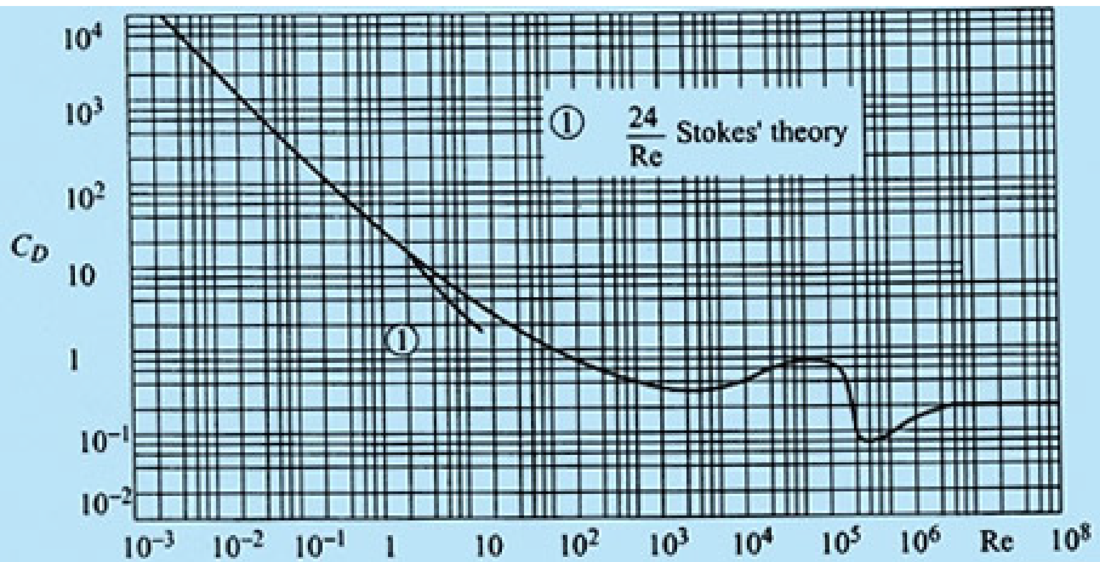
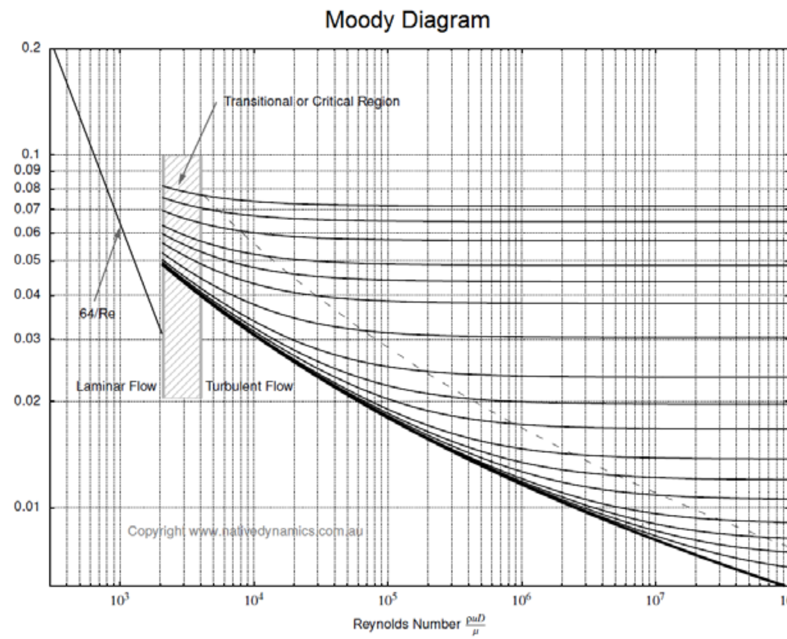
Interaction between mean flow and turbulence

Gregory Falkovich





Drag coefficient for a flow past a body and Friction factor for pipe flows

$C_D \to const\ at\ Re \to \infty$ $\qquad$ $f \to 1/\log Re\ \ at\ Re \to \infty$

KITP, January 2017

We cannot yet describe turbulence generated by a flow. Let us try to describe a flow generated by turbulence.
How to predict a flow generated by an inverse cascade in two dimensions?

G. Falkovich, **Interaction between mean flow and turbulence in two dimensions** *Proc. R. Soc. A 2016* ***472*** *20160287,* special feature 'Perspectives in Astrophysical and Geophysical Fluids'

Anna Frishman, Jason Laurie, Gregory Falkovich, **Jets or vortices - what flows are generated by an inverse turbulent cascade?** Arxiv:1608.04628 Phys Rev 2017

General problem: what are the forms of large scale flows created by inverse cascades in different domains?

Going all the way to the **lowest available wavenumber** – is that a guiding principle?

- **Box** with walls and aspect ratio *r* of order unity – central vortex and four counter-rotating corner vortices. What for $r>>1$?

- **Sphere**, rotating and non-rotating. Expectation: a flow around a sphere. What meridional profile?

- **Torus**. Expectation: lowest wavenumber corresponds to a flow going around in short and modulated along a long direction.

In a square box, the jets can be directed along either side and then one may expect a superposition of two sets of jets, which would look like a vortex dipole

Two-dimensional turbulence with forcing and friction

$$\partial_t \boldsymbol{v} + \alpha \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \boldsymbol{f} - \nabla p + \nu \Delta \boldsymbol{v}$$

$\epsilon = \langle [\mathbf{f} \cdot \mathbf{v} - \nu \mathbf{v} \cdot \Delta \mathbf{v}] \rangle$- positive and space-independent

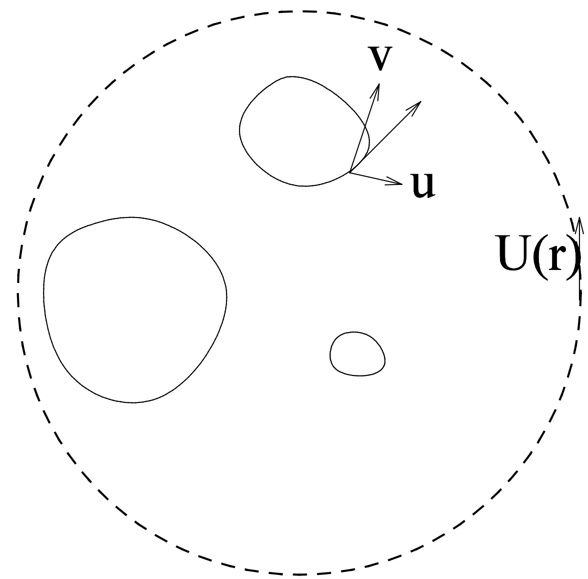
Condition for a strong mean flow

$$\epsilon^{-1/3} L^{2/3} \alpha \ll 1$$

Two-dimensional vortex sustained by small-scale forcing

$$v_t + v v_r + \frac{U+u}{r} v_\phi - \frac{(U+u)^2}{r} \text{ = force+dissipation}$$

$$u_t + v(U' + u_r) + \frac{U+u}{r} u_\phi + \frac{v(U+u)}{r} \text{ = force+dissipation}$$



$$r^{-1}\partial_r \left(r^2 \langle uv \rangle\right) = -\alpha r U$$

$$\epsilon - \frac{1}{r}\partial_r \left(r U \langle uv \rangle\right) = \alpha U^2$$

$$U^2 = 3\epsilon/\alpha\,, \qquad \tau = \langle uv \rangle = -r\sqrt{\epsilon\alpha/3}$$

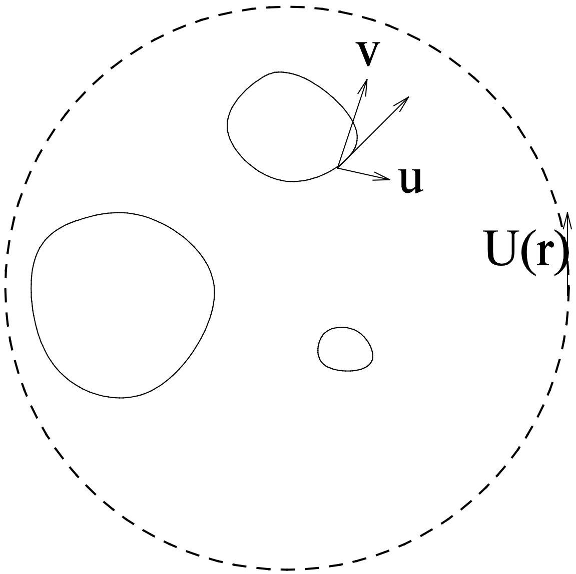
We thus got *U* (large) and τ (small), but not $< u^2 >, < v^2 >$. J Laurie et al *PRL* 2014

Two-dimensional vortex sustained by small-scale forcing

Consider inhomogeneous forcing: $\epsilon(r) \propto r^{-2\gamma}$.
For example, by a current $I(r) \propto r^{-1}$ running radially from the center

$$r^{-1}\partial_r\left(r^2\langle uv\rangle\right) = -\alpha r U$$

$$\epsilon - \frac{1}{r}\partial_r\left(rU\langle uv\rangle\right) = \alpha U^2$$



$$U(r) \propto r^{-\gamma}, \qquad < uv > \propto r^{1-\gamma}/(3-\gamma).$$

Frishman and GF, unpublished

Next step: describing correlation functions of turbulence

$$\partial_t \langle v^i(\boldsymbol{r_1}) v^j(\boldsymbol{r_2})\rangle + \langle v_1^l \nabla_1^l v_1^i v_2^j\rangle + \langle v_2^l \nabla_2^l v_2^j v_1^i\rangle + \langle \nabla_1^i p_1 v_2^j\rangle + \langle \nabla_2^j p_2 v_1^i\rangle$$

$$= \langle f_1^i v_2^j\rangle + \langle f_2^j v_1^i\rangle - 2\alpha \langle v^i(\boldsymbol{r_1}) v^j(\boldsymbol{r_2})\rangle$$

$$\frac{1}{r_1}\langle p_1 \partial_{r_2} r_2 v_2\rangle + \frac{1}{r_2}\langle p_2 \partial_{r_1} r_1 v_1\rangle = \epsilon + U_1 \langle u_2 \partial_{r_1} v_1\rangle + U_2 \langle u_1 \partial_{r_2} v_2\rangle$$

$$\left(\frac{U_1}{r_1} - \frac{U_2}{r_2}\right)\partial_{\phi_1}\langle v_1 v_2\rangle + \langle v_2 \partial_{r_1} p_1\rangle + \langle v_1 \partial_{r_2} p_2\rangle = \epsilon + \frac{2U_1}{r_1}\langle u_1 v_2\rangle + \frac{2U_2}{r_2}\langle u_2 v_1\rangle$$

$$\frac{U_2}{r_2}(\langle v_1 v_2\rangle - 2\langle u_1 u_2\rangle) + \left(U_2 \frac{r_1}{r_2} - U_1\right)\langle v_2 \partial_{r_1} v_1\rangle + \langle v_2 \frac{1}{r_1}\partial_{\phi_1} p_1\rangle + \langle u_1 \partial_{r_2} p_2\rangle = 0$$

$$-\Delta p = \frac{\partial v_i}{\partial x_k}\frac{\partial v_k}{\partial x_i} \approx 2U' \frac{\partial v}{\partial x}\,.$$

$$p(\boldsymbol{r}) = \frac{2\Gamma(d/2)}{(d-2)2\pi^{d/2}} \int \frac{U'(y')d\boldsymbol{r'}}{|\boldsymbol{r} - \boldsymbol{r'}|^{d-2}} \frac{\partial v(\boldsymbol{r'})}{\partial x'}$$

with A Frishman, work in progress

$$r_1^2 \triangle_1 r_1 r_2^2 \triangle_2 (\langle v_1 v_2 \rangle - 2\langle u_1 u_2 \rangle) + r_1^2 \triangle_1 r_2^2 \triangle_2 (r_1 - r_2) r_1 \partial_{r_1} \langle v_2 v_1 \rangle$$
$$+ 2 r_2^2 \triangle_2 r_2 r_1 \partial_{r_1} \partial_{\phi_1} \langle v_2 u_1 \rangle + 2 r_1^2 \triangle_1 r_1 (r_2 \partial_{r_2})^2 \langle u_1 u_2 \rangle = 0.$$

$$\langle v_1 v_2 \rangle = \sum_m V_m(r_1, r_2) e^{im(\phi_1 - \phi_2)}$$

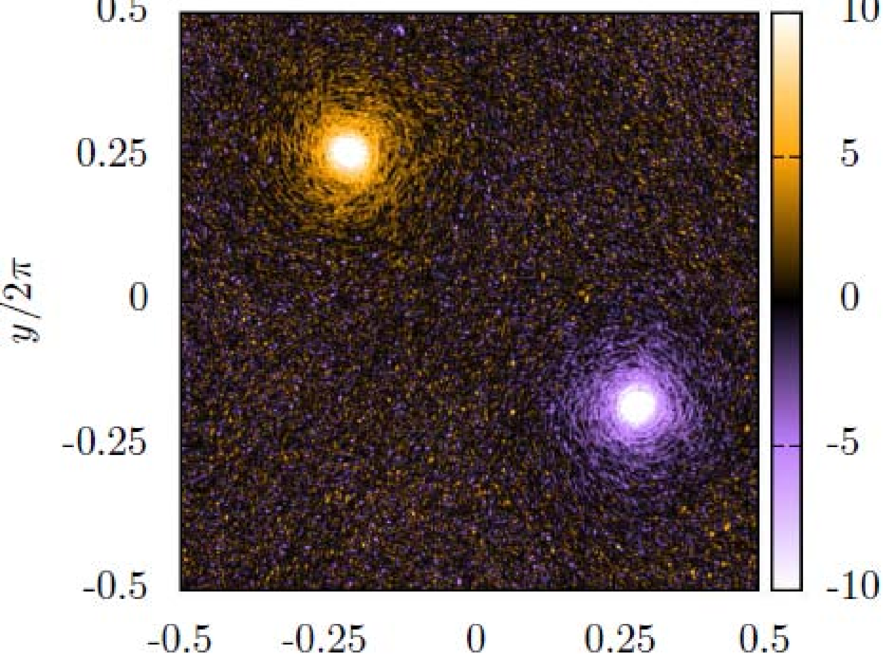
$$(\partial_x^2 - m^2)\partial_x(\partial_y^2 + 2\partial_y + 2 - m^2) e^x V_m(x,y) = (\partial_y^2 - m^2)\partial_x(\partial_x^2 + 2\partial_x + 2 - m^2) e^y V_m(x,y)$$

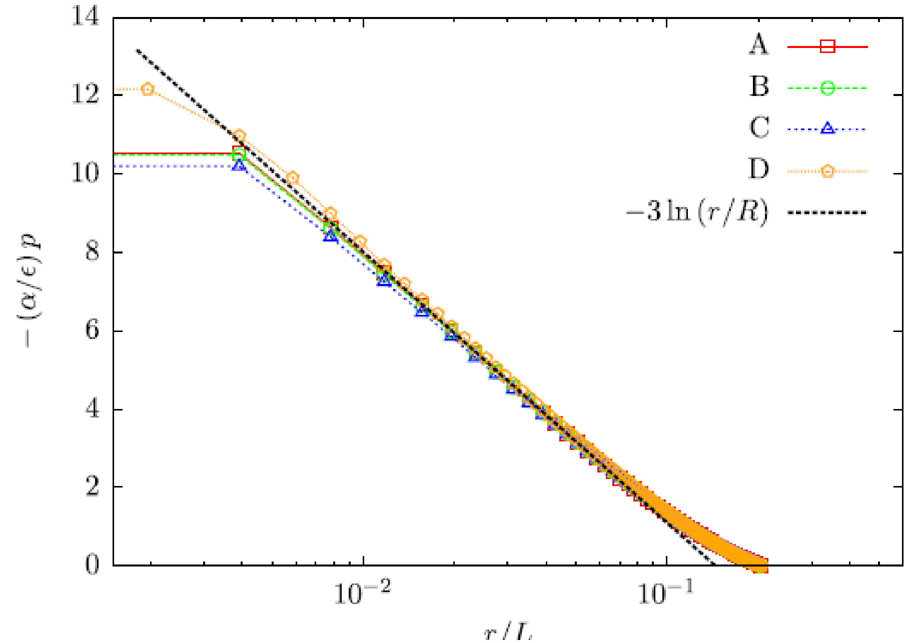
$$\langle v_1 v_2 \rangle = A_1 \cos \phi_{12} - B_1 \sin \phi_{12}$$

$$+ \sum_{m>1} A_m \left(\frac{r_1}{r_2}\right)^{\lambda_m} \left(\frac{1 + 2\lambda_m}{r_1} + \frac{1 - 2\lambda_m}{r_2}\right) \cos m\phi_{12}$$

$$\lambda_m = \sqrt{m^2 - 1}$$

C.Herbert, A. Frishman & GF, in preparation

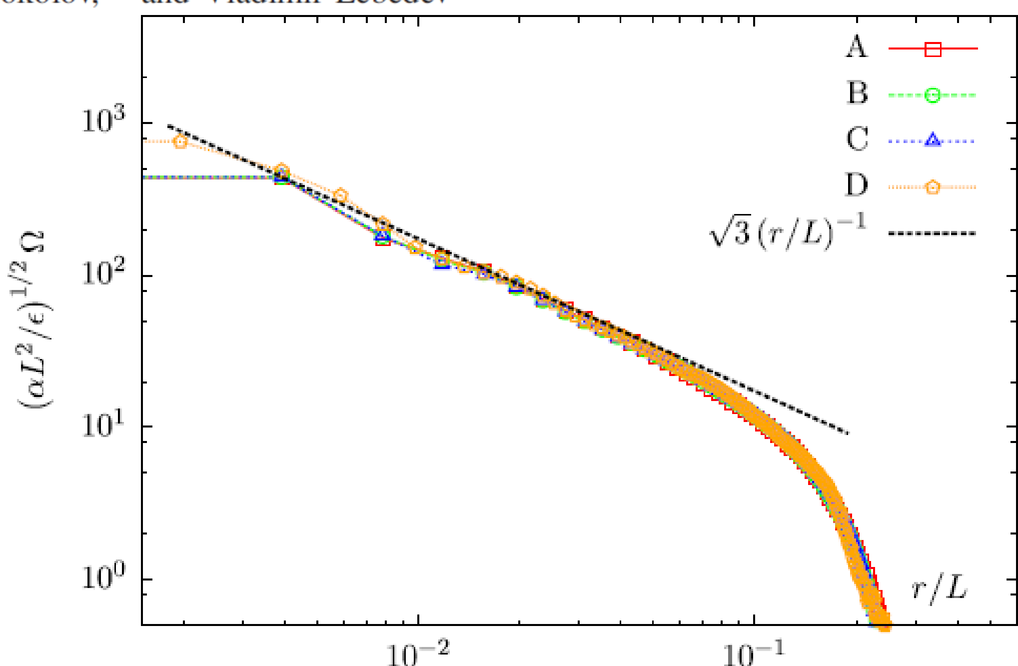


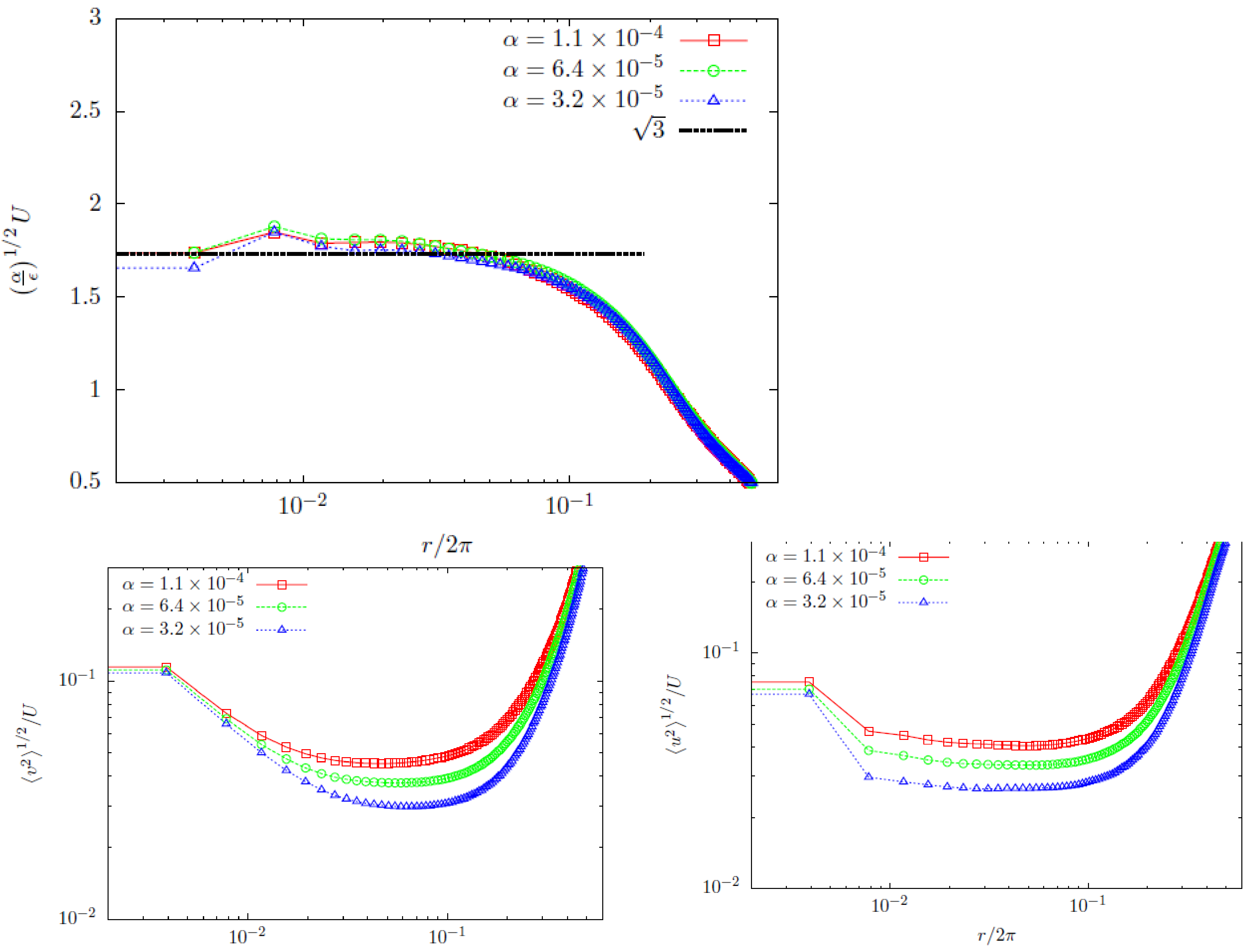


Universal Profile of the Vortex Condensate in Two-Dimensional Turbulence

Jason Laurie,[1,*] Guido Boffetta,[2] Gregory Falkovich,[1,3] Igor Kolokolov,[4,5] and Vladimir Lebedev[4,5]

PRL **113,** 254503 (2014)

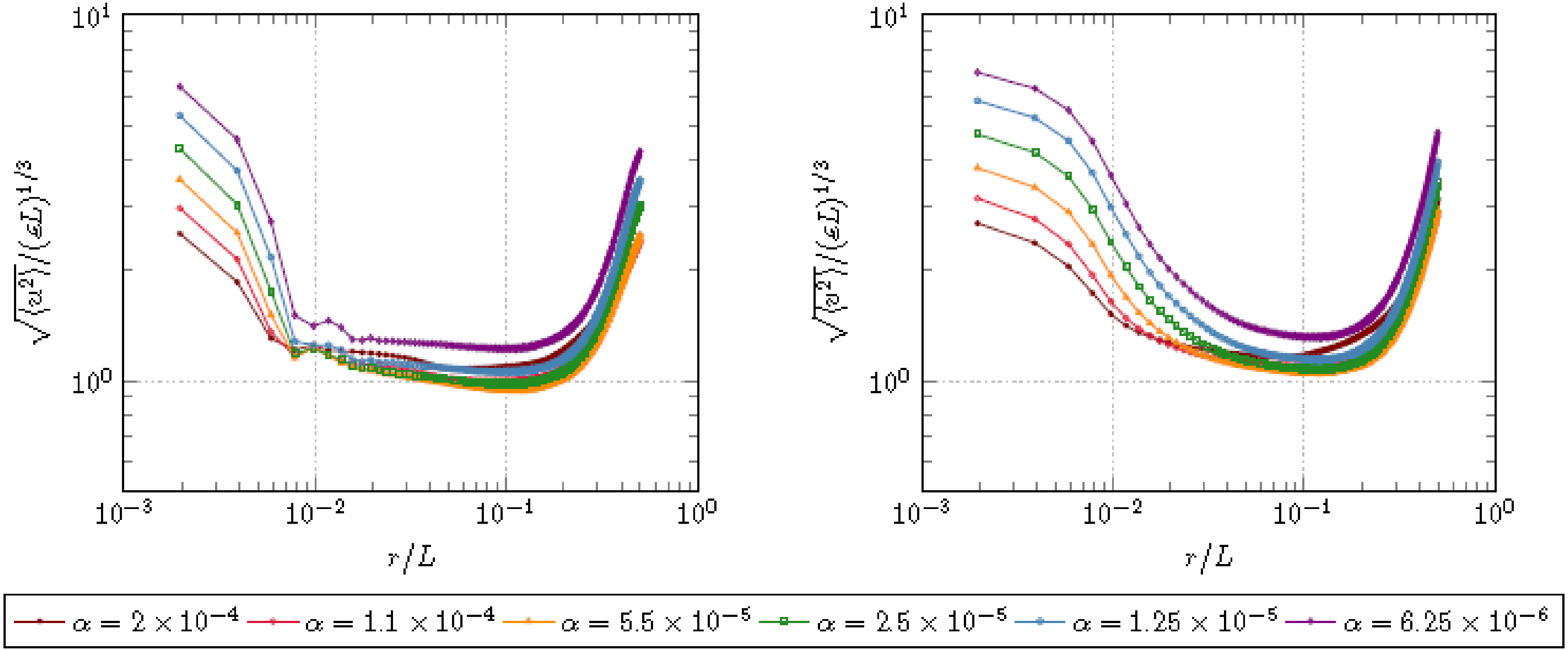




$\alpha = 1.1 \times 10^{-4}$
$\alpha = 6.4 \times 10^{-5}$
$\alpha = 3.2 \times 10^{-5}$
$\sqrt{3}$
$\left(\frac{\alpha}{\epsilon}\right)^{1/2} U$
$r/2\pi$
$\langle v^2 \rangle^{1/2} / U$
$\langle u^2 \rangle^{1/2} / U$

Fluctuating energy profile (DNS)

DNS: 512^2, $k_F = 100$, hyperviscosity, ~ 300000 turnover times.

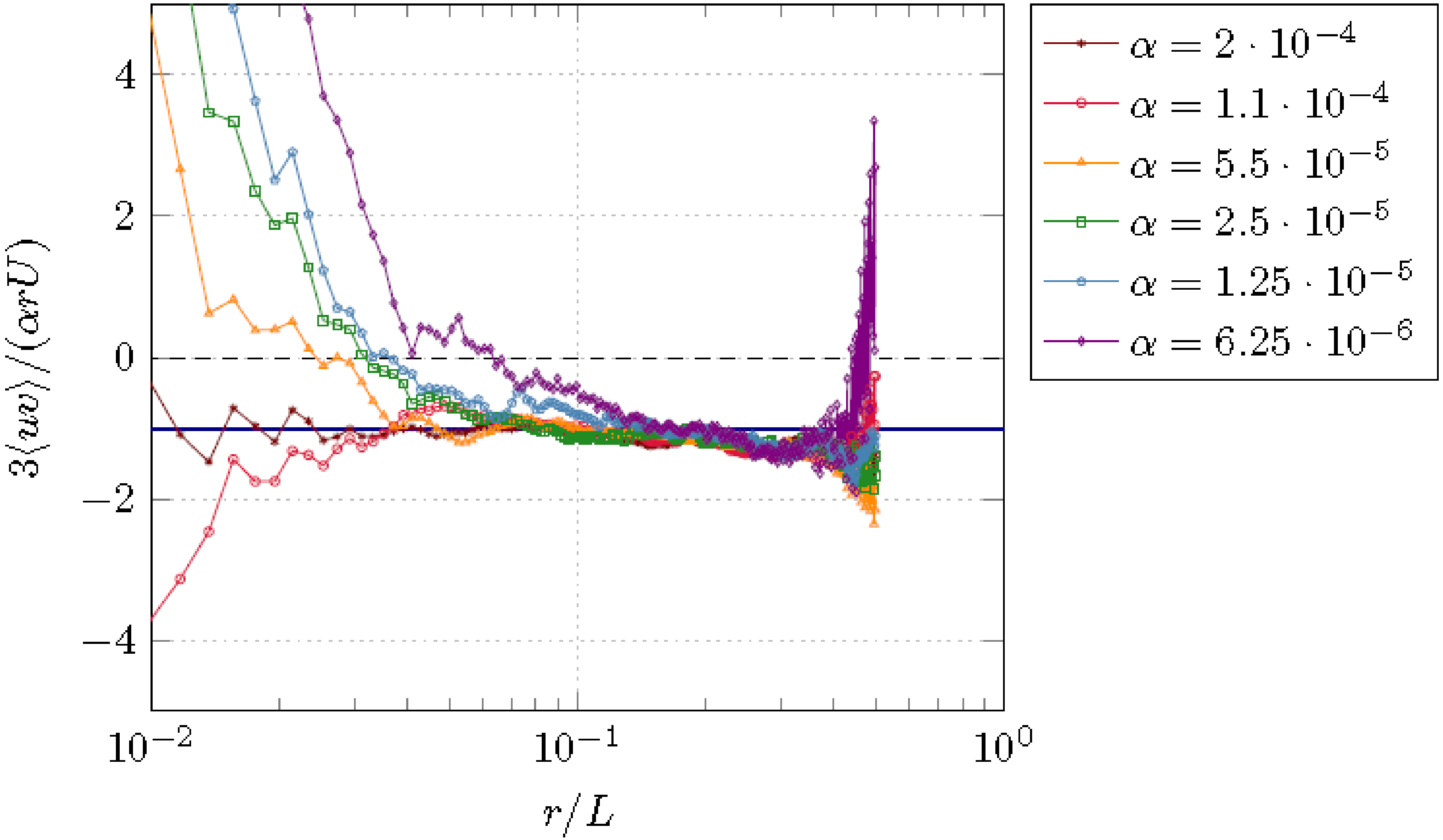


- Confirms that $\langle u^2 + v^2 \rangle / U^2 = O(\delta)$
- Independence from r comes from the first Fourier Harmonics

C.Herbert, A. Frishman & GF, in preparation

Momentum flux profile (DNS)

$\langle uv \rangle / U^2 = O(\delta^{3/2})$ and not sign definite.

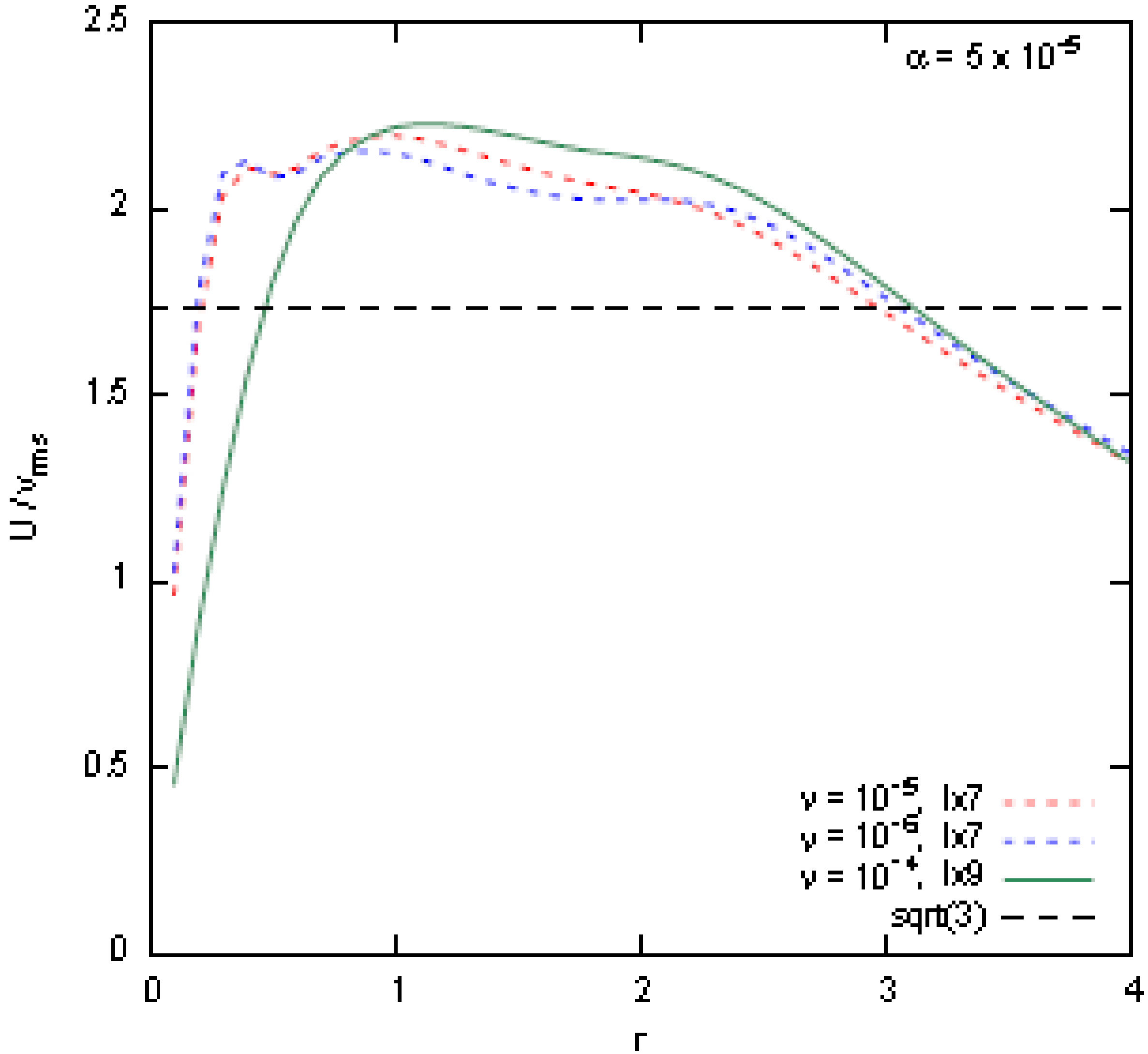


DNS: 512^2, $k_F = 100$, hyperviscosity, ~ 300000 turnover times.

C.Herbert, A. Frishman & GF, in preparation

Numerical modeling in a box with no-slip boundary

Natalia Vladimirova, work in progress



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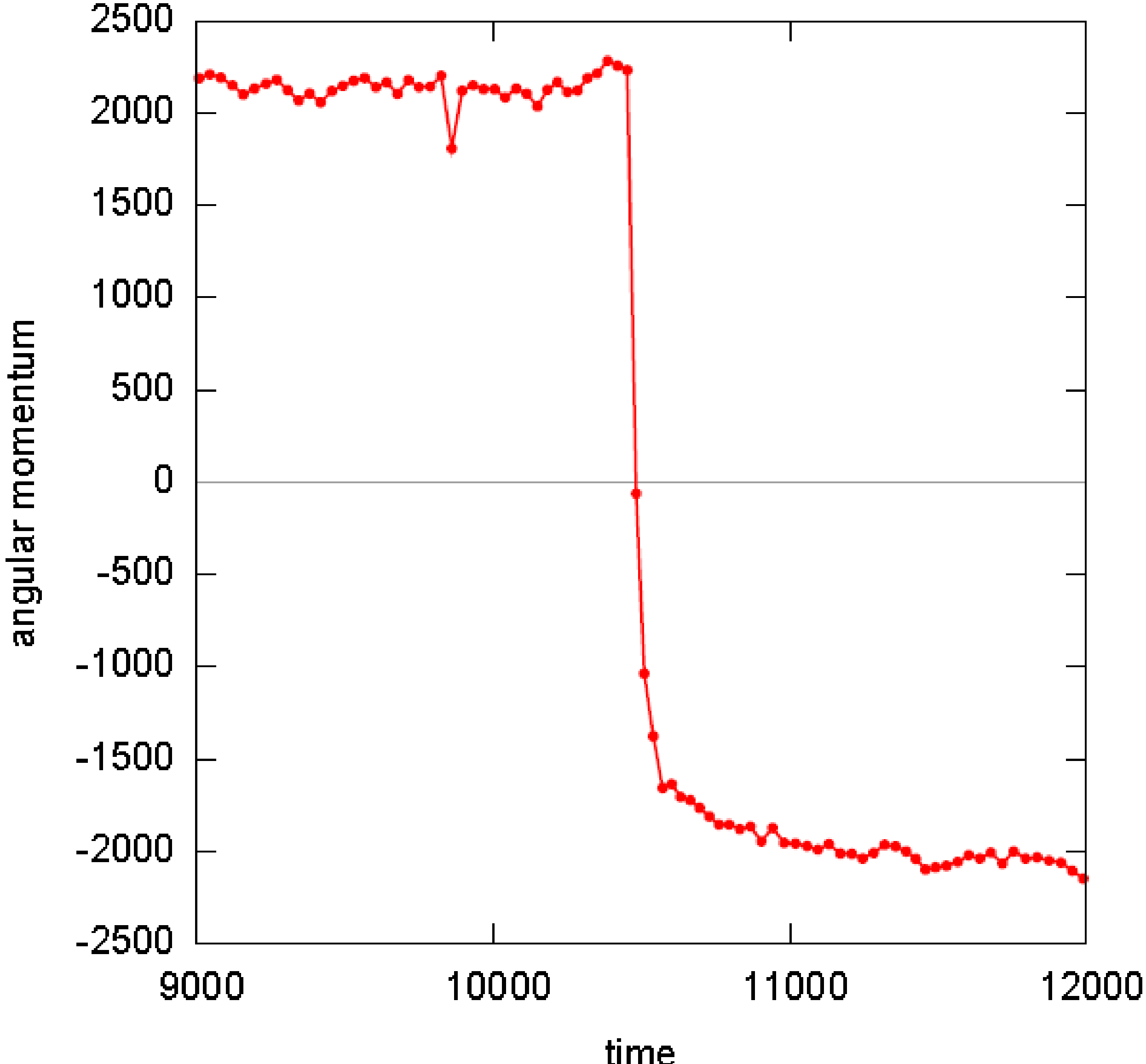

III. INVERSE CASCADE ON A SPHERE

Let us describe now the case where the closed solution is possible. Consider the forced Euler equation with a bottom friction on a rotating sphere, where one introduces the angular coordinates, latitude $\theta \in [-\pi/2, \pi/2]$ and longitude $\phi \in [0, 2\pi]$. Then the equations of motion for $u = v_\phi$ and $v = v_\theta$ take the form

$$\frac{\partial v}{\partial t} + \frac{v}{R}\frac{\partial v}{\partial \theta} + \frac{U+u}{R\cos\theta}\frac{\partial v}{\partial \phi} + \frac{(U+u)^2}{R\cot\theta} + 2\Omega(U+u)\sin\theta = f_\theta - \alpha v - \frac{\partial p}{R\partial\theta}\,, \tag{6}$$

$$\frac{\partial u}{\partial t} + \frac{v}{R}\frac{\partial (U+u)}{\partial \theta} + \frac{U+u}{R\cos\theta}\frac{\partial u}{\partial \phi} - \frac{(U+u)v}{R\cot\theta} - 2\Omega v\sin\theta = f_\phi - \alpha(U+u) - \frac{1}{R\cos\theta}\frac{\partial p}{\partial \phi}\,. \tag{7}$$

$$\alpha RU = 2\tau\tan\theta - \frac{\partial\tau}{\partial\theta} = -\frac{1}{\cos^2\theta}\frac{\partial}{\partial\theta}\tau\cos^2\theta = \langle v\omega\rangle R$$

$$(\epsilon - \alpha U^2)R = \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\cos\theta\left\langle v\left\{p + [(U+u)^2 + v^2]/2\right\}\right\rangle$$

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$$\alpha R U = 2\tau\tan\theta - \frac{\partial \tau}{\partial \theta} = -\frac{1}{\cos^2\theta}\frac{\partial}{\partial\theta}\tau\cos^2\theta = \langle v\omega\rangle R$$

$$(\epsilon - \alpha U^2)R = \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\cos\theta\left\langle v\left\{\not{p} + [(U+u)^2 + v^2]/2\right\}\right\rangle$$

$$2(U\tan\theta + \Omega\sin\theta)\tau - \epsilon R/2 + \langle v\partial_\theta p\rangle = 0\,,$$

$$\tau\partial_\theta U - (U\tan\theta + 2\Omega\sin\theta)\tau - \epsilon R/2 + \langle u\partial_\phi p\rangle/\cos\theta = 0$$

New closed equation on the momentum flux

$$\alpha R^2 \epsilon(\theta) = 2\tau^2 \frac{\sin^2\theta + 1}{\cos^2\theta} + \tau\tau' \tan\theta - \tau\tau''$$

For a constant ϵ

$$\tau = \langle uv \rangle = \pm\sqrt{\alpha\epsilon/3}R\cos\theta \quad \text{and} \qquad U = \pm\sqrt{3\epsilon/\alpha}\sin\theta$$

A turbulence source localized near equator

$$\epsilon(\theta) = \epsilon_0 \cos^4\theta \Rightarrow \tau = \pm\sqrt{\alpha\epsilon_0/4}\cos^2\theta \quad \text{and} \qquad U = \pm\sqrt{\epsilon_0/\alpha}\sin 2\theta$$

a source concentrated in mid-latitudes

$$\left.\begin{aligned}
&\epsilon(\theta) = 25\epsilon_0 \sin\theta(2\sin\theta + 3\sin 3\theta + \sin 5\theta) = 100\epsilon_0 \sin^2\theta\cos^4\theta \;\Rightarrow \\
&\tau = \pm\sqrt{\frac{\alpha\epsilon_0}{2}}4\sin\theta\cos^2\theta, \quad U = \pm\sqrt{\frac{\epsilon_0}{\alpha}}(\cos\theta - 5\cos 3\theta), \\
&= \pm 4\sqrt{\frac{\epsilon_0}{\alpha}}\cos\theta(4 - 5\cos^2\theta).
\end{aligned}\right\}$$

Can we have a plane flow out of a two-dimensional turbulence with forcing and friction?

$$\partial_t \boldsymbol{v} + \alpha \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \boldsymbol{f} - \nabla p + \nu\Delta \boldsymbol{v}$$

$\epsilon = \langle[\mathbf{f}\cdot\mathbf{v} - \nu\mathbf{v}\cdot\Delta\mathbf{v}]\rangle$ - positive and space-independent

$$\frac{\partial\langle uv\rangle}{\partial y} - \nu\frac{\partial^2 U}{\partial y^2} = -\frac{\partial\langle p\rangle}{\partial x} - \alpha U \qquad \tau' = -\alpha U$$

$$-\alpha\langle u^2\rangle = \frac{\partial\langle vu^2\rangle}{2\partial y} - \frac{\epsilon}{2} + \langle p v_y\rangle + U'\tau\ ,$$

$$-\alpha\langle v^2\rangle = \frac{\partial\langle v(p+v^2/2)\rangle}{\partial y} - \frac{\epsilon}{2} - \langle p v_y\rangle\ ,$$

$$-\alpha\langle u^2+v^2\rangle = \frac{\partial\langle v[p+(u^2+v^2)/2]\rangle}{\partial y} - \epsilon + U'\tau$$

$$\overline{\langle p v_y\rangle} \approx -\epsilon/2$$

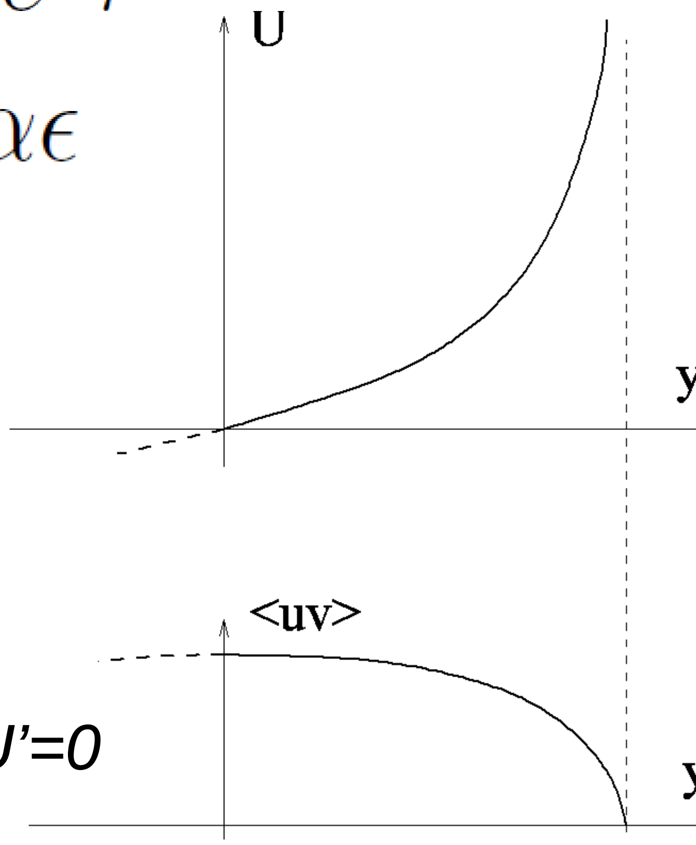
2d plane flow with small-scale forcing
let's again neglect $<vp>$

$$\epsilon = \partial_y U\tau - \nu U U'' + \alpha U^2 = U'\tau$$

$$\tau' = -\alpha U \qquad \tau''\tau = -\alpha\epsilon$$

$$\tau \to 0,\ \tau'' \to -\infty$$

$$[d\tau/dy]^2 = 2\alpha\epsilon \ln(\tau_0/\tau)$$



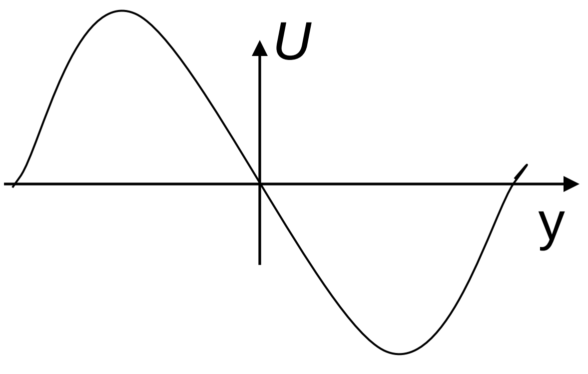
Real jet must have a maximum where *U'=0*

$$\frac{\partial \left\langle v\left[p + \left(u^2+v^2+w^2\right)/2\right]\right\rangle}{\partial y} + \epsilon + U'\tau = 0$$

Impossibility of jets + weak turbulence. Consider enstrophy balance.

$$-\langle v\omega\rangle U'' = -\alpha UU'' = Q - \partial_y \langle v\omega^2\rangle/2$$

$$Q(\theta) = \langle \omega\nabla \times \boldsymbol{f}\rangle - \nu\langle|\nabla\omega|^2 - \alpha\langle\omega^2\rangle$$

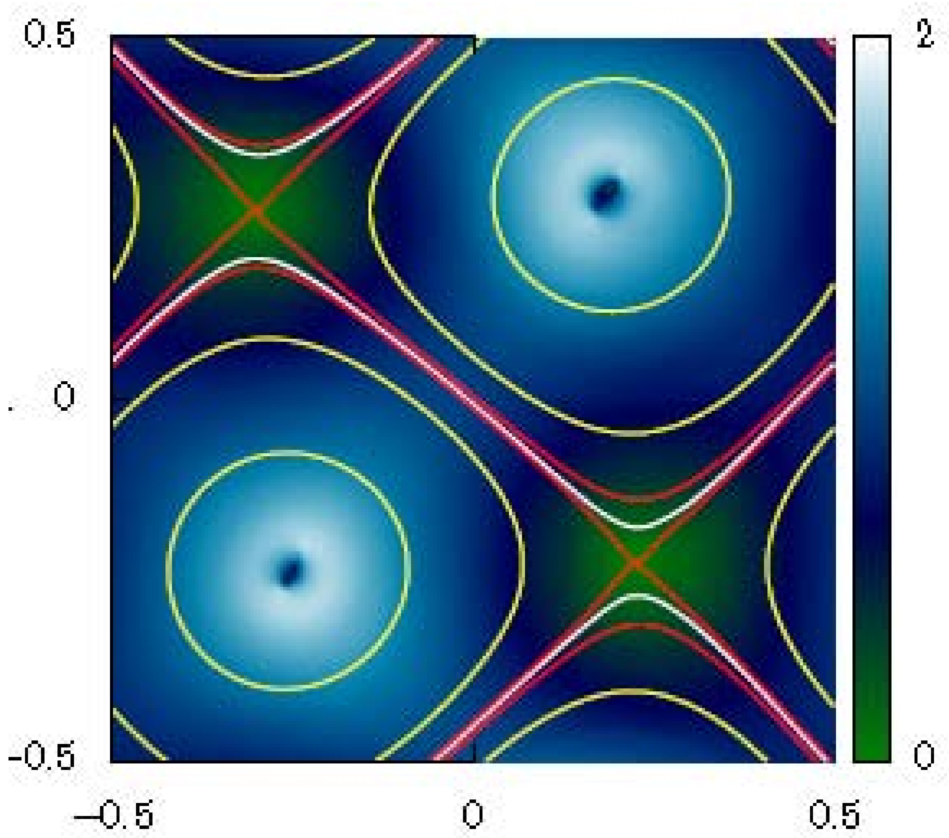


Consider *U”=0 and U=0*

Conclusion: one cannot neglect turbulent enstrophy flux.
Turbulent fluctuations are not weak.

Run	A	B	C	D
Res	512×256	512×256	512×256	512×256
dt	1×10^{-3}	5×10^{-4}	2×10^{-4}	2×10^{-4}
α	2×10^{-4}	1×10^{-4}	5×10^{-5}	3×10^{-5}
ν_p	1×10^{-36}	1×10^{-36}	1×10^{-36}	1×10^{-36}
k_f	100	100	100	100
ε_I	3.09×10^{-4}	3.09×10^{-4}	3.09×10^{-4}	3.09×10^{-4}
ε_α	2.35×10^{-4}	2.32×10^{-4}	2.31×10^{-4}	1.94×10^{-4}
η_ν	3.09	3.09	3.09	3.09
$\delta = \alpha L^{2/3} \varepsilon^{-1/3}$	1.10×10^{-2}	5.54×10^{-3}	2.77×10^{-3}	1.70×10^{-3}
No. of Frames	70000	60000	40000	40000

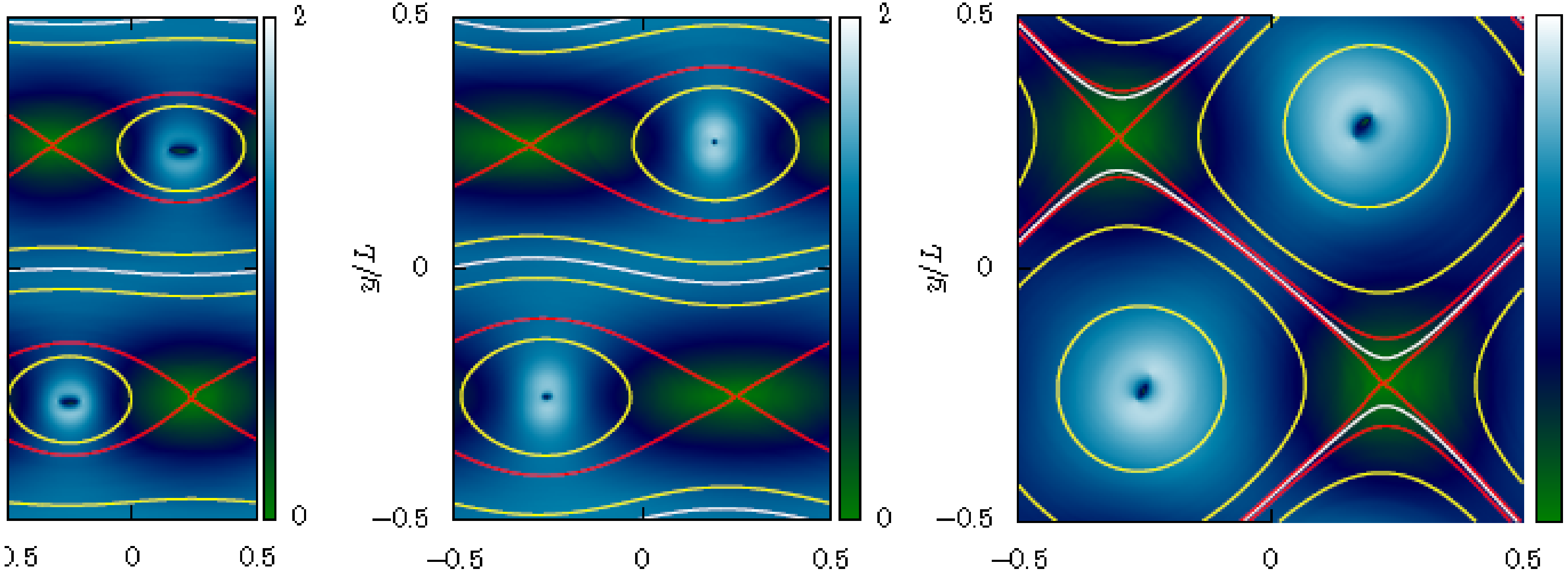
Inverse cascade on a torus: jets or vortices?



Velocity modulus averaged over short times (mean flow)
Vortices are effectively pinned. Aspect ration one.

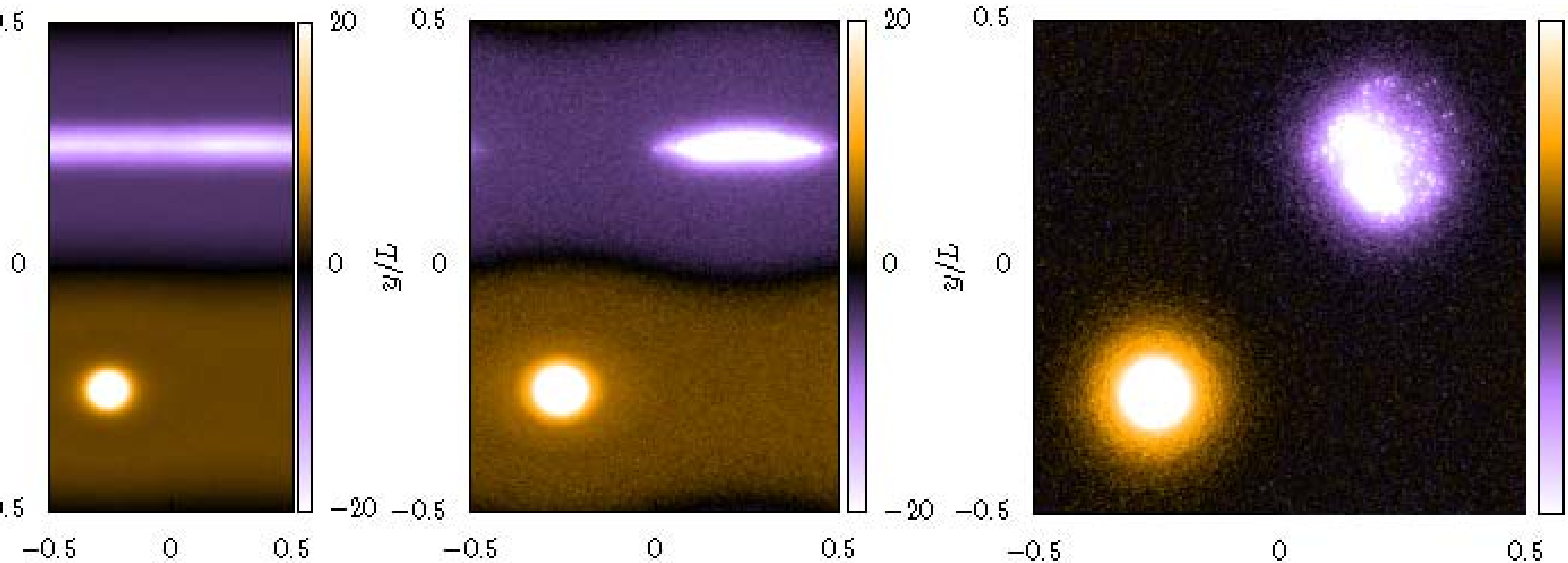
J Laurie, A. Frishman and GF Arxiv:1608.04628

Inverse cascade on a torus: jets or vortices?



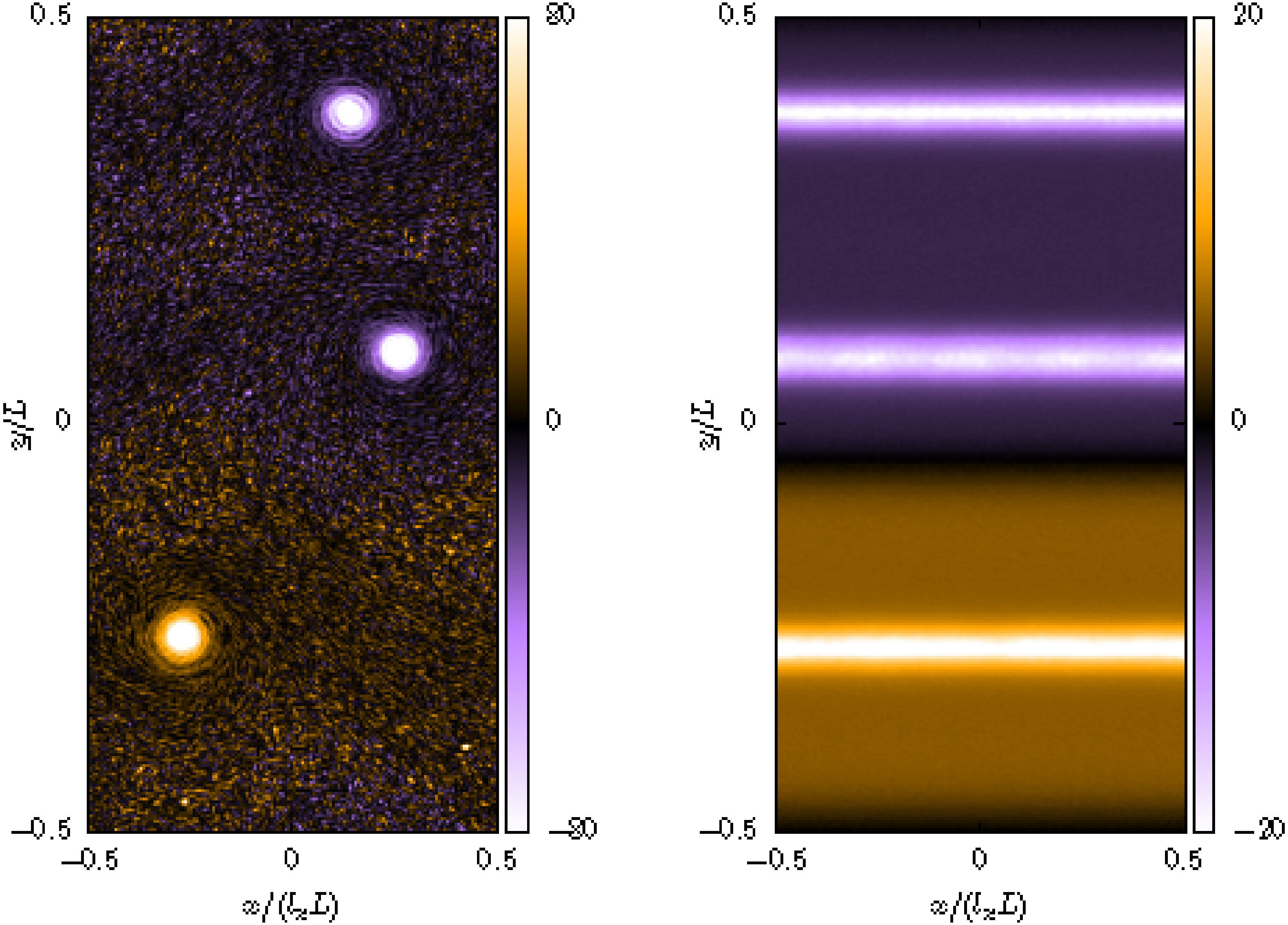
Velocity modulus averaged over short times (mean flow)
Vortices are effectively pinned. Different aspect ratios.

Inverse cascade on a torus: jets or vortices?



Velocity modulus averaged over time 1/α for different aspect ratios.

Even lower friction



Snapshot and long-time average

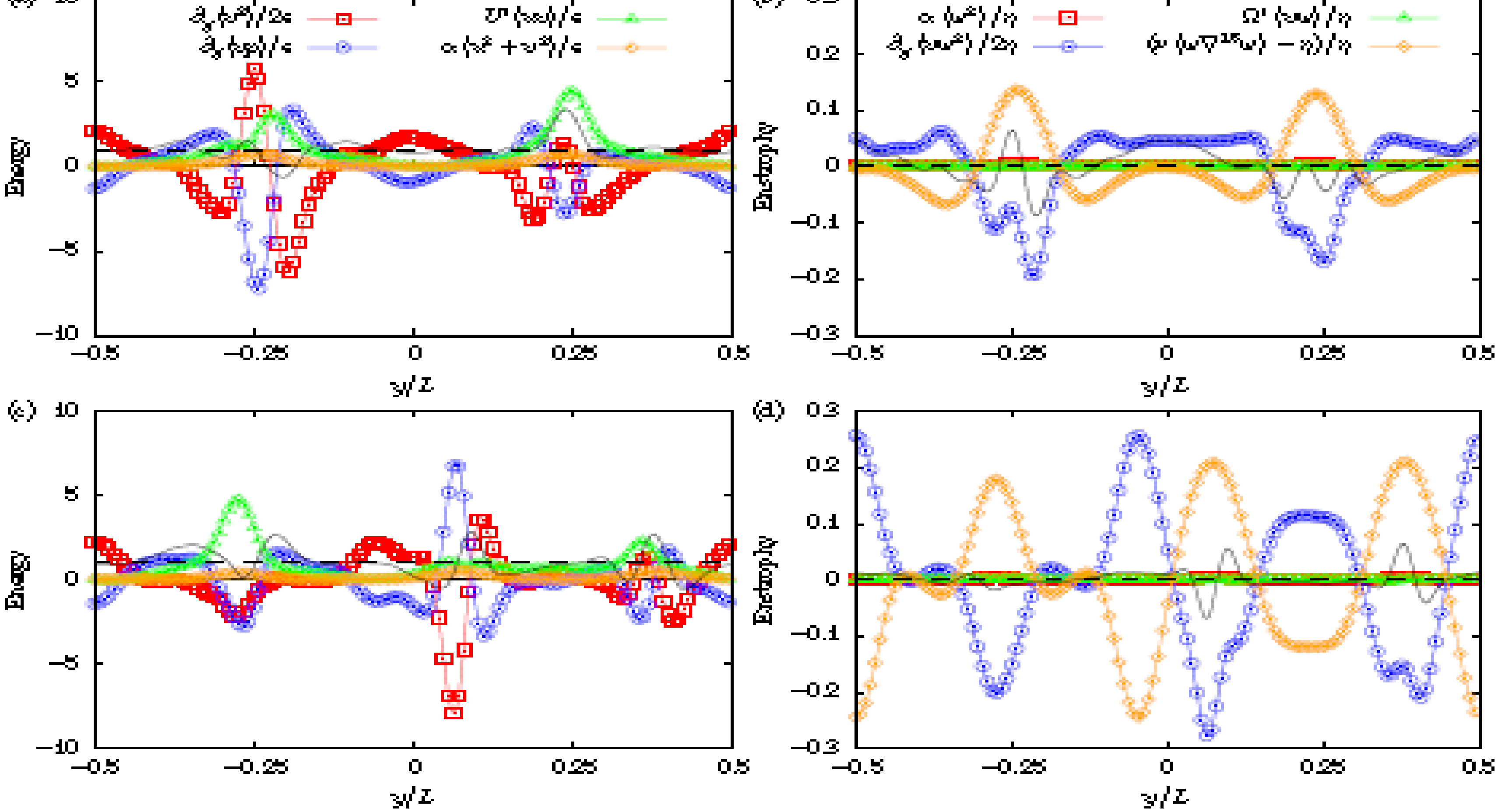
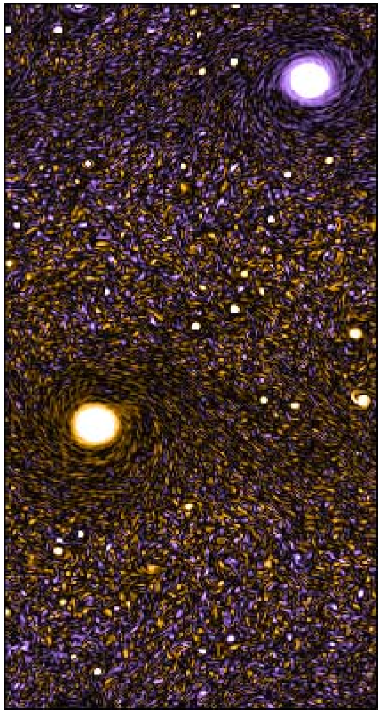
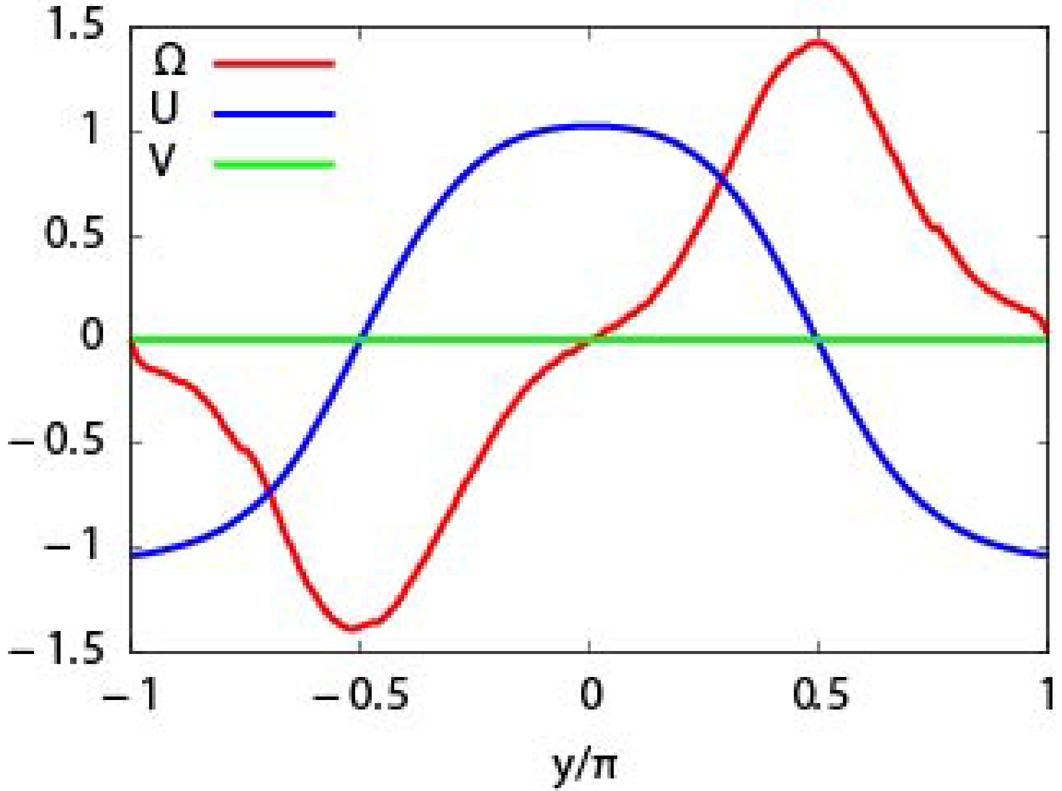
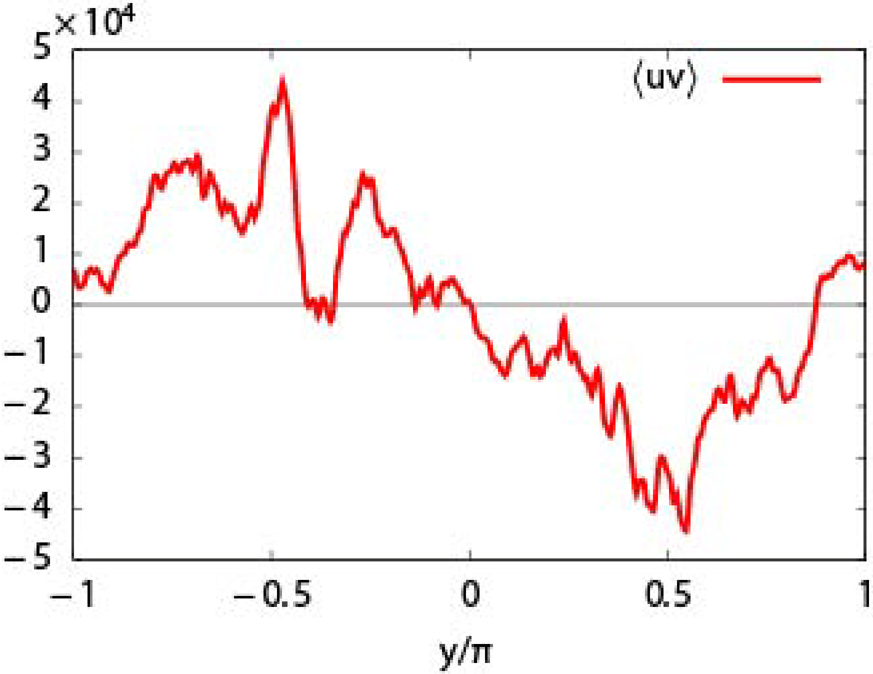


FIG. 5. (color online) Energy balance terms (left) and enstrophy balance terms (right) of the zonal flows for aspect ratio 1/2 with $\alpha = 1 \times 10^{-4}$ (top), and $\alpha = 5 \times 10^{-5}$ (middle). The bottom plots show the energy and enstrophy balance of a vortex mean flow in a square domain in polar coordinates. The energy and enstrophy balance terms should sum (given by the solid black curve) to one and zero respectively (dashed line).

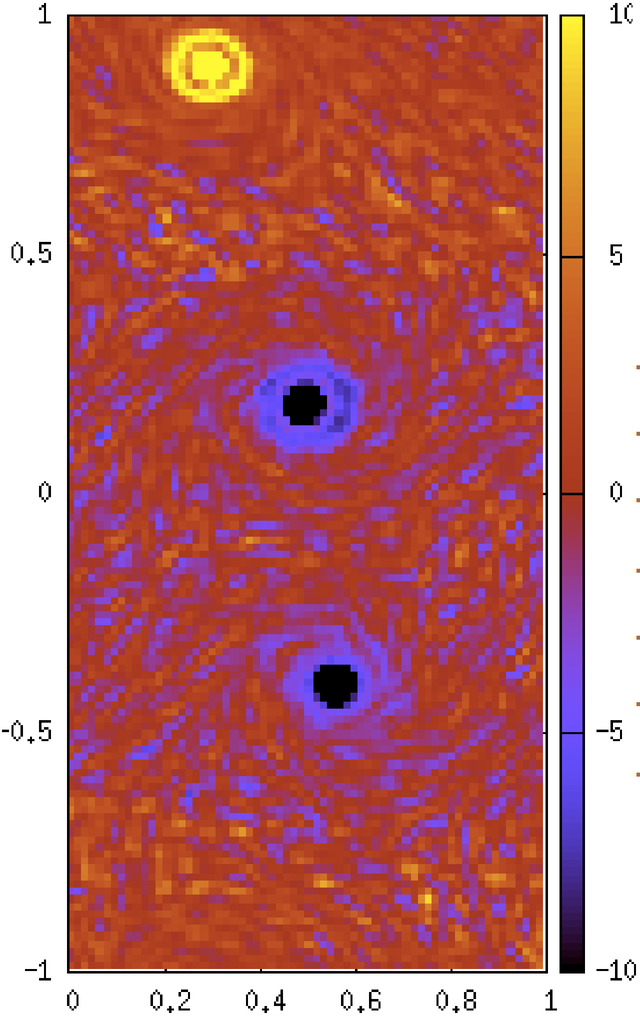


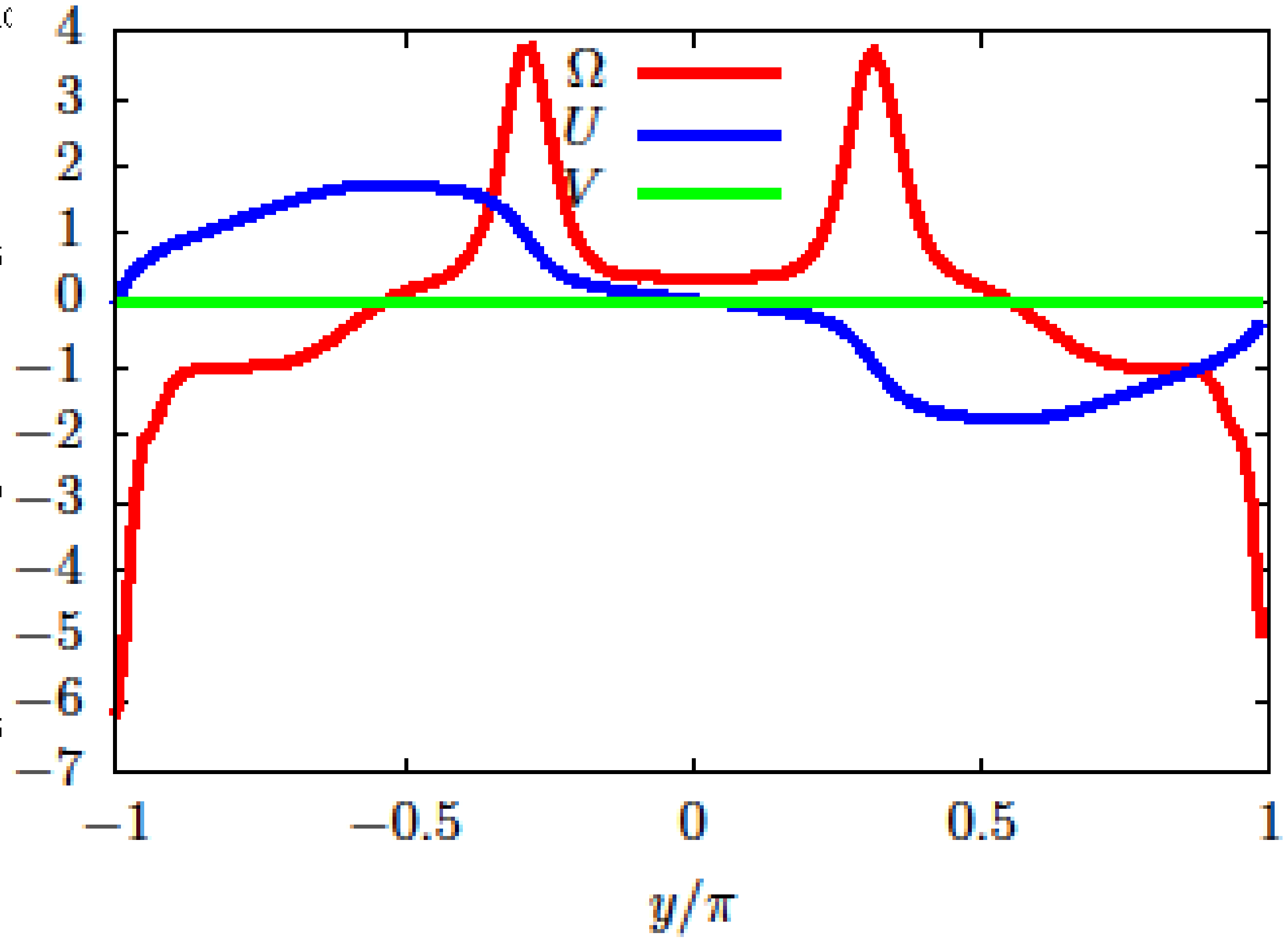


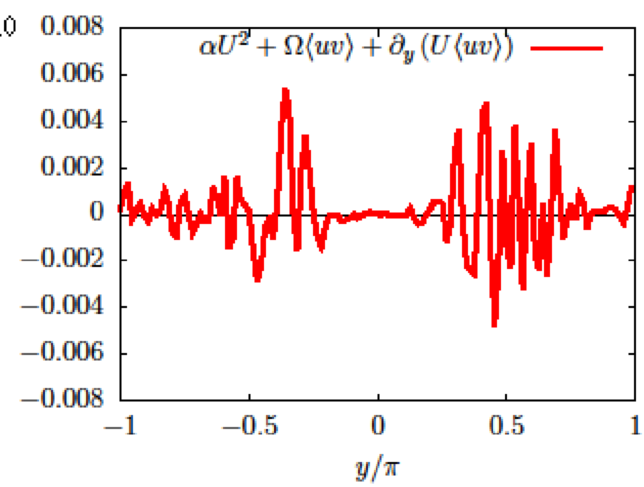
Mean profiles

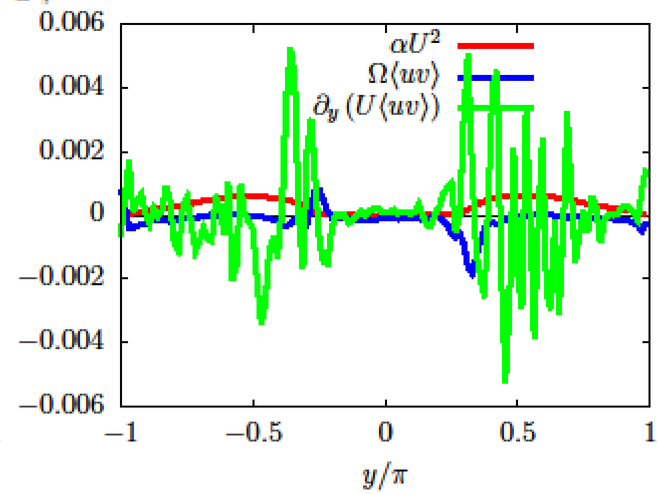


Mean momentum flux <uv>



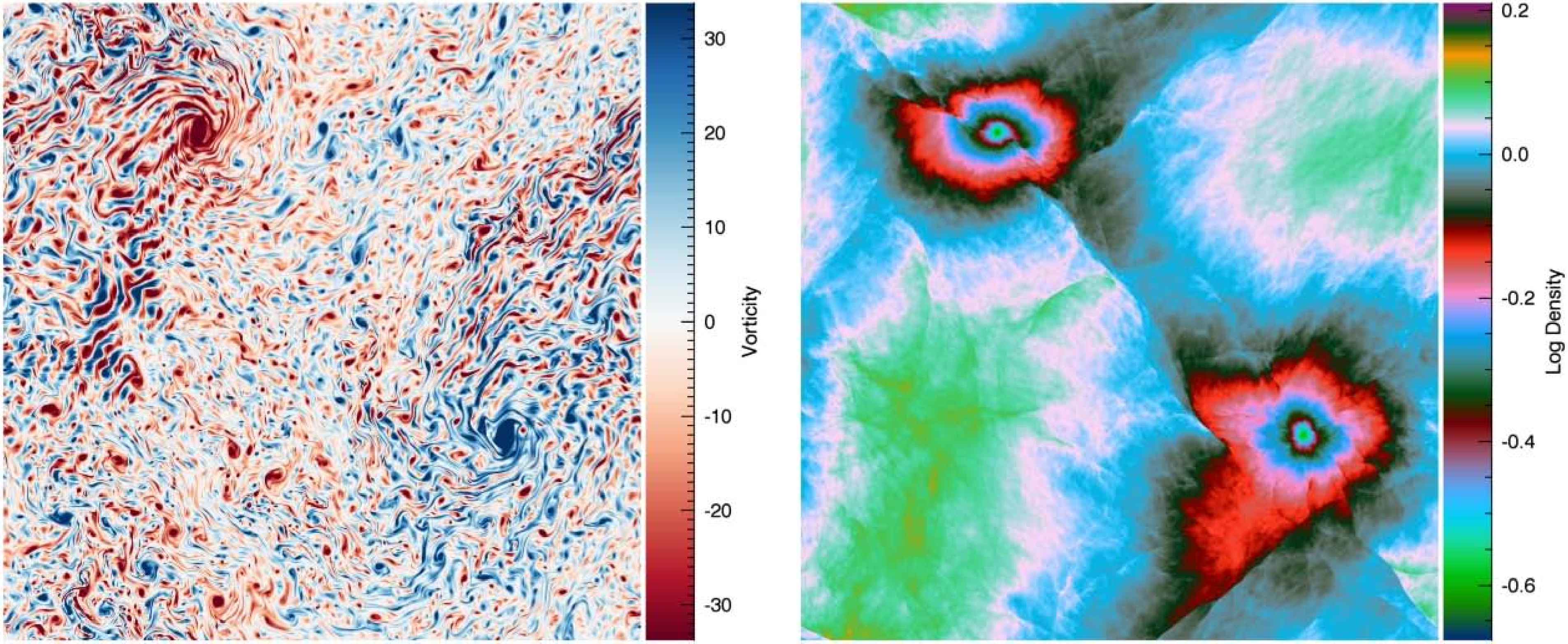




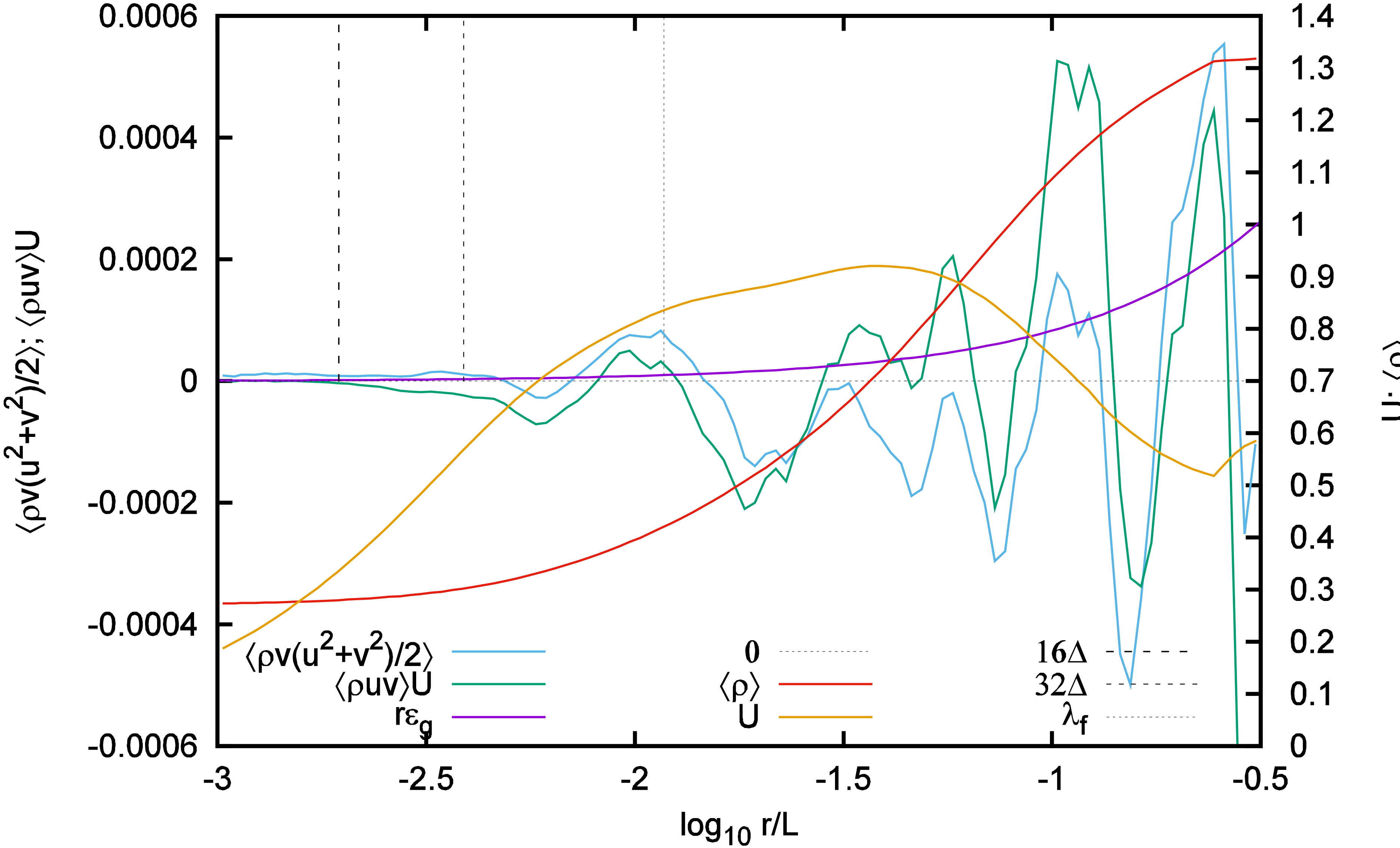


J Laurie, A Frishman

Inverse cascade and vortices in compressible 2d turbulence



Vorticity (left) and density (right) profiles in a mid-size box



Profile and fluxes in the condensate vortex in c compressible inverse cascade

Summary

It is a good time to revisit the classical approach to the analytic perturbative description of turbulent flow.

Several two-dimensional problems are consistently solvable when the geometry of the flow is trivial.

We have no idea what determines the form of the flow in more general cases like double-periodic rectangle.

Fluid Mechanics

The multi-disciplinary field of fluid mechanics is one of the most actively developing fields of physics, mathematics and engineering. In this book, the fundamental ideas of fluid mechanics are presented from a physics perspective.

Using examples taken from everyday life, from hydraulic jumps in a kitchen sink to Kelvin–Helmholtz instabilities in clouds, the book provides readers with a better understanding of the world around them. It teaches the art of fluid-mechanical estimates and shows how the ideas and methods developed to study the mechanics of fluids are used to analyse other systems with many degrees of freedom in statistical physics and field theory.

Aimed at undergraduate and graduate students, the book assumes no prior knowledge of the subject and only a basic understanding of vector calculus and analysis. It contains 32 exercises of varying difficulties, from simple estimates to elaborate calculations, with detailed solutions to help readers understand fluid mechanics.

Gregory Falkovich is a Professor in the Department of Physics of Complex Systems, Weizmann Institute of Science. He has researched in plasma, condensed matter, fluid mechanics, statistical and mathematical physics and cloud physics and meteorology, and has won several awards for his work.

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FALKOVICH Fluid Mechanics

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Fluid Mechanics

A Short Course for Physicists

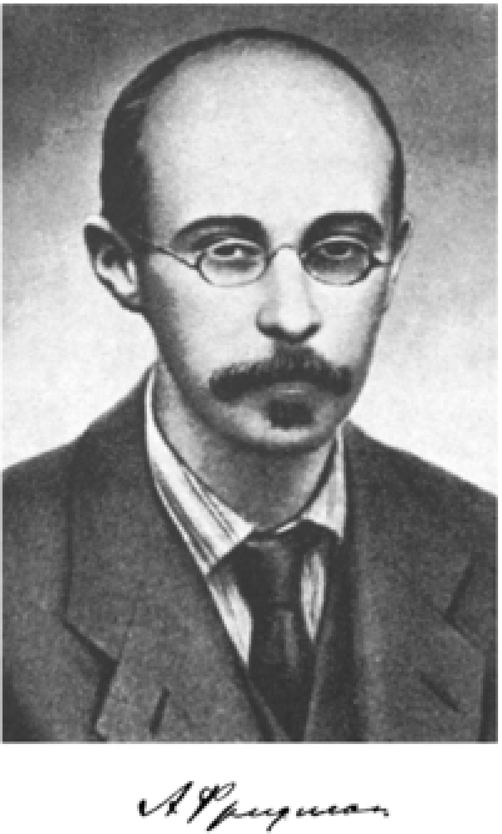
GREGORY FALKOVICH

CAMBRIDGE

From Kolmogorov back to Fridman

“At any moment, there exists a narrow layer between trivial and impossible where mathematical discoveries are made. Therefore, an applied problem is either solved trivially or not solved at all. It is altogether different story if an applied problem is found to fit (or made to fit!) the new formalism interesting for a mathematician.”

A. N. Kolmogorov, diary 1943



“… a goal to write down a closed set of equations for which an initial value problem for turbulent flow can be posed and solved” A. A. Fridman 1924

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = \text{ force}$$

$\partial\langle\mathbf{v}\rangle/\partial t$ ➡ $\langle(\mathbf{v}\nabla)\mathbf{v}\rangle$

correlation functions   Erhaltungsmomenten

$$\partial\langle\mathbf{v}_1\mathbf{v}_2\rangle/\partial t$$

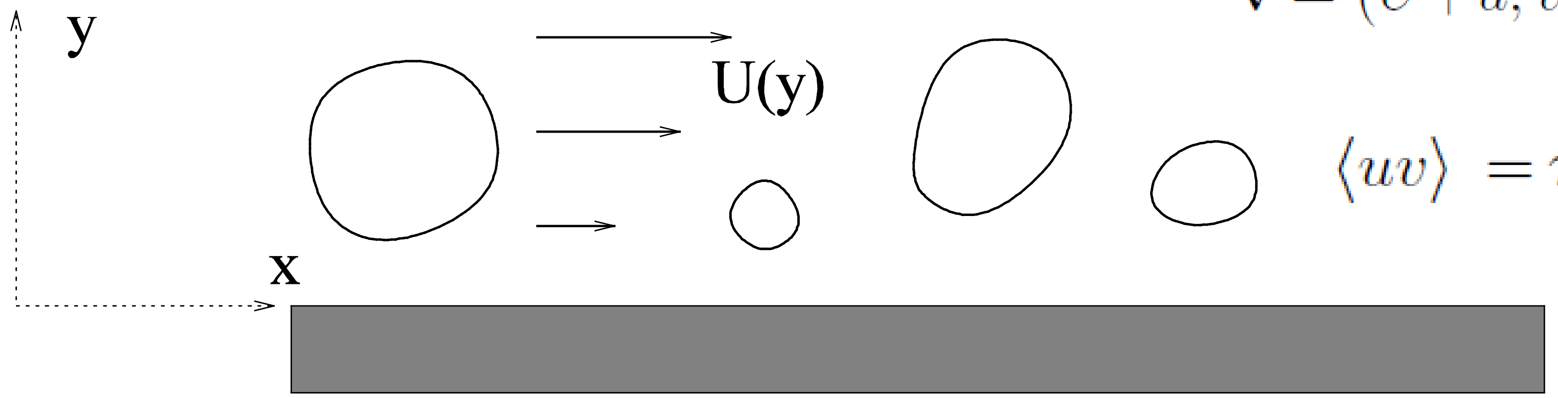
$$\langle v_1^i v_2^j v_2^k\rangle = \langle v_{1i}\rangle\langle v_{2j} v_{2k}\rangle + \ldots$$

A.A. Friedman introduced correlation functions in turbulence theory.

Plane flow

pipe, channel, boundary layer

$$\mathbf{v} = (U+u, v, w)$$



$$\langle uv \rangle = \tau$$

$$\frac{\partial \langle uv \rangle}{\partial y} - \nu \frac{\partial^2 U}{\partial y^2} = -\frac{\partial \langle p \rangle}{\partial x}$$

$$\partial_y \langle p + v^2 \rangle = 0 \ .$$

$$\partial_y \langle wv \rangle = 0 \ .$$

$$-\left\langle \frac{\partial u^2}{2\partial t} \right\rangle = \frac{\partial \langle vu^2 \rangle}{2\partial y} + \epsilon_u + \langle p(v_y + w_z) \rangle + U'\tau \,,$$

$$-\left\langle \frac{\partial v^2}{2\partial t} \right\rangle = \frac{\partial \langle v(p + v^2/2) \rangle}{\partial y} + \epsilon_v - \langle p v_y \rangle \,,$$

$$-\left\langle \frac{\partial w^2}{2\partial t} \right\rangle = \frac{\partial \langle vw^2 \rangle}{2\partial y} + \epsilon_w - \langle p w_z \rangle \,,$$

$$-\left\langle \frac{\partial (u^2 + v^2 + w^2)}{2\partial t} \right\rangle = \frac{\partial \langle v[p + (u^2 + v^2 + w^2)/2] \rangle}{\partial y} + \epsilon + U'\tau \,,$$

$$\epsilon = -\nu \langle \mathbf{v} \Delta \mathbf{v} \rangle = \epsilon_u + \epsilon_v + \epsilon_w$$

$$-\frac{\partial \langle uv \rangle}{\partial t} = \frac{\partial \langle v^2 u \rangle}{\partial y} + \epsilon' + U' \langle v^2 \rangle + \langle u p_y + v p_x \rangle \,.$$

$$\epsilon' = -\nu \langle u \Delta v - v \Delta u \rangle$$

For a steady turbulence, after the integration over space we have

$$\bar{\epsilon} = \bar{\epsilon}_u + \bar{\epsilon}_v + \bar{\epsilon}_w = -\overline{U' \langle uv \rangle} \,,$$

$$\bar{\epsilon}_v = \langle \overline{p v_y} \rangle > 0 \,, \quad \bar{\epsilon}_w = \langle \overline{p w_z} \rangle > 0 \,, \quad \langle \overline{p u_x} \rangle < 0 \,.$$

Yet another "derivation" of log-profile

Let's consider statistical steady state, make quasi-linear approximation and neglect cubic term in the energy balance:

$$\epsilon = -U'\tau$$

$$v_* = \sqrt{\tau}$$

$$\epsilon(y) \simeq v_*^3/y$$

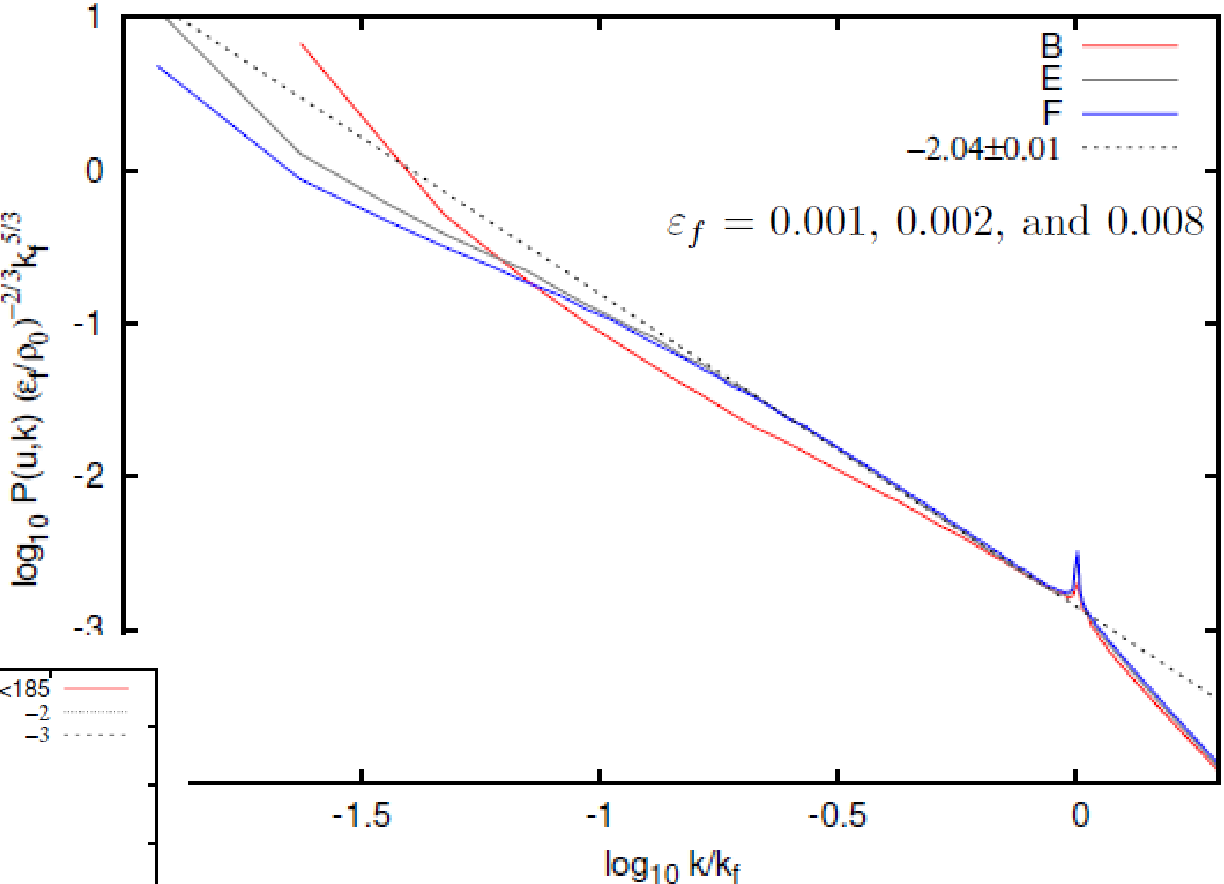
$$\partial_y U = -\epsilon(y)/\tau \simeq v_*/y$$

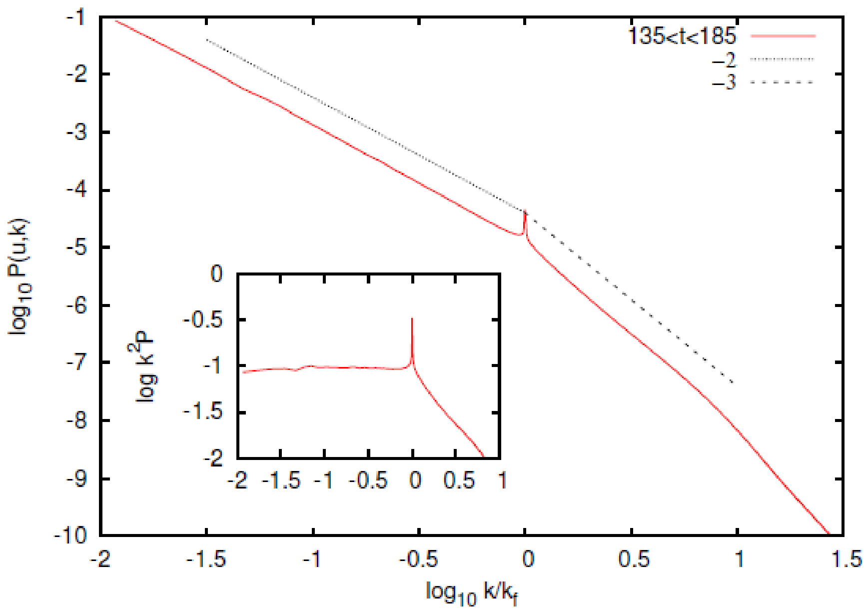
$$\frac{U(z,V,L,\nu)}{V} = \frac{\log(L/y)}{\log Re_*} \text{ for } y \geq L/Re_*$$

$$= 1 - \frac{yRe_*}{L} \text{ for } y \ll L/Re_*$$

$$Re_* = v_* L/\nu = VL/\nu \log Re_*$$

Arresting inverse cascade before it reaches the box size





Operator product expansion and universality of turbulence

$$\langle O_i(\mathbf{r}_1)O_j(\mathbf{r}_2)\rangle = \sum \langle O_k\big((y_1+y_2)/2\big)\rangle C_{ij}^k(\mathbf{r}_{12})$$

the structure functions $C_{ij}^k(\mathbf{r}_{12})$ are those of isotropic homogeneous turbulence

$$v^i(x_1)v^j(x_2) = v^i v^j + D^{ij}(x_{12})\epsilon_2 + \frac{x_{12}^k}{2}\big[\nabla_k(v^i v^j)$$
$$+(v^i\nabla_k v^j - v^j\nabla_k v^i)\big] + G_{kl}^{ij}(x_{12})(\nabla_k v^l + \nabla_l v^k)\,.$$

$$\langle u_2 v_3\rangle = \langle u_2 v_2\rangle - \epsilon_2(y_2)x_{23}y_{23}r_{23}^{\zeta_2-2} + y_{23}\partial_{y_2}\langle u_2 v_2\rangle$$
$$-c_2U'(y_2)[1+4x_{23}y_{23}r_{23}^{-2}/3 - 8x_{23}^2y_{23}^2r_{23}^{-4}]r_{23}^{4/3}\,.$$

Gregory Falkovich[1,2] **and Alexander Zamolodchikov**[2,3] J. Phys. A: Math. Theor. **48** (2015) 18FT02