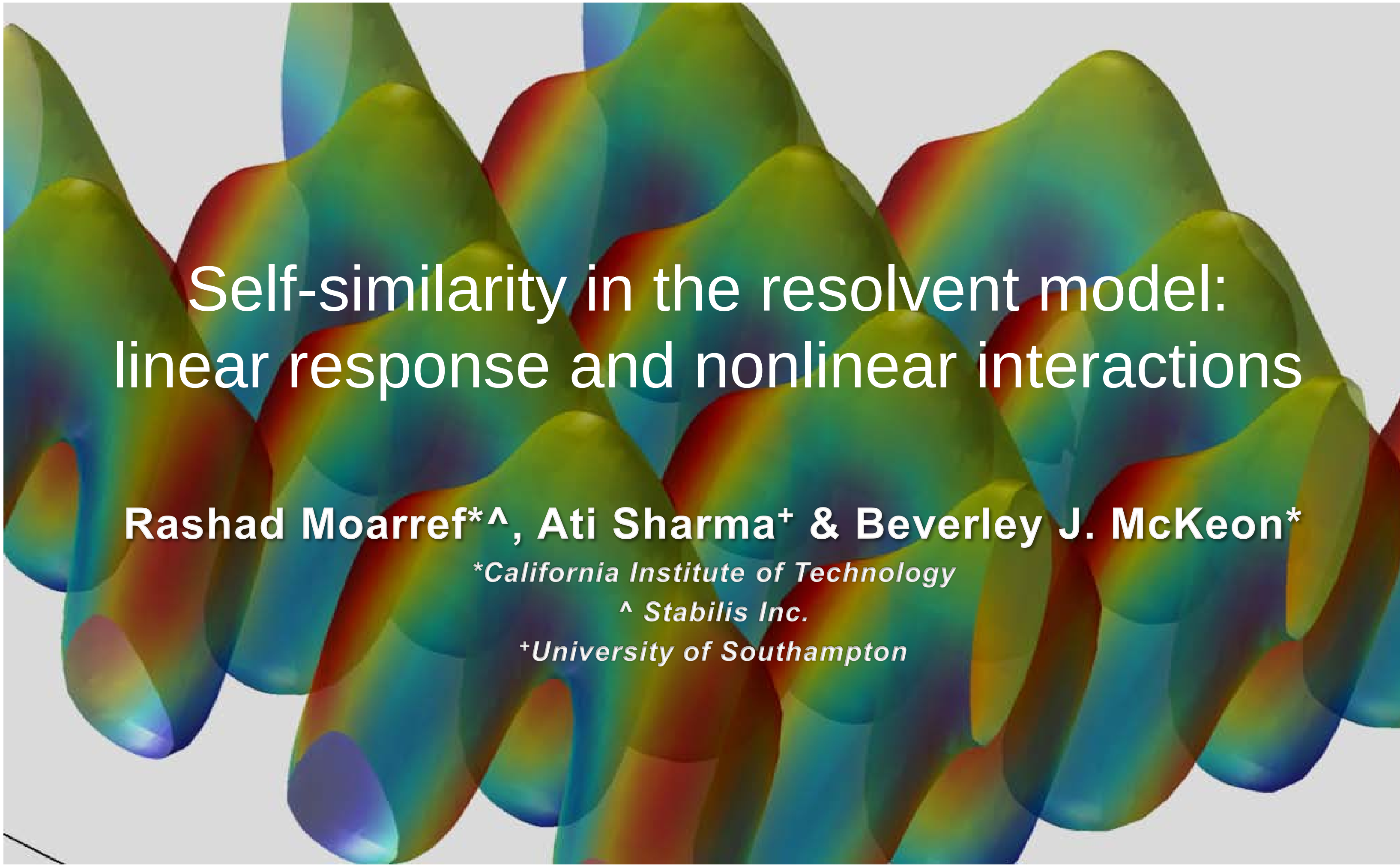

Self-similarity in the resolvent model: linear response and nonlinear interactions

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**California Institute of Technology*

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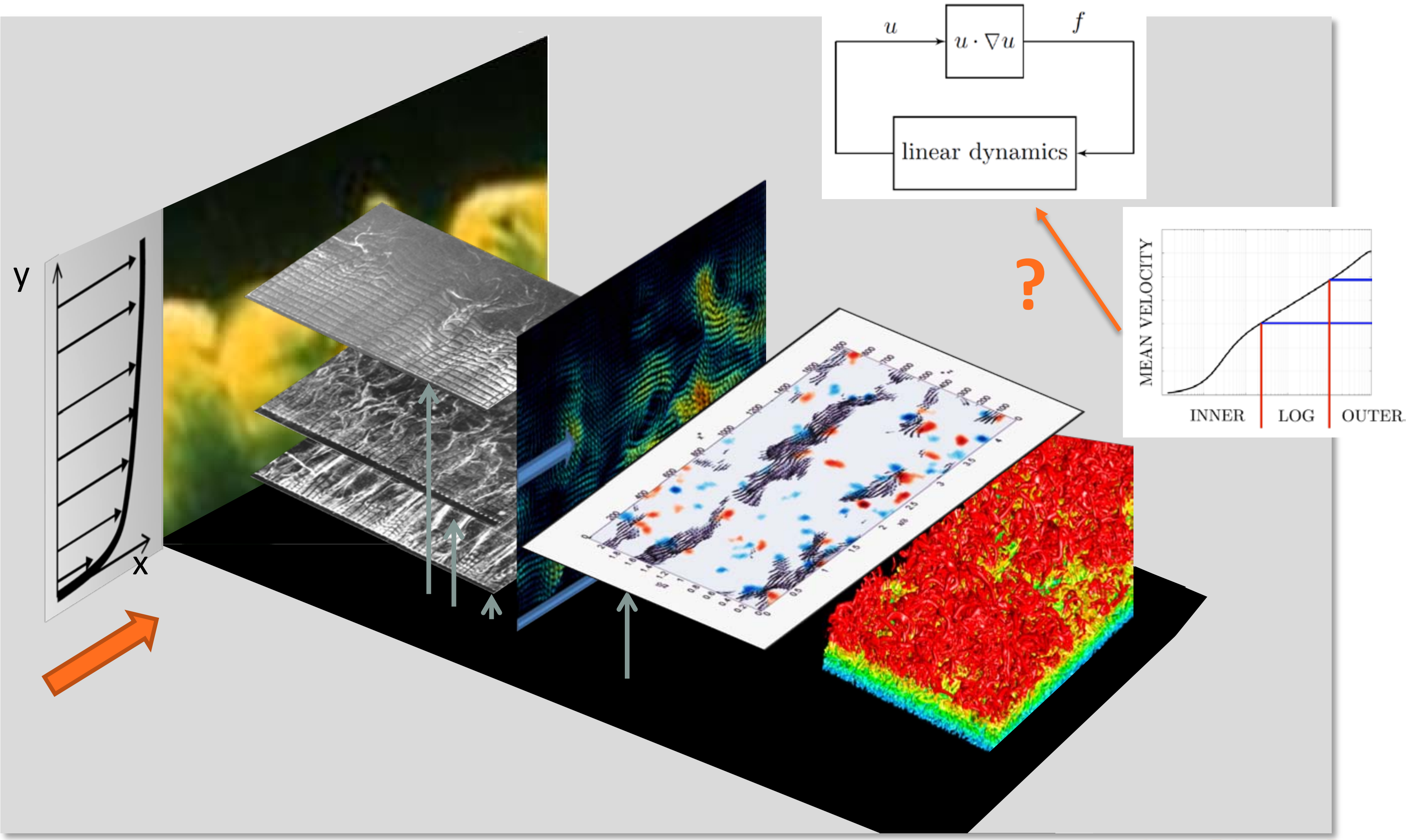
OUTLINE



von Karman (1930)
Millikan (1938)
Coles (1952)

Gad-el-Hak
http://efluids.com

Kline, Reynolds, Schraub & Runstadler
J. Fluid Mechanics (1967)

LeHew, Guala & McKeon
Expts. in Fluids (2011)

Wu & Moin
http://ctr.stanford.edu

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SUMMARY OF RESOLVENT ANALYSIS

1. Fourier decomposition in homogeneous directions

$$\mathbf{u}(x,y,z,t) \;=\; \iiint_{-\infty}^{\infty} \hat{\mathbf{u}}(y;\kappa_x,\kappa_z,\omega)\,\mathrm{e}^{\mathrm{i}(\kappa_x x \,+\, \kappa_z z \,-\, \omega t)}\,\mathrm{d}\kappa_x\,\mathrm{d}\kappa_z\,\mathrm{d}\omega$$

$$\mathbf{u}_t \,+\, (\mathbf{u}\cdot\nabla)\mathbf{U} \,+\, (\mathbf{U}\cdot\nabla)\mathbf{u} \,+\, \nabla p \,-\, (1/Re_\tau)\nabla^2\mathbf{u} \;=\; \overbrace{-(\mathbf{u}\cdot\nabla)\mathbf{u}}^{\mathbf{f}}$$

resolvent operator

$$\hat{\mathbf{u}}(y;\kappa_x,\kappa_z,\omega) \;=\; H(\kappa_x,\kappa_z,\omega)\,\hat{\mathbf{f}}(y;\kappa_x,\kappa_z,\omega)$$

2. singular value decomposition in y for any $\kappa_x, \kappa_z, \omega$

$$\hat{\mathbf{f}}(y) \;=\; \hat{\phi}_j(y) \quad \xrightarrow{\;\;H\;\;} \quad \hat{\mathbf{u}}(y) \;=\; \sigma_j\,\hat{\psi}_j(y)$$

$$\hat{\mathbf{f}}(y) \;=\; \sum_{j=1}^{\infty} \chi_j\,\hat{\phi}_j(y) \quad \xrightarrow{\;\;H\;\;} \quad \hat{\mathbf{u}}(y) \;=\; \sum_{j=1}^{\infty} \chi_j\,\sigma_j\,\hat{\psi}_j(y)$$

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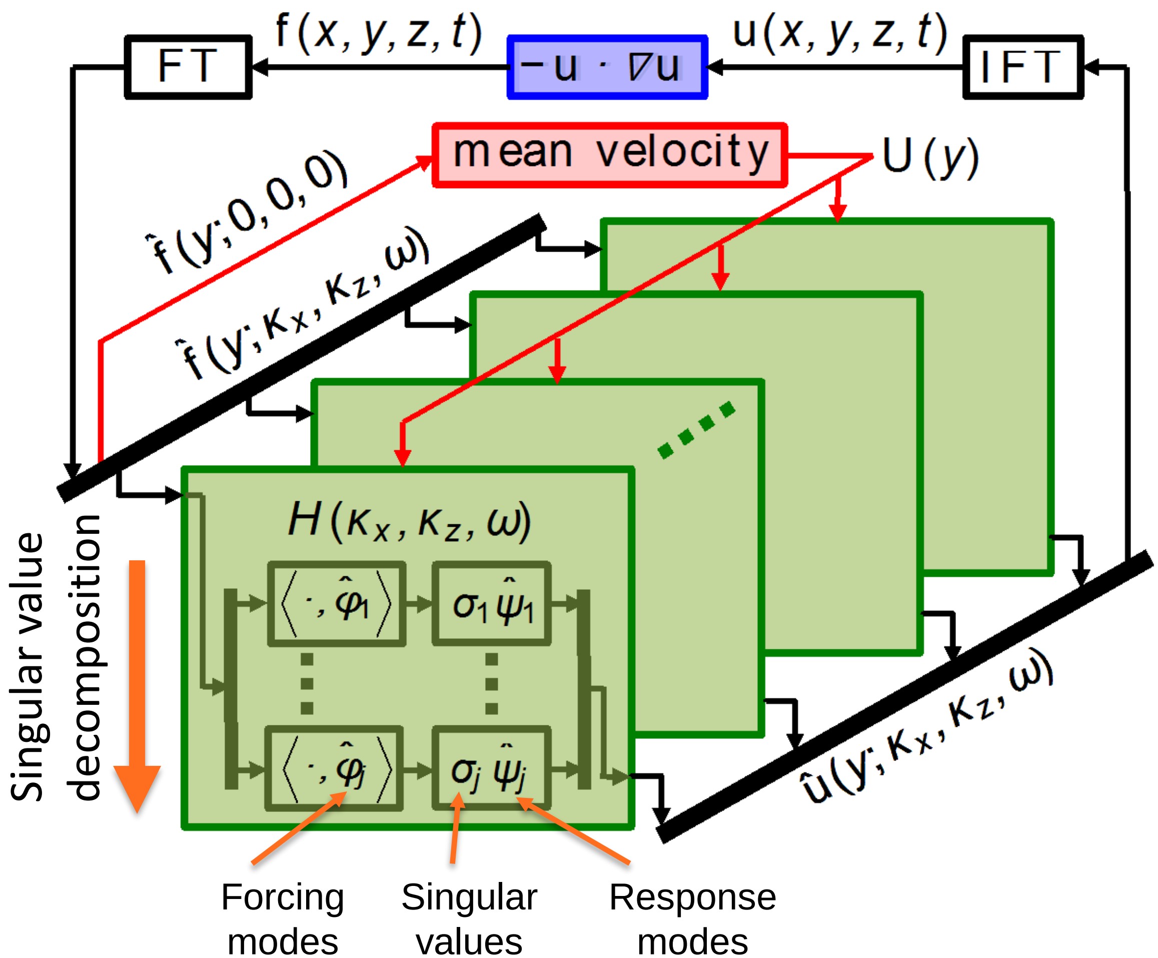
(randomized scheme by Halko, Martinsson & Tropp, SIAM Rev., 2011)

Resolvent analysis – systems representation

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{u} + \nabla p - (1/Re_\tau)\nabla^2\mathbf{u} = \overbrace{-(\mathbf{u}\cdot\nabla)\mathbf{u}}^{\mathbf{f}}$$



Boundary conditions:
$u_w=0,\ v_w=0,\ w_w=0$

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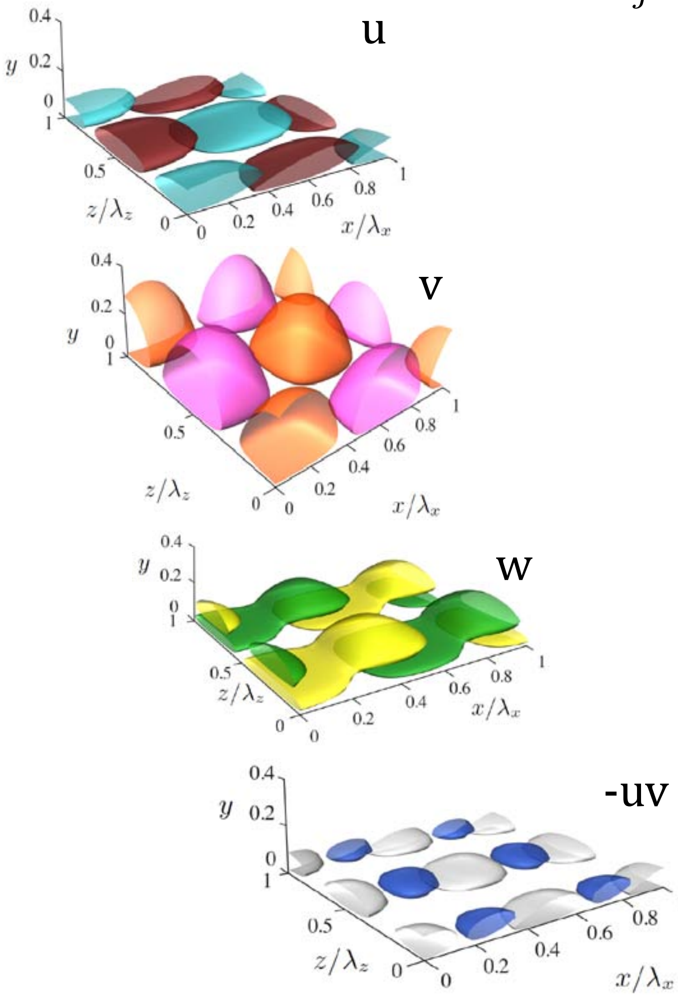
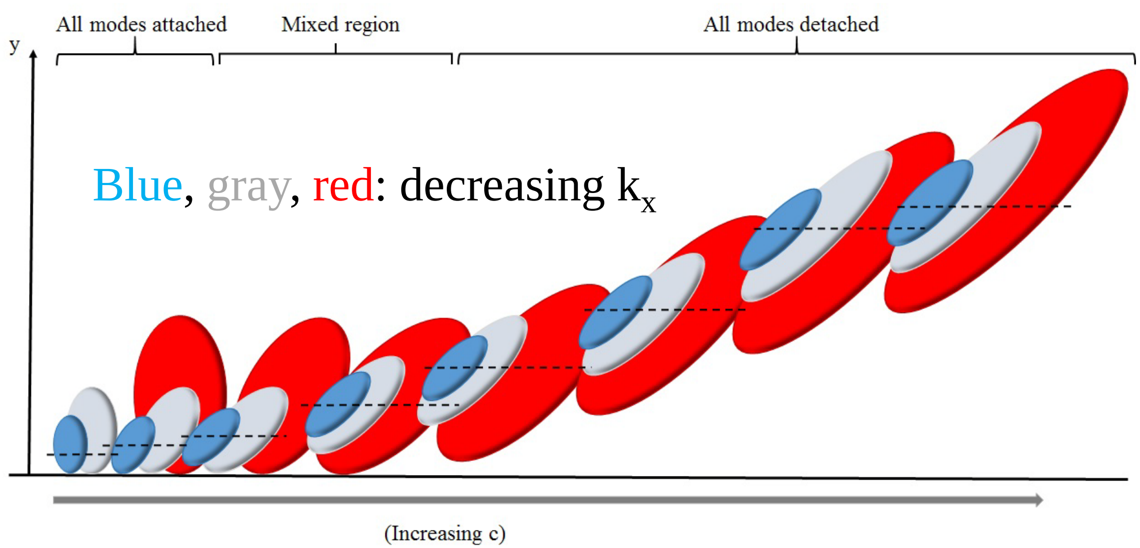
RESOLVENT MODE SHAPES

$$\hat{\mathbf{u}}(y;\kappa_x,\kappa_z,\omega) = \sum_{j=1}^{\infty} \chi_j(\kappa_x,\kappa_z,\omega)\,\sigma_j(\kappa_x,\kappa_z,\omega)\,\hat{\psi}_j(y;\kappa_x,\kappa_z,\omega)$$





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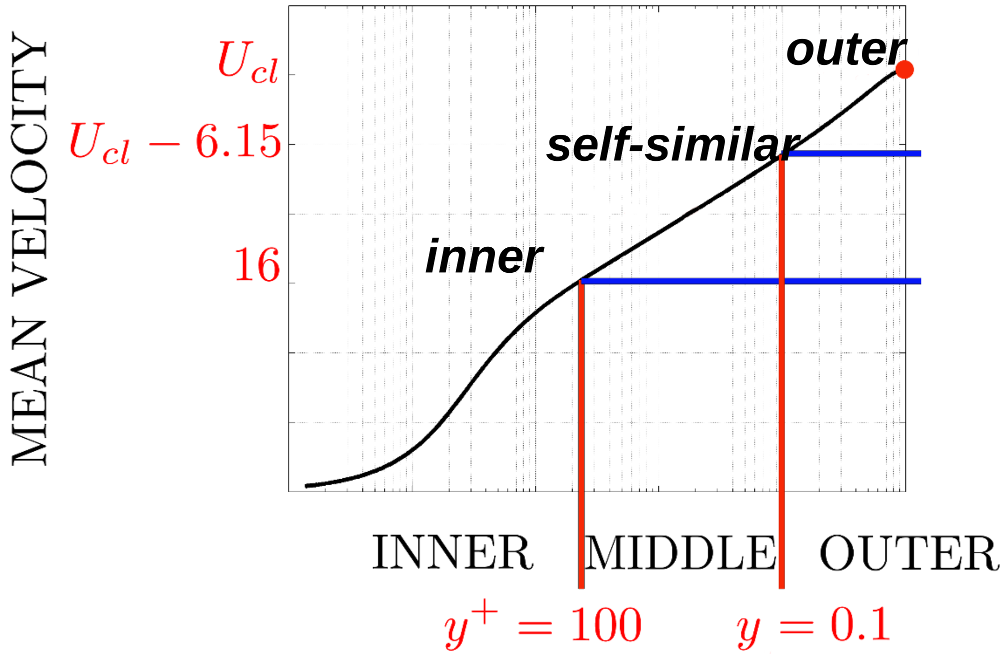
INFLUENCE OF THE MEAN VELOCITY

$$H(\kappa_x, \kappa_z, \omega) = C(\kappa_x, \kappa_z) R(\kappa_x, \kappa_z, \omega) C^\dagger(\kappa_x, \kappa_z)$$

$$R = \begin{bmatrix} \Delta^{-1}\left(\mathrm{i}\kappa_x (U - c)\Delta - U''\right) - (1/Re_\tau)\Delta^2) & 0 \\ \mathrm{i}\kappa_z U' & \mathrm{i}\kappa_x (U - c) - (1/Re_\tau)\Delta \end{bmatrix}^{-1}$$



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Moarref , Sharma, Tropp & McKeon, J. Fluid Mech. (2013)

Requirements for self-similarity

$$H(\kappa_x,\kappa_z,\omega) \;=\; C(\kappa_x,\kappa_z)R(\kappa_x,\kappa_z,\omega)C^{\dagger}(\kappa_x,\kappa_z)$$

$$R \;=\; \begin{bmatrix} \Delta^{-1}\left(\mathrm{i}\kappa_x\,(U-c)\Delta \;-\; U''\right) \;-\; (1/Re_\tau)\Delta^2\right) & 0 \\ \mathrm{i}\kappa_z U' & \mathrm{i}\kappa_x(U-c) \;-\; (1/Re_\tau)\Delta \end{bmatrix}^{-1}$$

condition 1: $U - c$ should be independent of Re_τ
yields: scaling in y

condition 2: wall-normal locality of resolvent modes
yields: range of c

condition 3: $\mathrm{i}\kappa_x(U-c)$ balances $(1/Re_\tau)\Delta$
yields: scaling in x, z

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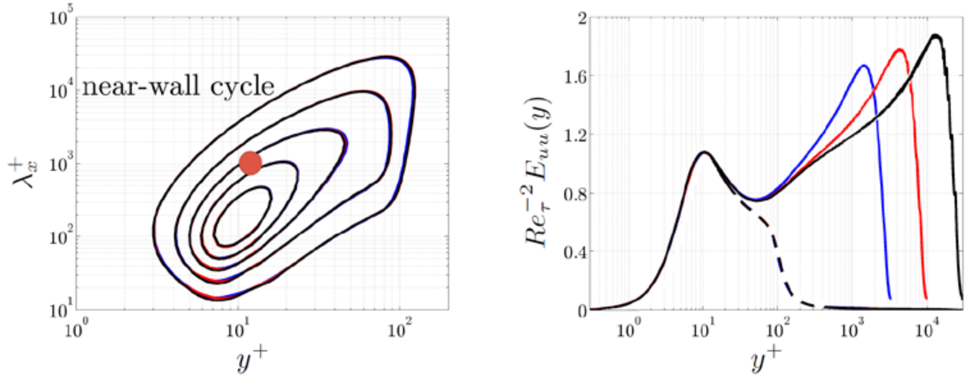
INNER AND OUTER SIMILARITY

$$E_{uu}(y;\kappa_x,\kappa_z,c) = \kappa_x^2\kappa_z(\sigma_1(\kappa_x,\kappa_z,c)|u_1|(y;\kappa_x,\kappa_z,c))^2$$

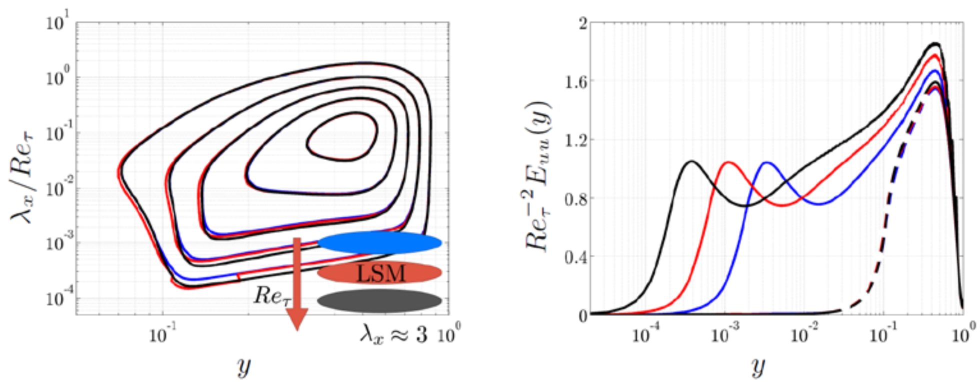
$$E_{uu}(y,\kappa_x,\kappa_z) = \int_2^{16} E_{uu}(y,\kappa_x,\kappa_z,c)\,\mathrm{d}c + \int_{16}^{U_{cl}-6.15} E_{uu}(y,\kappa_x,\kappa_z,c)\,\mathrm{d}c + \int_{U_{cl}-6.15}^{U_{cl}} E_{uu}(y,\kappa_x,\kappa_z,c)\,\mathrm{d}c$$

$$E_{uu}(y) = \iiint_{\mathscr{S}} E_{uu}(y;\kappa_x,\kappa_z,c)\,\mathrm{d}\log(\kappa_x)\,\mathrm{d}\log(\kappa_z)\,\mathrm{d}c$$

Inner
(near-wall)



Outer
(wake)



$R_\tau = 3333$ $R_\tau = 10000$ $R_\tau = 30000$

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REQUIREMENTS FOR GEOMETRIC SELF-SIMILARITY

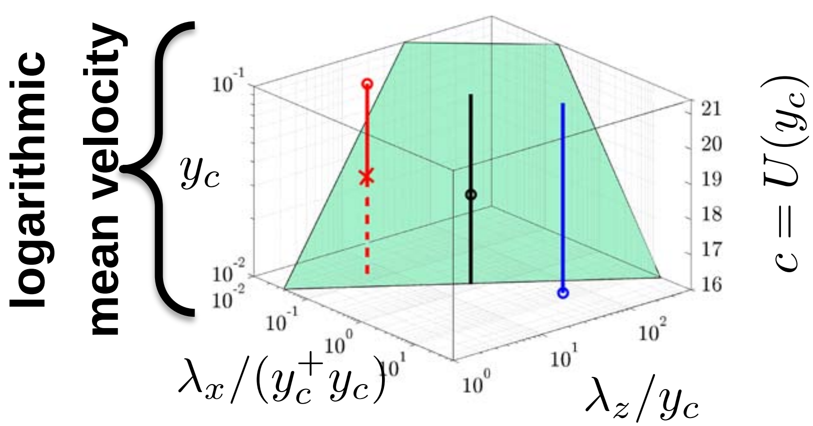
$$H(\kappa_x, \kappa_z, \omega) \;=\; C(\kappa_x, \kappa_z) R(\kappa_x, \kappa_z, \omega) C^{\dagger}(\kappa_x, \kappa_z)$$

$$R \;=\; \begin{bmatrix} \Delta^{-1}\left(\mathrm{i}\kappa_x\,(U - c)\Delta \,-\, U''\right) \,-\, (1/Re_\tau)\Delta^2\right) & 0 \\ \mathrm{i}\kappa_z U' & \mathrm{i}\kappa_x (U - c) \,-\, (1/Re_\tau)\Delta \end{bmatrix}^{-1}$$

condition 1: $U - c$ should be scalable in y $\quad U(y) - c \;=\; g(y/y_c)$

$$\left.\begin{aligned} U(y) \;&=\; d_1 \,+\, d_2 \log_{d_3}(y) \\ c \;&=\; U(y_c) \end{aligned}\right\} \quad U(y) - c \;=\; d_2\, \log_{d_3}(y/y_c)$$

condition 2, condition 3: as before



$$\begin{aligned} y \;&\propto\; y_c \\ \lambda_z \;&\propto\; y_c \\ \lambda_x \;&\propto\; y_c^+ y_c \\ \lambda_x \;&>\; \gamma\,\lambda_z \end{aligned}$$

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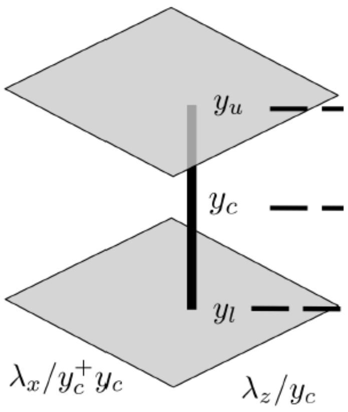
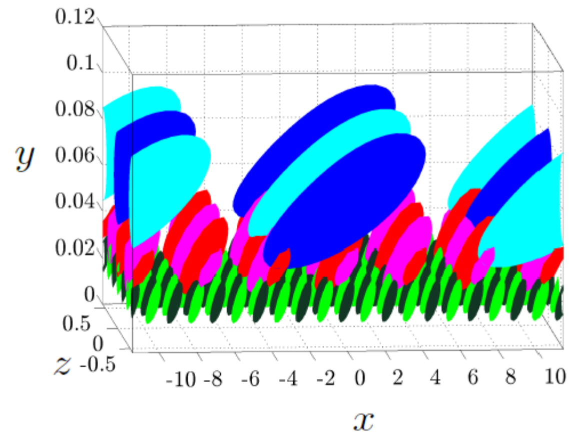
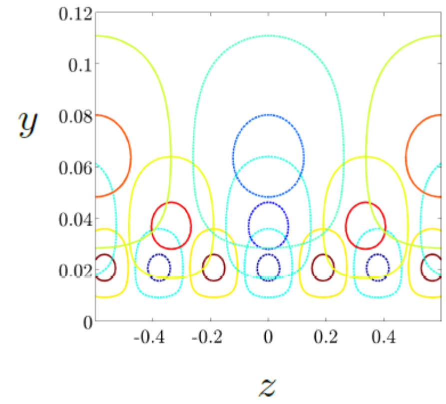
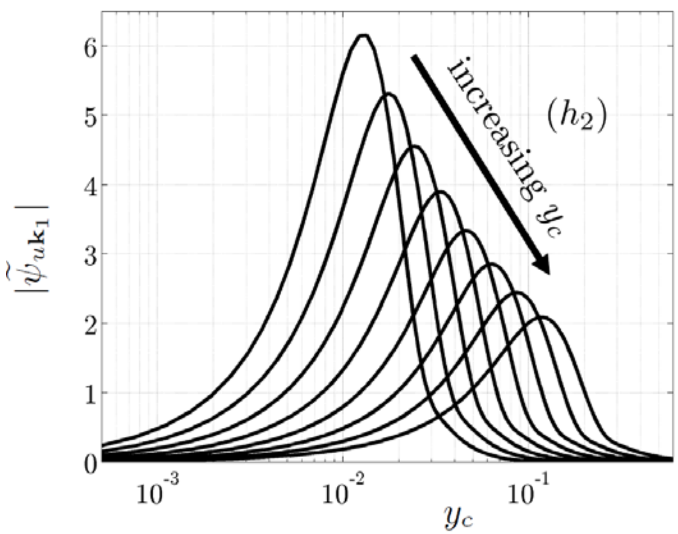
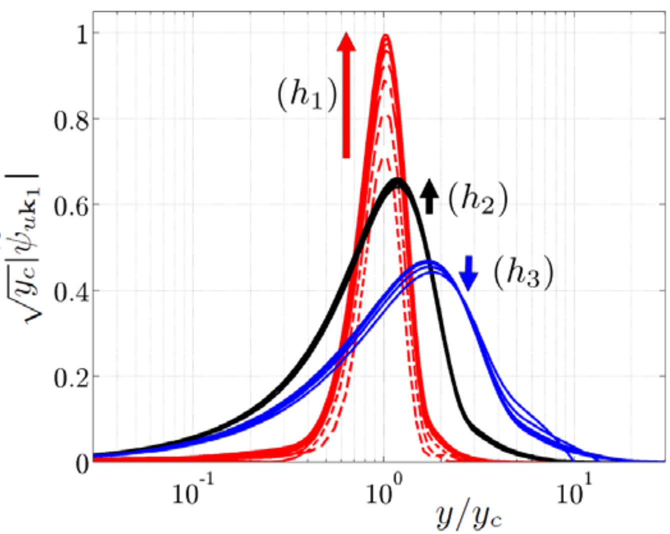
SELF-SIMILAR HIERARCHIES











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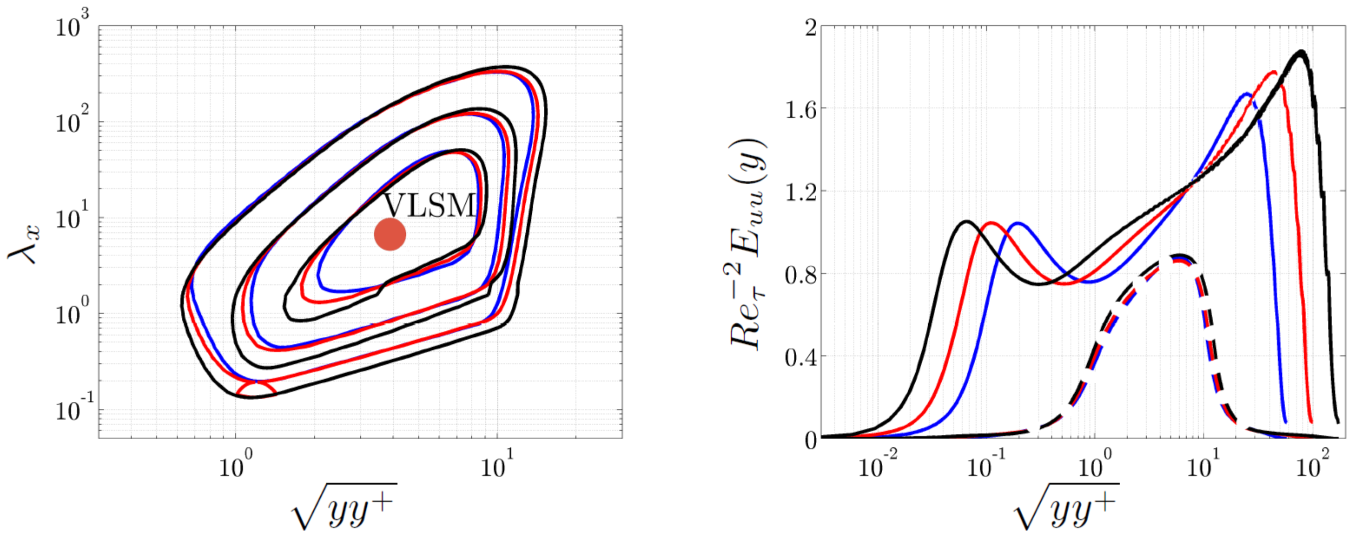
Geometric self-similarity

$$E_{uu}(y;\kappa_x,\kappa_z,c) = \kappa_x^2\kappa_z(\sigma_1(\kappa_x,\kappa_z,c)|u_1|(y;\kappa_x,\kappa_z,c))^2$$

$$E_{uu}(y,\kappa_x,\kappa_z) = \int_2^{16} E_{uu}(y,\kappa_x,\kappa_z,c)\,\mathrm{d}c + \int_{16}^{U_{cl}-6.15} E_{uu}(y,\kappa_x,\kappa_z,c)\,\mathrm{d}c + \int_{U_{cl}-6.15}^{U_{cl}} E_{uu}(y,\kappa_x,\kappa_z,c)\,\mathrm{d}c$$

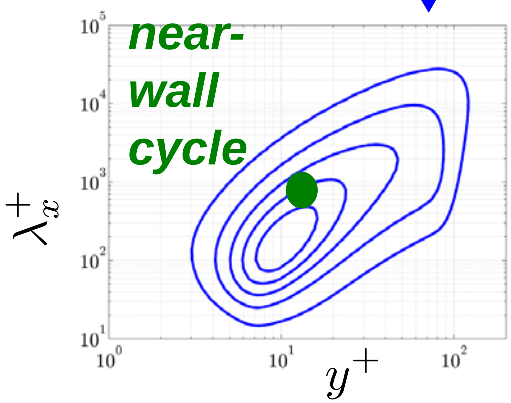
$$E_{uu}(y) = \iiint_{\mathscr{S}} E_{uu}(y;\kappa_x,\kappa_z,c)\,\mathrm{d}\log(\kappa_x)\,\mathrm{d}\log(\kappa_z)\,\mathrm{d}c$$

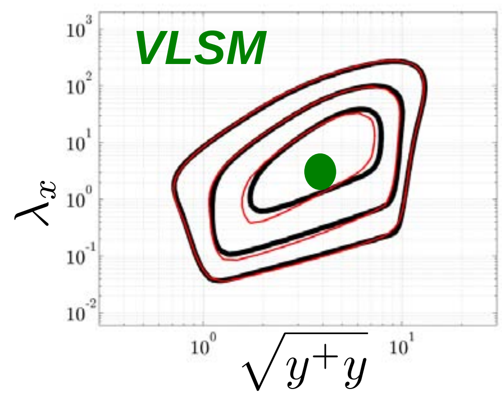


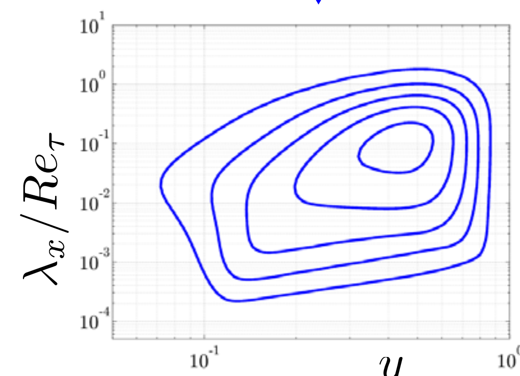
Similarity under broadband forcing

INNER (UNIVERSAL) | SELF-SIMILAR (ANALYTICAL) | OUTER (UNIVERSAL)

$$E_{uu}(y;\kappa_x) = \int_0^{16} E_{uu}(y;\kappa_x,c)\,\mathrm{d}c + \int_{16}^{U_{cl}-6} E_{uu}(y;\kappa_x,c)\,\mathrm{d}c + \int_{U_{cl}-6}^{U_{cl}} E_{uu}(y;\kappa_x,c)\,\mathrm{d}c$$





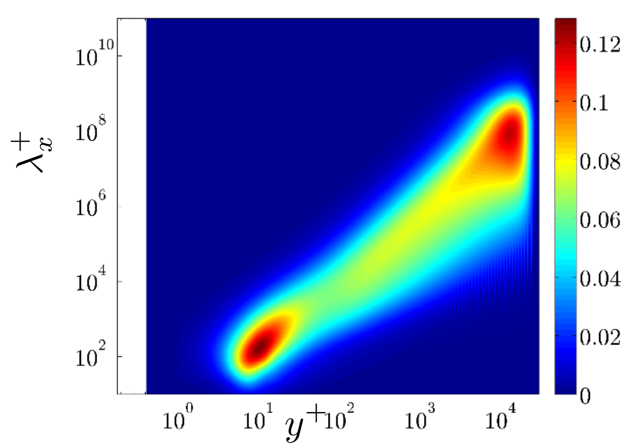


$E_{uu}(y,\lambda_x)/Re_\tau^2$

$Re_\tau = 3,333$

$Re_\tau = 10,000$

$Re_\tau = 30,000$



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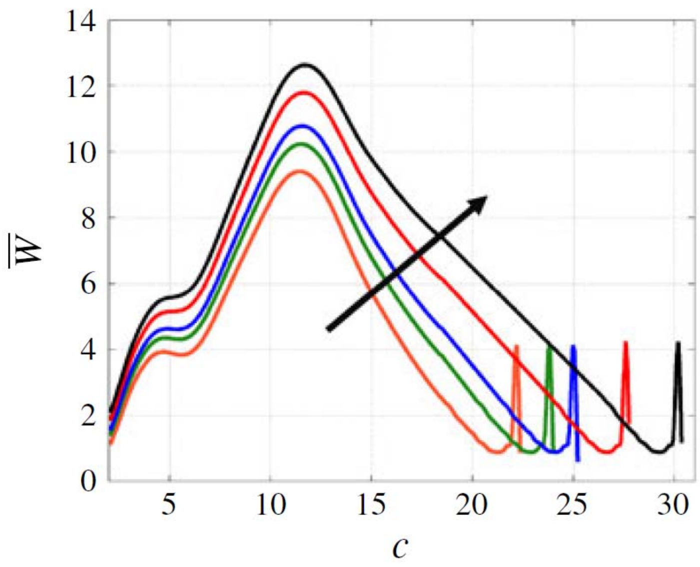
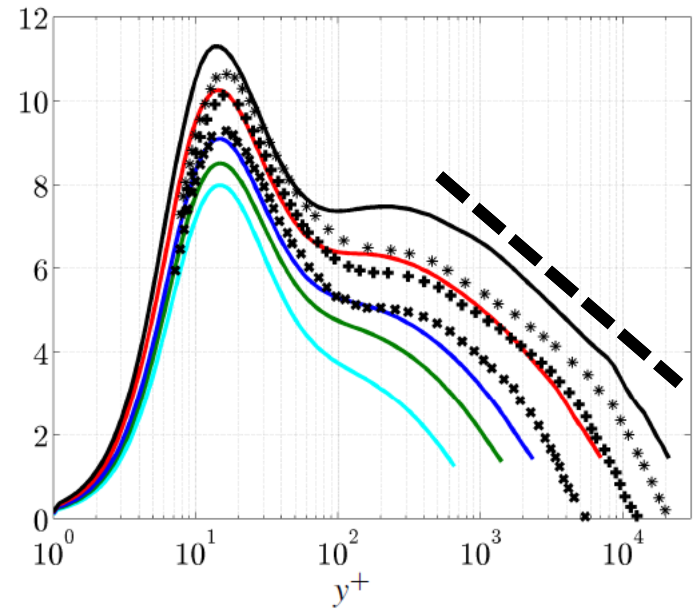
PREDICTIONS UP TO ATMOSPHERIC RE

$$E_{uu,W}(y) = \int_{2}^{U_{cl}} W(c)\, E_{uu}(y,c)\, \mathrm{d}c,$$





$R_\tau = 934$ $R_\tau = 2003$ — DNS

$R_\tau = 3333$ $R_\tau = 10000$ $R_\tau = 30000$ — prediction

boundary layers [Kunkel & Marusic, 06]: $R_\tau = 5813; \times$ $13490; +$ $23013; \star$

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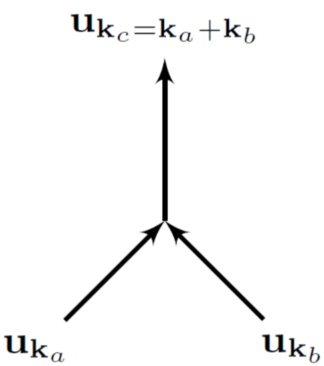
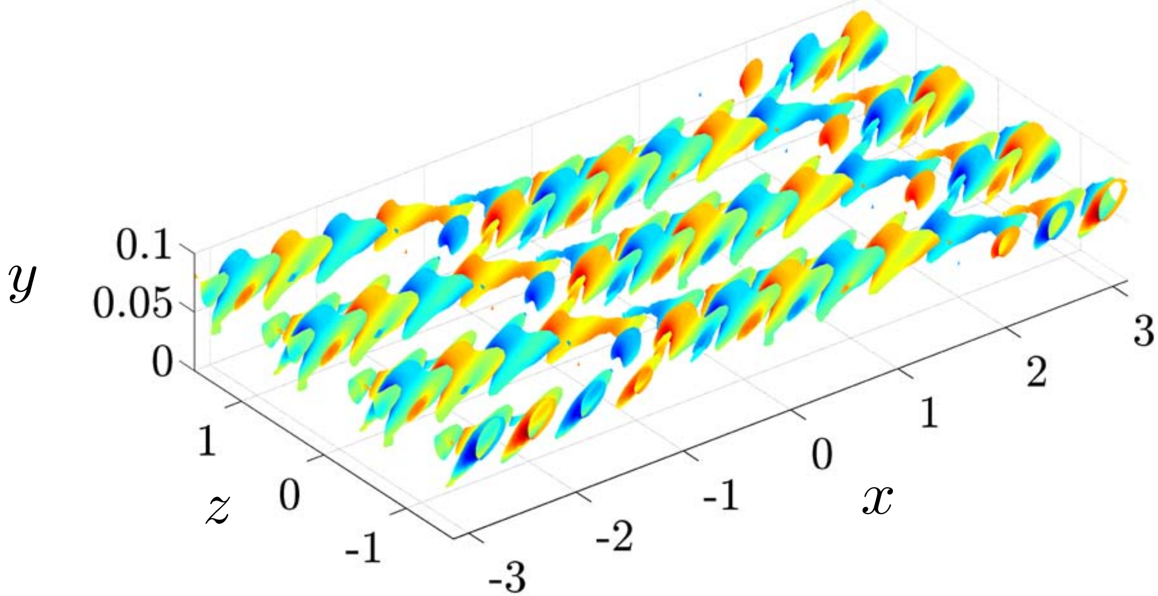
A TURBULENCE "KERNEL"



$$\kappa_{x,1} + \kappa_{x,2} + \kappa_{x,3} = 0$$
$$\kappa_{z,1} + \kappa_{z,2} + \kappa_{z,3} = 0$$
$$\omega_1 + \omega_2 + \omega_3 = 0$$

	$\pm\kappa_x$	$\pm\kappa_z$	c	A
m_1	6	6	16	$0.4\mathrm{e}^{-2.6\mathrm{i}}$
m_2	1	6	16	1.8
m_3	7	12	16	$0.3\mathrm{e}^{-2\mathrm{i}}$

$$A_1 m_1 \; + \; A_3 m_3 \; + \; A_2 m_2 \; + \; U$$



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Sharma & McKeon, J. Fluid Mech., 2013

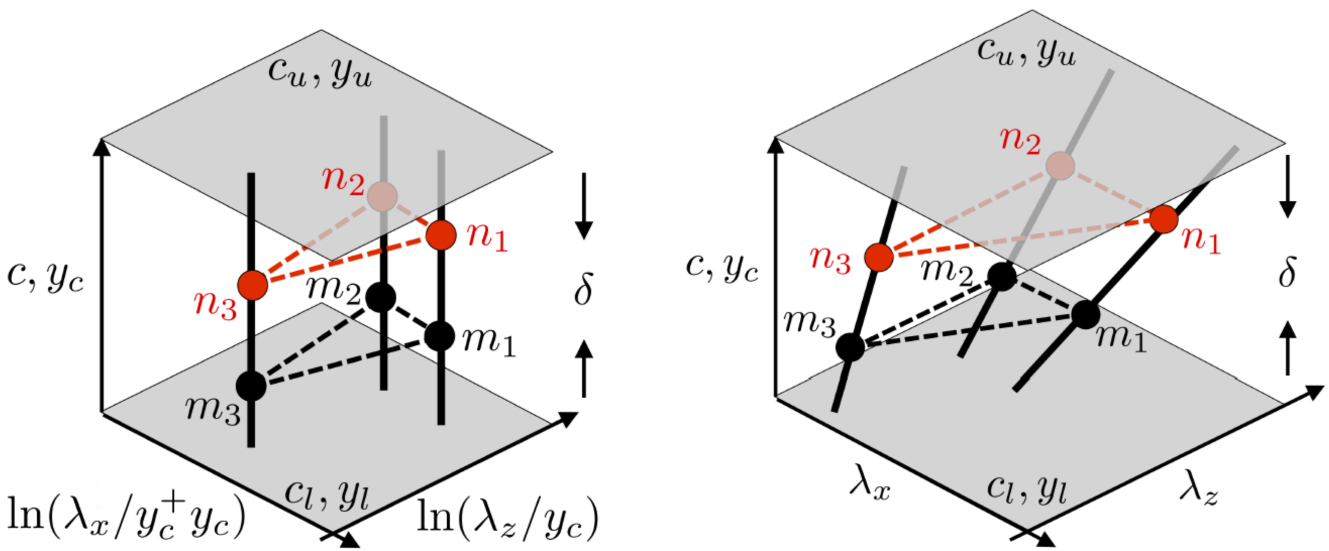
Nonlinearly interacting hierarchies

$$\chi_1 = \sum_{m_2, m_3} \mathcal{N}_{123}\, \chi_2\, \chi_3$$

$$\mathcal{N}_{123} = \sigma_2\, \sigma_3 \left\langle \hat{\psi}_2 \cdot \nabla \hat{\psi}_3,\ \hat{\phi}_1 \right\rangle$$

scaling of the interaction coefficient

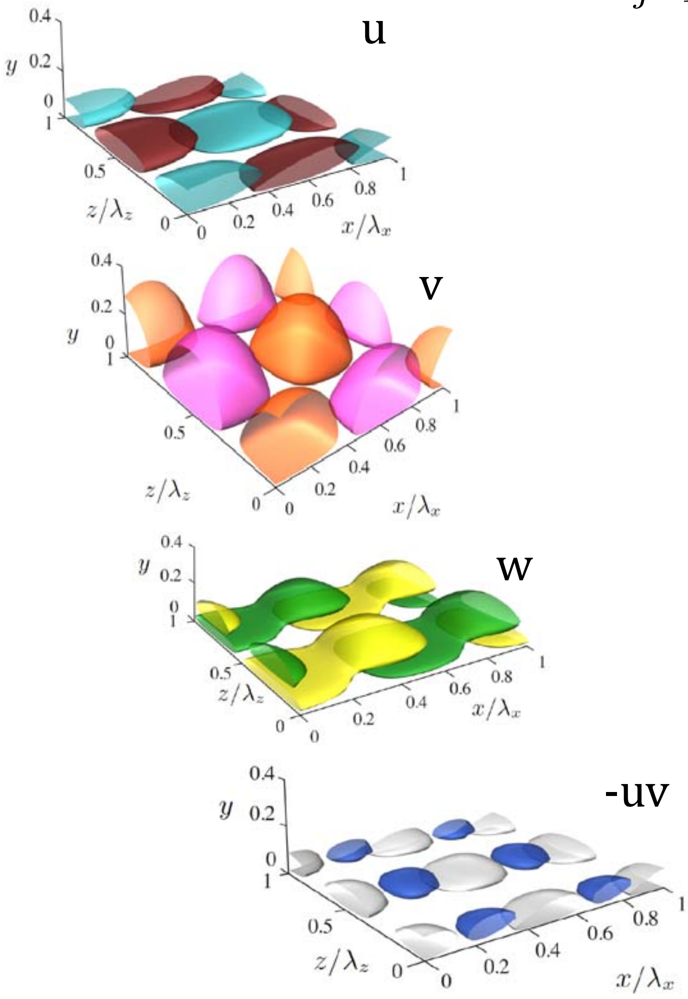
$$\mathcal{N}_{123} = \mathrm{e}^{-2.5\kappa(c_n - c_m)}\, \mathcal{N}_{123}$$

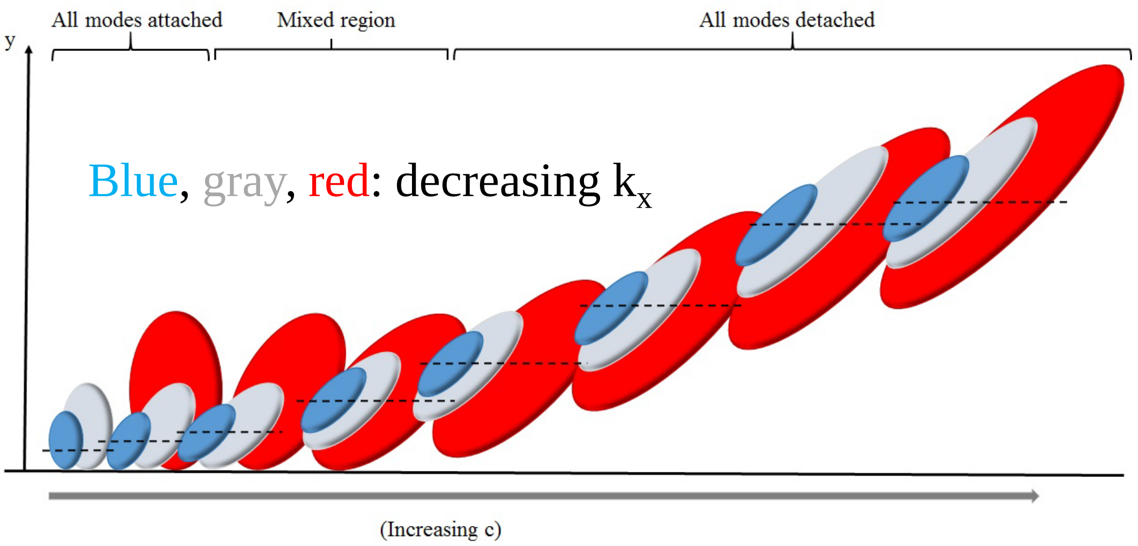


Sharma, Moarref & McKeon, PTRSA, 2017 (to appear)

RESOLVENT MODE SHAPES

$$\hat{\mathbf{u}}(y;\kappa_x,\kappa_z,\omega) = \sum_{j=1}^{\infty} \chi_j(\kappa_x,\kappa_z,\omega)\, \sigma_j(\kappa_x,\kappa_z,\omega)\, \hat{\psi}_j(y;\kappa_x,\kappa_z,\omega)$$





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SOME CONCLUSIONS AND IMPLICATIONS

- Resolvent analysis as a tool to study structure and self-sustaining mechanisms
- By starting with a self-similar mean profile, similarity of the resolvent can be identified
  - Similarity of modes with convection velocities corresponding to inner, wake and overlap (log) regions
  - Geometric self-similarity in log region, impressed on nonlinear interactions

- Implications for “reassembly” of invariant solutions
- Implications for computation of near-wall cycle
- Further structure to exploit…

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