Telling the time: Using the clockwork of turbulence to answer open questions in fluid dynamics

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Engineering and Physical Sciences Research Council

BP INSTITUTE FOR MULTIPHASE FLOW

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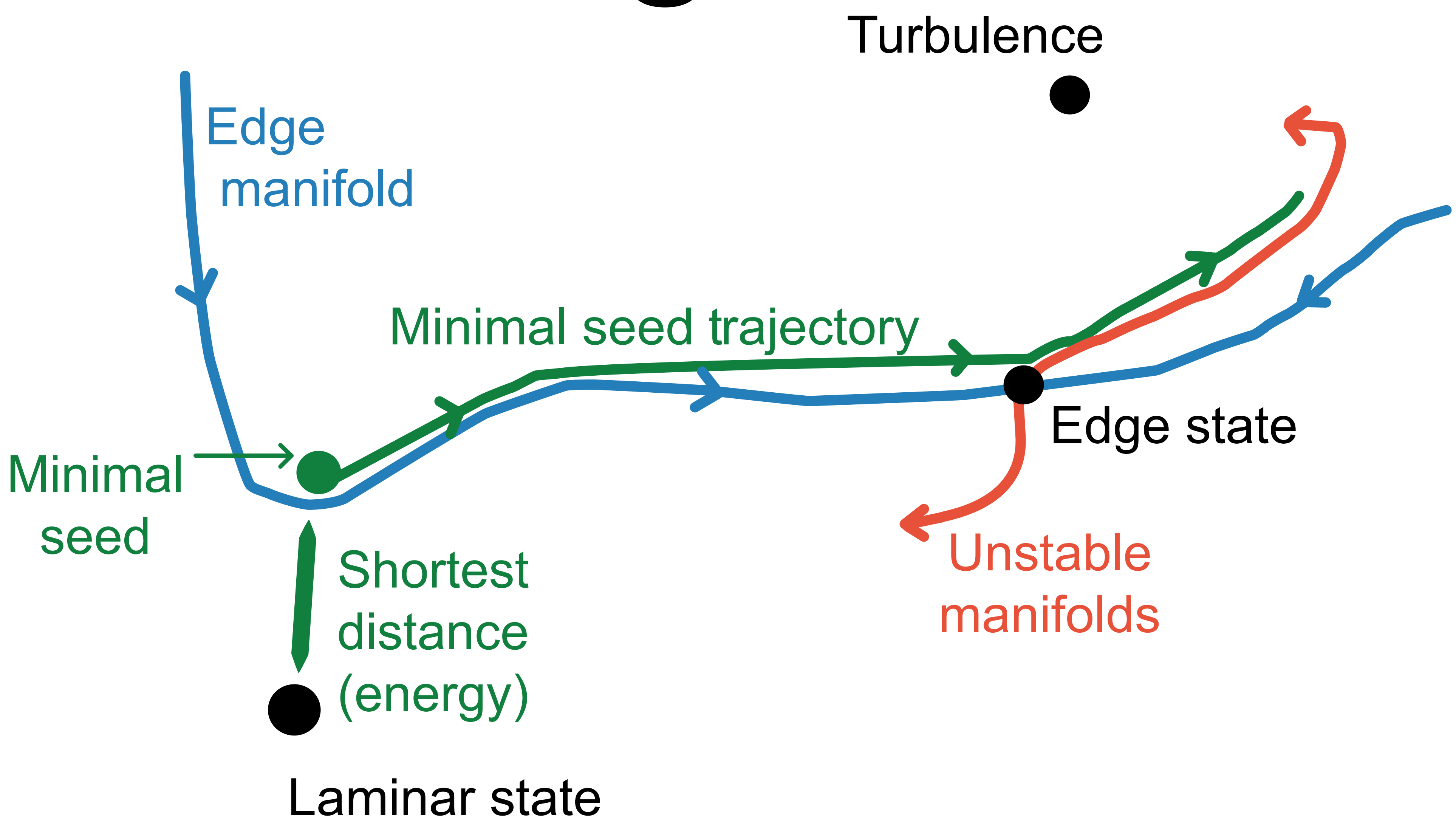
Overview

1. Motivation:
   - Turbulence "obviously" important...
   - "Structures not statistics" appealing viewpoint
   - But can these structures actually give insight to real problems?
2. Approach: (HOW?) Two examples:
   A. Can Koopman modes give insight into edge state/transition?
   B. Can UPOs give insight into layering/mixing in stratified flows? (Recurrence, self-organization and the dynamics of turbulence)
3. Conclusions...time to be quantitative...

A: Seeds, edges & modes

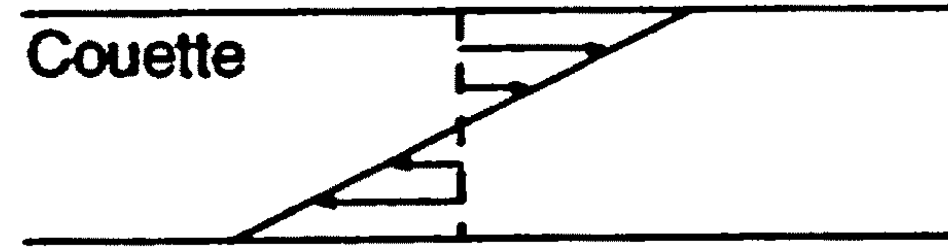


- Turbulent and laminar state separated by an “edge”
- Can we find the lowest energy state to get across the edge?
- Does “seed” follow special route to the “edge state” (1 unstable)?

A: Seeds, edges & modes

- Linear Optimal Perturbations are “easy”: Linear Algebra available
- Variational formulation also possible (Schmid 07)
- Using “adjoint” operators (Hill 95, Corbett & Bottaro 00)
- Linear adjoints are “nice”: completely decoupled...
- But nonlinear adjoints can be defined...and calculated
- Pringle & Kerswell 10, Cherubini et al 11, Monokrousos et al 11
- Rabin et al 2012 Duguet et al 2013 Kerswell et al 2014...
- Perturbation can feed back: what does the fluid want to do?

Formulation



- Fix ideas: consider plane Couette flow:

- Butler & Farrell geometry:

$$L_x = 13.66,\ L_y = 2,\ L_z = 3.11,\ U = y\mathbf{e}_x,\ Re = \frac{1}{\nu} = 1000$$

- Hypothesis: Maximize gain of (nonlinear) perturbation $\hat{u} = u - U$

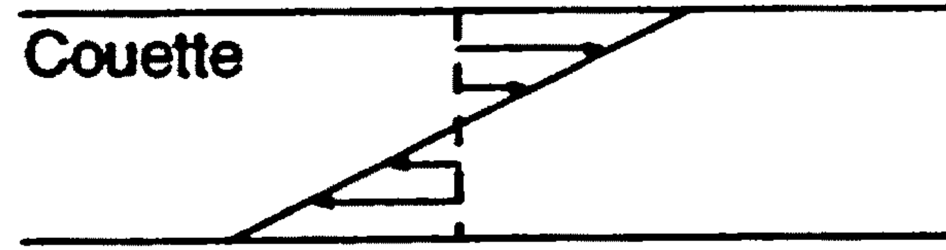
$$G(T) = \frac{E(T)}{E(0)};\ E(T) = \frac{1}{2}\langle \hat{u}^2 + \hat{v}^2 + \hat{w}^2 \rangle$$

- Across all time horizons & amplitudes E(0) (note 1/2...!)

- Define spatial and temporal averages (dagger is c.c. transpose):

$$\langle \mathbf{b}, \mathbf{a} \rangle = \frac{1}{V}\int_{\mathcal{D}} \mathbf{b}^\dagger \mathbf{a} \mathrm{d}V,\ [\mathbf{b}, \mathbf{a}] = \frac{1}{VT}\int_0^T \int_{\mathcal{D}} \mathbf{b}^\dagger \mathbf{a} \mathrm{d}V \mathrm{d}t$$

Lagrangian



- Constrained variational problem:

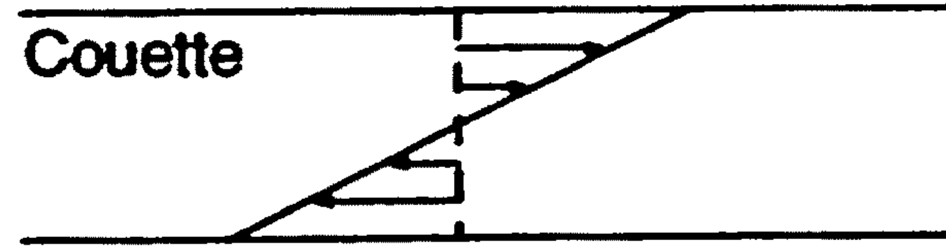
$$\mathcal{L} = \langle \mathbf{q}(T), \mathbf{q}(T) \rangle - [\partial_t \mathbf{q} + N(\mathbf{q}) + \nabla p, \mathbf{a}] + [\nabla . \mathbf{u}, b]$$
$$- (\langle \mathbf{q}_0, \mathbf{q}_0 \rangle - E(0))c + \langle \mathbf{q}_0 - \mathbf{q}(0), \mathbf{a}_0 \rangle$$

$$\mathbf{q} = \hat{u}, \qquad N(q_i) = U_j \partial_j q_i + q_i \partial_i U_j + q_j \partial_j q_i - \nu \partial_j \partial_j q_i$$

- Final (perturbation) energy to be maximized
- Vector L-multiplier a imposes (nonlinear) Navier-Stokes for all t
- b imposes incompressibility (pressure?)
- c imposes initial energy perturbation
- a0 imposes initial perturbation (structure)

$$\langle \mathbf{b}, \mathbf{a} \rangle = \frac{1}{V} \int_{\mathcal{D}} \mathbf{b}^\dagger \mathbf{a} \mathrm{d}V, \; [\mathbf{b}, \mathbf{a}] = \frac{1}{VT} \int_0^T \int_{\mathcal{D}} \mathbf{b}^\dagger \mathbf{a} \mathrm{d}V \mathrm{d}t$$

Adjoint



- "Obviously" variations with respect to a, b,c, a0

$$\mathcal{L} = \langle \mathbf{q}(T), \mathbf{q}(T) \rangle - [\partial_t \mathbf{q} + N(\mathbf{q}) + \nabla p, \mathbf{a}] + [\nabla . \mathbf{u}, b]$$
$$- (\langle \mathbf{q}_0, \mathbf{q}_0 \rangle - E(0))c + \langle \mathbf{q}_0 - \mathbf{q}(0), \mathbf{a}_0 \rangle$$

$$\mathbf{q} = \hat{u}, \qquad N(q_i) = U_j \partial_j q_i + q_i \partial_i U_j + q_j \partial_j q_i - \nu \partial_j \partial_j q_i$$

- Recover NS/incompressibility/ICs...but what about:

$$\frac{\delta \mathcal{L}}{\delta \mathbf{q}} = \partial_t \mathbf{a} + N^\dagger(\mathbf{a}, \mathbf{q}) + \nabla b - (\mathbf{a} + 2\mathbf{q})|_{t=T} + (\mathbf{a} - \mathbf{a}_0)|_{t=0} = 0$$

$$\frac{\delta \mathcal{L}}{\delta p} = \nabla . \mathbf{a} = 0$$

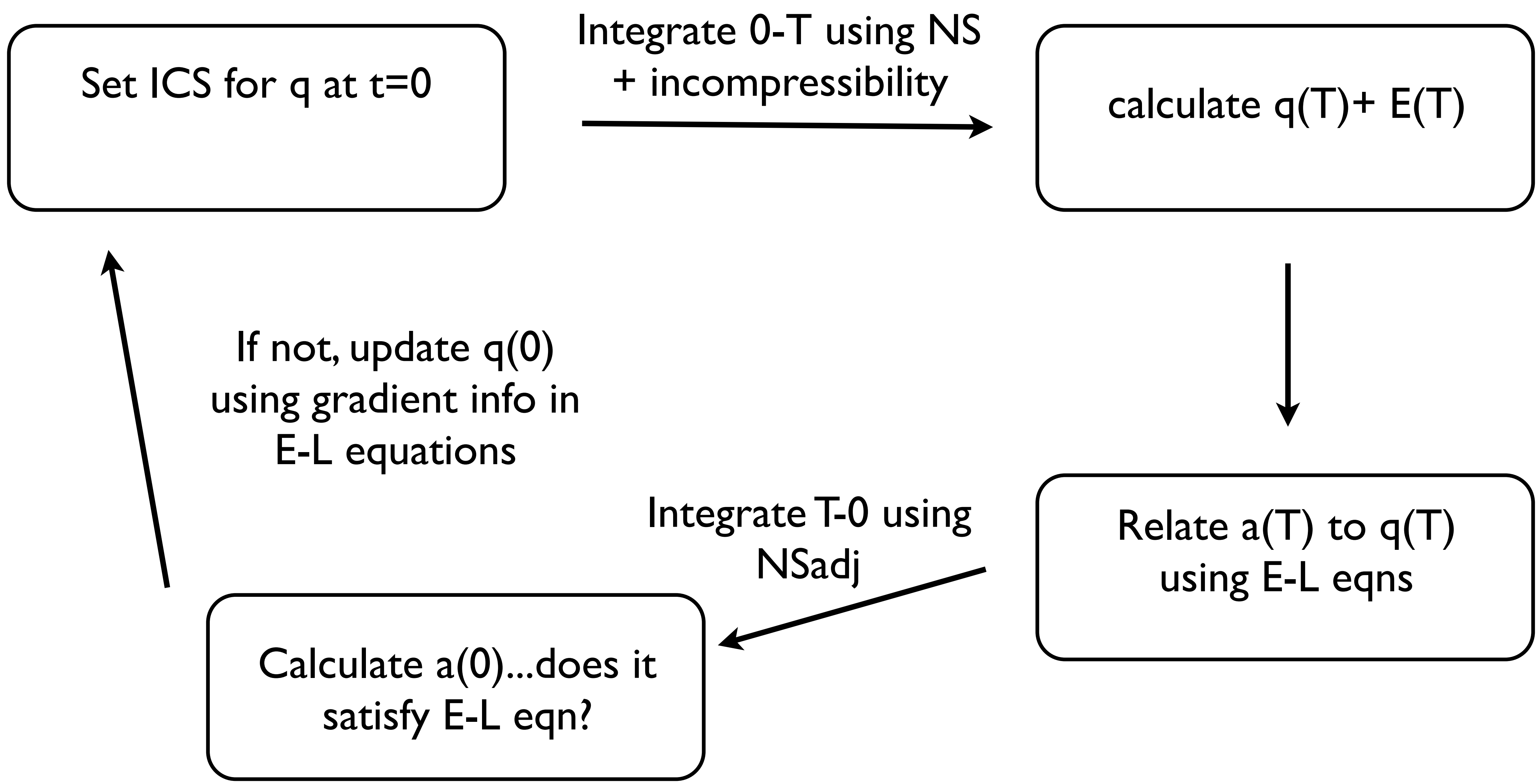
$$\frac{\delta \mathcal{L}}{\delta \mathbf{q}_0} = \mathbf{a}_0 - 2\mathbf{q}_0 c = 0 \qquad \frac{\delta \mathcal{L}}{\delta T} = \frac{\delta E}{\delta T} = 0$$

$$N^\dagger(a_i, \mathbf{q}) = \partial_j \left(q_j a_i \right) - a_j \partial_i q_j + \partial_j \left(U_j a_i \right) - a_j \partial_i U_j + \nu \partial_j \partial_j a_i$$

- Adjoint equation inevitably coupled to q as well as a...

Algorithm

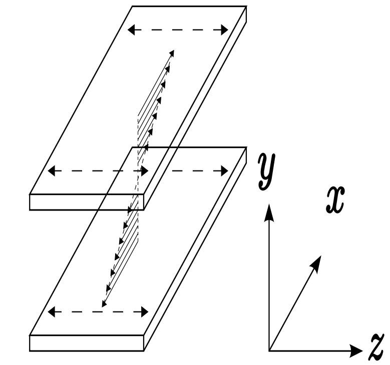
- And now the algorithm is obvious (diffusion has opposite sign!)



- For small amplitudes: recover Quasi-Linear-Optimal Perturbations

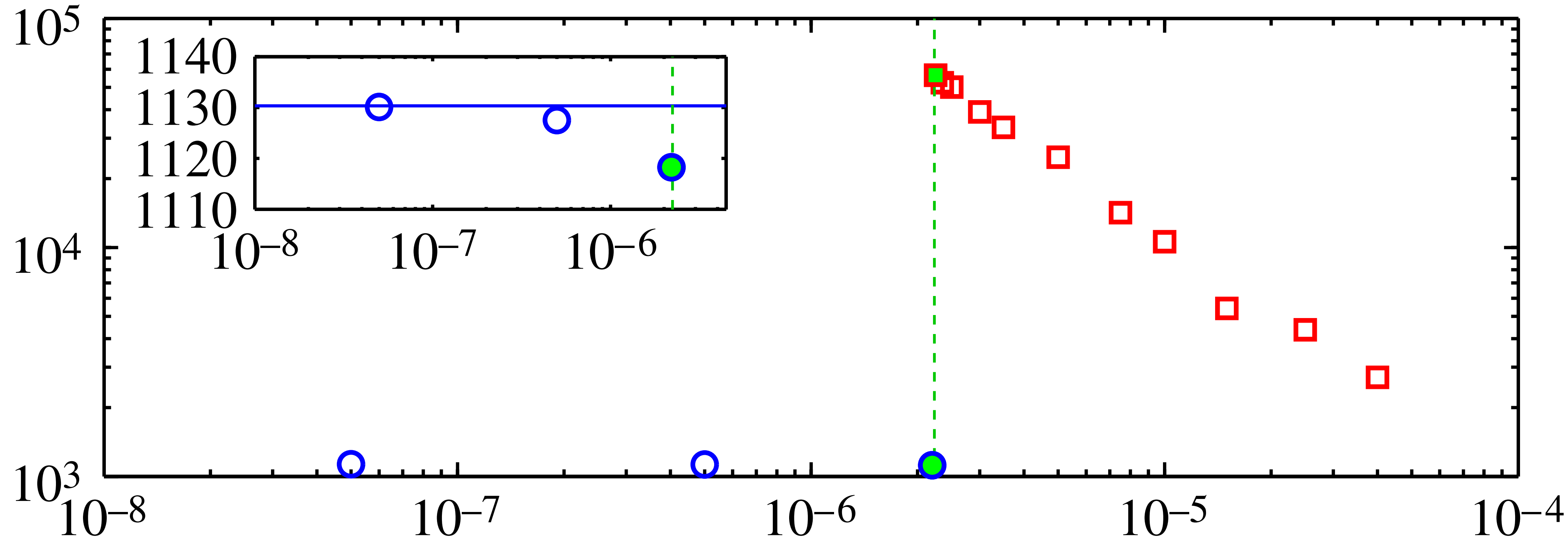
- What happens as the amplitude increases...Re is high enough?

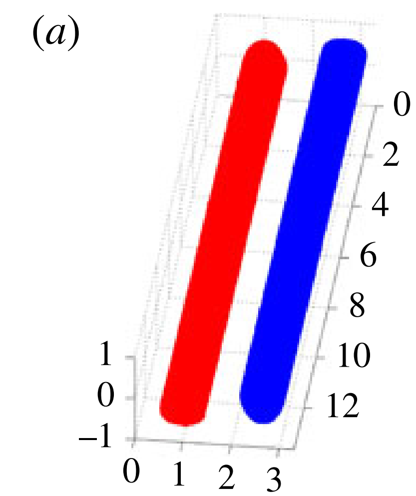
Minimal seeds



$E_0 \simeq 2.2 \times 10^{-6}; Re = 1000$

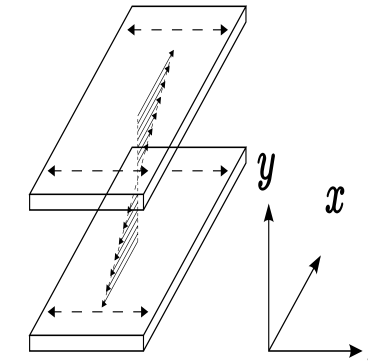
(a)





- For small initial energy...agrees very closely with linear optimals in gain

Minimal seeds



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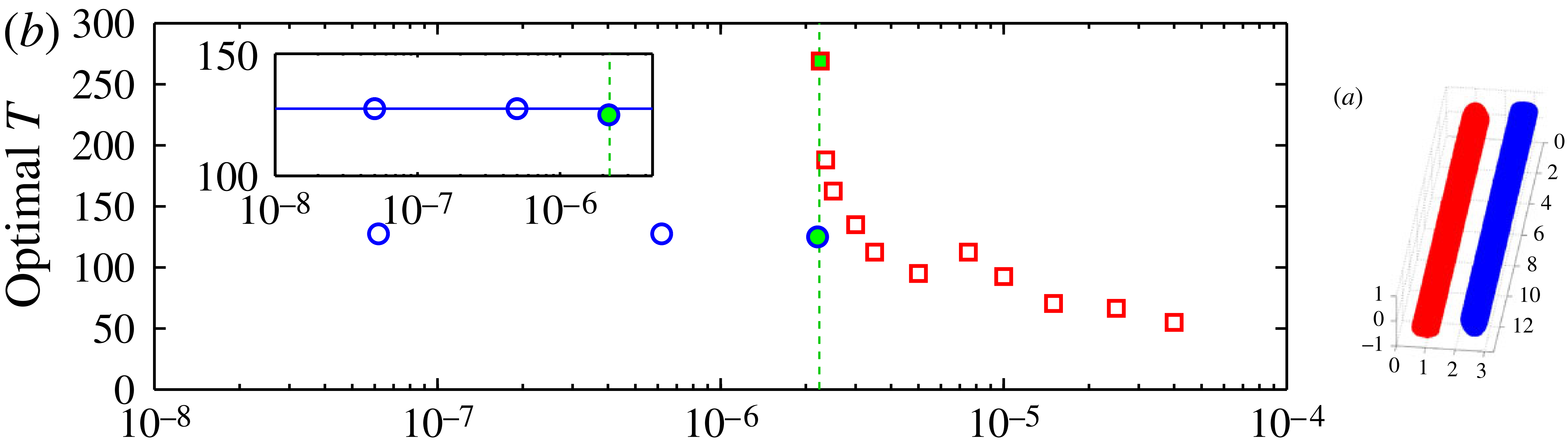
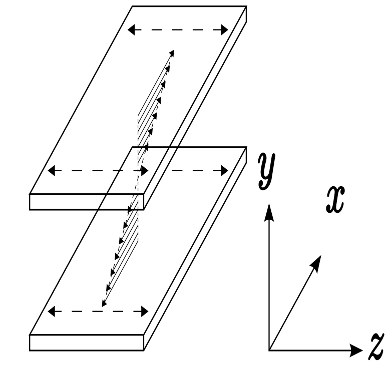


- For small initial energy...agrees very closely with linear optimals and T

Minimal seeds



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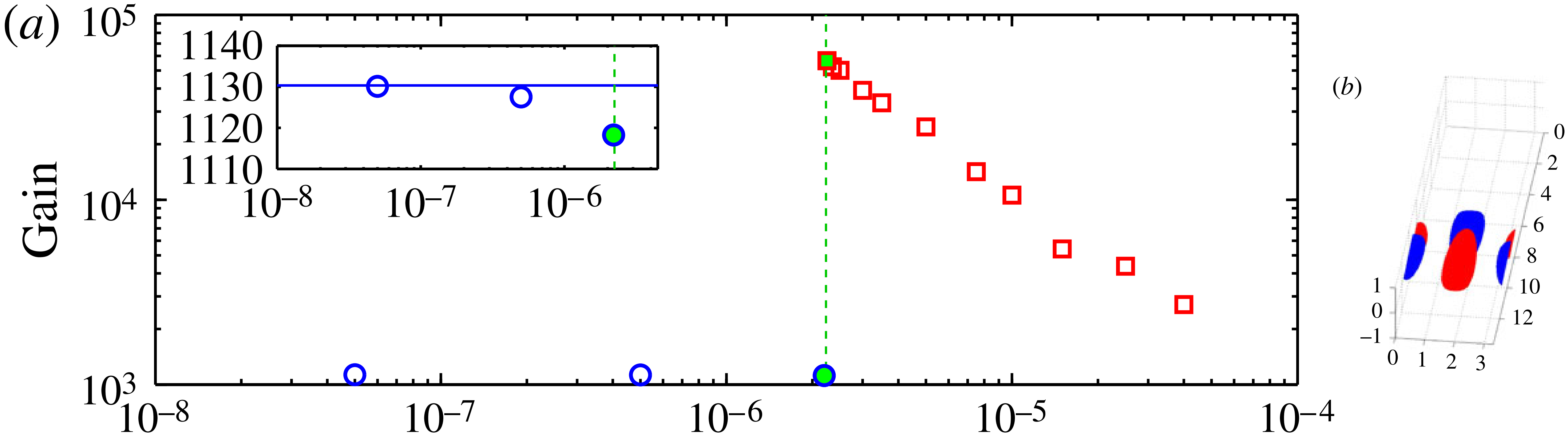

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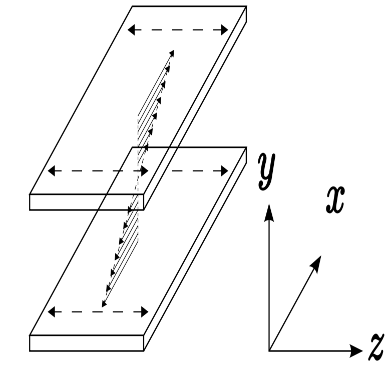
- But sudden change at high enough energy....
- Inherently localised structure

Minimal seeds



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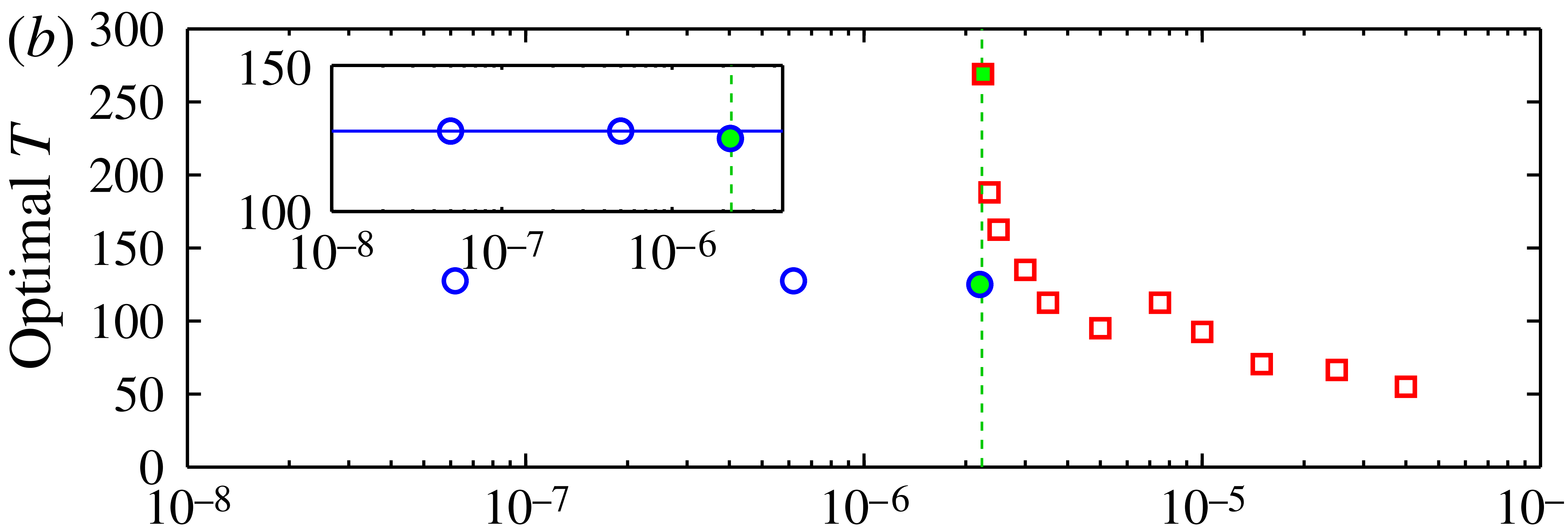
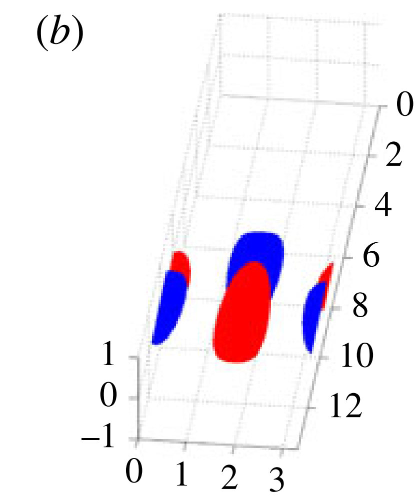
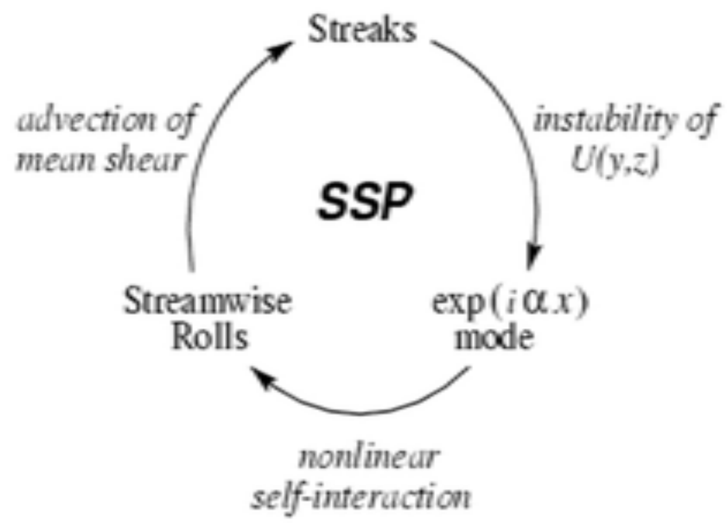




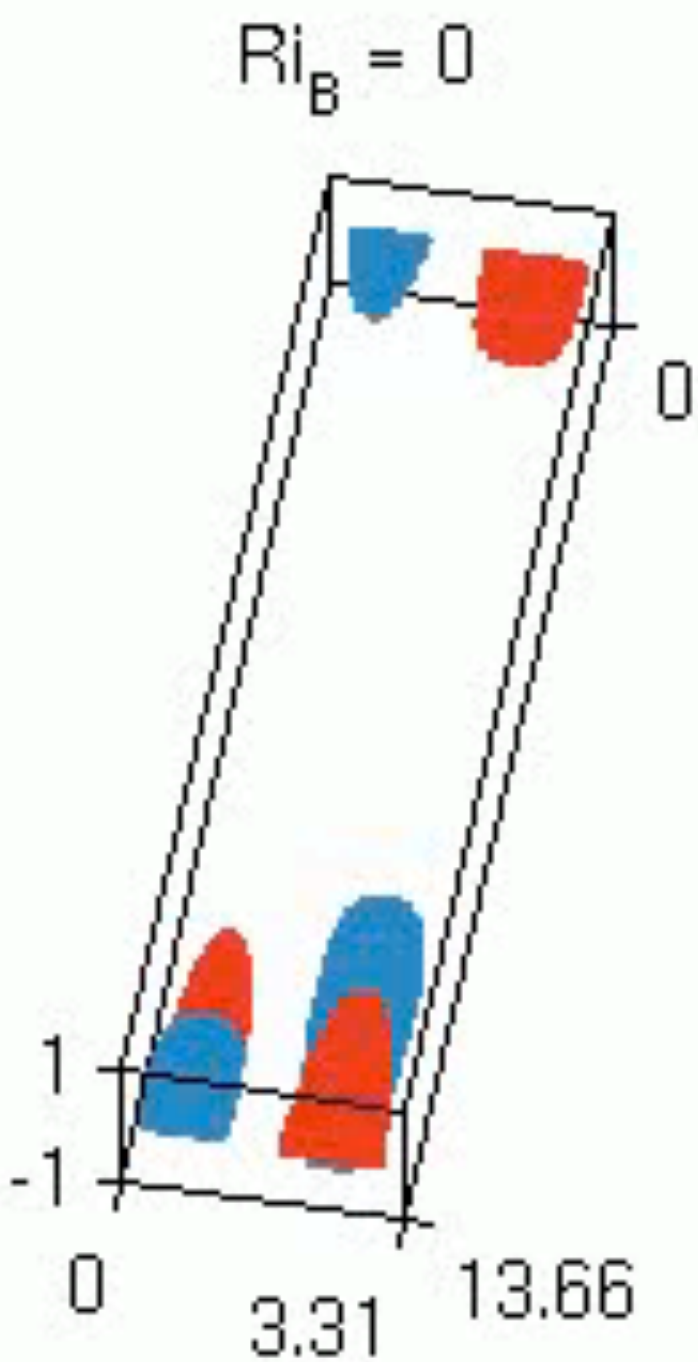
- For small initial energy...agrees very closely with linear optimals
- But sudden change at high enough energy....
- Inherently localised structure...with turbulence at late time

Minimal seeds



Fabian Waleffe, *Physics of Fluids*, **9**, 1997

- SSP: Waleffe/VWI: Hall
- Hierarchy of growth mechanisms
- Bootstrapping inherently nonlinearly
- Duguet et al 2013
- Kerswell et al 2014
- Unpack to streaks to bending waves
- to breakdown...



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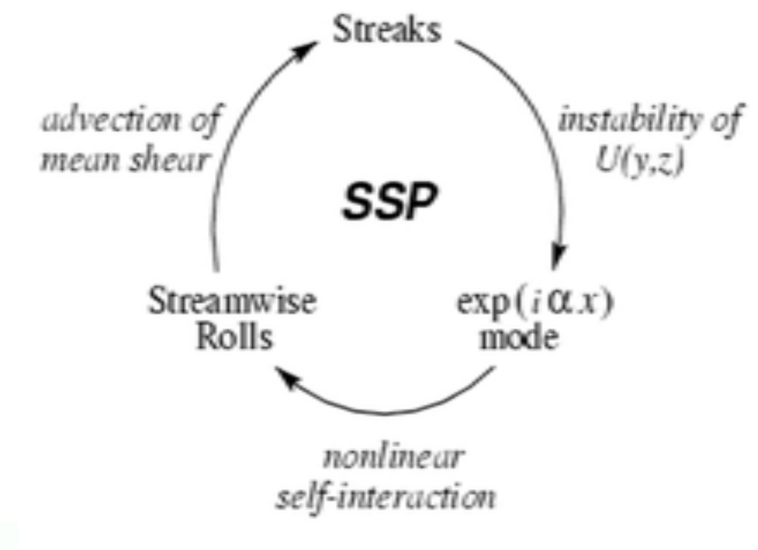

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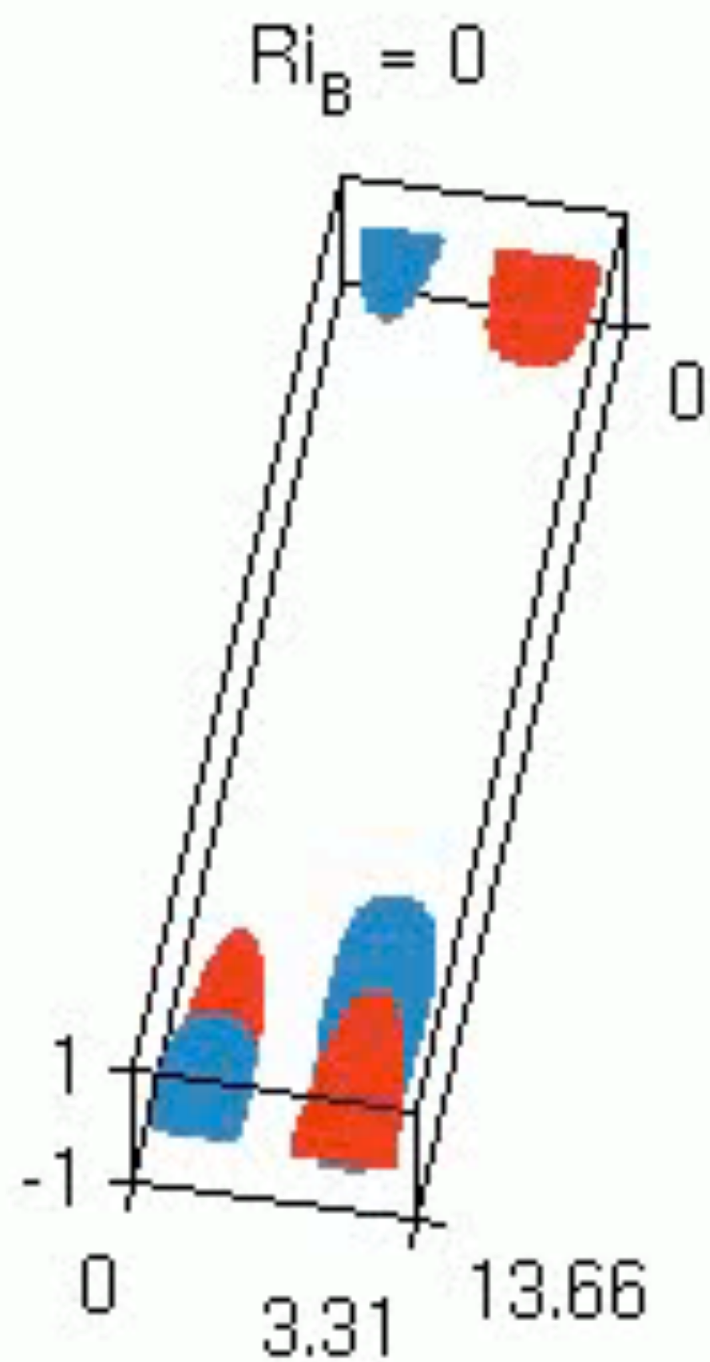
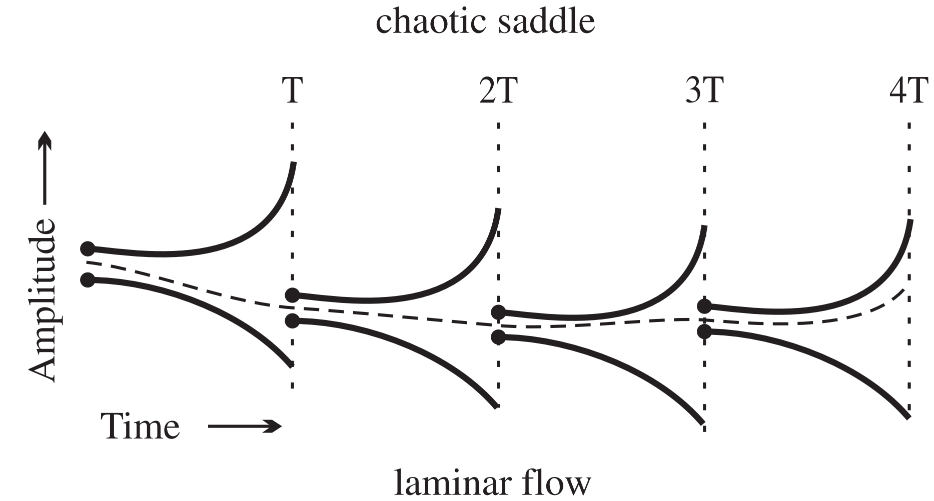
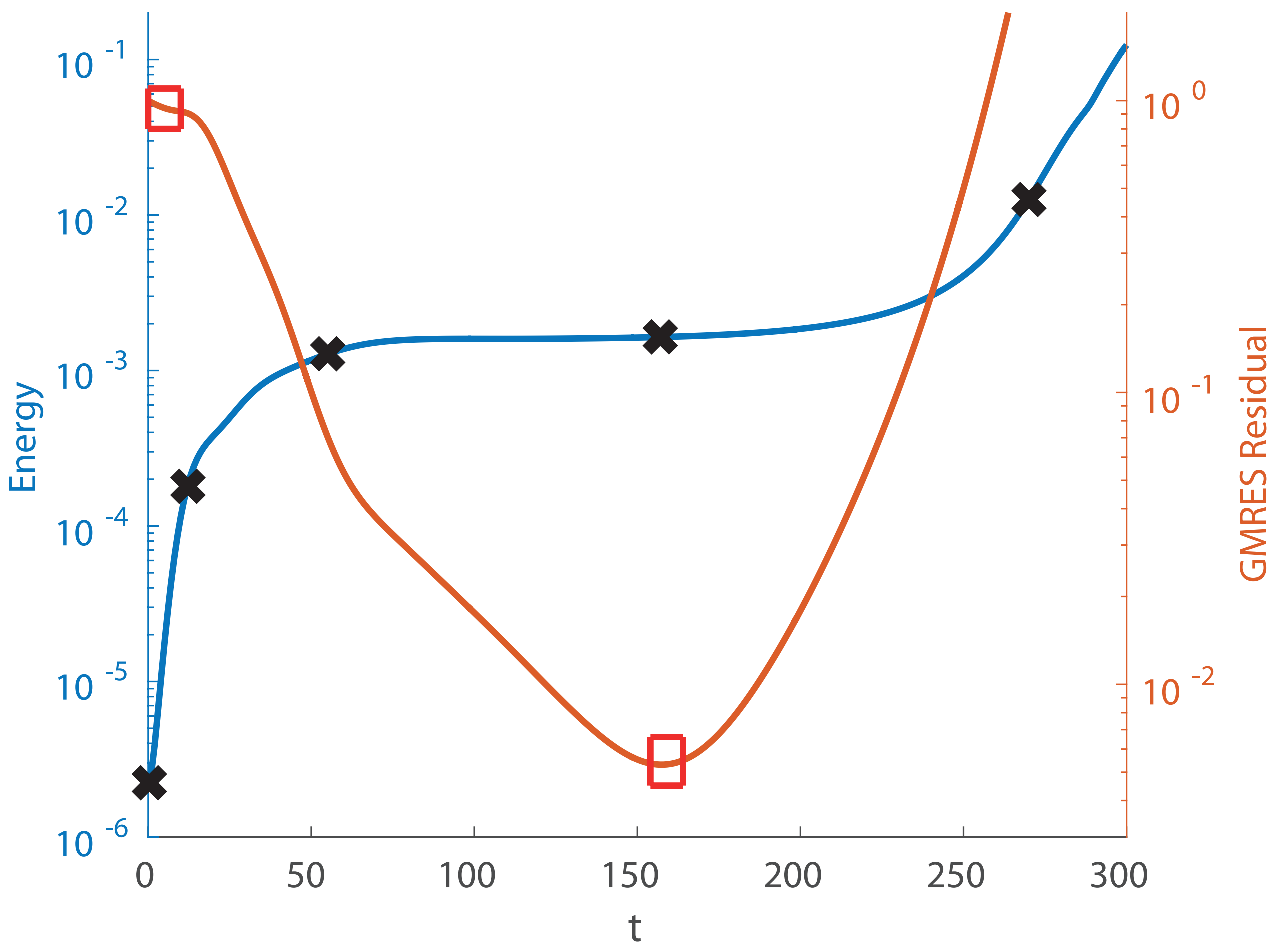
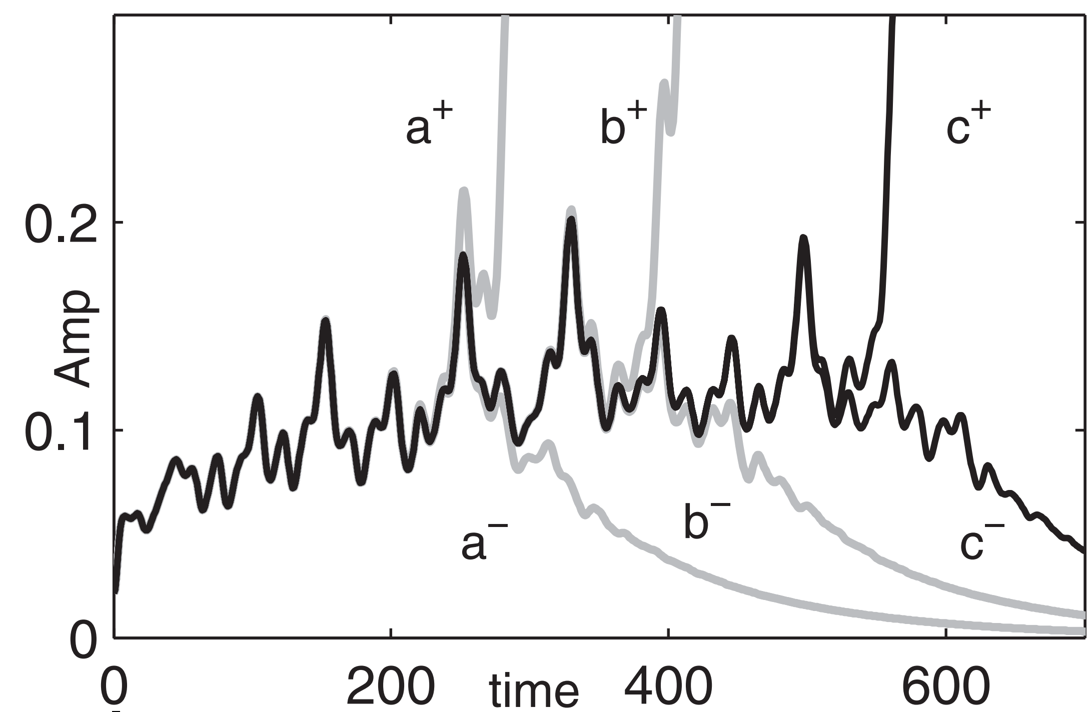


Relation to edge?



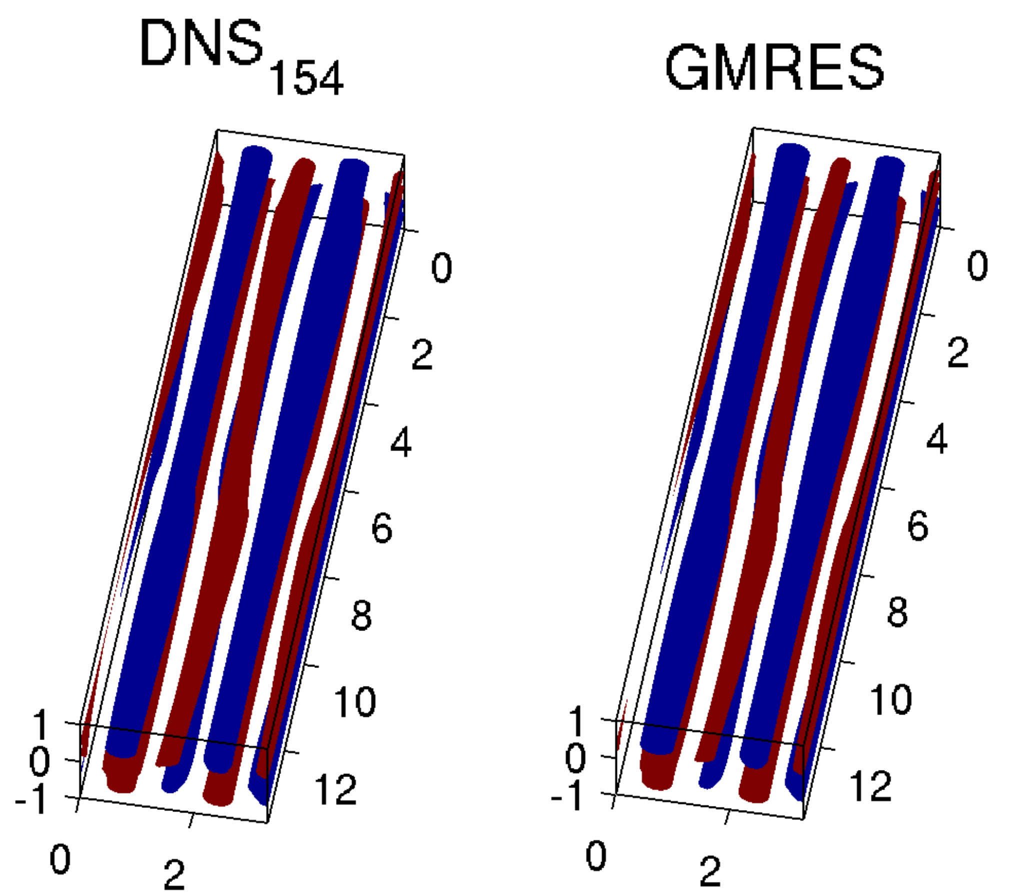


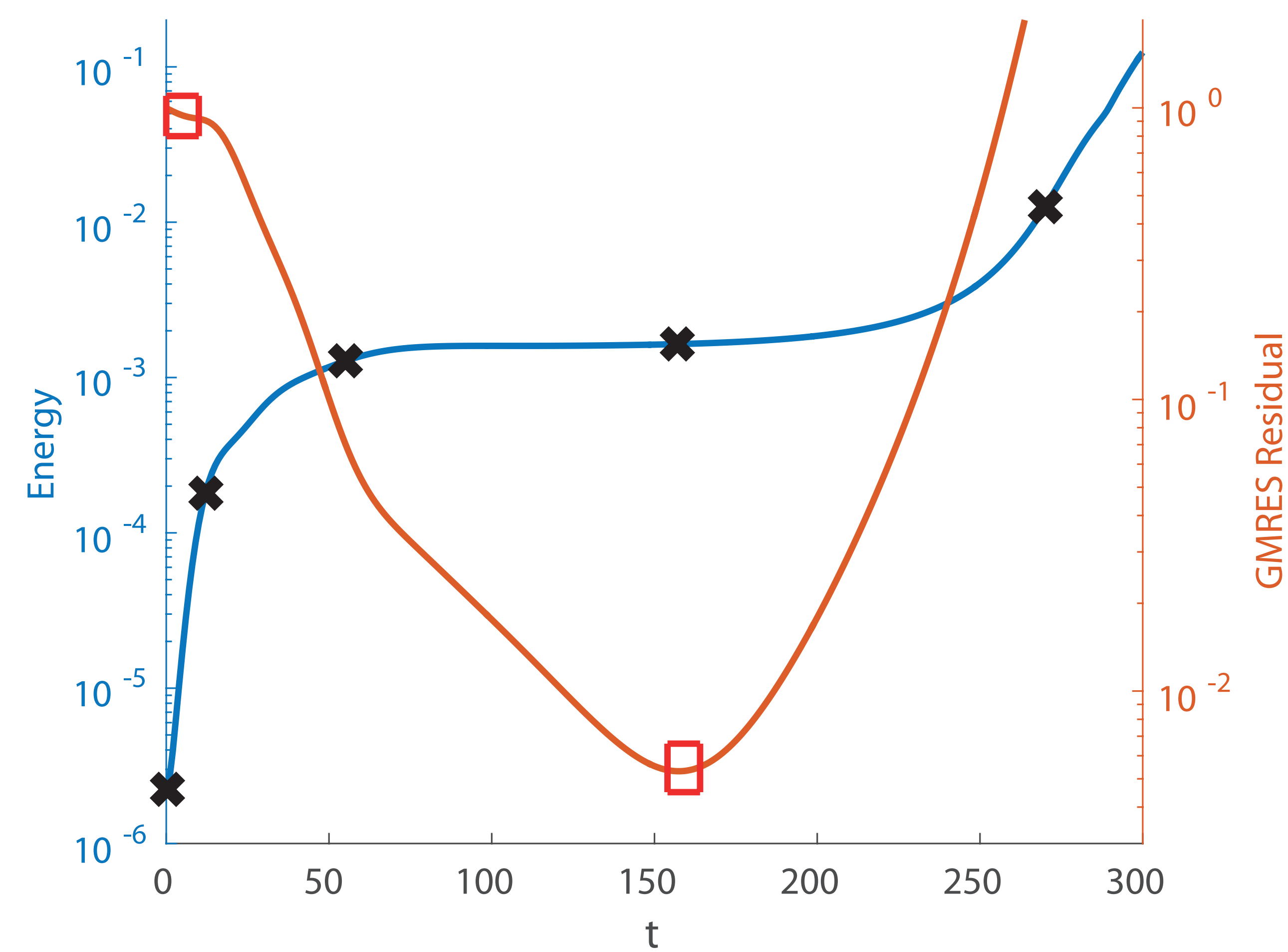


- Energy stays close to constant for a long time....
- Suggestive of being a specific way to tiptoe along the edge
- Suggesting approach on stable manifold of edge state Skufca et al 06 etc

Seeding the edge

$$R(t) = \frac{|\mathbf{u}(\mathbf{x},t) - \mathbf{u}_{GMRES}(\mathbf{x})|^2}{|\mathbf{u}(\mathbf{x},t)|^2}$$





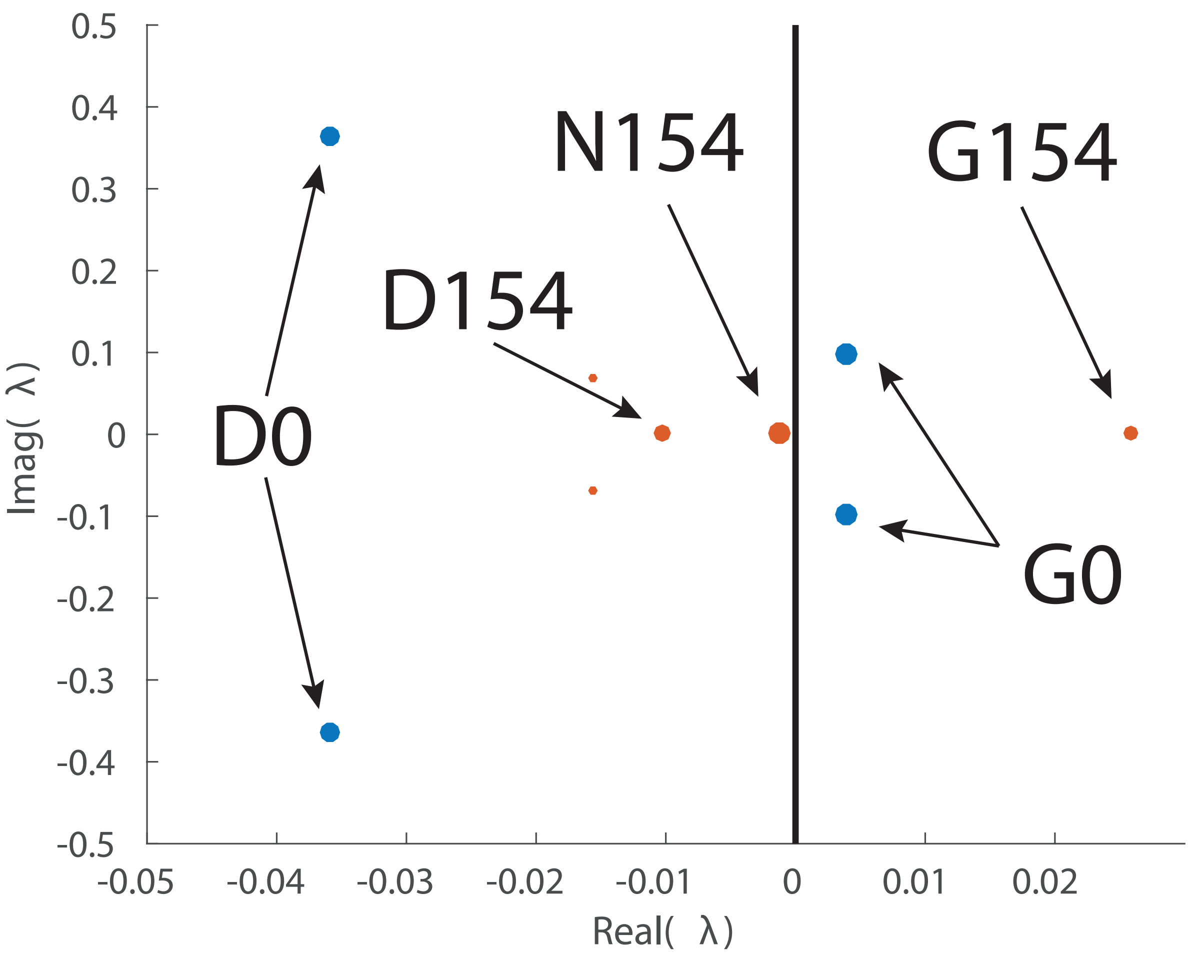
- Use DNS results as guess for Generalized Minimal Residual method
- Finds a steady edge state with very clear structure
- Is it possible to identify reduced description of approach/departure?

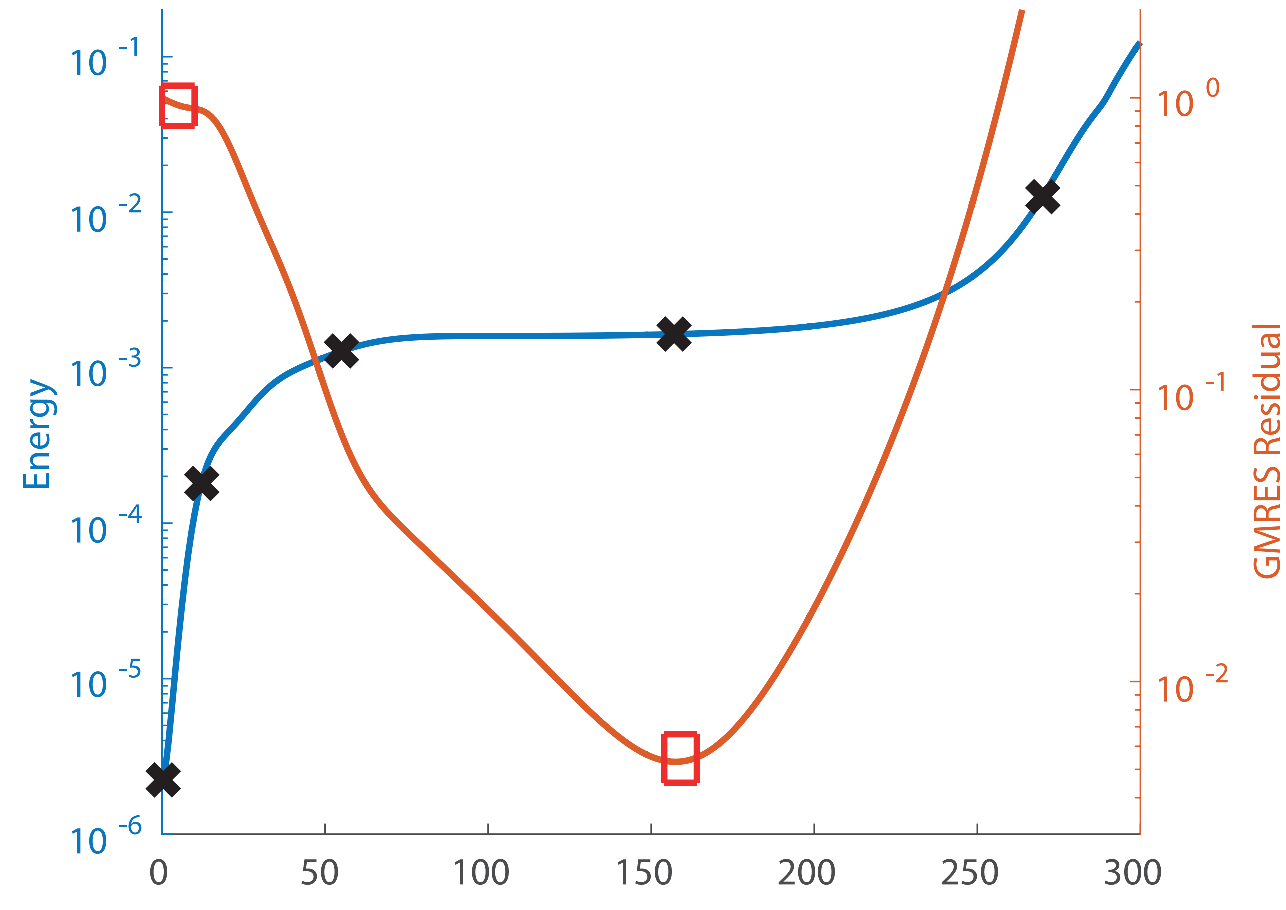
Koopman/DMD Modes

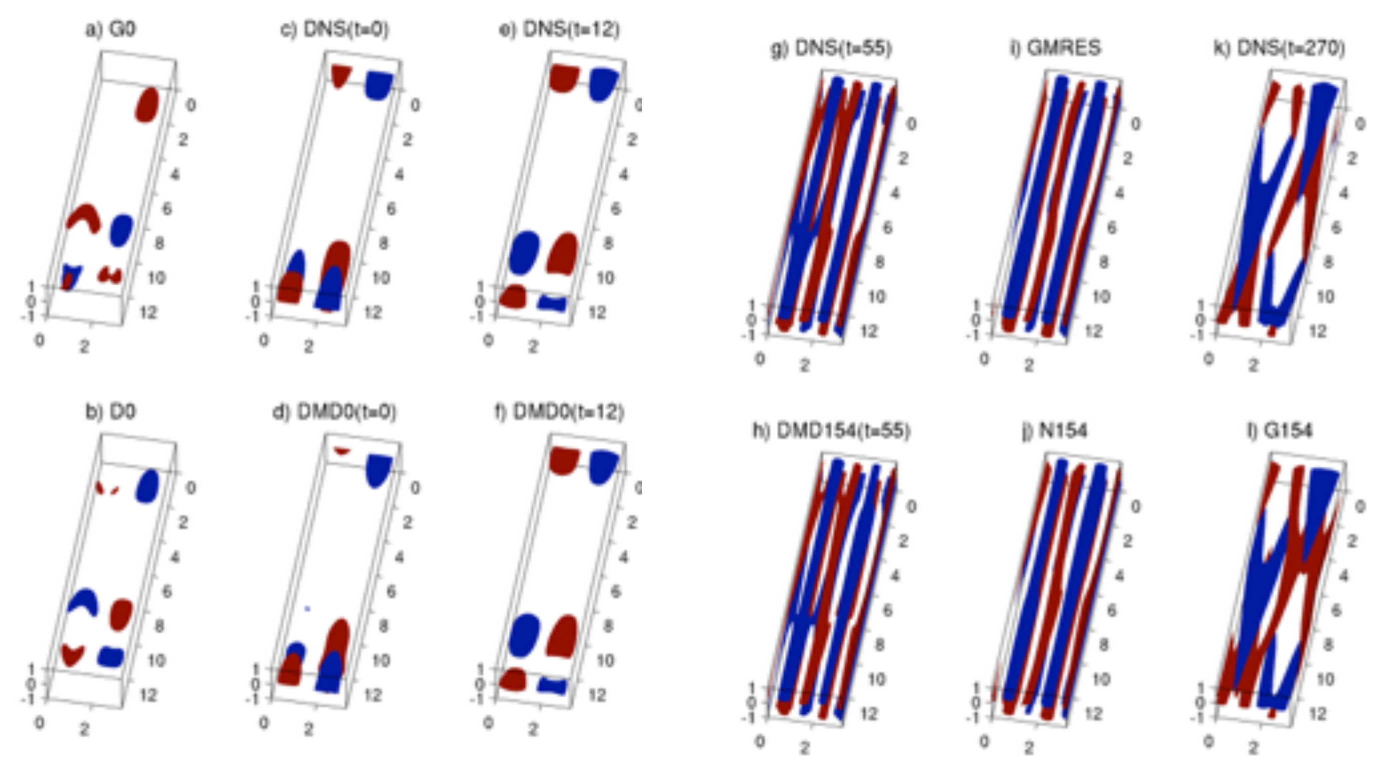
- Dynamical information about flow: take snapshots $\mathbf{q}_n = \mathbf{q}(\mathbf{x}, t_n)$
- Assume (for every n...system not changing much...): $\mathbf{q}_{n+1} = \mathbf{N}_0 \mathbf{q}$
- Use DMD (Schmid 2010) to find spectrum of $\mathbf{N}_0$
- Decompose (amplitudes "fit" snapshots): $\mathbf{q}(\mathbf{x}, t) = \sum a_n \mathbf{m}_n(\mathbf{x}) \exp(\lambda_n t)$
- Finite-dimensional/fixed in time approximation of Koopman operator
- Mezic (2005) linear in the observable: $\mathcal{K}_t \mathbf{q}(\mathbf{x}, 0) = \mathbf{q}(\mathbf{x}, t)$: efns etc!
- On attractor: $\mathbf{q}(\mathbf{x}, t) - \overline{\mathbf{q}}(\mathbf{x}) = \sum a_n \mathbf{m}_n(\mathbf{x}) \exp(\lambda_n t) + \int e^{2\pi i \alpha t} [dE(\alpha)\mathbf{q}]$
- Does this yield a low-dimensional description of the flow?

DMD: early and edge

- 11 snapshots: 0-10 and 154-164:
- Spectrum: 4 (early) 3 (late) mod

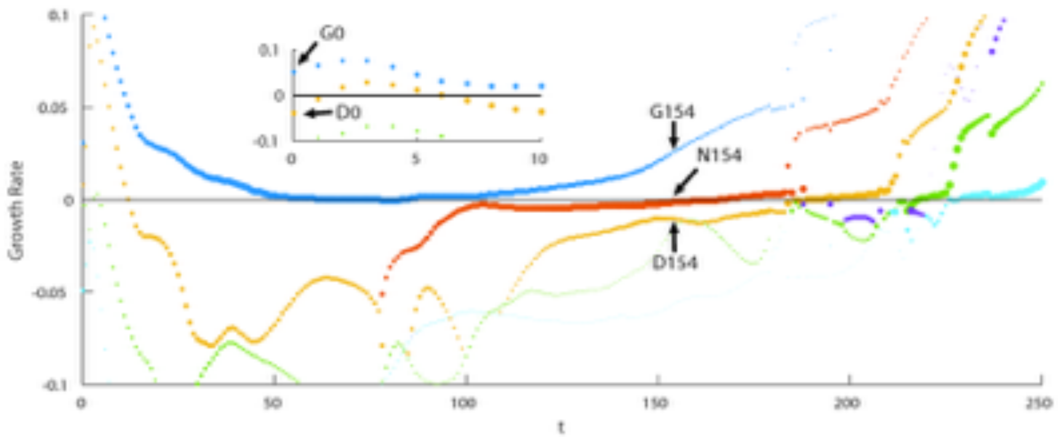


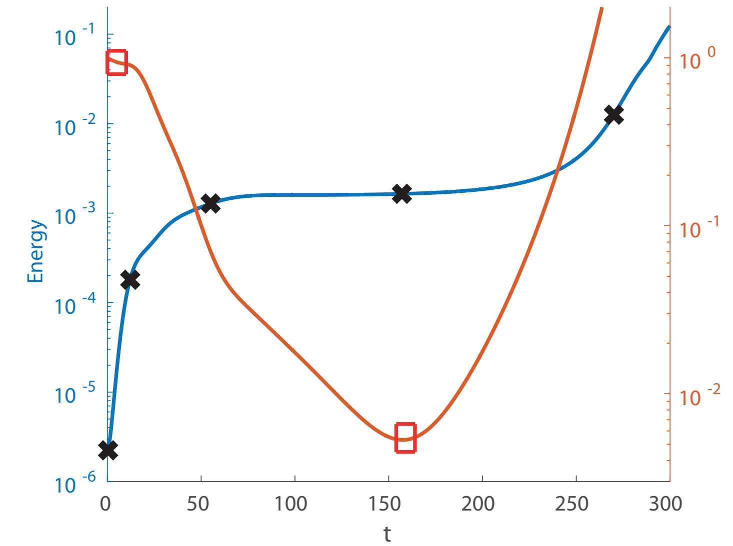


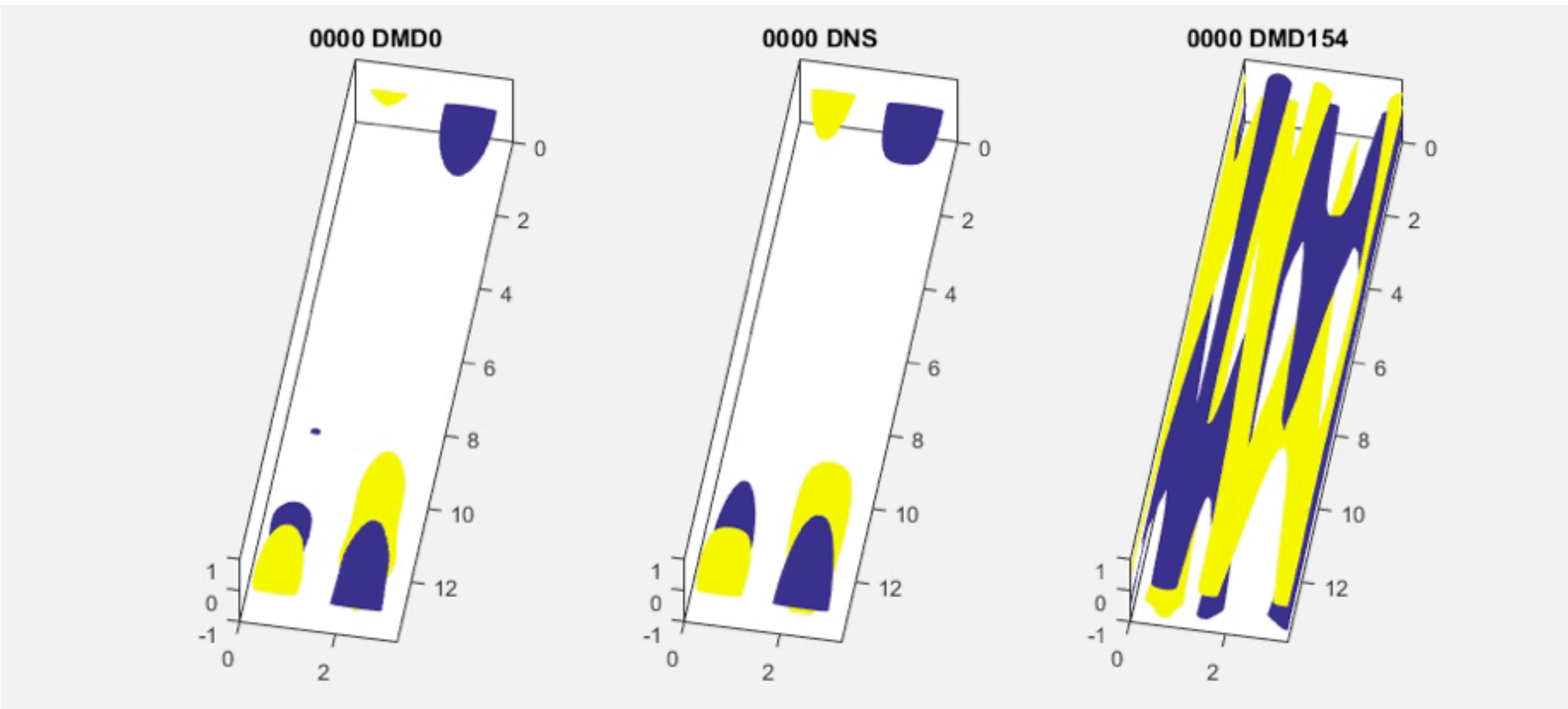


- Captures early & late evoluti
- Using constant eigenvalues

DMD: early and edge

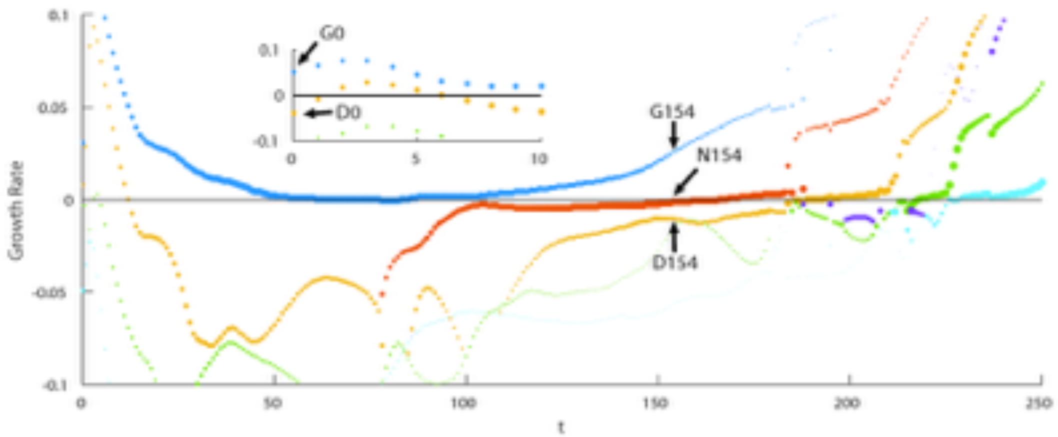


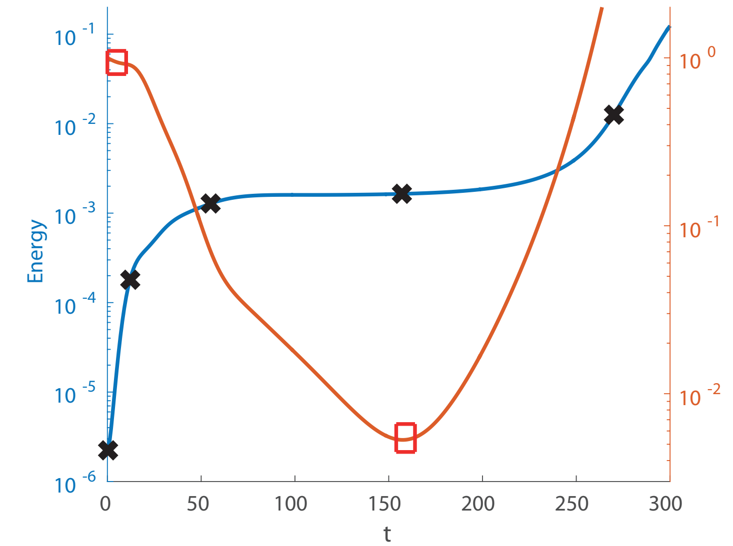


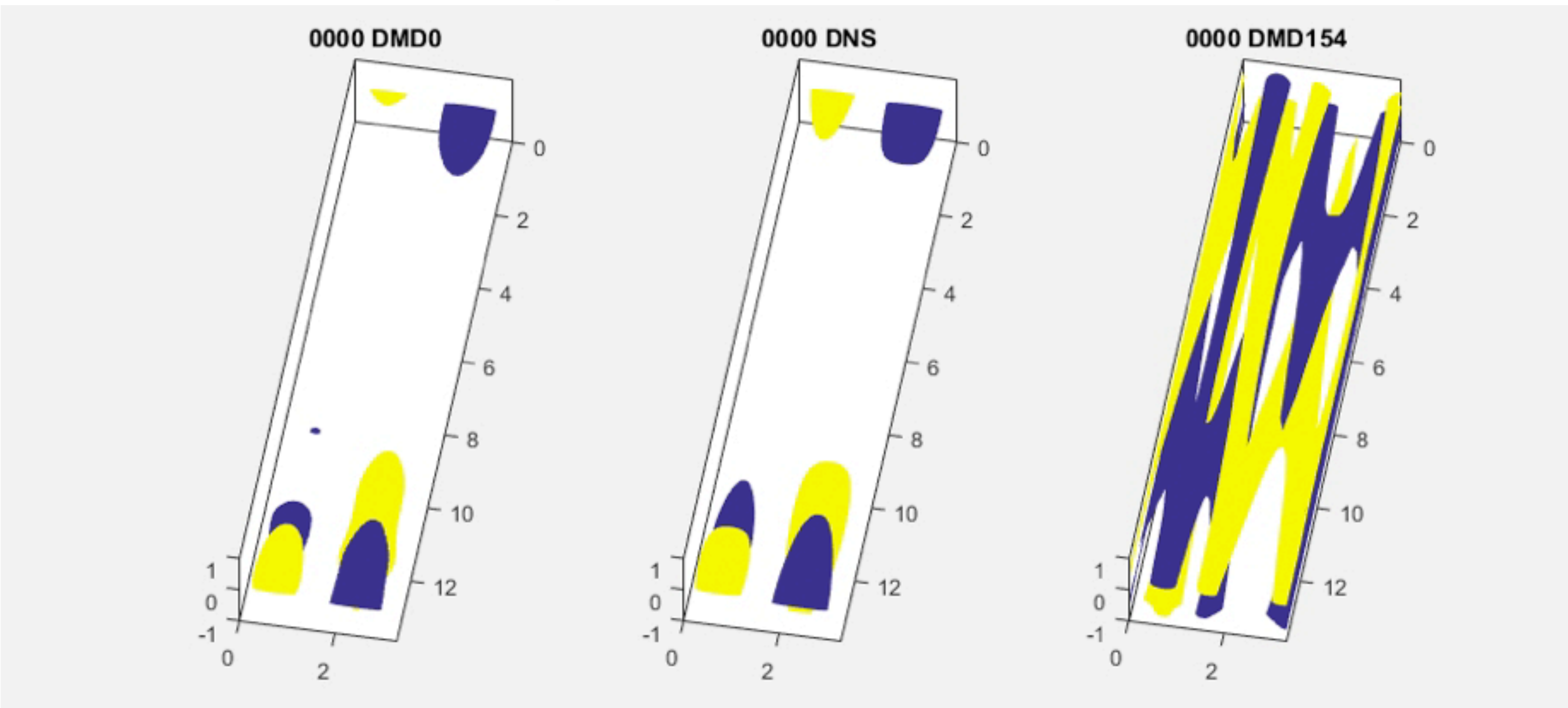


4 mode DMD0 | Full DNS | 3 mode DMD154

DMD: early and edge







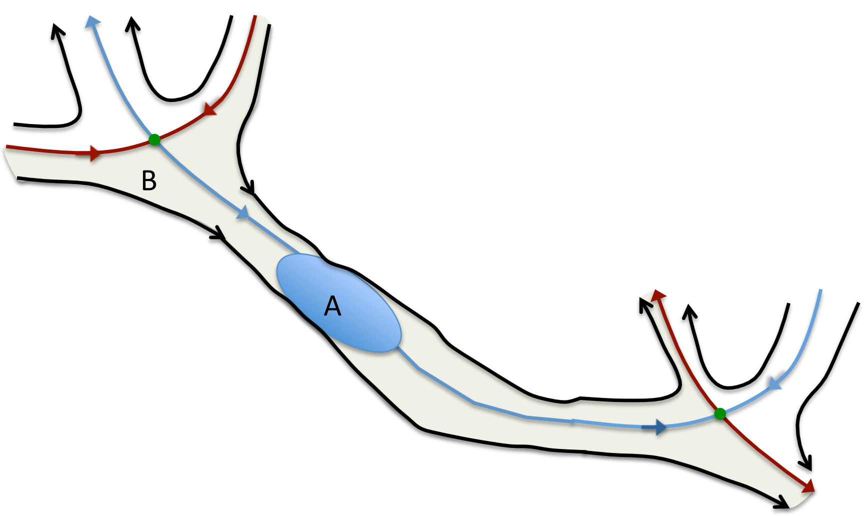
4 mode DMD0

Full DNS

3 mode DMD154

A: Conclusions

- Direct-Adjoint Looping method rides along edge to edge state
- DAL method gives specific route on stable manifold
- DAL method leaves edge state along unstable manifold
- Koopman modes appear to be the natural description
- Can we use this process to play pinball turbulence?



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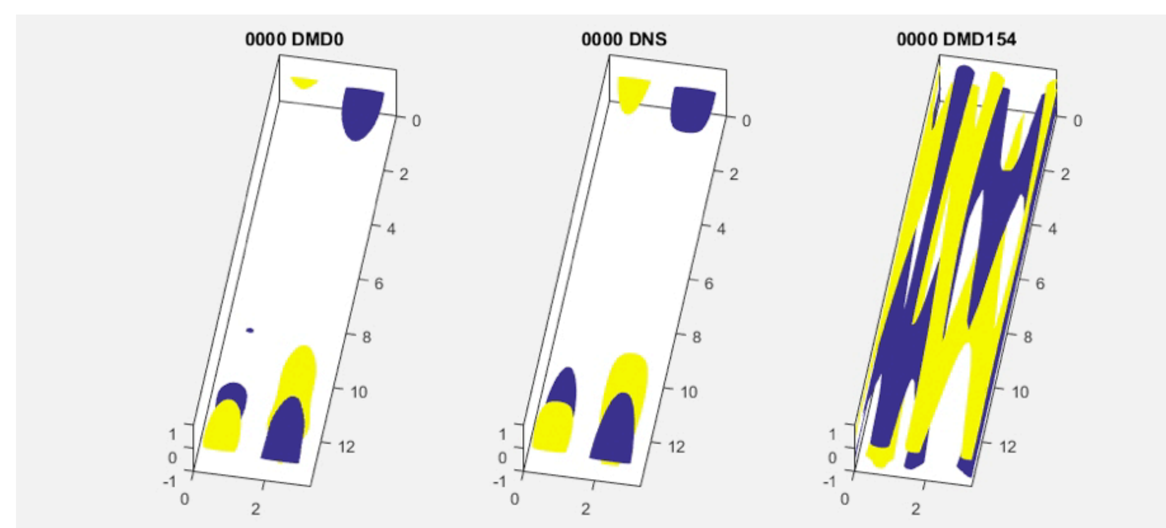

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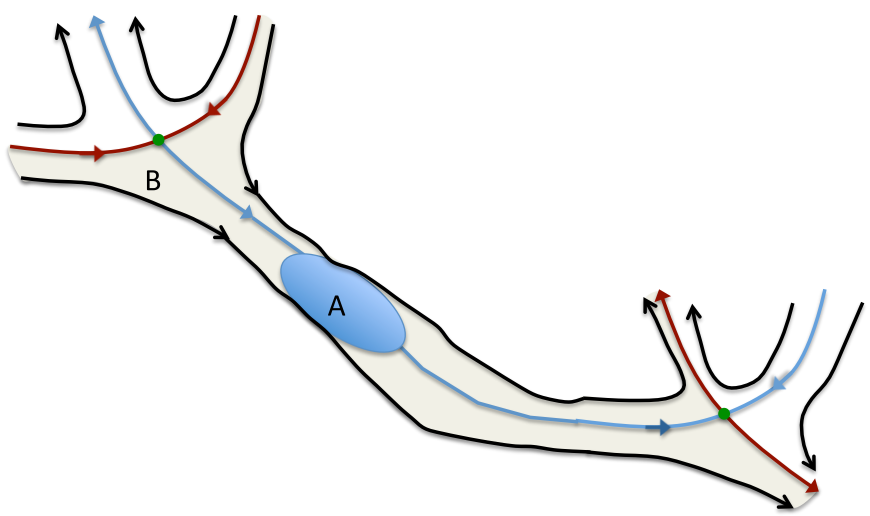

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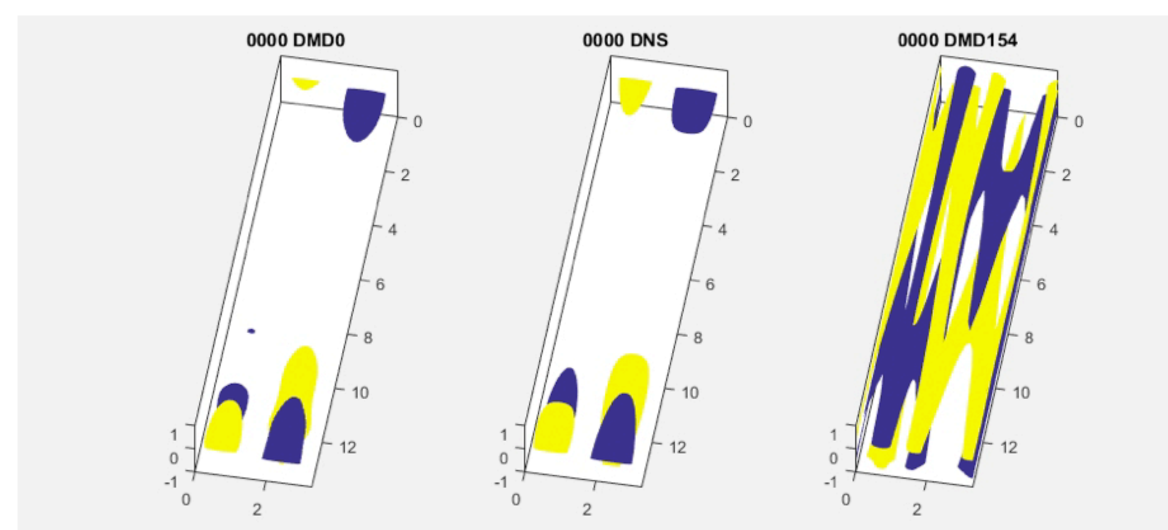
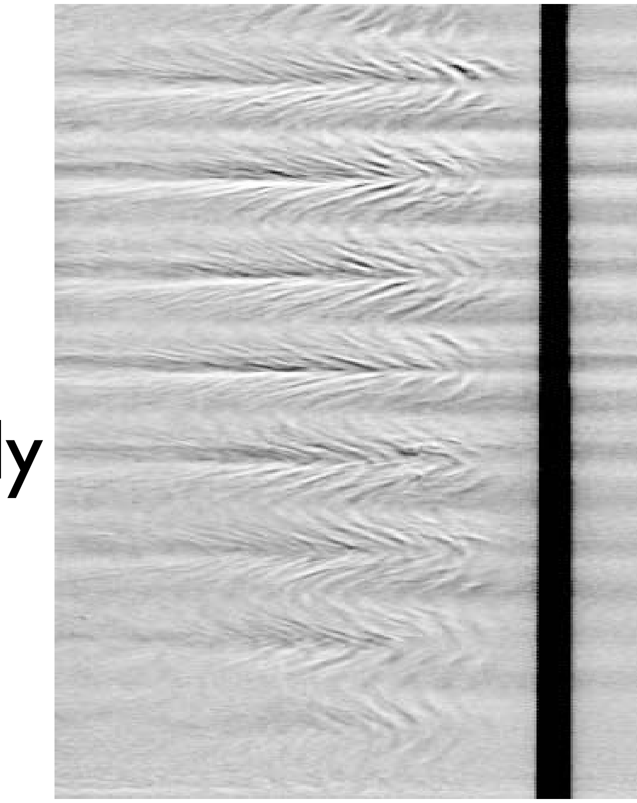
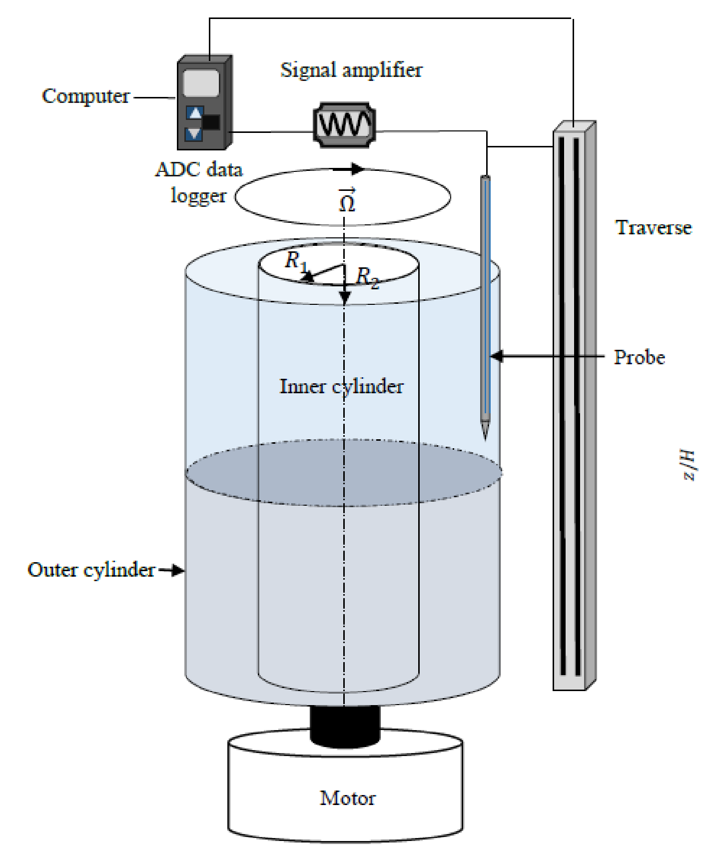


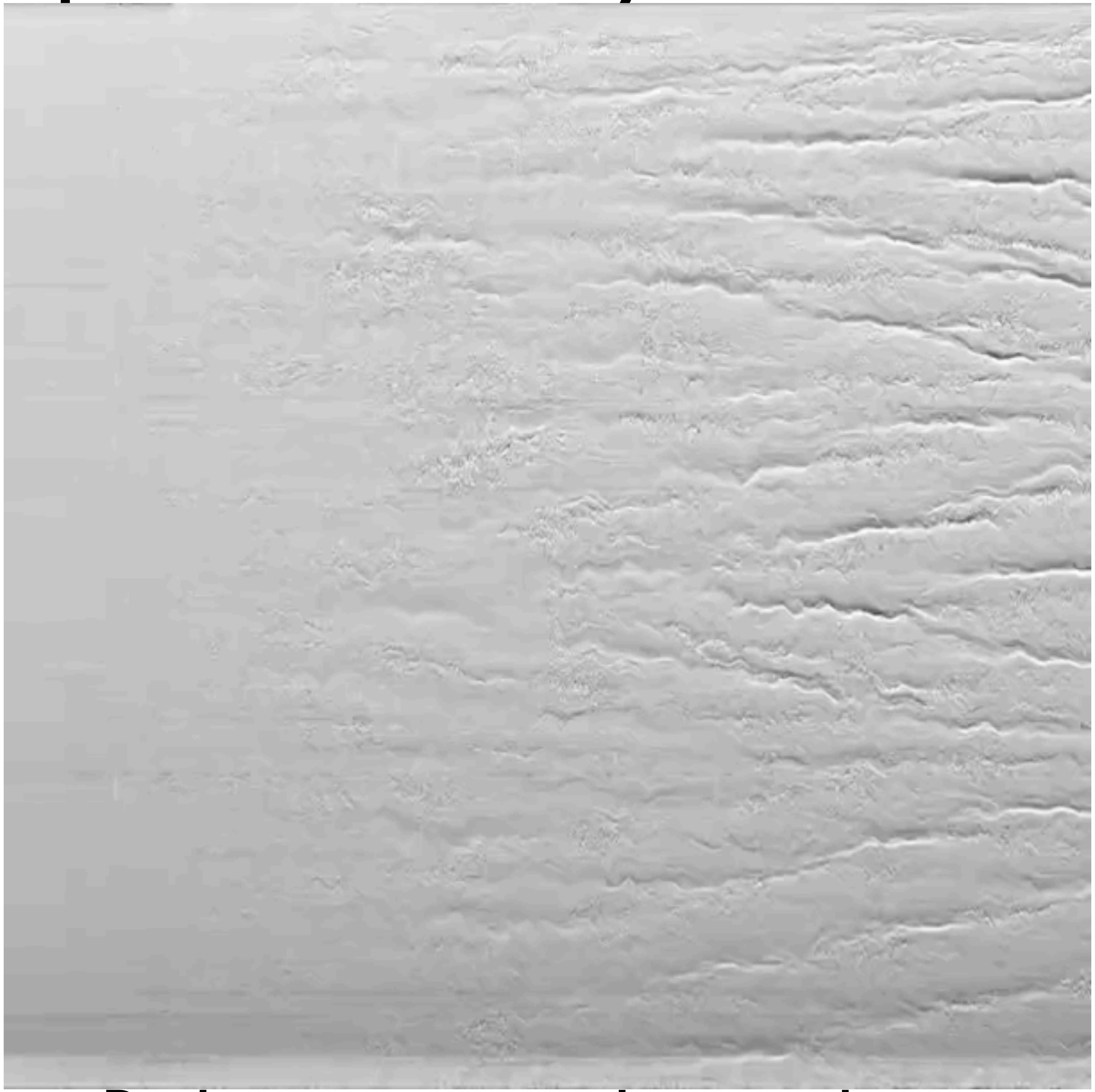
B: Layering & UPOs



- Initially linearly stratified flows layer spontaneously
- “anti-diffusion” Phillips(1972) Park et al (1994)
- Holford & Linden (1998) vertical rod spontaneously layers:
- Horizontal shear can also be important: stratified T-C flow?
- Outer cylinder 24.7 cm: inner 5/10/15 cm
- Radius ratio: $\eta = \frac{R_i}{R_0} = 0.208, 0.417, 0.625$
- Re > 10000, initially linearly stratified
- Lots of horizontal shear: visualize with shadow



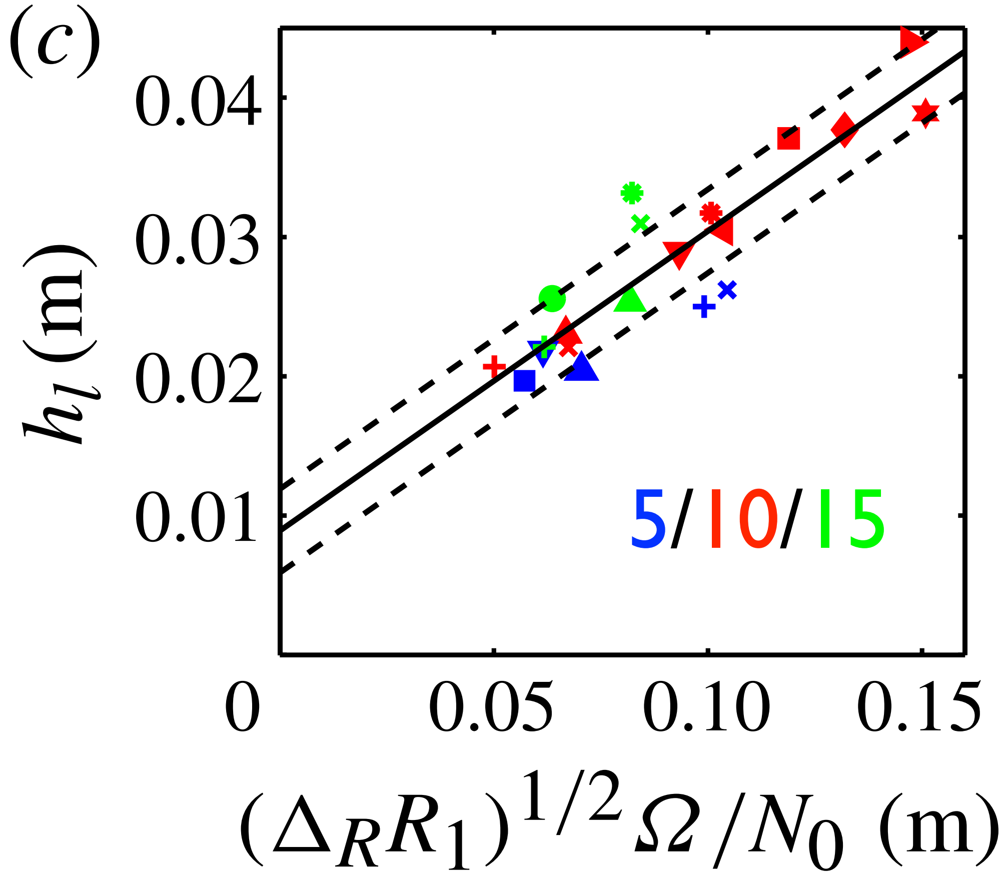
Spontaneous layers: NOT Taylor vortices



$$Ri = \frac{N^2}{\Omega^2} \simeq 3$$ Jamie Partridge (MUST)

$$Re = \frac{\Omega R_i \Delta R}{\nu} \simeq 14000$$

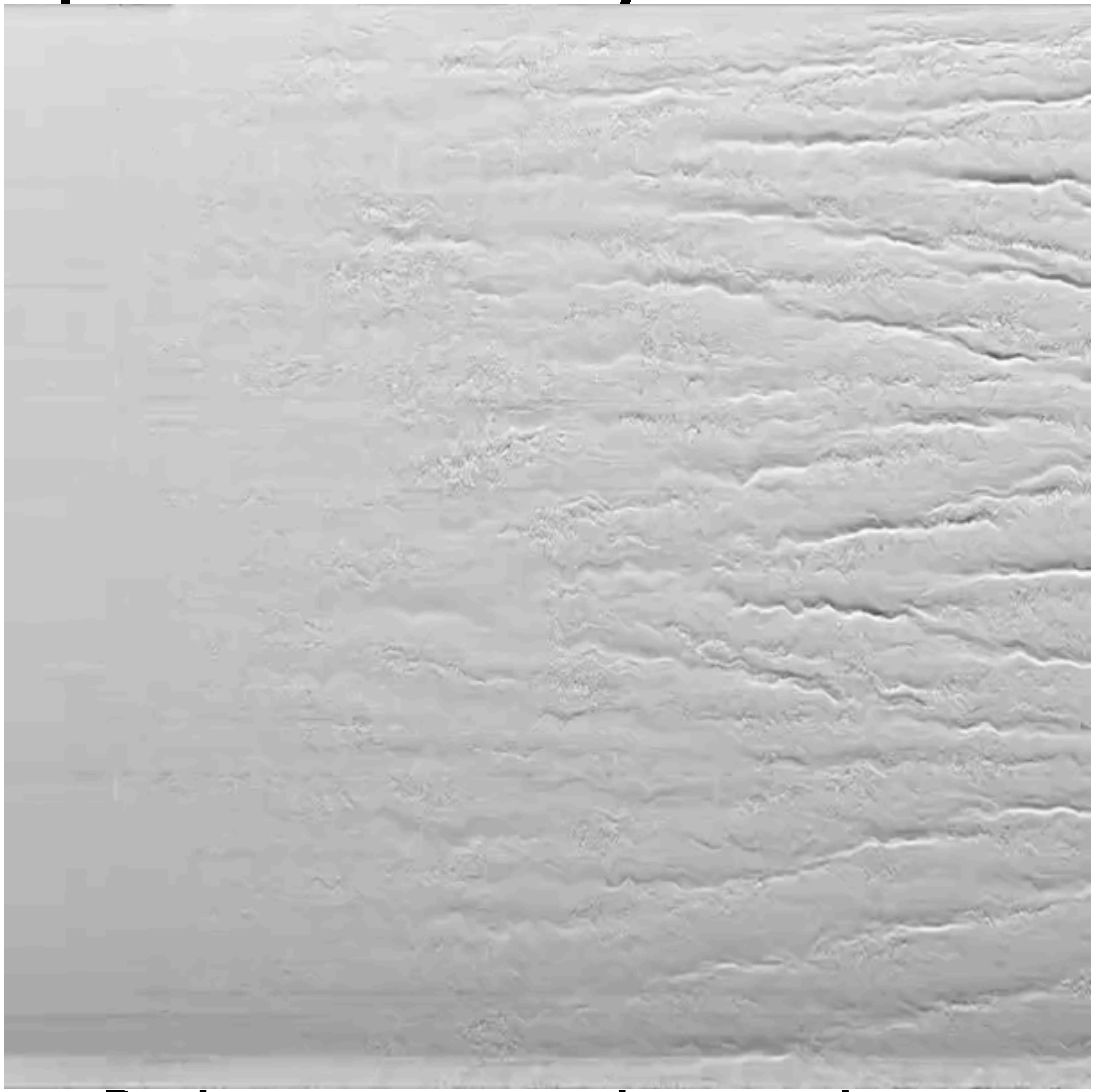
$$h \propto \frac{\Omega}{N}\sqrt{R_i \Delta R} \propto \frac{\Omega}{N}\sqrt{\eta(1-\eta)}$$



Oglethorpe , CPC & Woods 2013

Rich quasi-periodicity: what sets layer scale/mixing?

Spontaneous layers: NOT Taylor vortices



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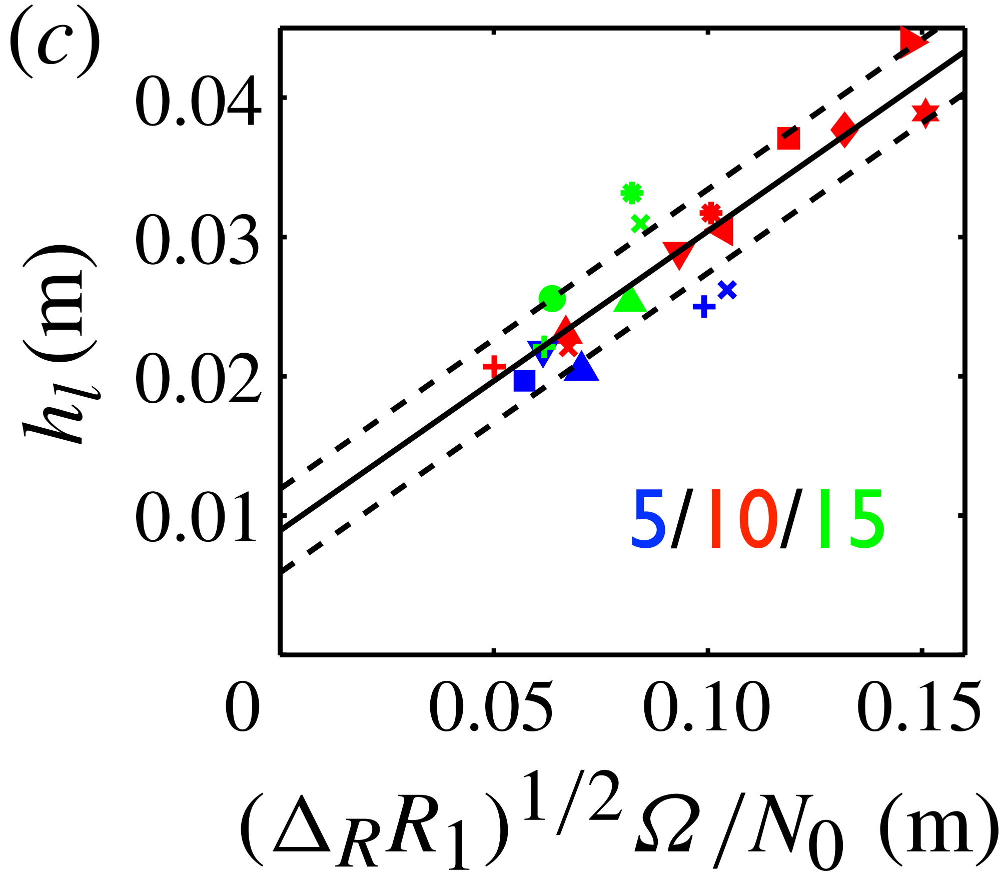


Oglethorpe , CPC & Woods 2013

Rich quasi-periodicity: what sets layer scale/mixing?

Layers, zig-zags & connections

- Layer depth scales like U/N for characteristic velocity U...
- Reminiscent of zig-zag instability of Billant & Chomaz (2000a,b)

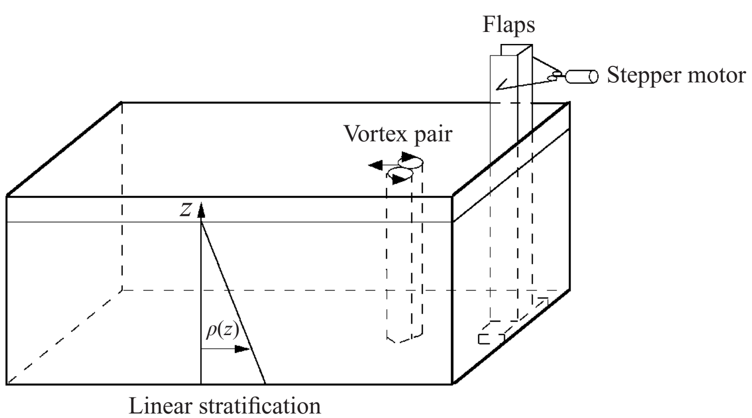
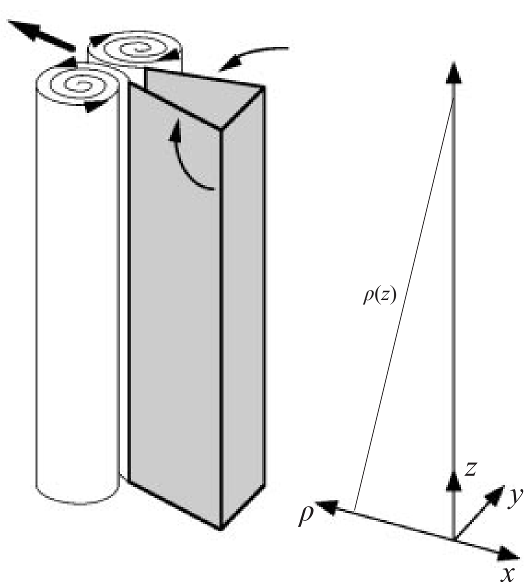
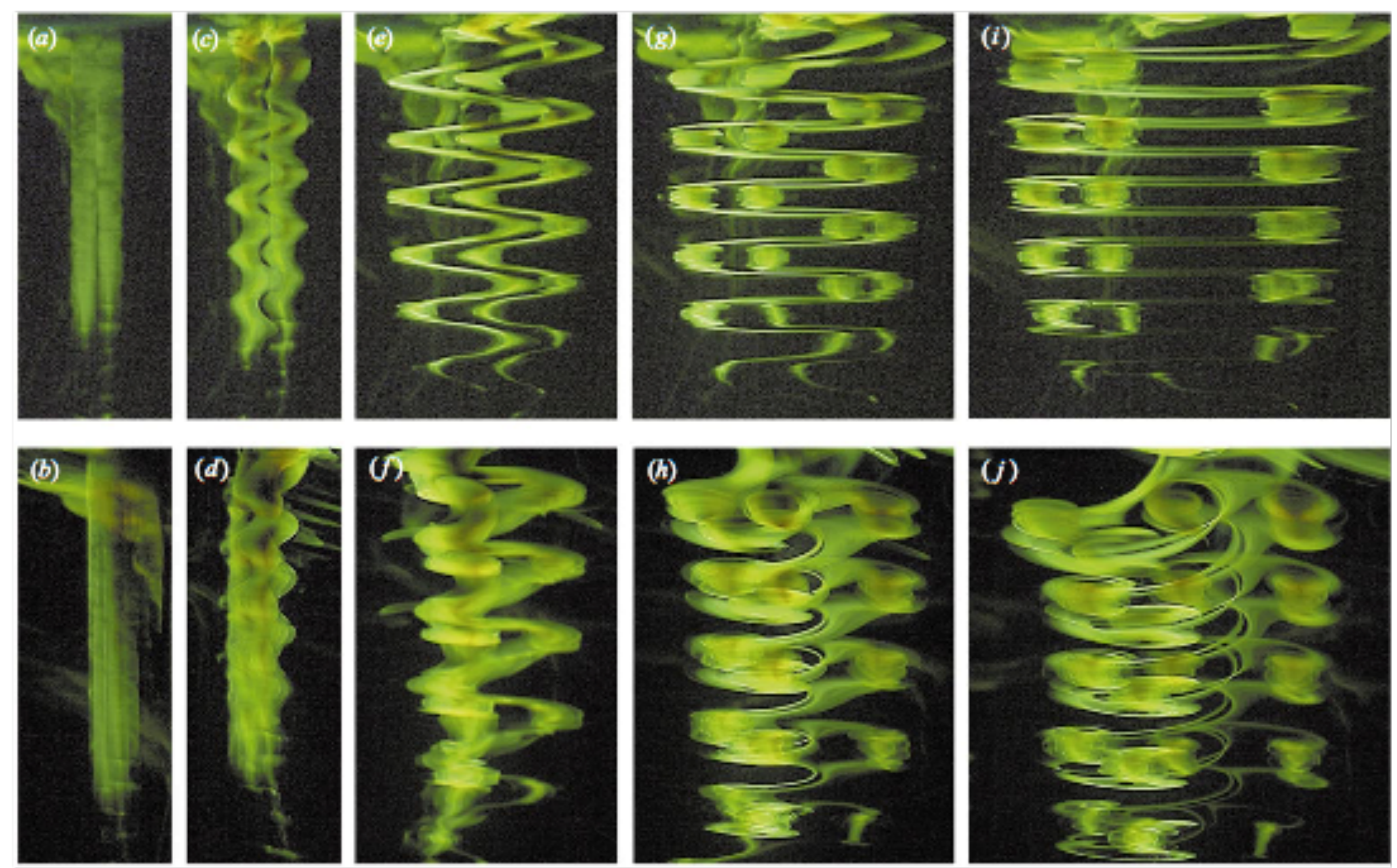
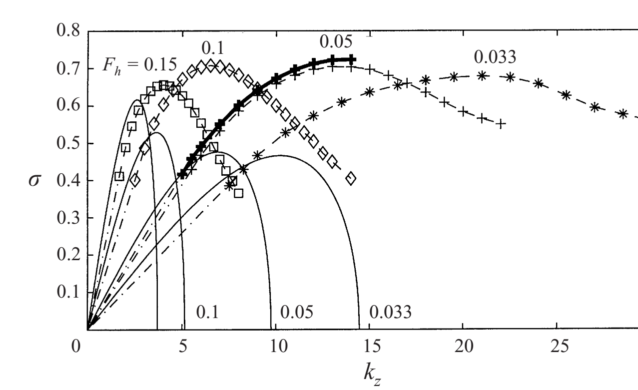


FIGURE 1. Sketch of experimental set-up.

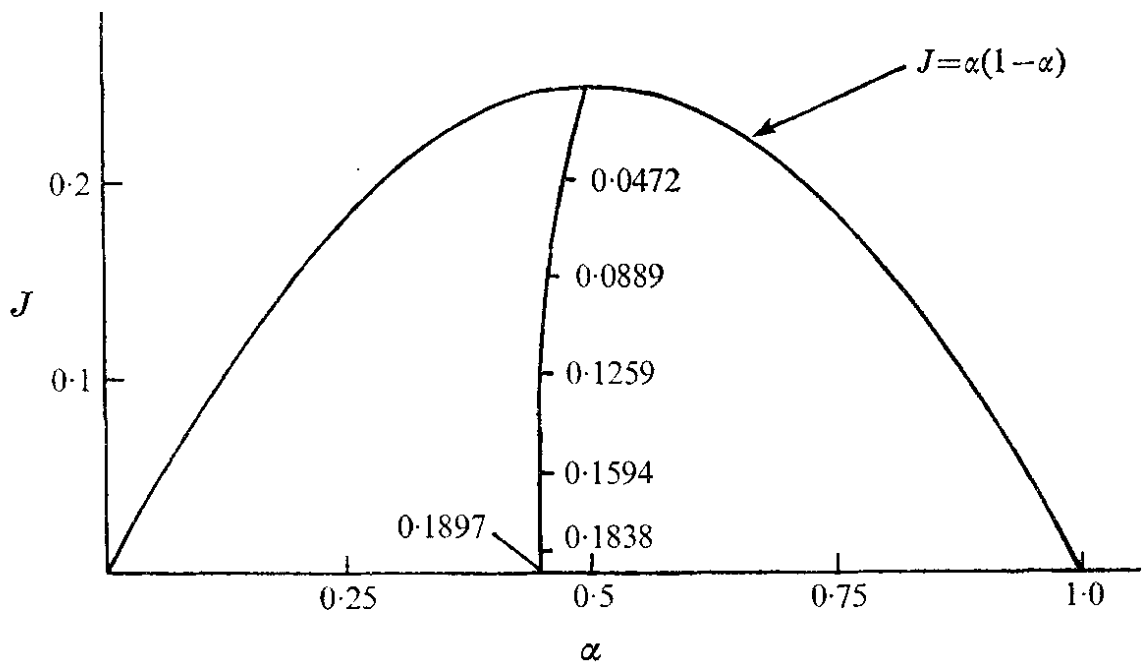


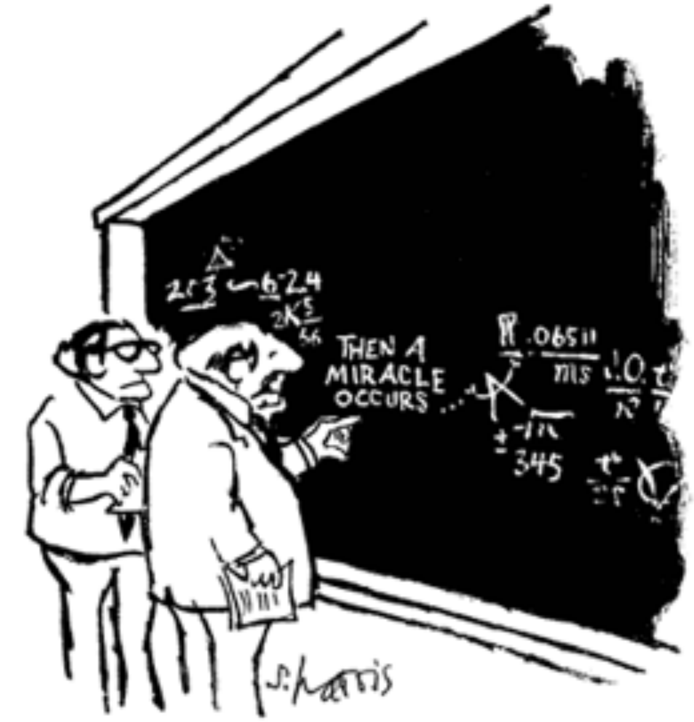




- Experimental miracle: linear mode to structure?

Connecting linear modes to self-organization?





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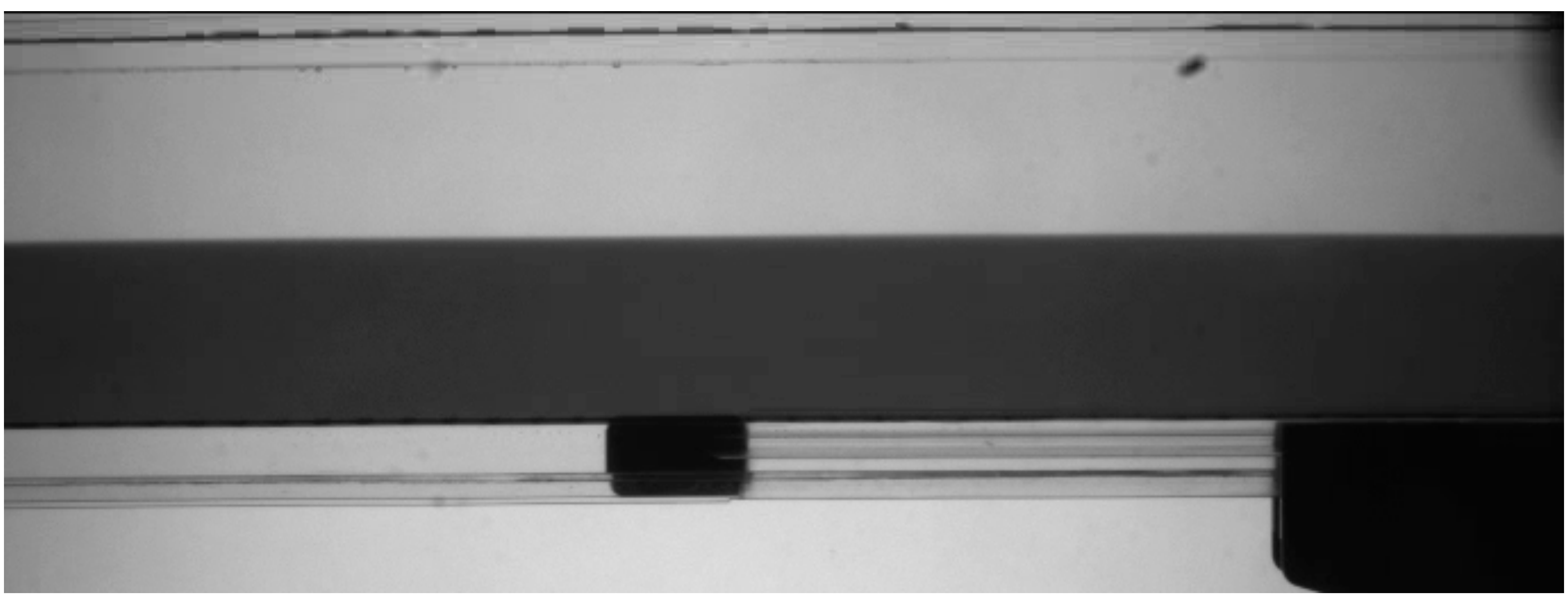

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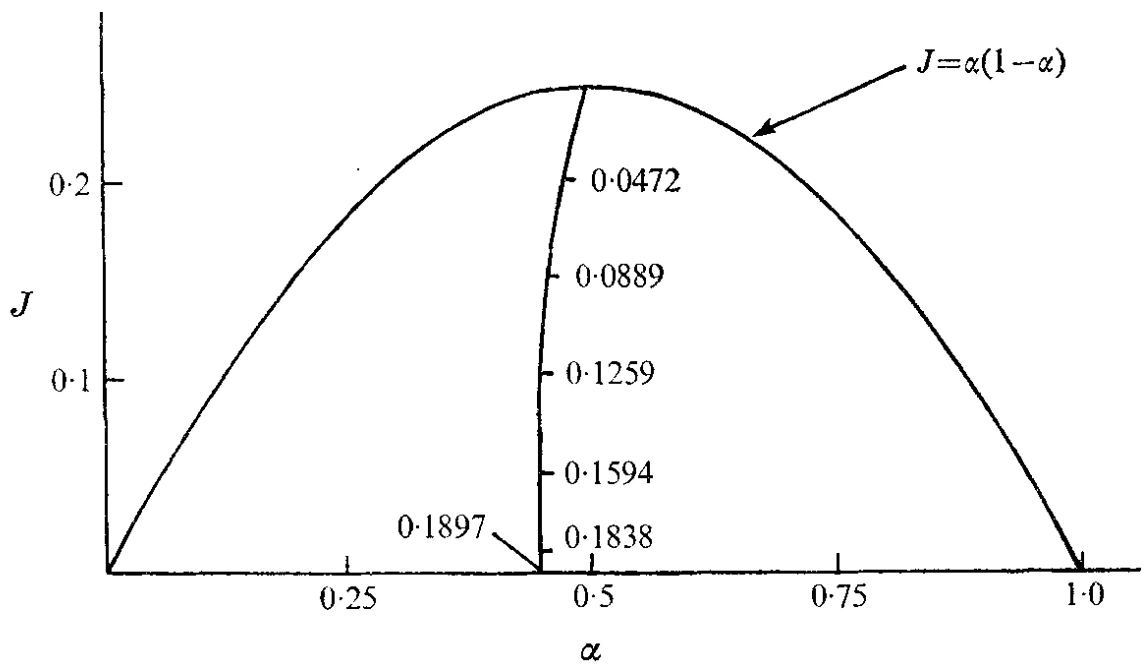

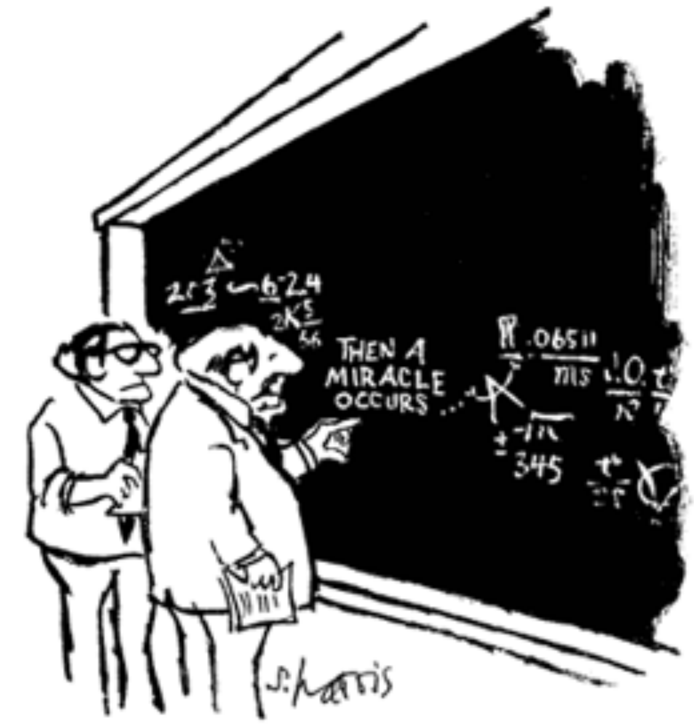

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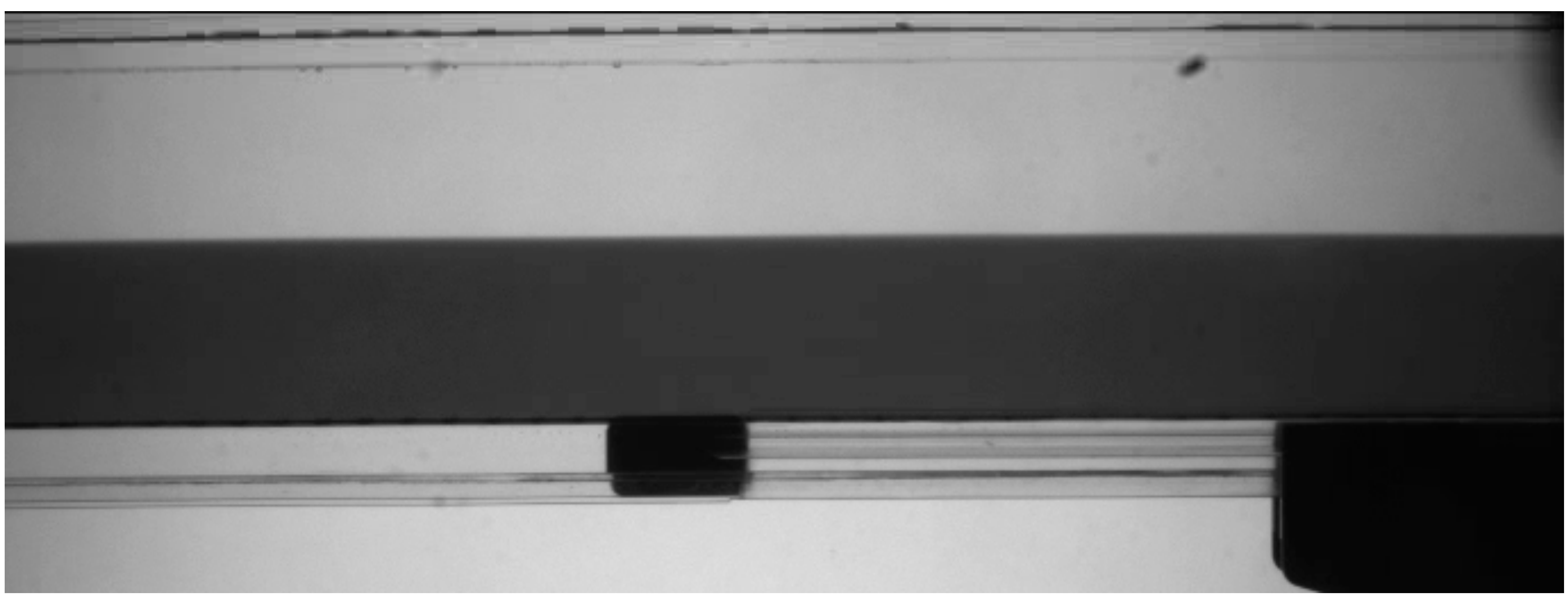


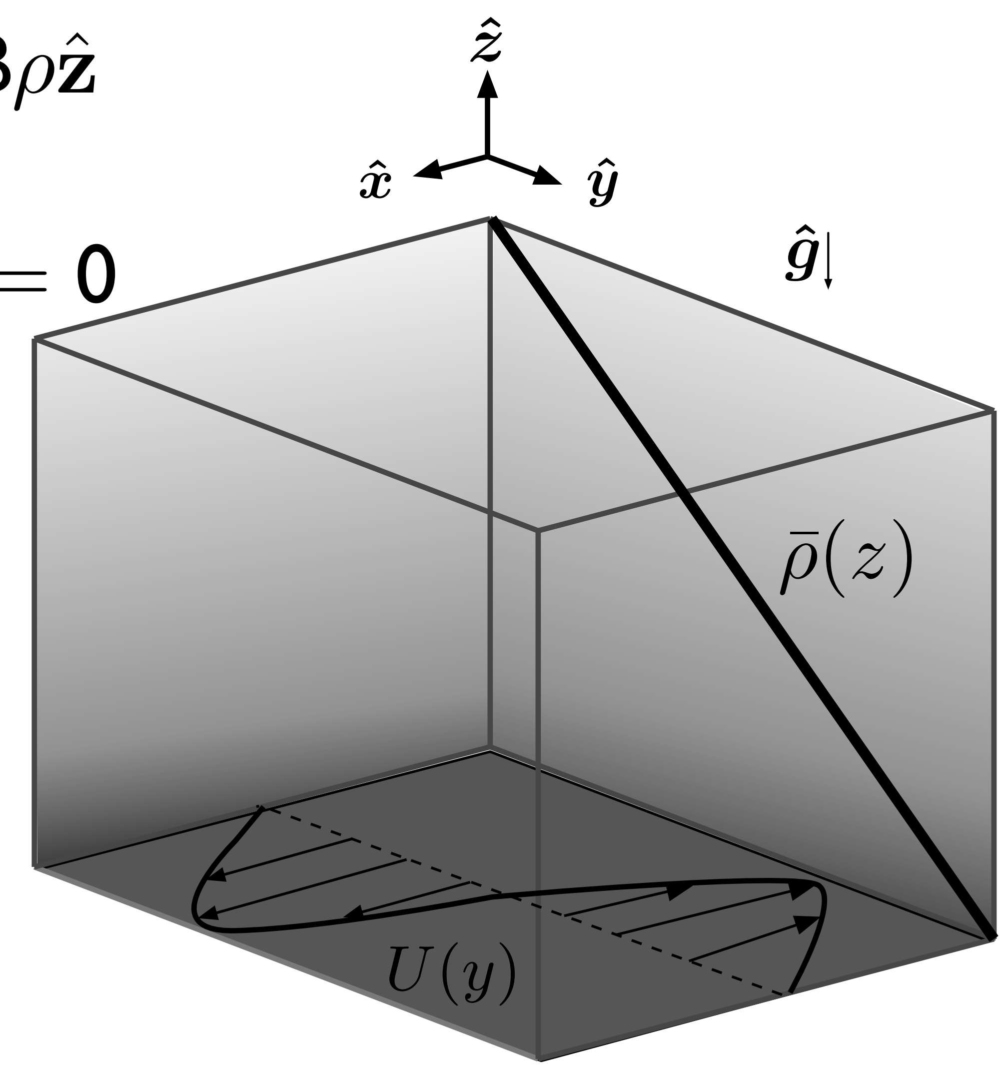
HS Kolmogorov Flow

- Horizontal Kolmogorov flow $U \equiv u_{lam} = Re\sin(y)\hat{\mathbf{x}},\ \rho_B = -z$

- Forced back to laminar flow with three control parameters

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \frac{1}{Re}\Delta\mathbf{u} + \sin(y)\hat{\mathbf{x}} - B\rho\hat{\mathbf{z}}$$

$$\frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\nabla\rho = w + \frac{1}{Re}\Delta\rho, \quad \nabla\cdot\mathbf{u} = 0$$



- Reynolds number Re

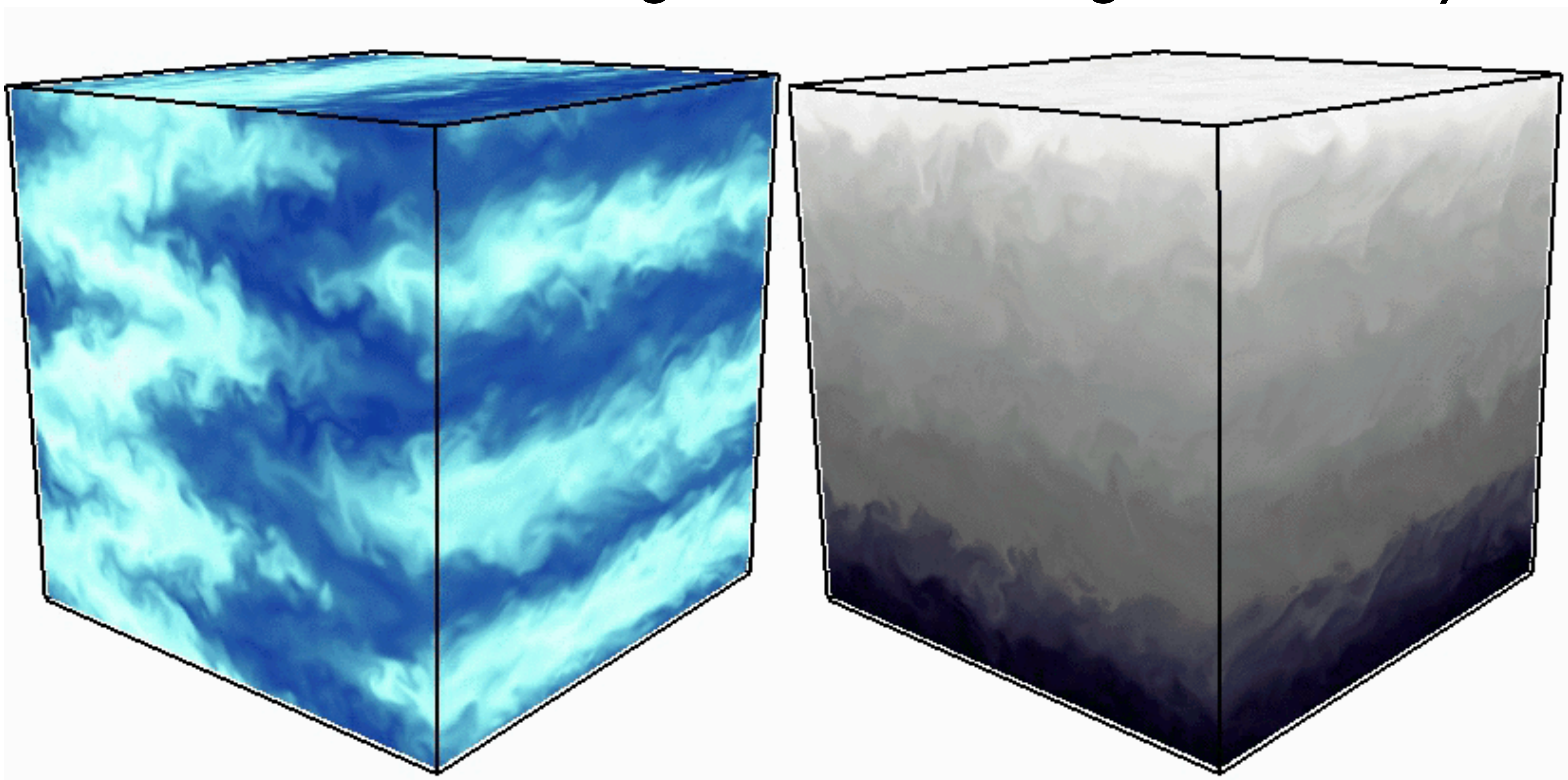
- Buoyancy parameter B

- Aspect ratio $\alpha = L_y/L_x = L_z/L_x$

- Horizontal shear so vertical vorticity...

HS Kolmogorov Flow

- Horizontal stratified Kolmogorov flow self-organizes into layers!

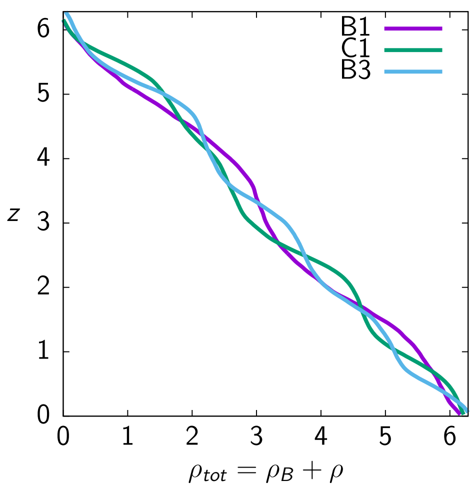


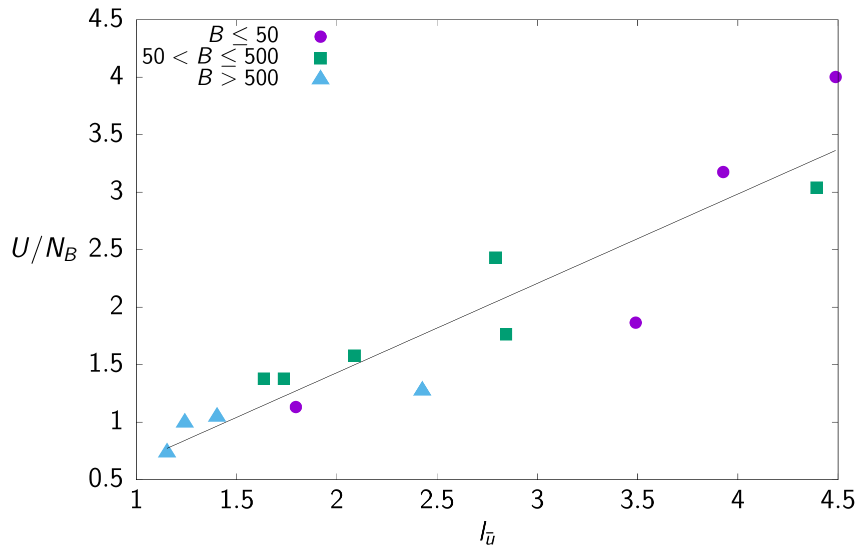
streamwise u $\qquad\qquad$ $\rho_B + \rho$

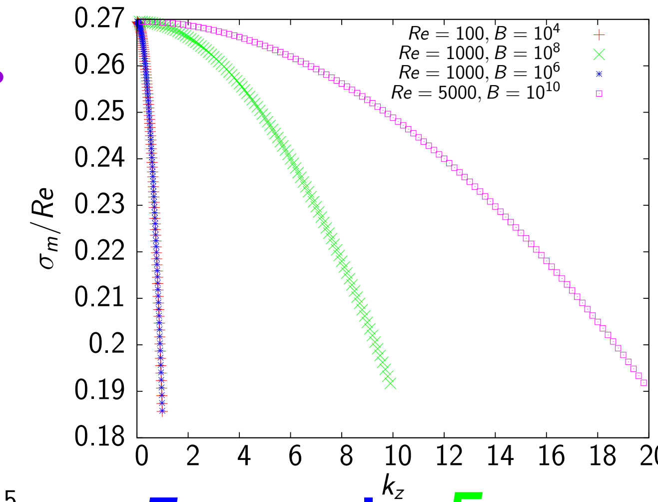
- Any connection between linear stability and nonlinear dynamics?

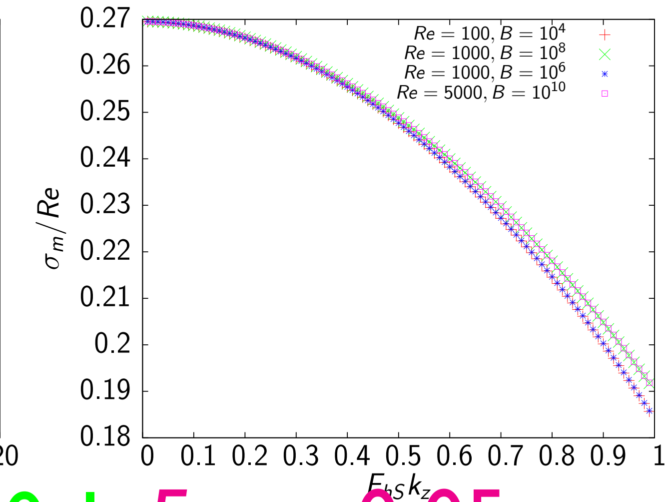
Layer scaling

- Layer scaling consistent with U/N...and zig-zag like linear instability







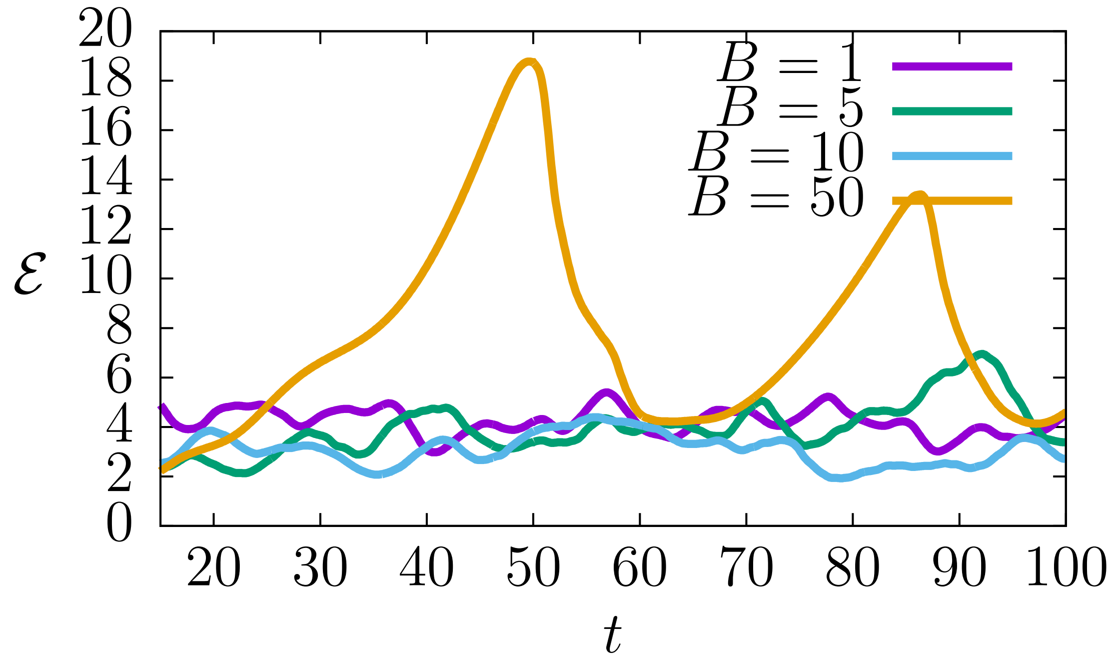


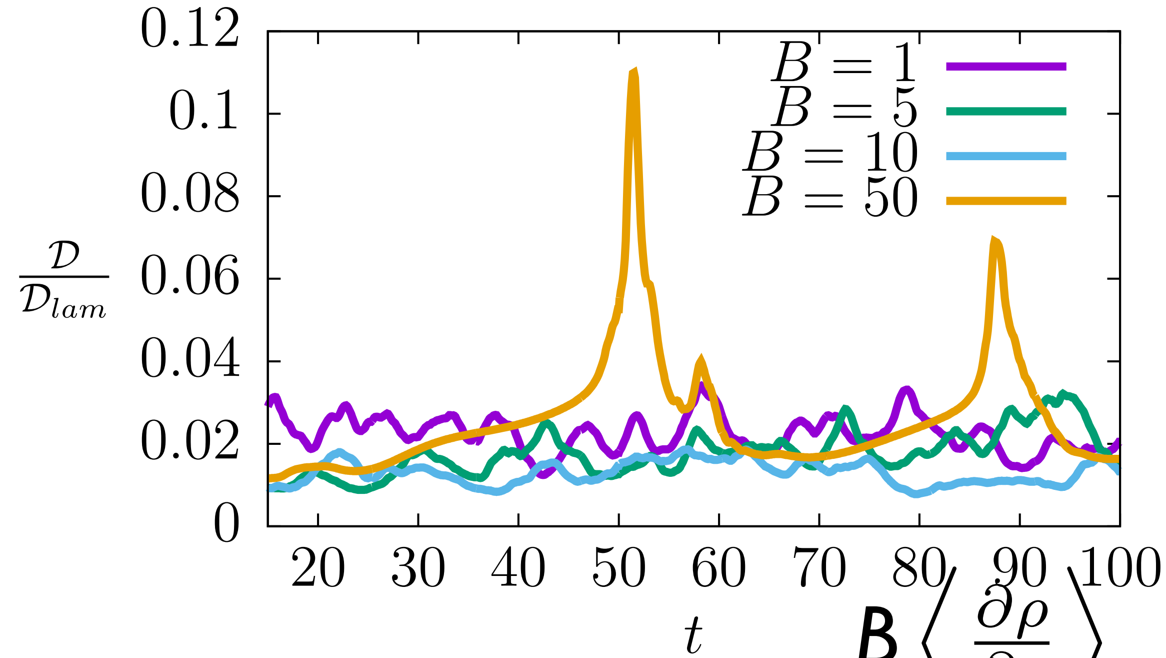
$F_{hS} = 1$ $F_{hS} = 0.1$ $F_{hS} = 0.05$

- Vertical wavenumber collapses with Froude number: $F_{hS} = \dfrac{U}{LN} = \dfrac{Re}{\sqrt{B}}$
- Scaling for $F_{hS} \ll 1; \; Re \gg 1$
- Stratified turbulence scaling of Billant & Chomaz/Lindborg etc
- Stability properties just like zigzag/Deloncle et al 2007
- Nonlinear properties, particularly mixing?

Mixing: UPOs?

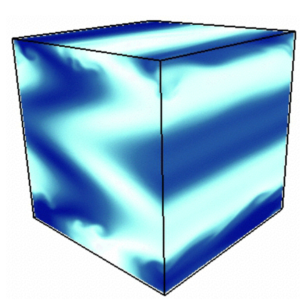
- Particularly in strongly stratified flow: mixing very intermittent



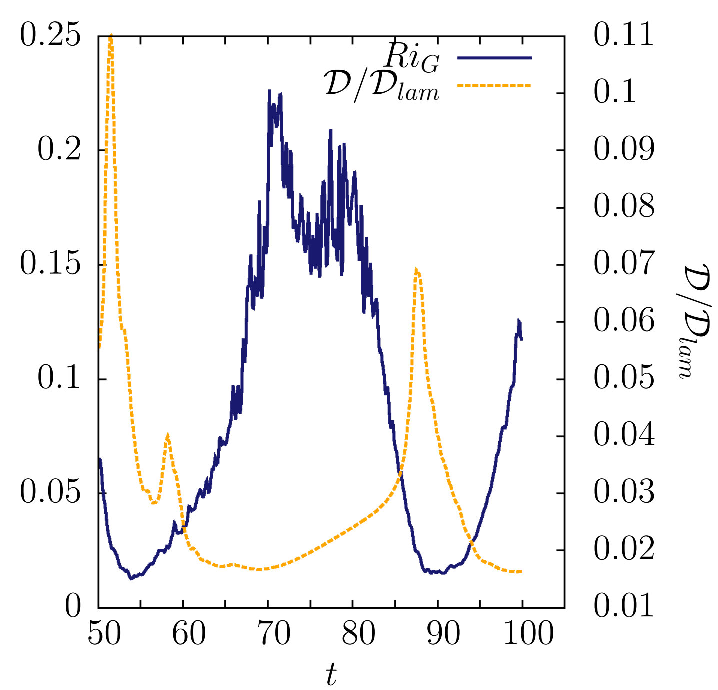


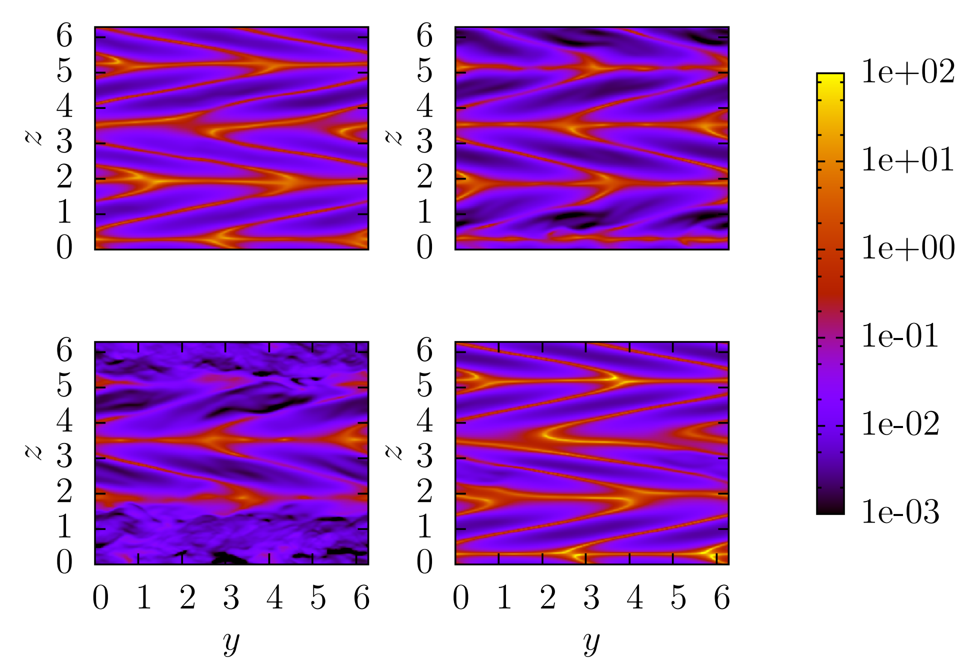
- Complex spatio-temporal structure of $Ri_G(y,z,t) = -\frac{B\left\langle \frac{\partial \rho}{\partial z}\right\rangle_x}{\left\langle \left(\frac{\partial u_h}{\partial z}\right)^2\right\rangle_x}$

- t=70,88,90,100
- <Ri> drops...



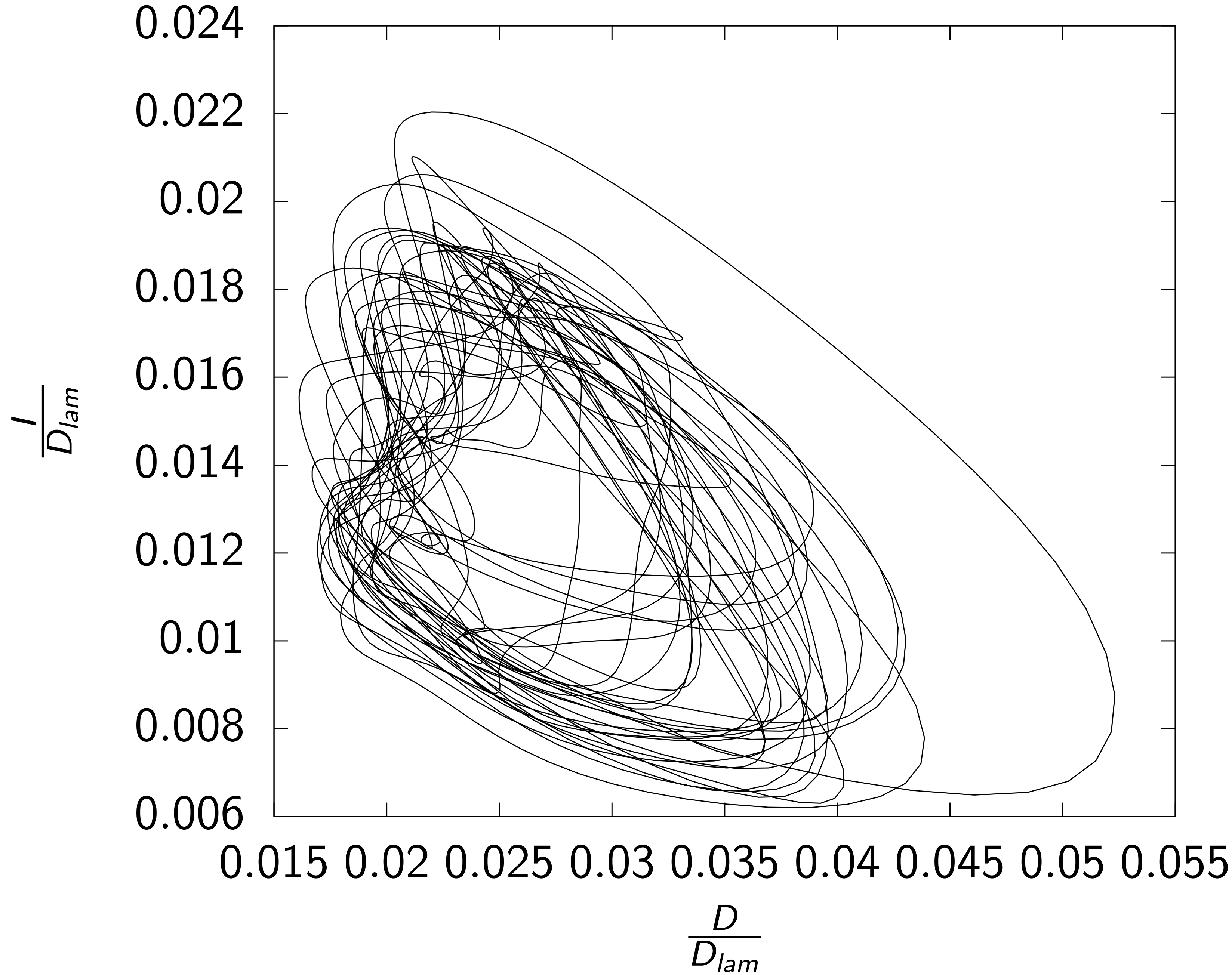
- Then turbulence...then
- Ri up again: UPO??





Stratified UPOs

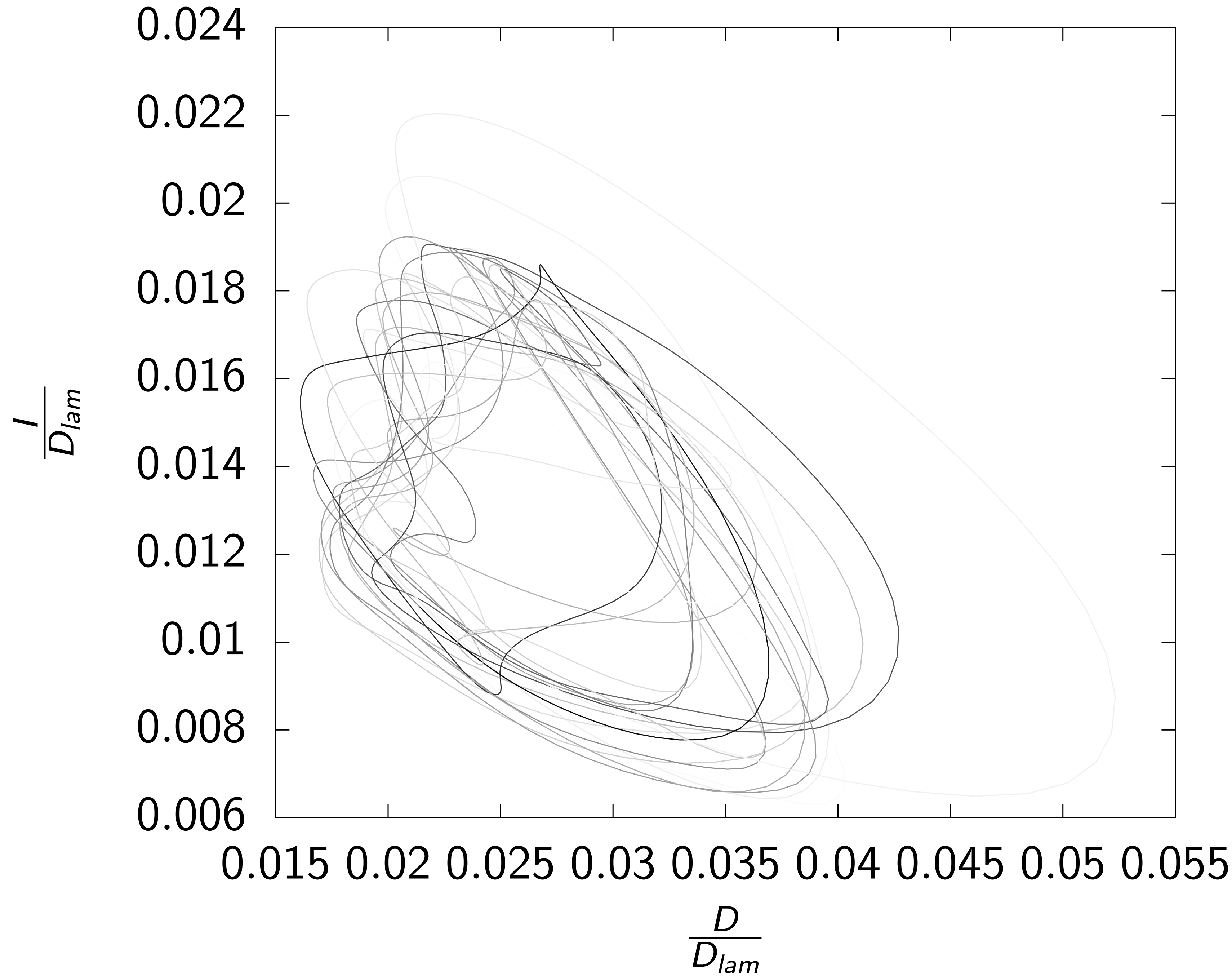
- Project trajectory onto plane:



- Looks an unholy mess....but look closer...

Stratified UPOs

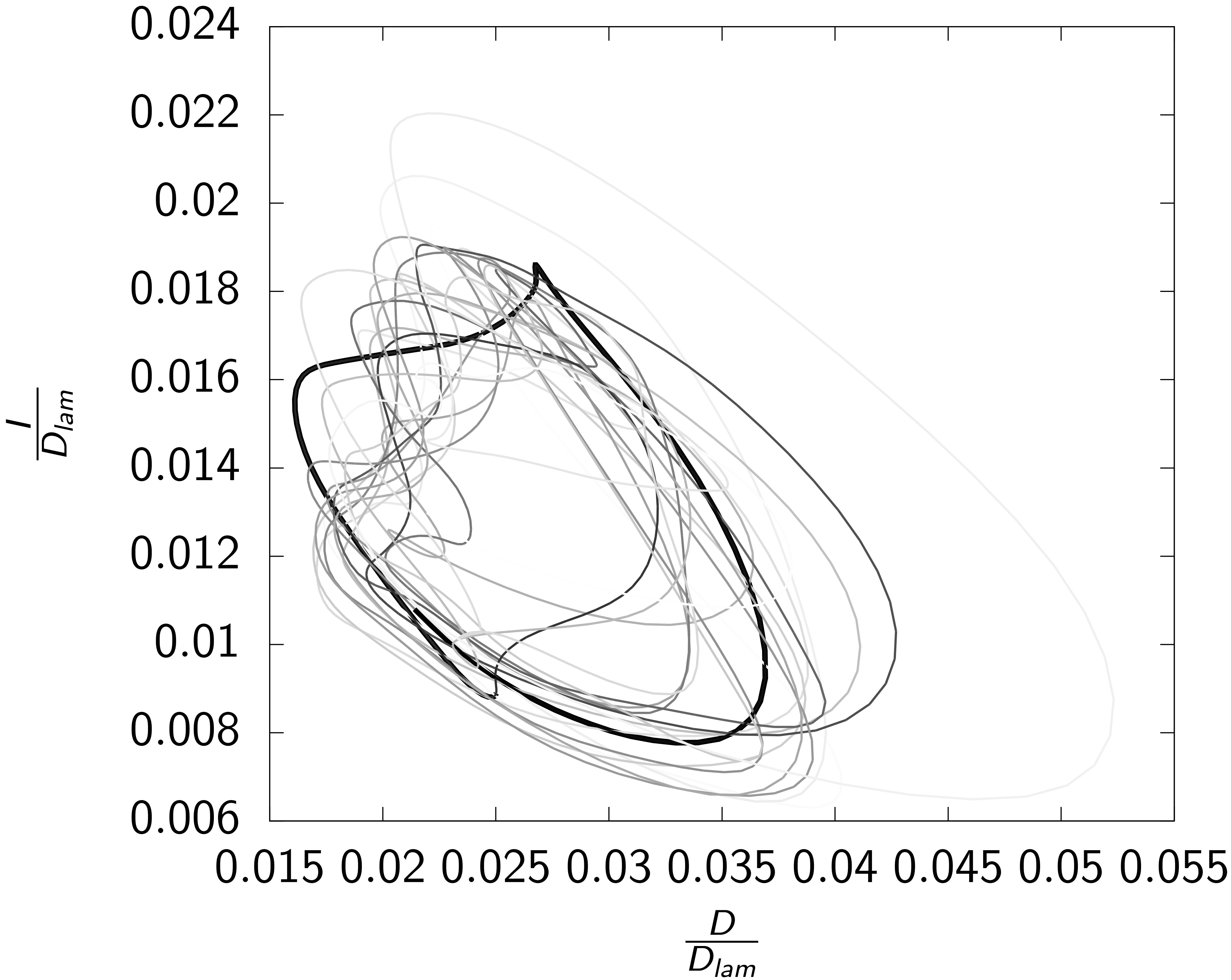
- Closer inspection reveals nearly recurrent episodes



- If you look really closely...

Stratified UPOs

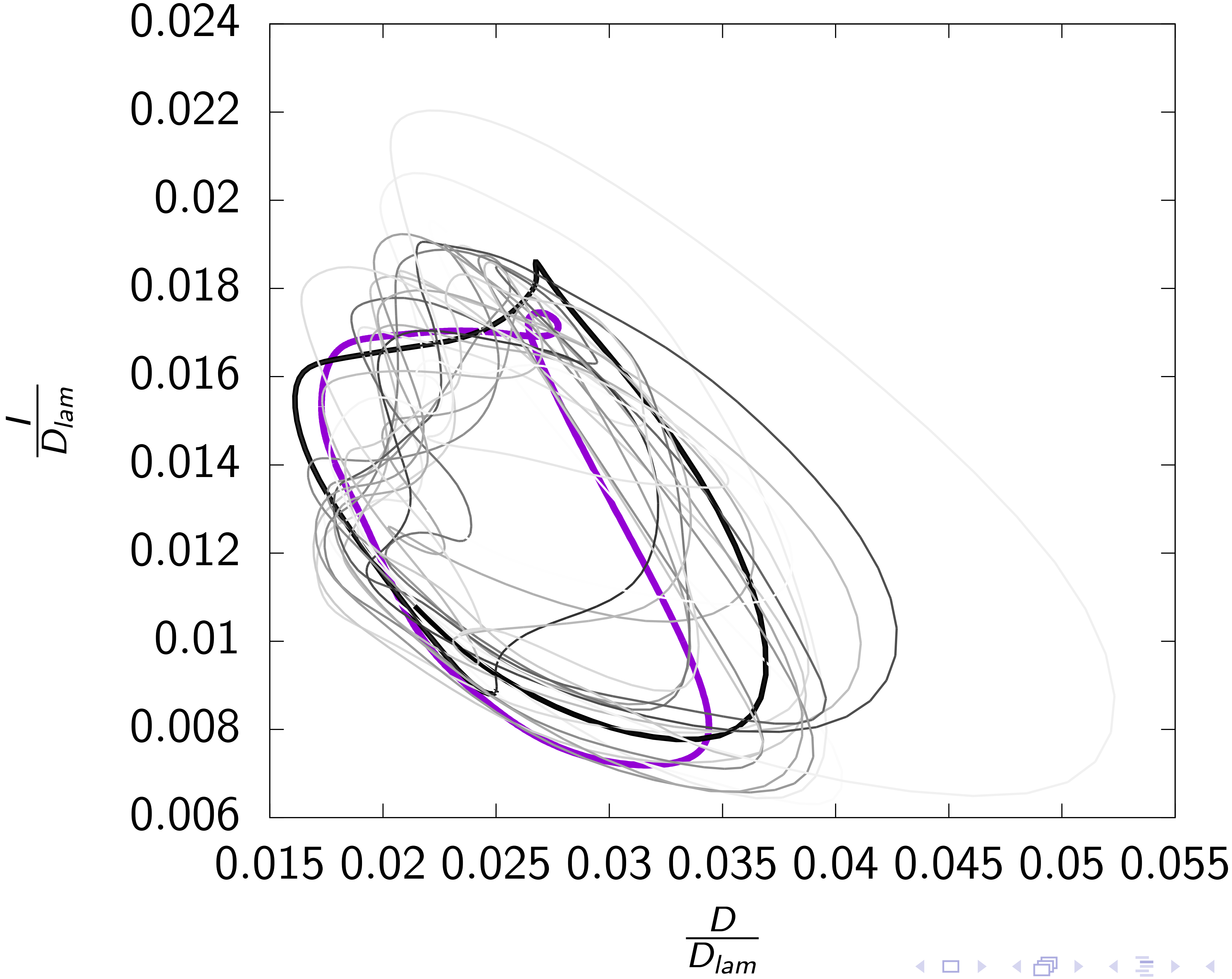
- Closer inspection reveals nearly recurrent episodes

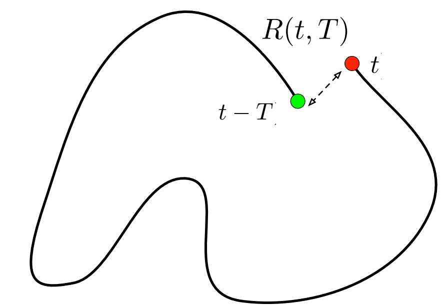


- Which can be used as original guesses for recurrent flow analysis...

Stratified UPOs

- Which can be "polished" to find UPOs!





$$\omega(\mathbf{x},t) = \sum_{\mathbf{k}} \Omega_{\mathbf{k}}(t) e^{i(\mathbf{k}\cdot\mathbf{x})}$$

$$R(t,T) := \min_{\mathbf{s}} \frac{\sum_{\mathbf{k}} \left|\Omega_{\mathbf{k}}(t)e^{i(\mathbf{k}\cdot\mathbf{s})} - \Omega_{\mathbf{k}}(t-T)\right|^2}{\sum_{\mathbf{k}} |\Omega_{\mathbf{k}}(t)|^2} < R_{thres}$$

$$\mathbf{F}(\Omega_0, s, T) := \hat{\Omega}_s(\Omega_0, T) - \Omega_0 = 0$$

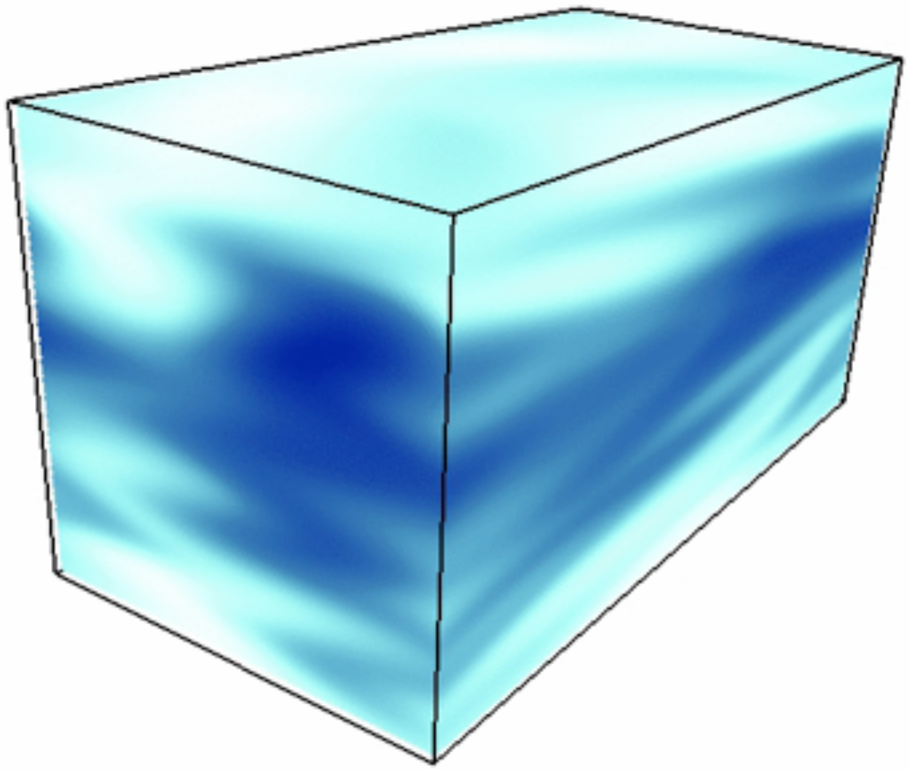
$$\left[\begin{array}{cc|c|c} \frac{\partial\hat{\Omega}_s}{\partial\Omega_0} - \mathrm{I} & & \vdots & \vdots \\ & & \frac{\partial\hat{\Omega}_s}{\partial s} & \frac{\partial\hat{\Omega}_s}{\partial T} \\ & \ddots & \vdots & \vdots \\ \hline (\frac{\partial\Omega_0}{\partial s})^{\mathrm{T}} & \cdots & 0 & 0 \\ \hline (\frac{\partial\Omega_0}{\partial T})^{\mathrm{T}} & \cdots & 0 & 0 \end{array}\right] \begin{bmatrix} \vdots \\ \delta\Omega \\ \vdots \\ \delta s \\ \delta T \end{bmatrix} = - \begin{bmatrix} \vdots \\ \mathbf{F}(\Omega_0, s, T) \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

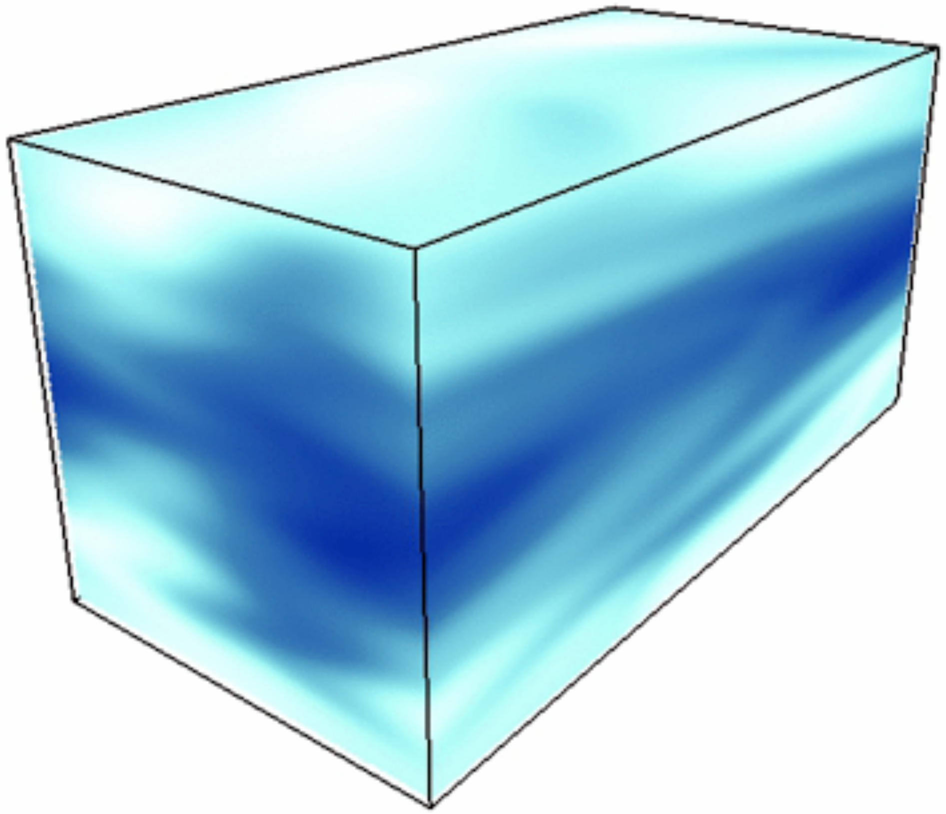
$$R = 0 \leftrightarrow A\delta x = b$$

- See Lucas & Kerswell 2015/16 for Newton Solve/Hookstep/GMRES etc

Stratified UPOs

- Can find stratified UPOs, and they really "look" like the flow:





$Re = 50, B = 50, \alpha = 0.5$

- But can they tell us something quantitative about the flow behaviour?

Stratified UPOs

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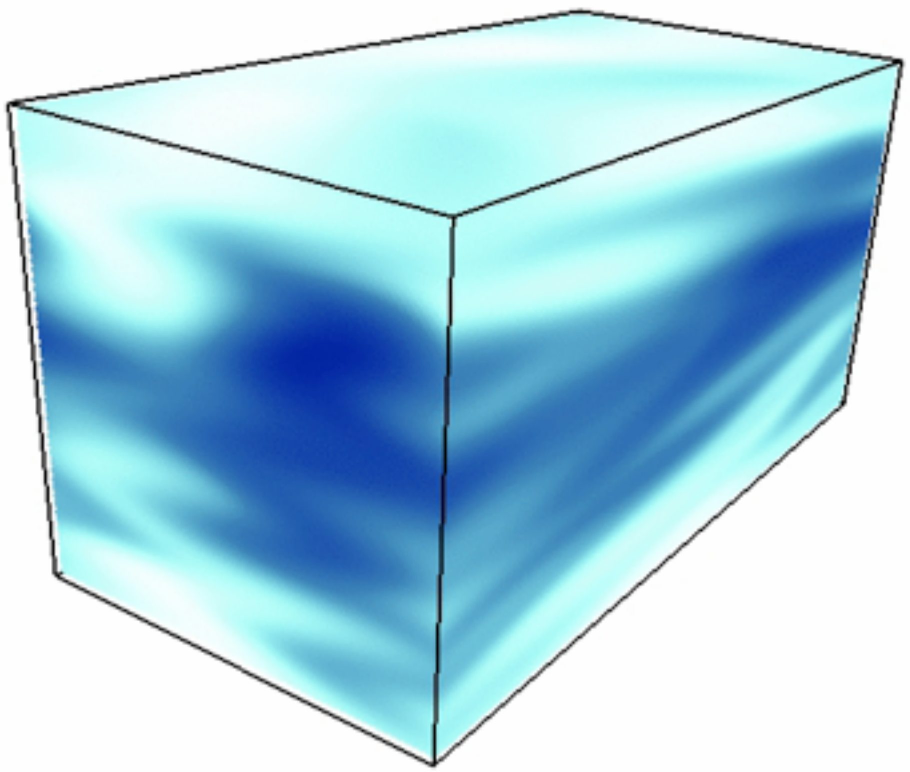

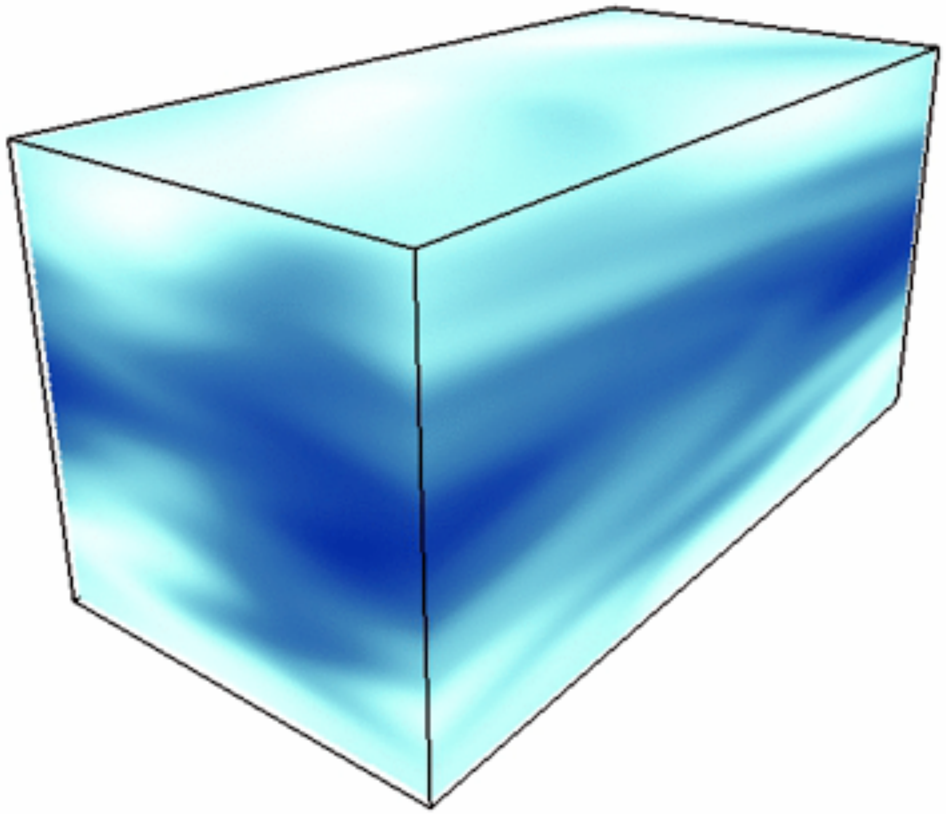

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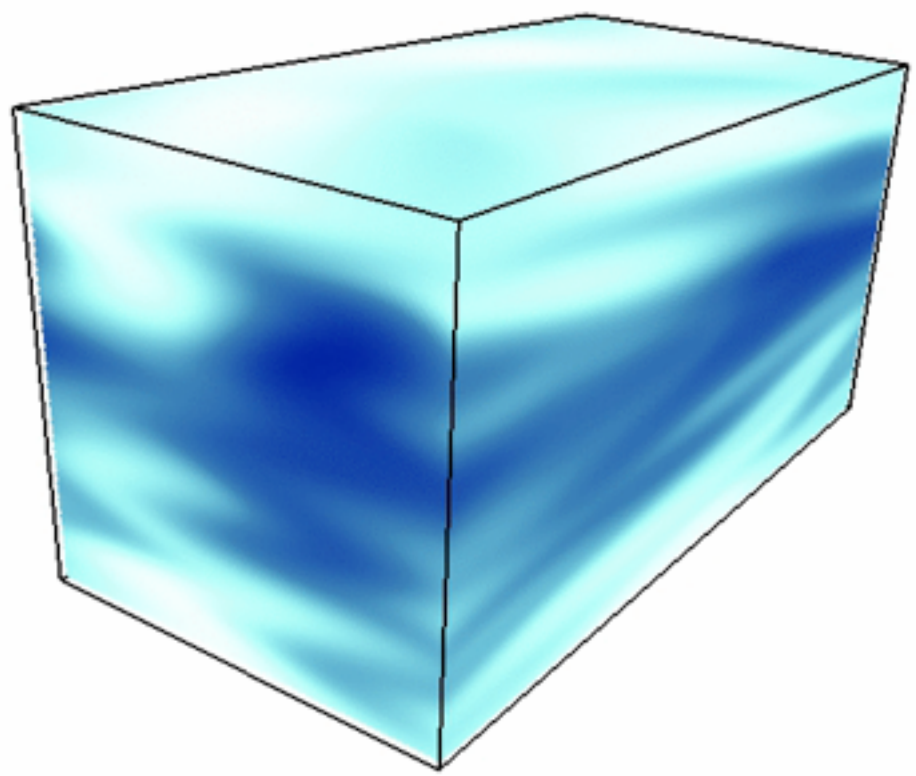
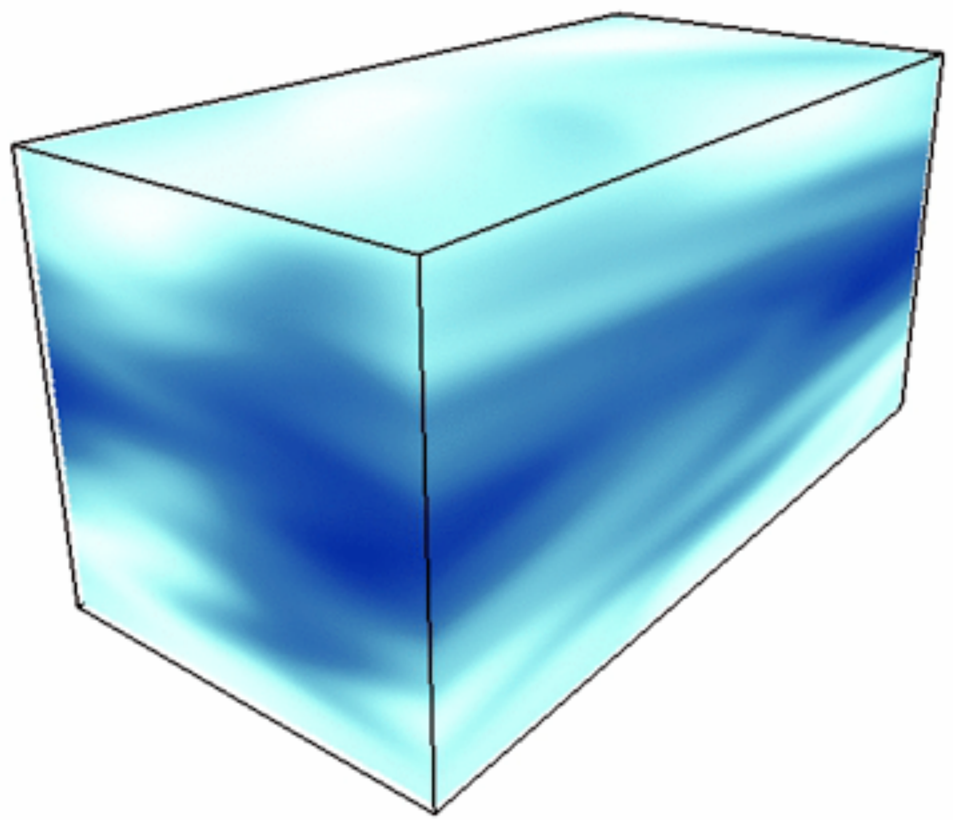


DNS



UPO

$Re = 50, B = 50, \alpha = 0.5$

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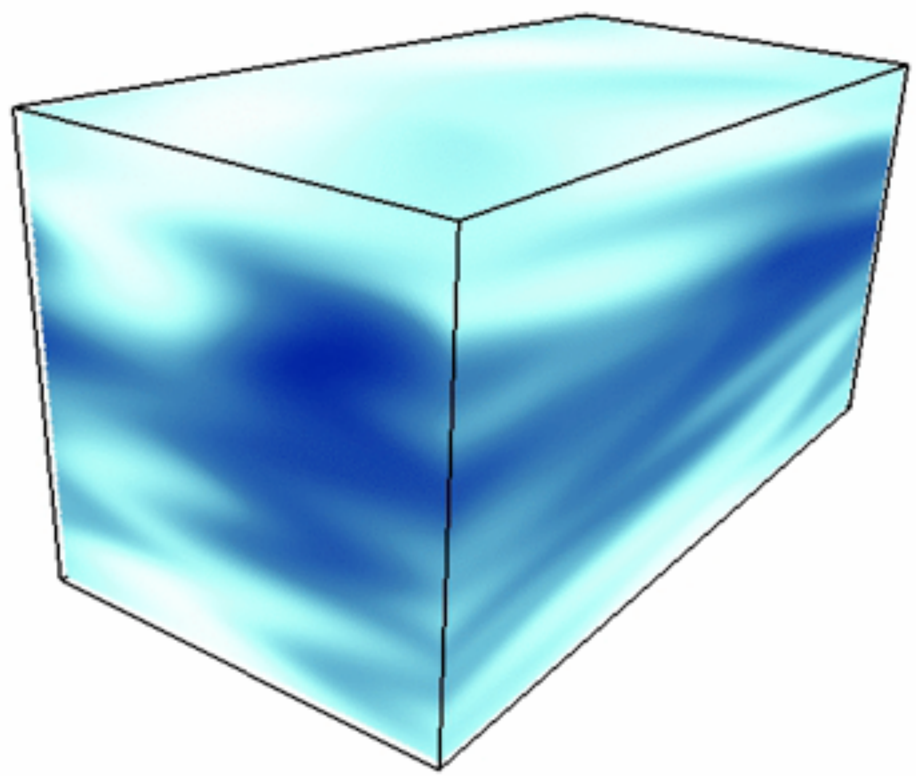

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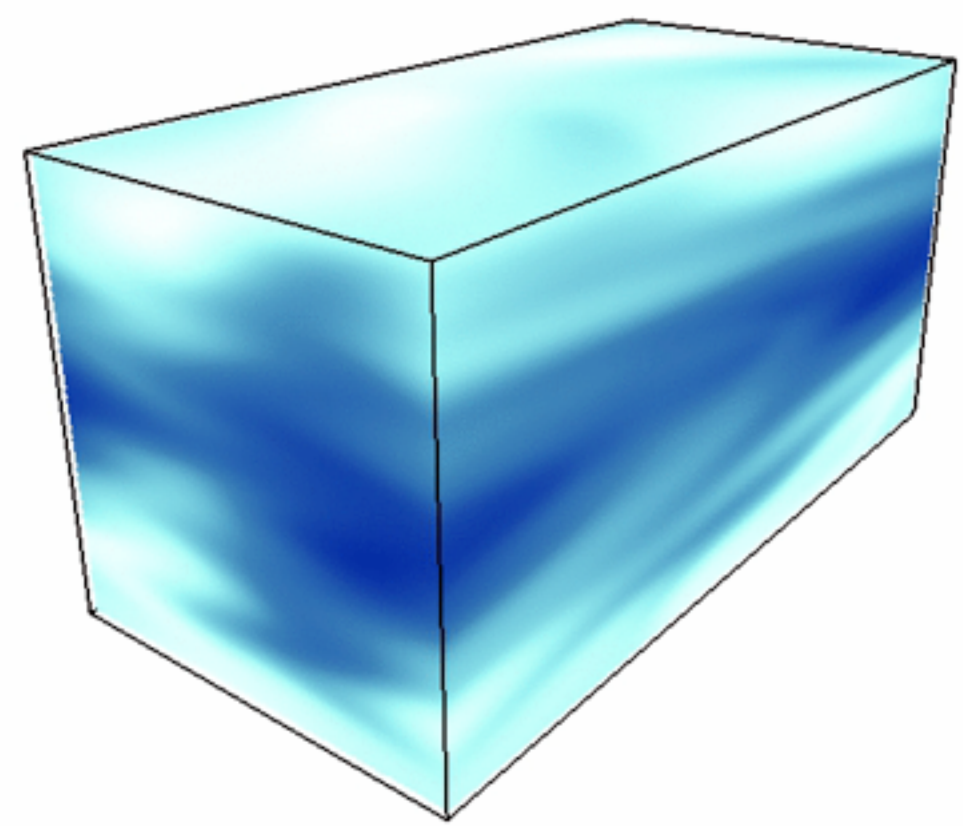


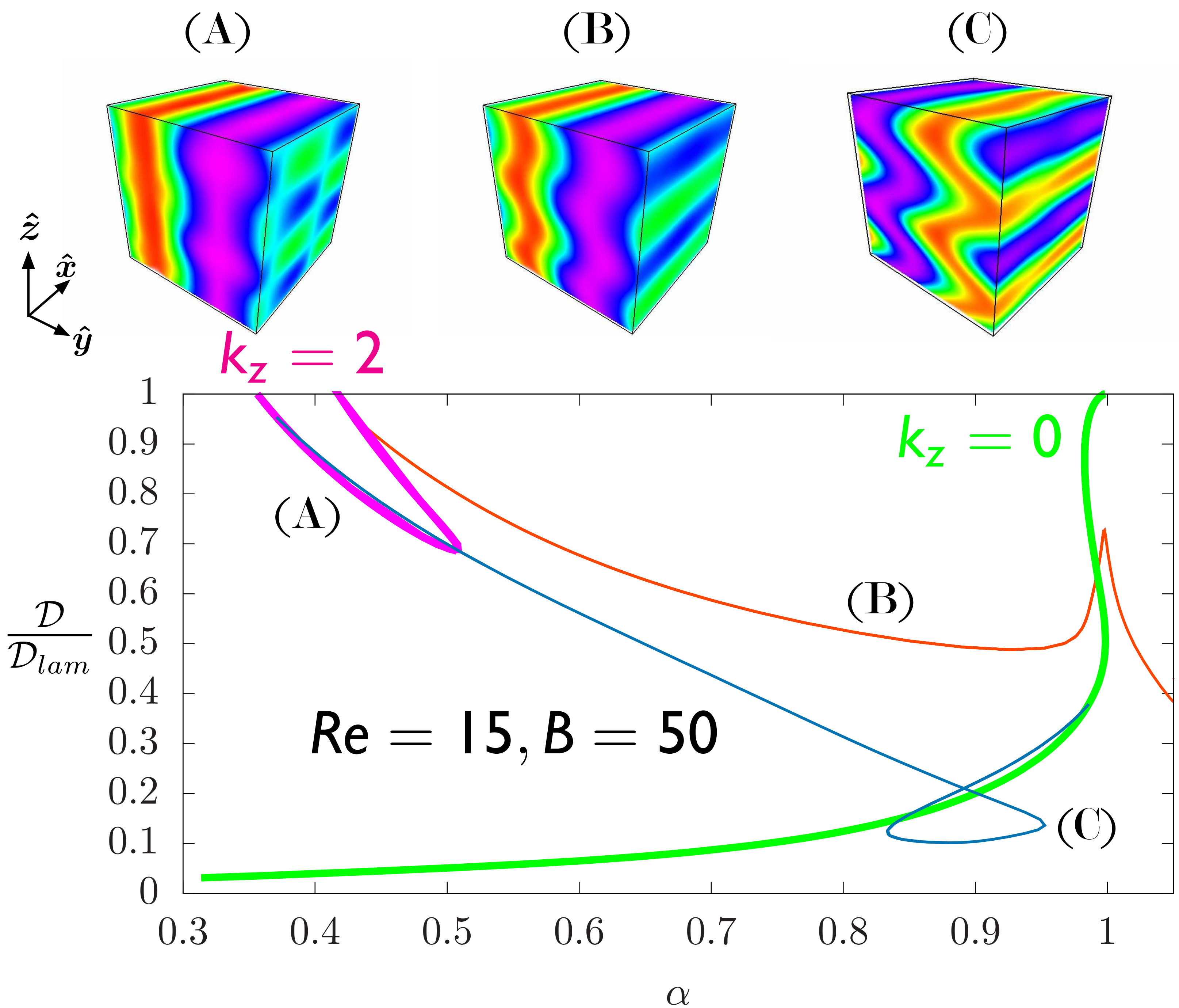
UPO

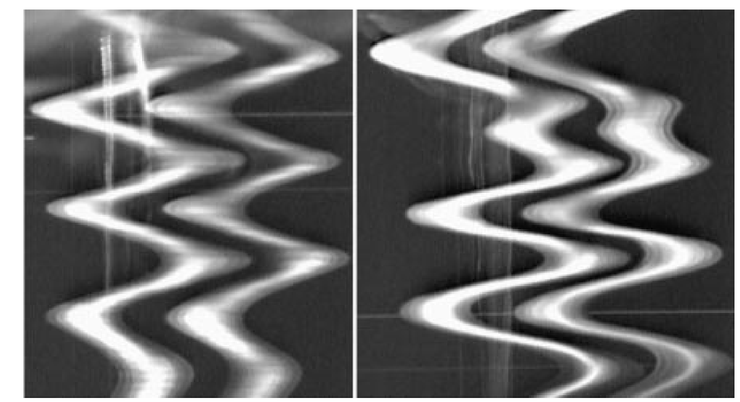
$$Re = 50, B = 50, \alpha = 0.5$$

- But can they tell us something quantitative about the flow behaviour?

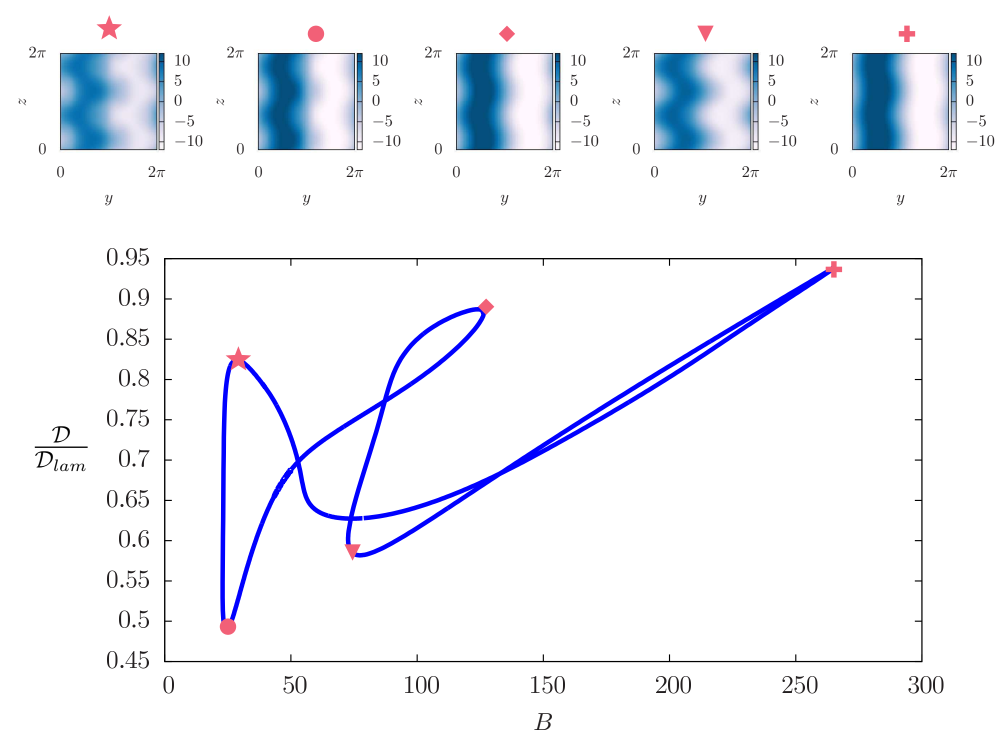
Connection to instabilities

- Continuation in parameters allows us to connect to linear instabilities





Billant & Chomaz 2000 experiment

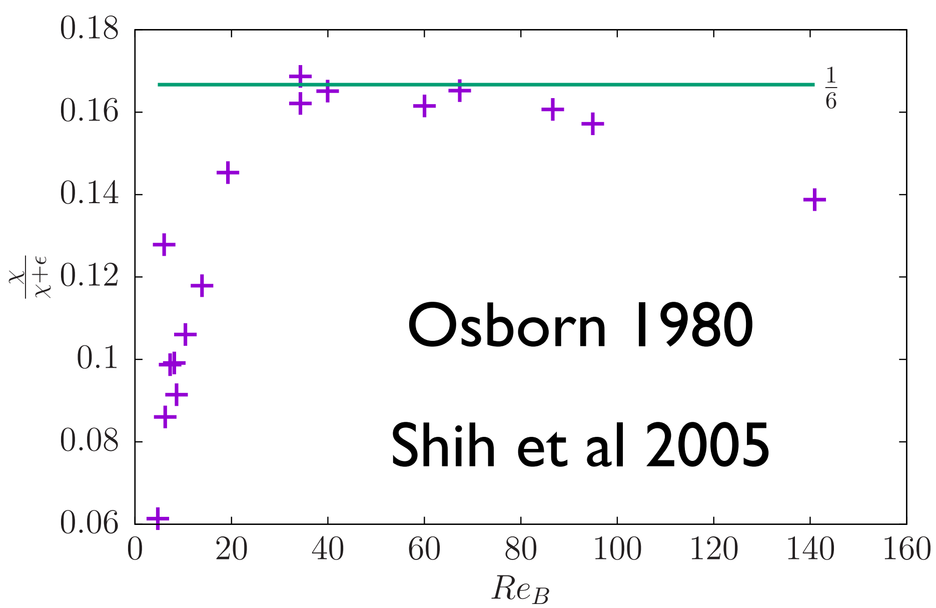


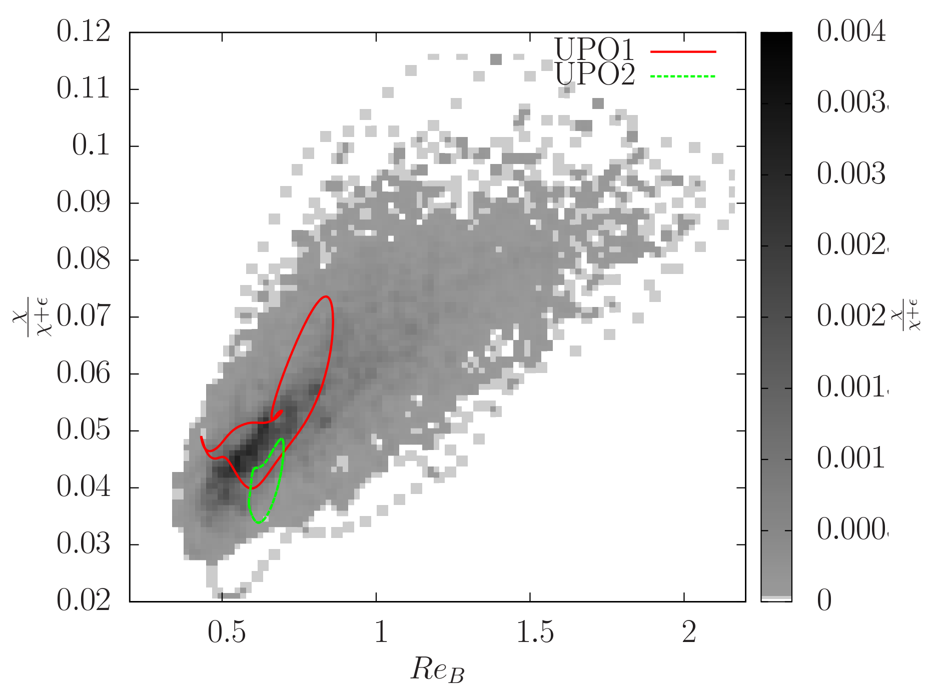
$Re = 15, \alpha = 0.9$

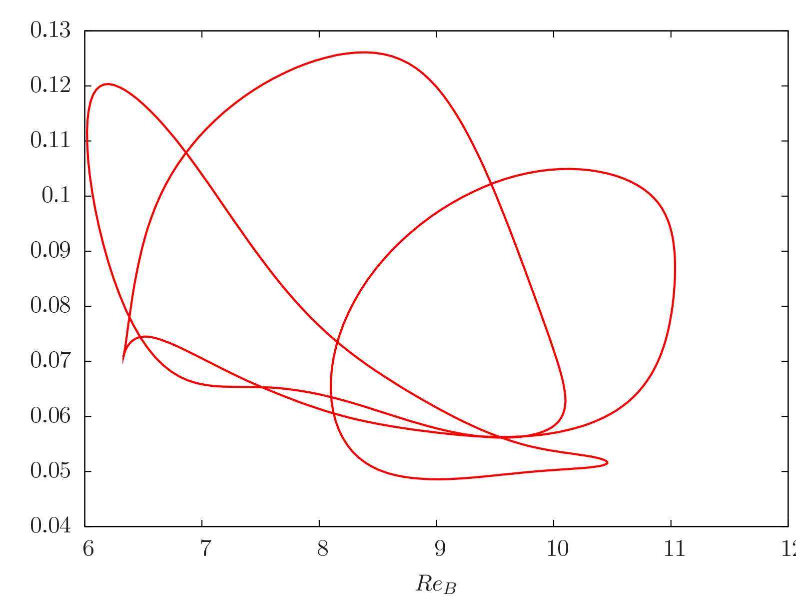
- Chevron state is a secondary bifurcation from both primary instabilities!

Mixing?

- In stratified turbulence, potential energy sink as well as dissipation
- Question: how much into mixing/how much into dissipation: efficiency
- Buoyancy versus momentum dissipation: $\chi = B\langle|\nabla\rho|^2\rangle,\ \epsilon = \frac{1}{Re}\langle|\nabla \mathbf{u}|^2\rangle$
- Flow is consistent with standard models of other flows with $Re_B = \frac{\epsilon^*}{\nu N_B^2}$



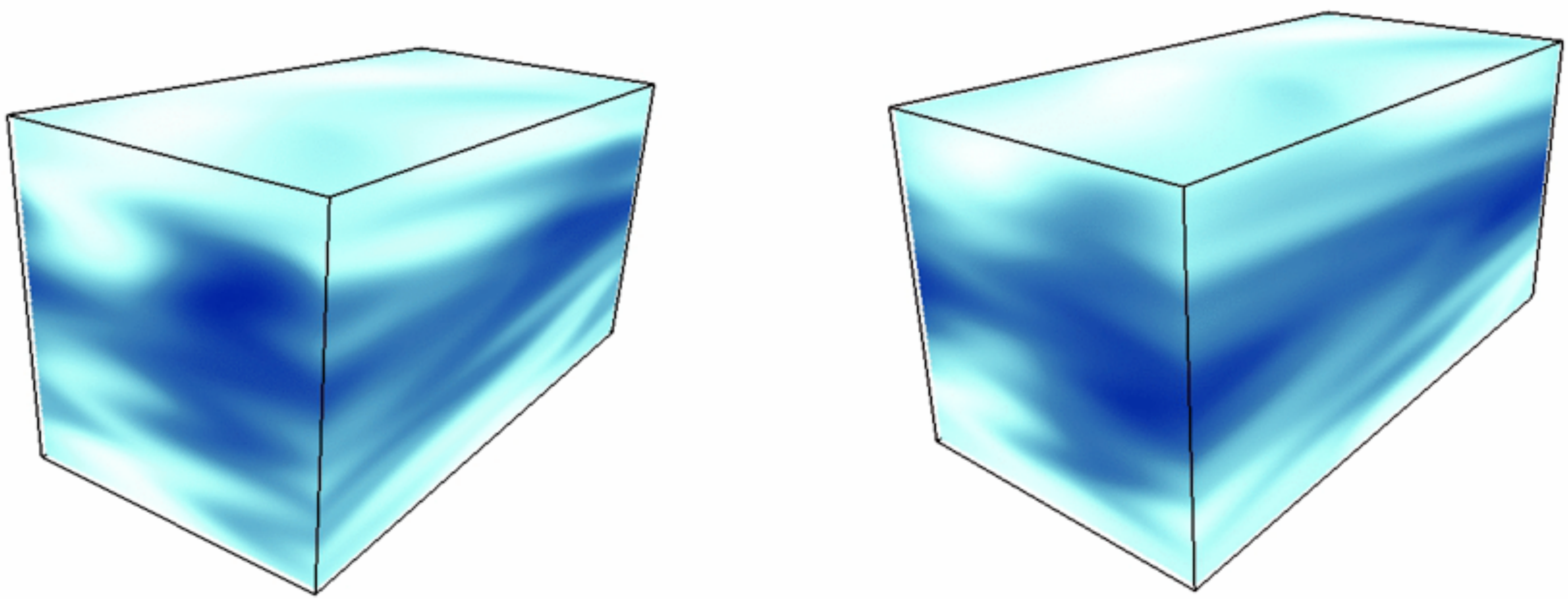




- ECS has the quantitative mixing property of the full flow...higher Re?

B: Conclusions

- Recurrent flow analysis applicable to stratified shear turbulence
- ECS/UPOs: RECURRENCE & SELF-ORGANIZATION
- Finite amplitude bifurcations from linear instabilities
- Layering structure and mixing DYNAMICS set quantitatively
- Can they be used as "modes" for reduced models/explanations?



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- Recurrent flow analysis applicable to stratified shear turbulence
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