UNIVERSITY OF Southampton

Resolvent modes and invariant solutions

Conference on Recurrence, Self-Organization, and the Dynamics Of Turbulence
Kavli Institute of Theoretical Physics
University of California, Santa Barbara

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new equilibria: with **MA Ahmed**
resolvent model: with **BJ McKeon**
pipe/channel projections: with **M Graham, BJ McKeon, R Moarref, JS Park, A Willis**

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January 2017

'To the flows observed in the long run after the influence of the initial conditions has died down there correspond certain solutions of the Navier-Stokes equations. These solutions constitute a certain manifold $\mathfrak{M} = \mathfrak{M}(\mu)$ in phase space invariant under the phase flow. Presumably owing to viscosity μ has a finite number $N = N(\mu)$ of dimensions.'

Hopf (on turbulence), 1948

The essential idea is to approximate $\mathfrak{M}$ with a basis derived from NSE, allowing low-dimensional truncations — to 'sketch the attractor'

The periodic table of turbulence

vs.

The optimal basis for turbulence

What should a model be/do?

- simpler than Navier-Stokes equations
- capture the ‘essential features’
- provide an approximating basis
- *gracefully degrade* with truncation
- may be extracted from data, but ideally derived

Outline

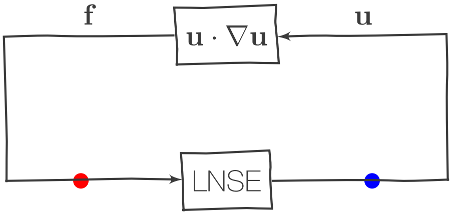
We will see how the resolvent framework gives a low-dimensional space in which Navier-Stokes dynamics approximately evolve.

We'll use it to find new invariant solutions of NSE.

We are entering an age where UPOs etc will be useful in *application*;

we need methods to find them easily

Solutions to NSE are intersection of graph from (r) to (b) with (b) to (r).



Can insert a symmetry transformation to restrict the space of solutions.

Zames, IEEE TAC 1966

Sharma, Limebeer, McKeon, Morisson, AIAA, 2006

An expansion around the turbulent attractor I

Let $\mathbf{u}(t)$ be the state, and the Navier-Stokes equations be written

$$\dot{\mathbf{u}}(t) = \mathbf{f}\left(\mathbf{u}(t)\right). \tag{1}$$

In the "long run", decompose the state as

$$\mathbf{u}(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{i\omega t}\hat{\mathbf{u}}(\omega)d\omega.$$

Notice that the equation corresponding to $\omega = \mathbf{0}$ is the mean equation

$$0 = \int_{-\infty}^{\infty} \mathbf{f}(\mathbf{u})dt$$

with $\hat{\mathbf{u}}(0)$ the mean.

An expansion around the turbulent attractor II

The expansion of (1) about this mean (and subtracting this mean equation) is

$$\begin{aligned}\dot{\tilde{\mathbf{u}}}(t) &= \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\bar{\mathbf{u}}} \tilde{\mathbf{u}}(t) + \mathcal{O}(2) \\ &= \mathbf{L}\tilde{\mathbf{u}}(t) + \tilde{\mathbf{g}}(t)\end{aligned}$$

where $\tilde{\mathbf{g}}$ represents the second-order terms in the expansion of $\mathbf{f}$.

At a particular $\omega \neq 0$ we then have

$$\hat{\mathbf{u}}(\omega) = (\omega I - \mathbf{L})^{-1}\,\hat{\mathbf{g}}(\omega).$$

The second-order terms, rather than being truncated, act to excite the state.

A low-rank basis

The resolvent $\mathbf{H}(\omega) = (\omega I - \mathbf{L})^{-1}$ is well-approximated by a projection $\Pi(\omega)$,

$$\hat{\mathbf{u}}(\omega) = \mathbf{H}(\omega)\hat{\mathbf{g}}(\omega) \simeq \Pi(\omega)\hat{\mathbf{g}}(\omega).$$

The SVD gives the optimal N-dimensional basis on which the velocity field evolves, in sense that $\|\mathbf{H}(\omega) - \Pi_N(\omega)\|_F$ is minimised. [1]

The flow “lives” mostly in Π.

The ‘error’ is $\hat{\mathbf{u}}^{\perp} = \Pi_N^{\perp}\hat{\mathbf{g}}$.

[1] McKeon & Sharma, JFM, 2010.

A basis for optimal projection; resolvent modes

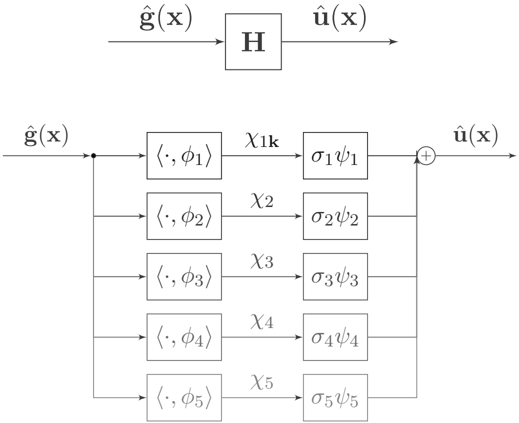
$$\mathbf{H}\hat{\mathbf{g}} = \sum_{m=1}^{\infty} \psi_m(\mathbf{x})\sigma_m \langle \phi_m^*(\mathbf{x}), \hat{\mathbf{g}}(\mathbf{x}) \rangle$$

$$\langle \psi_m, \psi_{m'} \rangle = \delta_{m,m'}$$

$$\langle \phi_m, \phi_{m'} \rangle = \delta_{m,m'}$$

$$\sigma_1 \geq \sigma_2 \geq \ldots$$

Each σ_m is a (real) gain, σ_1 is the maximum gain.
Velocity field response is $\psi_m(\mathbf{x})$.

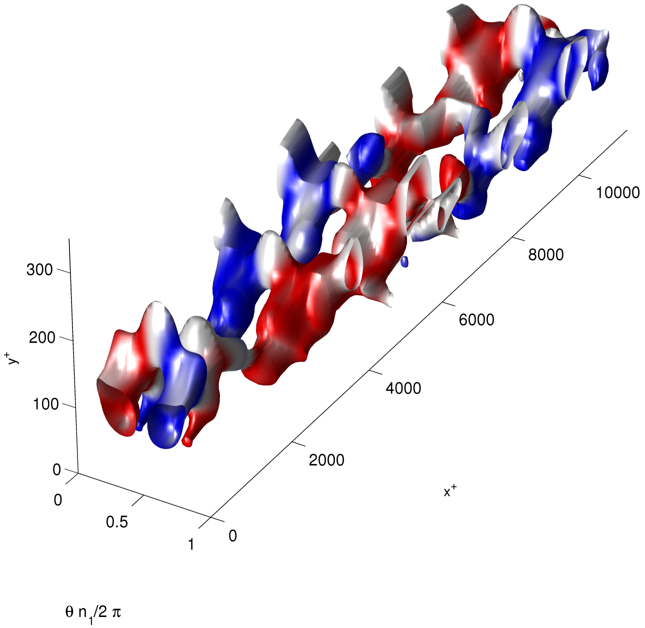


Self-interacting structure

- Structure / kernel in log layer can self-interact
- Self-sustaining mode combo requires solving coefficients
- Relates to Hwang / Cossu log-layer SSP?

	k	n	c
'hairpin'	6	±6	2/3
'VLSM'	1	±6	2/3
beating mode	7	±12	2/3

Sharma & McKeon 2013



Problem: fixing coefficients in nonlinear optimisation

We have optimal basis $\{\psi_i\}$ for velocity field; projecting Navier-Stokes onto these equations gives a quadratic optimisation problem in the coefficients $\{\chi_i\}$:

$$\chi_a = \sum_{b,c} N_{abc} \chi_b \chi_c.$$

N_{abc} is $\sim$ “eddy-eddy” interaction.

Solving this optimisation is solving turbulent flow in a (projected) low-dimensional space.

For turbulence this is still very high-dimensional. We would like a lower-dimensional problem to develop the techniques on.

How are invariant solutions and resolvent modes linked?

- Symmetric recurrent solutions thought to give shape to turbulent state-space; cause coherent structure
- Resolvent framework designed to capture coherent structure
- RM should somehow capture low-dimensional structure of invariant solutions
- These provide a nice testbed for RM, since single c

Projection of solutions onto model modes

- 15 pipe solutions provided by A Willis (Sheffield), $Re_B = 5300$
- Also channel solutions provided by M Graham, JS Park (Wisonsin-Madison)
- S and N solutions presented, upper and lower branch*, †
- modes generated using $\mathbf{u}_0$ of solution

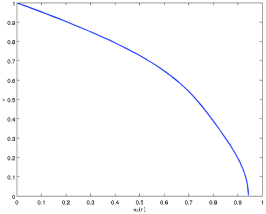
**original pipe solutions continued from Pringle et al, Phil. Trans. R. Soc. A, 2009*

†S have shift-reflect, N also have mirror, rotational symmetries

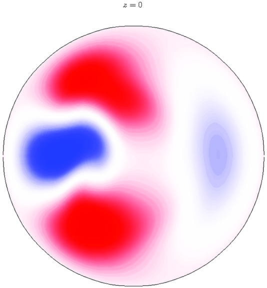
Sharma, Moarref, McKeon, Park, Graham, Willis, Phys. Rev. E 2016

S1 solution 3403.0007

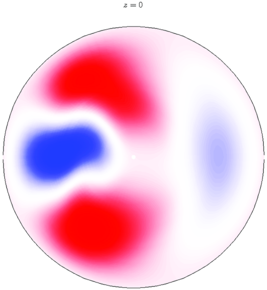
close to laminar; well represented with one mode per k



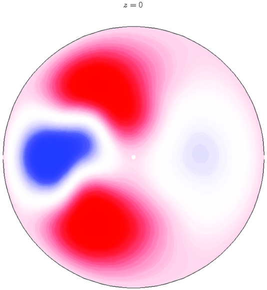
actual solution



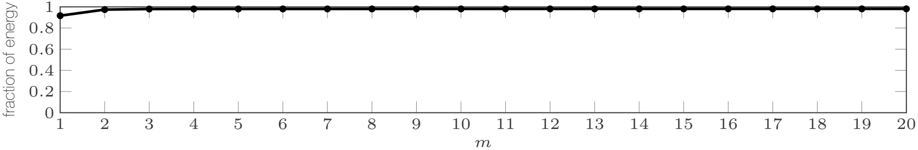
$m = 1 \ldots 5$



$m = 1$

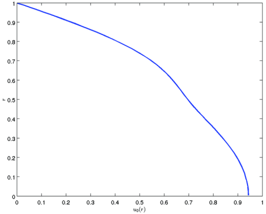


fraction of solution energy, keeping m singular values per Fourier mode

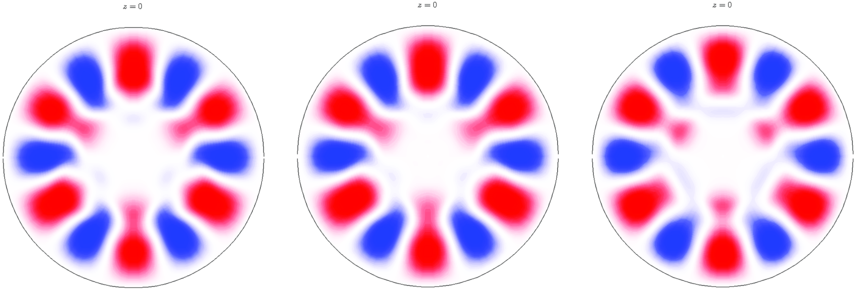


N3L solution 6507.1000

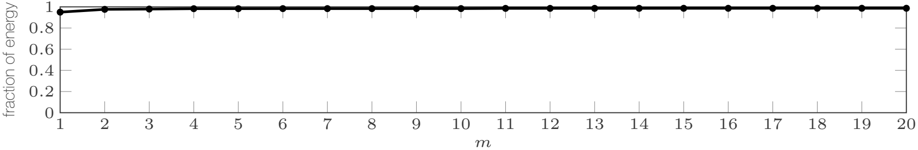
lower branch; close to laminar; well represented



actual solution | $m = 1 \ldots 5$ | $m = 1$

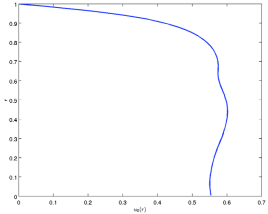


fraction of solution energy, keeping m singular values per Fourier mode

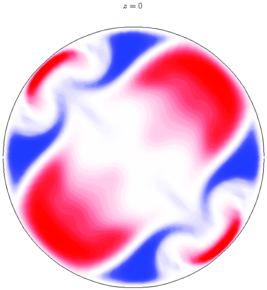


N2U solution 6502.0050

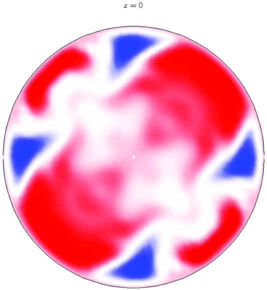
more ‘turbulent’; less well represented



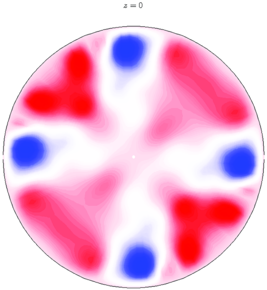
actual solution



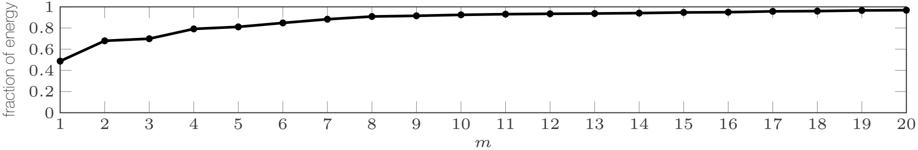
$m = 1 \dots 5$



$m = 1$

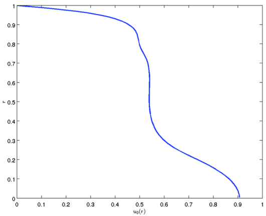


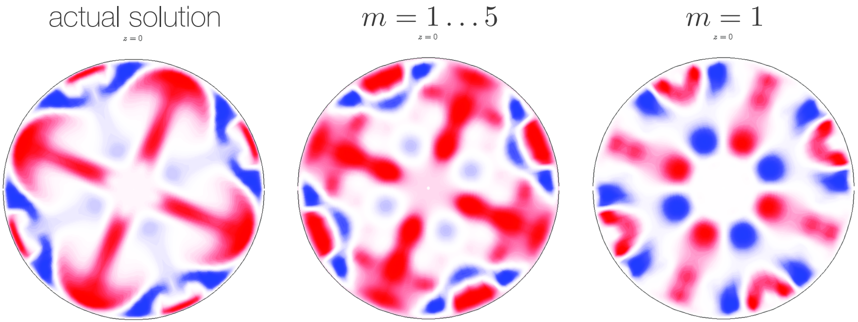
fraction of solution energy, keeping m singular values per Fourier mode



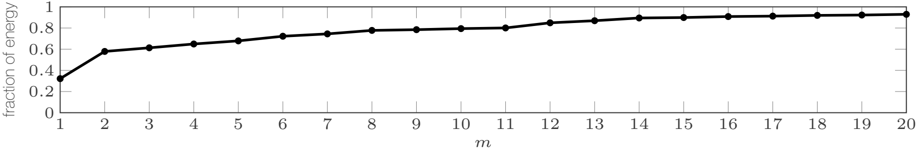
N4U solution 6512.1000

more ‘turbulent’; less well represented; not visited in turbulent DNS



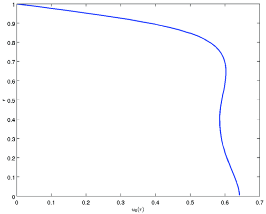


fraction of solution energy, keeping m singular values per Fourier mode

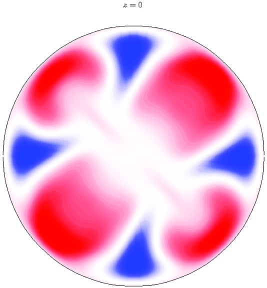


N2U solution 6502.0001

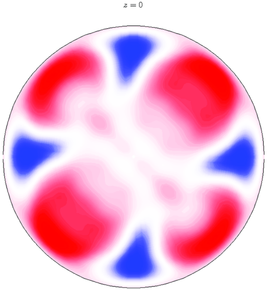
more 'turbulent'; but well represented ?!



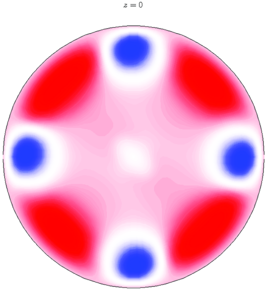
actual solution



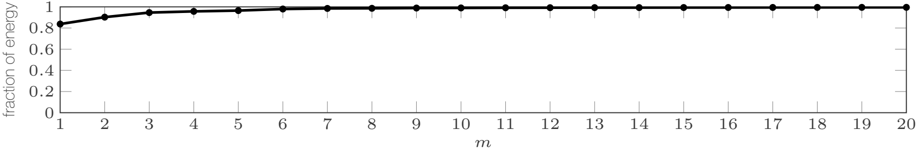
$m = 1 \dots 5$



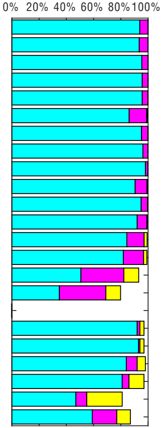
$m = 1$



fraction of solution energy, keeping m singular values per Fourier mode

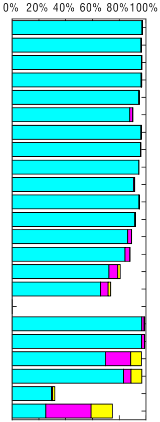


class	α	Re_τ	$c/2U_B$
S1	0.78	106	0.81
S1	0.34	107	0.78
S2b	1.24	107	0.76
S2b	0.95	108	0.76
S2b	0.39	110	0.71
S2b	0.24	122	0.62
N2L	1.73	109	0.83
N2L	1.24	109	0.82
N2L	0.55	111	0.78
N2L	0.15	117	0.70
N3L	1.25	111	0.74
N4L	1.70	114	0.69
N2U	1.25	151	0.66
N2U	1.01	156	0.66
N2U	0.80	177	0.64
N4U	1.70	214	0.55
class	**Re_B**	**Re_τ**	**$c/2U_B$**
P1L	3200	75	0.58
P3L	4533	85	0.52
P4L	3677	85	0.58
P1U	2267	75	0.49
P3U	1733	85	0.35
P4U	2200	85	0.55



Fluctuation energy

class	Re_τ	ε
S1	106	1.07
S1	107	1.09
S2b	107	1.09
S2b	108	1.09
S2b	110	1.15
S2b	122	1.41
N2L	109	1.11
N2L	109	1.12
N2L	111	1.16
N2L	117	1.30
N3L	111	1.16
N4L	114	1.25
N2U	151	2.17
N2U	156	2.31
N2U	177	3.02
N4U	214	4.49
class	**Re_τ**	**ε**
P1L	75	1.19
P3L	85	1.08
P4L	85	1.32
P1U	75	1.62
P3U	85	2.77
P4U	85	2.18



Dissipation

Upper set: pipe (all at $Re_B = 5300$), 1, 5, 10 modes.
Lower set: channel (at a range of Re_B), 1, 2, 5 mode pairs.

Sharma, Moarref, McKeon, Park, Graham, Willis, Phys. Rev. E 2016

An observation

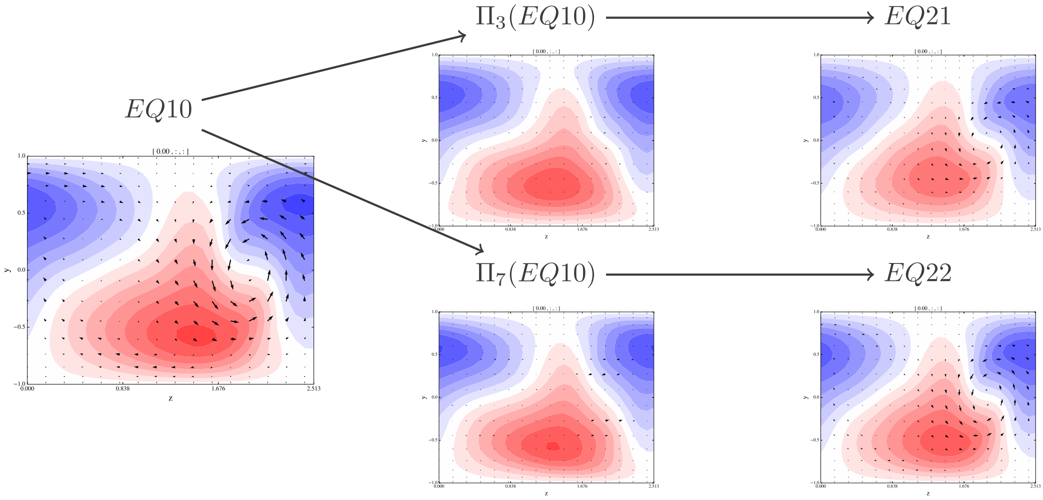
- JS Park noticed that the projection of P4U looks like its lower-branch counterpart *
- Why?
- Can we use this to generate new solutions?

Project and search

- Use projection of known PCF solutions* as seed for Newton search of new solutions (using *channelflow*)

* *Gibson, Halcrow, Cvitanovic, JFM 2008*

What does project-and-search do?



A symmetry operation σ is an operation which commutes with forward integration,

$$\sigma\dot{\mathbf{u}} = \sigma f(\mathbf{u}) = f(\sigma\mathbf{u})$$

Plane Couette flow admits seven discrete symmetries plus e,

$$\theta_1[u,v,w](x,y,z) = [-u,-v,w](-x,-y,z),$$
$$\theta_2[u,v,w](x,y,z) = [u,v,-w](x,y,-z+L_z/2),$$
$$\theta_3[u,v,w](x,y,z) = [-u,-v,-w](-x,-y,-z+L_z/2),$$
$$\theta_4[u,v,w](x,y,z) = [u,v,-w](x+L_x/2,y,-z),$$
$$\theta_5[u,v,w](x,y,z) = [-u,-v,-w](-x+L_x/2,y,-z),$$
$$\theta_6[u,v,w](x,y,z) = [u,v,w](x+L_x/2,y,z+L_z/2),$$
$$\theta_7[u,v,w](x,y,z) = [-u,-v,w](-x+L_x/2,-y,z+L_z/2)$$

subgroups:

Γ = all continuous symmetries,
$$\Sigma = \{e,\theta_4,\theta_7,\theta_3\},$$
$$\Sigma_n = \{e,\sigma_n\},$$
$$\Theta = \{e,\tau_{xz}\}\times\Sigma,$$
$$K = \{e,\Theta_1,\Theta_2,\Theta_3\}.$$

$\sigma_1 = \theta_4$ is shift-reflect
$\sigma_2 = \theta_7$ is shift-rotate
$\sigma_3 = \theta_3 = \sigma_1\sigma_2$ is PWI

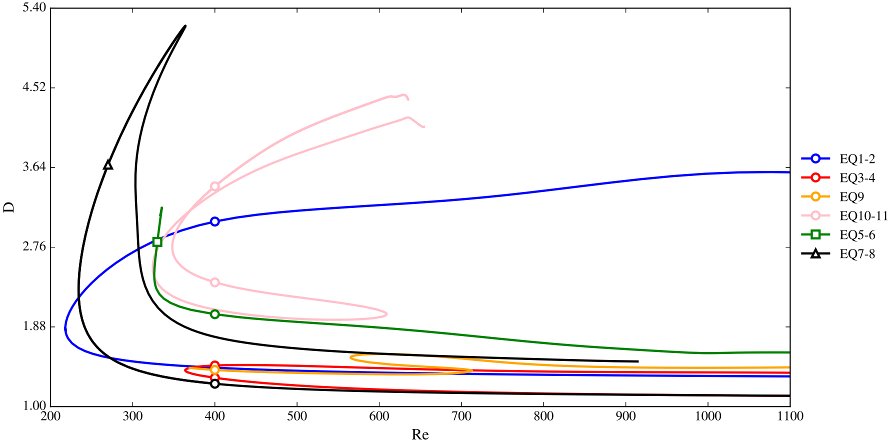
Root	Re	EQ	$\|\tilde{\mathbf{u}}\|$	E	D	H	d(W^u)	d(W^u_H)	Acc.	Occ.
	400	mean	0.2997	0.1016	2.6017	$\{e\}$				
		0	0.0000	0.1667	1.0000	Γ	0	0		199
	400	3	0.1259	0.1382	1.3177	Σ	4	2	10^{-4}	109
	400	4	0.1681	0.1243	1.4537	Σ	6	3	10^{-6}	75
	400	1	0.2091	0.1363	1.4293	Σ	1	1	10^{-6}	163
	400	5	0.2186	0.1073	2.0201	Σ	11	4	10^{-3}	33
	400	2	0.3858	0.0780	3.0437	Σ	8	2	10^{-4}	158
	400	7	0.0936	0.1469	1.2523	Θ	3	1	10^{-4}	110
	400	9	0.1565	0.1290	1.4048	Σ_3	5	3	10^{-4}	80
	400	10	0.3285	0.1080	2.3721	Σ_3	10	7	10^{-4}	49
	400	11	0.4049	0.0803	3.4322	Σ_3	13	10	10^{-3}	1
$\Pi_5(\mathrm{EQ9})$	400	20	0.2405	0.1289	1.6034	Σ_3	3	2	10^{-5}	146
$\Pi_3(\mathrm{EQ10})$	400	21	0.2683	0.1242	1.7630	Σ_3	4	3	10^{-6}	122
$\Pi_7(\mathrm{EQ10})$	400	22	0.3037	0.1160	2.0713	Σ_3	8	6	10^{-5}	65
$\Pi_7(\mathrm{EQ11})$	400	23	0.4014	0.0759	3.2474	Σ_3	10	4	10^{-5}	56
$\Pi_{10}(\mathrm{EQ11})$	400	24	0.4049	0.0813	3.3612	Σ_3	15	9	10^{-5}	9
	330	6	0.2751	0.0972	2.8185	Σ	19	6	10^{-3}	22
$\Pi_6(\mathrm{EQ6})$	330	13	0.2168	0.1337	1.4705	Σ	1	1	10^{-3}	126
$\Pi_{40}(\mathrm{EQ6})$	330	14	0.2375	0.1052	2.2785	Σ	15	6	10^{-2}	124
$\Pi_2(\mathrm{EQ6})$	330	12	0.1145	0.1410	1.3433	Θ	3	1	10^{-3}	94
$\Pi_{67}(\mathrm{EQ6})$	330	16	0.2348	0.1063	2.3047	Σ_3	15	8	10^{-2}	6
$\Pi_{42}(\mathrm{EQ6})$	330	15	0.2674	0.0988	2.6947	Σ_3	18	9	10^{-2}	64
$\Pi_{31}(\mathrm{EQ15})$	330	25	0.2331	0.1292	1.5650	Σ_3	3	2	10^{-3}	100
$\Pi_{66}(\mathrm{EQ15})$	330	26	0.2707	0.0975	2.6274	Σ_3	17	9	10^{-2}	21
	270	8	0.3466	0.0853	3.6719	Θ	15	2	10^{-3}	43
$\Pi_{11}(\mathrm{EQ8})$	270	18	0.2292	0.1294	1.5415	Σ	1	1	10^{-2}	99
$\Pi_3(\mathrm{EQ8})$	270	17	0.1546	0.1301	1.5530	Θ	5	1	10^{-2}	70
$\Pi_{38}(\mathrm{EQ8})$	270	19	0.3148	0.0904	3.1529	K	12	0	10^{-2}	76
$\Pi_2(\mathrm{EQ18})$	270	27	0.2297	0.1292	1.5444	Σ_3	2	2	10^{-2}	128

H is the isotropy subgroup

d(W^u) is the dimension of unstable manifold

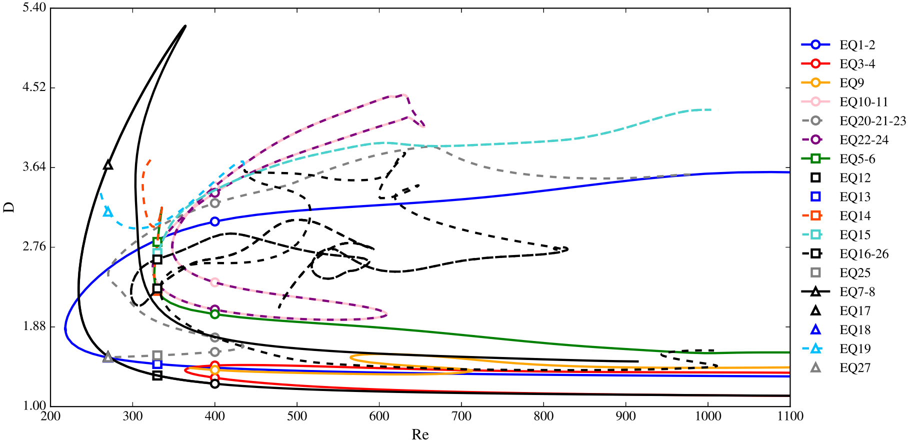
d(W^u_H) is the dimension of the unstable manifold within the H-invariant subspace

Known branches (equilibria)



dissipation (D) continuations;
solid: known equilibria

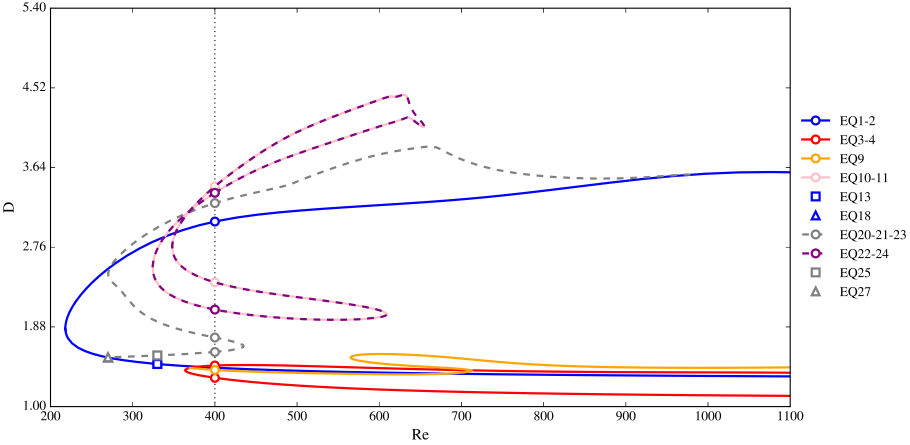
New branches (equilibria)



dissipation (D) continuations;

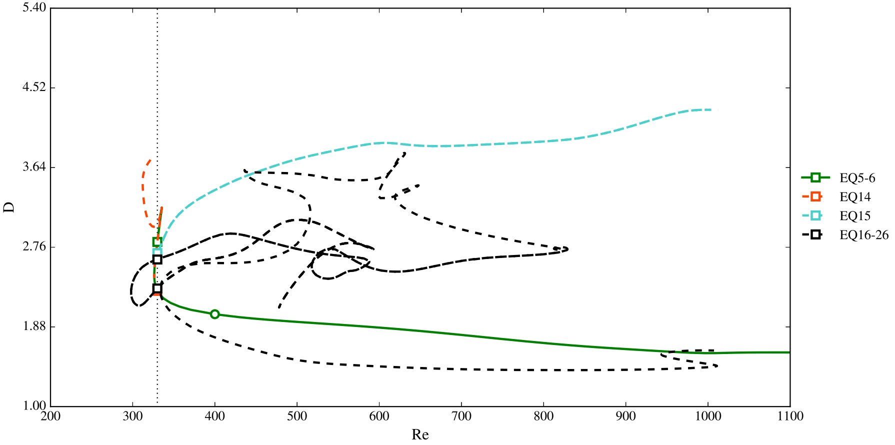
solid: known equilibria, dashed: new equilibria

New branches ($Re = 400$)



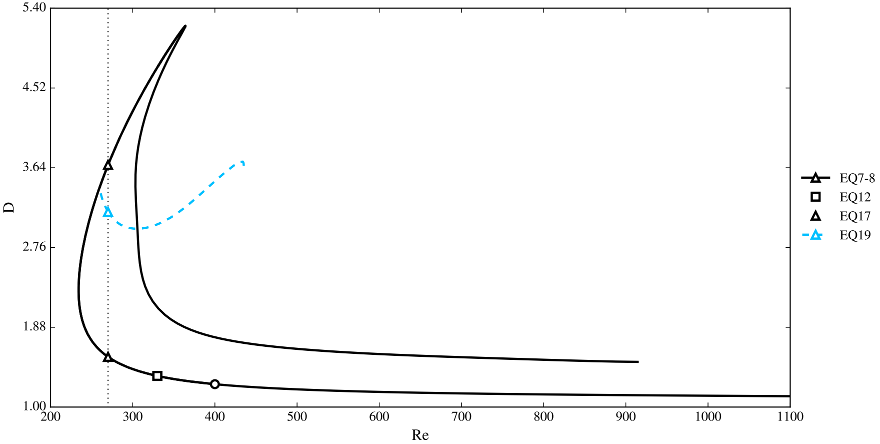
dissipation (D) continuations;
solid: known equilibria, dashed: new equilibria

New branches ($Re = 330$)



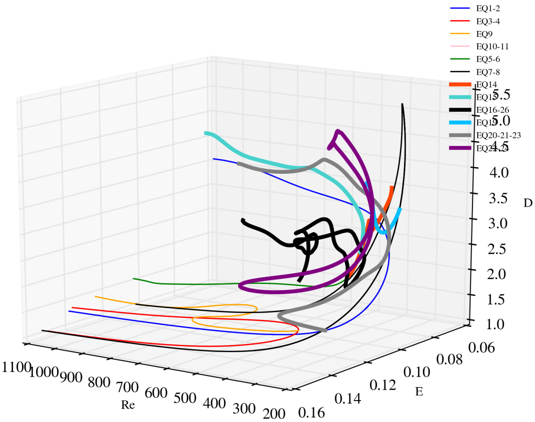
dissipation (D) continuations;
solid: known equilibria, dashed: new equilibria

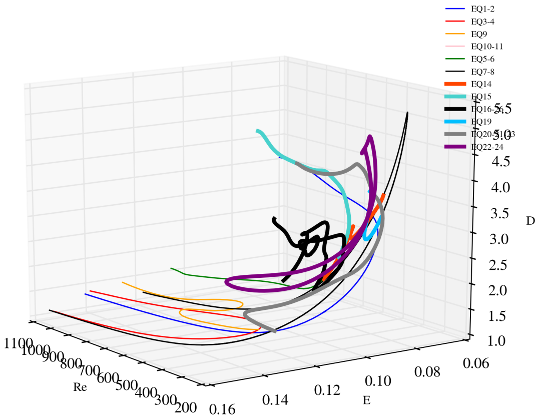
New branches ($Re = 270$)



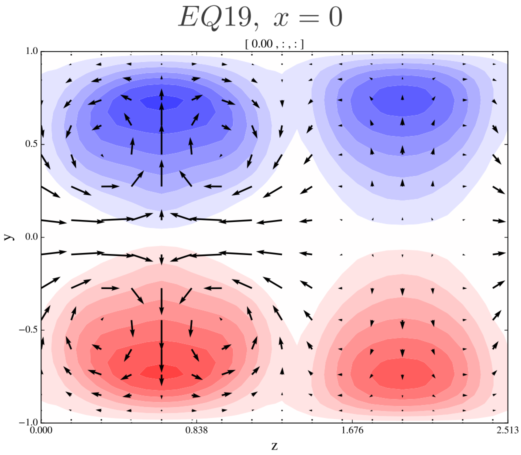
dissipation (D) continuations;

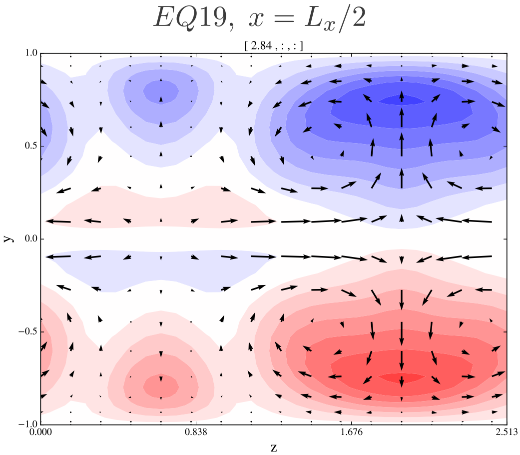
solid: known equilibria, dashed: new equilibria


EQ1-2
EQ3-4
EQ9
EQ10-11
EQ5-6
EQ7-8
EQ14
EQ15
EQ16-26
EQ19
EQ20-21-23
EQ22-24
D
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
5.5
E
0.06
0.08
0.10
0.12
0.14
0.16
Re
1100
1000
900
800
700
600
500
400
300
200


EQ1-2
EQ3-4
EQ9
EQ10-11
EQ5-6
EQ7-8
EQ14
EQ15
EQ16-26
EQ19
EQ20-21-23
EQ22-24
D
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
5.5
E
0.06
0.08
0.10
0.12
0.14
0.16
Re
1100
1000
900
800
700
600
500
400
300
200

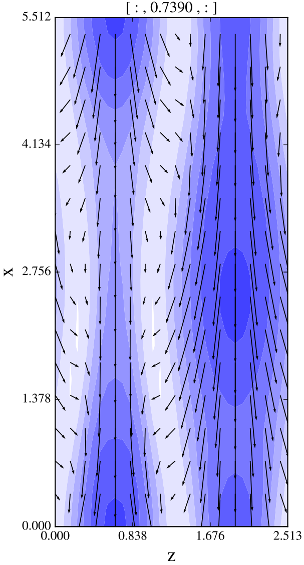
movies/project-and-search/3D.mp4

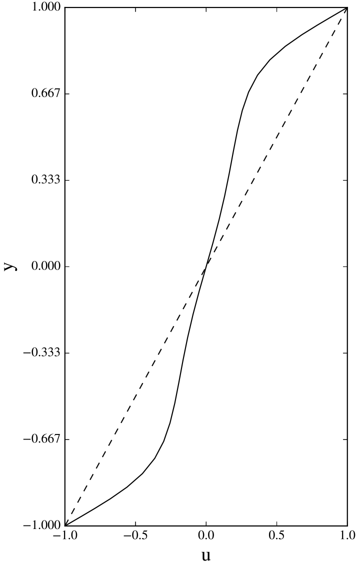


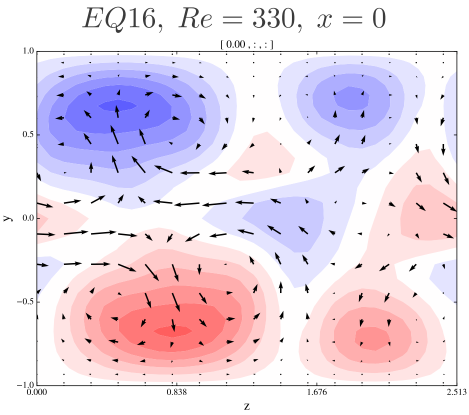


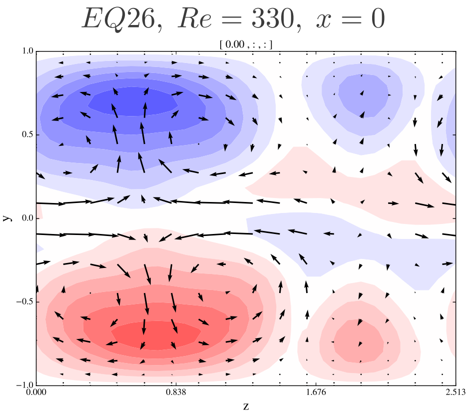
- new subgroup K (breaks half-cell shift in x)
- arises from pitchfork bifurcation from EQ7-8 curve

$EQ19,\ y = 0.73$

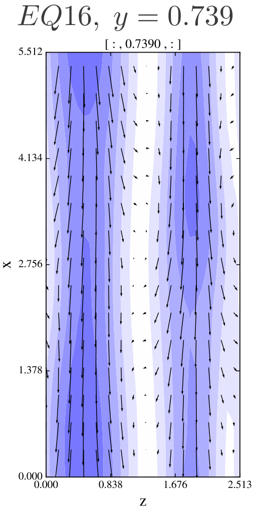

[: , 0.7390 , :]
5.512
4.134
2.756
1.378
0.000
x
0.000
0.838
1.676
2.513
z

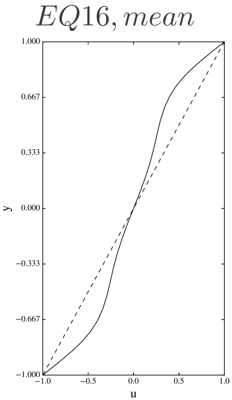

1.000
0.667
0.333
0.000
−0.333
−0.667
−1.000
y
−1.0
−0.5
0.0
0.5
1.0
u

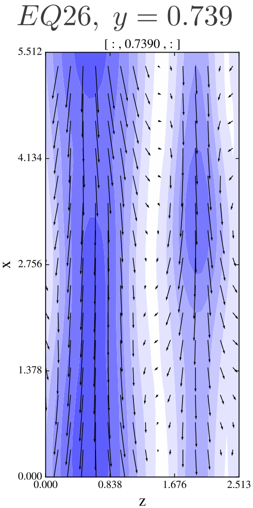


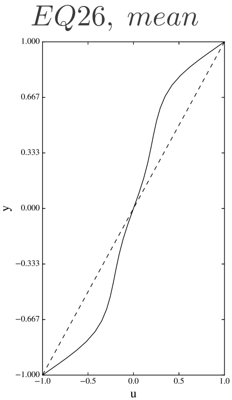


- arise from *two* saddle node bifurcations at $Re = 298.1$ and $Re = 517.9$
- upper and lower branches swap at higher Re


$EQ16,\ y = 0.739$
[: , 0.7390 , :]
5.512
4.134
2.756
1.378
0.000
x
0.000
0.838
1.676
2.513
z


$EQ16, mean$
1.000
0.667
0.333
0.000
−0.333
−0.667
−1.000
y
−1.0
−0.5
0.0
0.5
1.0
u


$EQ26,\ y = 0.739$
[: , 0.7390 , :]
5.512
4.134
2.756
1.378
0.000
x
0.000
0.838
1.676
2.513
z


$EQ26,\ mean$
1.000
0.667
0.333
0.000
−0.333
−0.667
−1.000
y
−1.0
−0.5
0.0
0.5
1.0
u

EQ16-26 following bifurcation curve

movies/project-and-search/EQ16-26_whole_curve_movie.mp4

To do

- Heteroclinic connections between equilibria
- Periodic orbits etc
- Coefficient solver

Conclusions

- Gain-optimal basis from NSE captures turbulent structure and invariant solutions
- We may generate good initial guesses at will
- A ‘force multiplier’ when finding equilibria
- Extends trivially to periodic orbits etc.
- **We are entering an age where UPOs etc will be useful in *application*;**
- **‘industrialisation’ is becoming key**