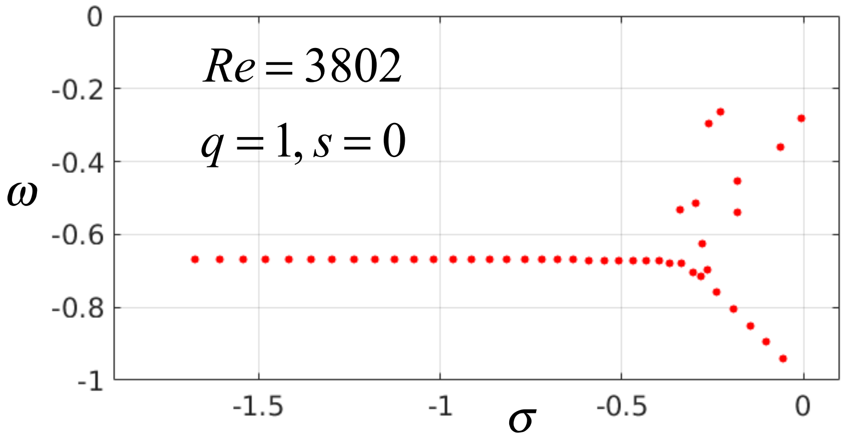
$$\partial_t u + U\partial_x u \approx Re^{-1}(\partial_y^2 u + \partial_z^2 u)$$

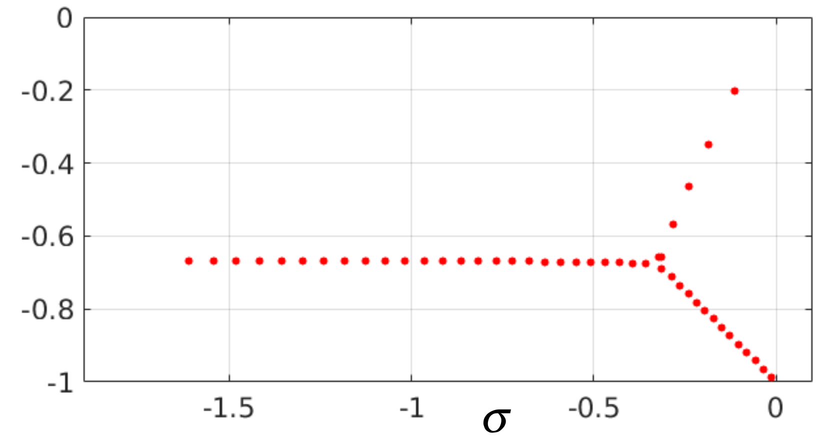
Brand & Gibson (2014)



Look for solutions (REq) in the form: $u(x,y,z,t) = \tilde{u}(y)e^{i\gamma z - s(x-ct)}$

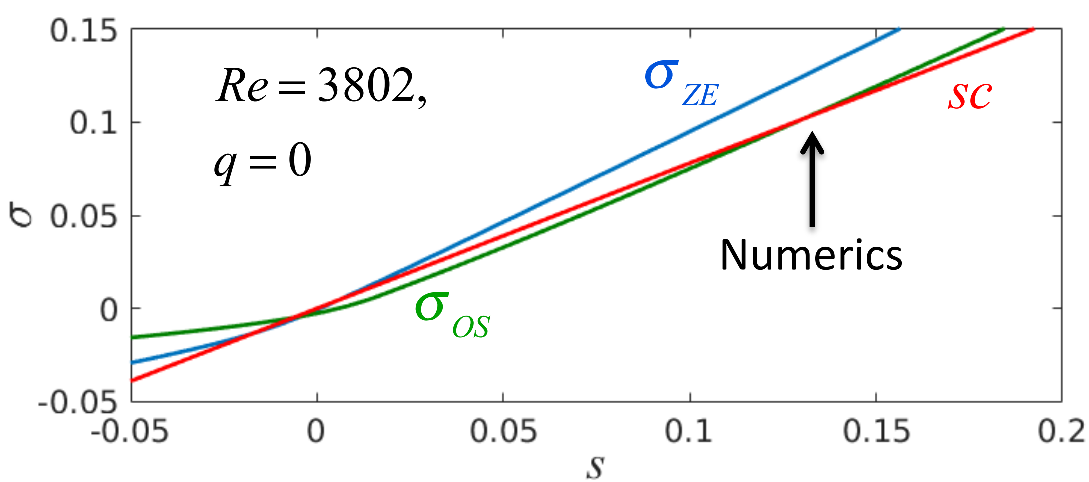
Eigenvalue spectra





$$(\partial_t + U\partial_x - Re^{-1}\nabla^2)\nabla^2 v = U''\partial_x v$$

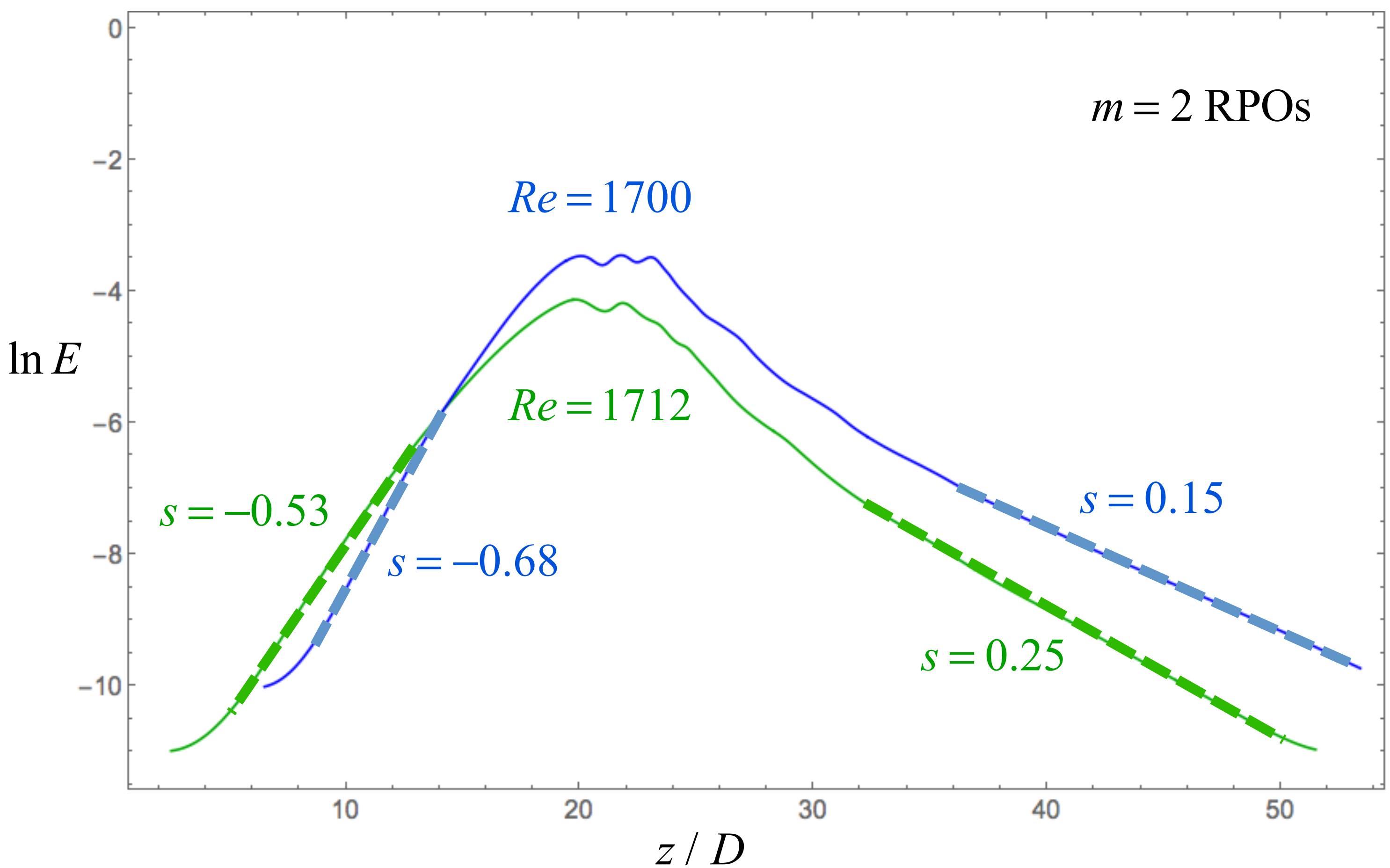
$$\partial_t u + U\partial_x u \approx Re^{-1}(\partial_y^2 u + \partial_x^2 u)$$



$$\sigma(q,s) - sc = 0$$

$$\omega(q,s) + qc = \frac{2\pi}{T}n$$

Pipe flow



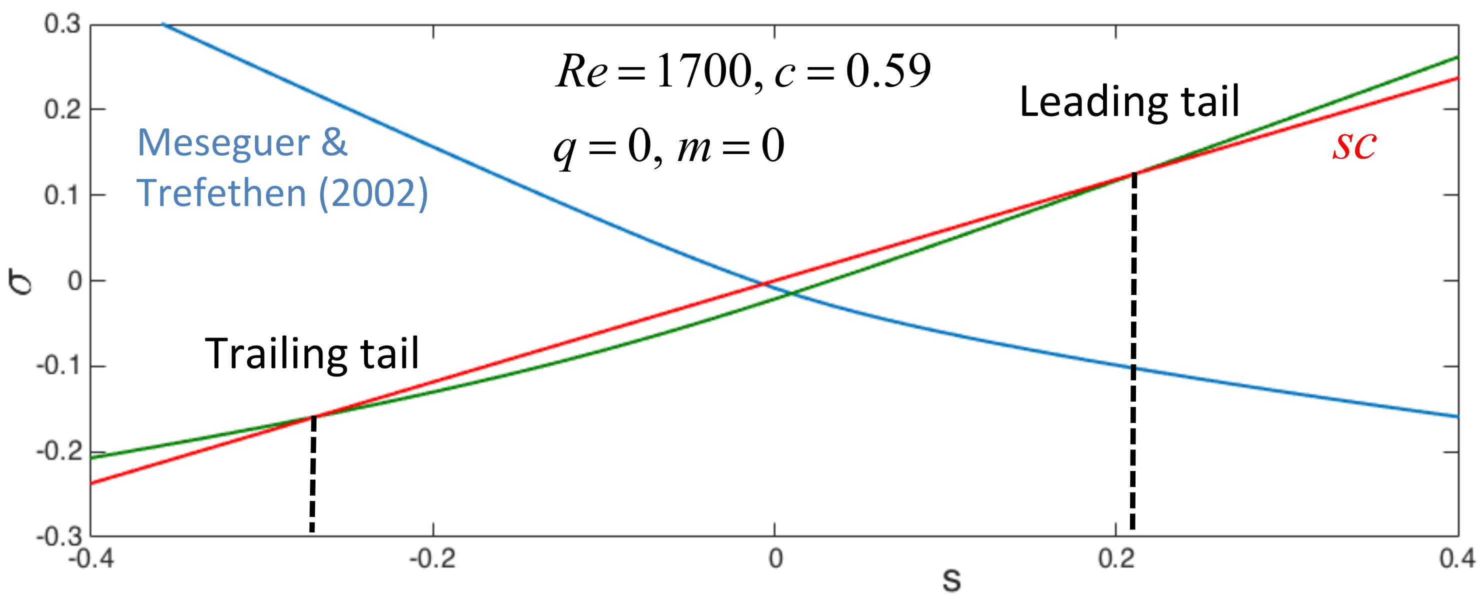
m = 2 RPOs
Re = 1700
Re = 1712
ln E
z / D
s = −0.53
s = −0.68
s = 0.15
s = 0.25
0
−2
−4
−6
−8
−10
10
20
30
40
50

Linearization

The $m = 0$ mode: $\mathbf{u} = r^{-1}\partial_z\phi\hat{\mathbf{r}} - r^{-1}\partial_r\phi\hat{\mathbf{z}}, \quad \phi(r,z) = f(r)e^{i\alpha z}$

Boundary value problem:

$$\lambda\left[\alpha^2 rf + f' - rf''\right] = i\alpha(1-r^2)(\alpha^2 rf - rf'' + f')$$
$$-\mathrm{Re}^{-1}\left[-\alpha^4 r^3 f + (3-2\alpha^2 r^2)(f' - rf'') + 2r^2 f''' - r^3 f''''\right]$$



Conclusions

- Orr-Sommerfeld equation predicts exponential scaling of leading and trailing asymptotics with *excellent* accuracy for RPOs (and REQs)
- Same approach can be extended to describe spatial localization of solutions in other shear flows (e.g., pipe flow).
- In 2D PPF, the length of the leading tail is essentially *Re*-independent, trailing tail extends as $Re^{1.75}$.
- The group speed c is controlled by a *nonlinear* mechanism (pushed vs. pulled front) and is *different* for different solutions
- Pressure gradient terms are important, generally cannot be neglected

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