Extreme Events in Turbulent Flows: A Variational Approach

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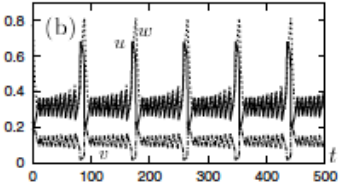
Joint work with Themis Sapsis (MIT)

KITP - Program on Recurrent Flows

February 9, 2017

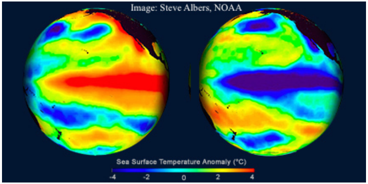
Examples of Extreme Events

J. Moehlis, Nonlin. Sci.



Chemical Reactions

Steve Albers, NOAA



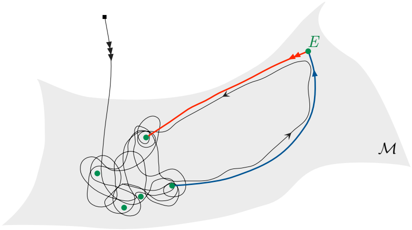
El Niño/La Niña Oscillations

Rogue Waves

M. Farazmand and T. Sapsis, J. Comput. Phys., In Revision (2017)

Intermittent Bursts in Turbulence

Close passages to steady states and traveling waves



M. Farazmand, An adjoint-based approach for finding invariant solutions of Navier–Stokes equations, J. Fluid Mech. (795), 2016

Adjoint Method

Find $u \in \mathbb{R}^n$ such that

$$F(u) = 0$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$.

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Idea: Design an ODE

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = G(u)$$

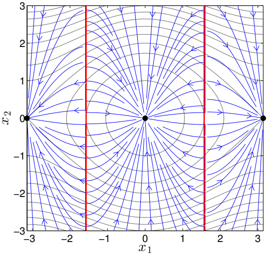
such that

$$\lim_{\tau \to \infty} \|F(u(\tau))\| \to 0$$

Adjoint Method

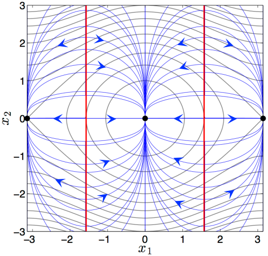
$$\frac{\mathrm{d}}{\mathrm{d}\tau}\|F(u(\tau))\|^2 = 2\langle \nabla F(u)G(u), F(u)\rangle$$
$$= 2\langle G(u), [\nabla F(u)]^\top F(u)\rangle$$

$$\nabla F(u)G(u) = -F(u)$$



Newton

$$G(u) = -[\nabla F(u)]^\top F(u)$$



Adjoint

Navier–Stokes equation

$$\partial_t u = -u \cdot \nabla u - \nabla p + \nu \Delta u + f$$

$$\nabla \cdot u = 0$$

Vorticity, $Re = 100$, $k_f = 4$

Velocity field: $u : \Omega \times \mathbb{R}^+ \to \mathbb{R}^2$

Pressure field: $p : \Omega \times \mathbb{R}^+ \to \mathbb{R}$

Reynolds number: $Re = \nu^{-1}$

Kolmogorov forcing: $f(x, y) = \sin(k_f y) e_1$

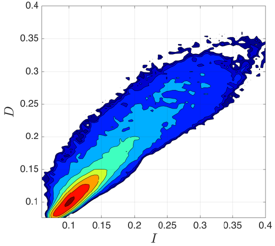
Energy considerations

Kinetic Energy: $E(u) = \frac{1}{|\Omega|} \int_\Omega \frac{|u|^2}{2} dx$

Energy Dissipation: $D(u) = \frac{\nu}{|\Omega|} \int_\Omega |\nabla u|^2 dx$

Energy Input: $I(u) = \frac{1}{|\Omega|} \int_\Omega u \cdot f dx$

$$\dot{E} = I - D$$



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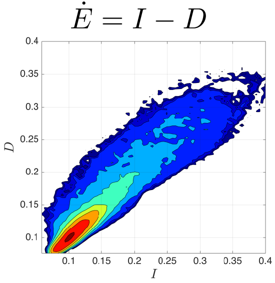
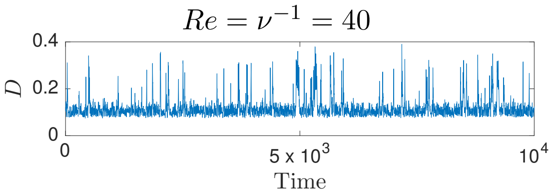
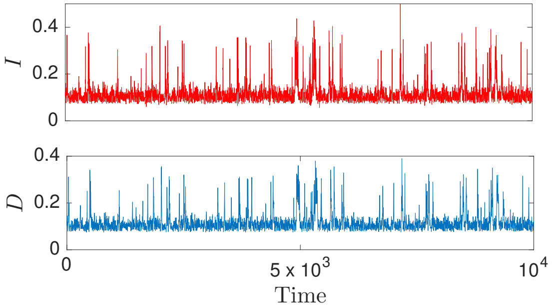
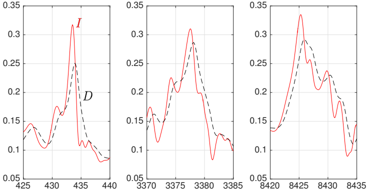




Energy dissipation vs. energy input

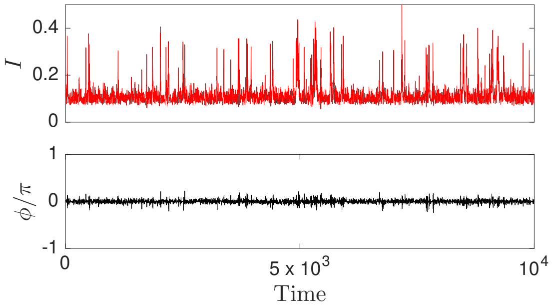


Bursts in I precede the bursts in D



$$I(u) = \frac{1}{|\Omega|}\int_\Omega u \cdot f \, \mathrm{d}x$$
$$\sim \alpha_0(t)\cos(\phi(t))$$

Energy dissipation vs. energy input



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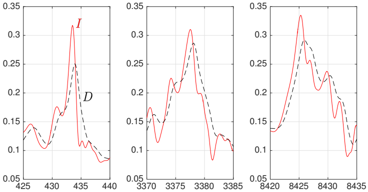
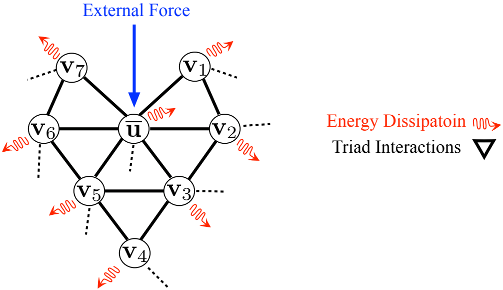


$$I(u) = \frac{1}{|\Omega|}\int_{\Omega} u \cdot f \, \mathrm{d}x$$
$$\sim \alpha_0(t)\cos(\phi(t))$$

Nonlinear energy transfers

$$\partial_t u = -u \cdot \nabla u - \nabla p + \nu \Delta u + f, \quad \nabla \cdot u = 0$$



$$u(x,t) = \overline{u}(x,t) + \sum_{j=1}^{\infty} \alpha_j(t) v_j(x)$$

$$\overline{u}(x,t) = \alpha_0(t) f(x)$$

A. J. Majda, PNAS (112), 2015

Why a new method?

- Symmetry-breaking bifurcation Y.-C. Lai, Phys. Rev. E (53) 1996
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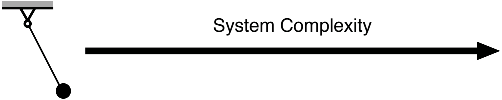
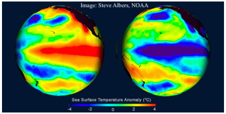




Problem formulation (set-up)

State of the system:

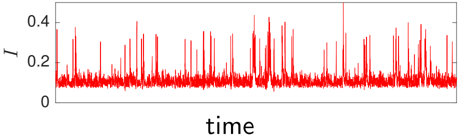
$$u(\cdot, t) \in X$$

Governing equations:

$$\partial_t u = \mathcal{N}(u)$$
$$\mathcal{K}(u) = 0$$
$$u(\cdot, t_0) = u_0$$

Observable:

$$I : X \to \mathbb{R}$$



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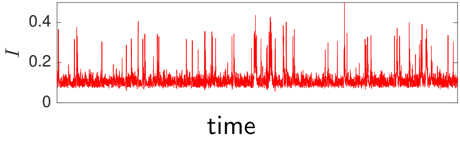
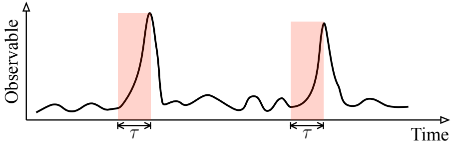
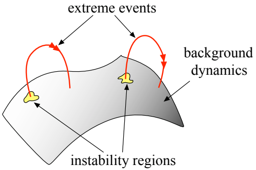


Goal: Find initial states u_0 that trigger the extreme growth of observable

Problem formulation (finite time)





$$\sup_{u_0 \in X} \left[I(u(t_0+\tau)) - I(u(t_0)) \right]$$

$$\text{constraints} \begin{cases} u(t) \text{ satisfies } \partial_t u = \mathcal{N}(u), \; \mathcal{K}(u) = 0 \\ C(u_0) = c_0 \end{cases}$$

- Singular value decomposition:

$$\max_{\|v\|=1} \|Av\|$$

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- Transition to turbulence

C. C. T. Pringle and R. R. Kerswel, Phys. Rev. Lett. (105), 2010

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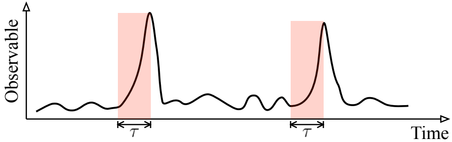
  C. C. T. Pringle and R. R. Kerswel, Phys. Rev. Lett. (105), 2010

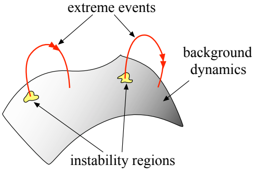
- Singularity formation in Navier–Stokes

  D. Ayala and B. Protas, J. Fluid Mech. (742), 2014

Problem formulation (instantaneous)

$$0 < \tau \ll 1$$





$$\sup_{u_0 \in X} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} I(u(t))$$

constraints $\begin{cases} u(t) \text{ satisfies } \partial_t u = \mathcal{N}(u),\ \mathcal{K}(u) = 0 \\ C(u_0) = c_0 \end{cases}$

Equivalent problem

M. Farazmand and T. Sapsis, A variational method for probing extreme events in turbulent dynamical systems, preprint (2017)

Lemma

The instantaneous problem is equivalent to

$$\sup_{u\in X} J(u), \quad \text{constraints} \begin{cases} \mathcal{K}(u) = 0 \\ C(u) = c_0 \end{cases}$$

where

$$J(u) = \mathrm{d}I(u; \mathcal{N}(u))$$

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Remark

If

(i) $J : X \to \mathbb{R}$ is continuous

(ii) $S = \{u \in X : \mathcal{K}(u) = 0, C(u) = c_0\}$ is compact

then J attains its maximum (and minimum) on S.

Euler–Lagrange Equations

M. Farazmand and T. Sapsis, A variational method for probing extreme events in turbulent dynamical systems, preprint (2017)

Theorem

Assume

(i) X is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_X$,

(ii) $\mathcal{K}$ is a linear operator.

Then the optimizers satisfy

$$J'(u) + \mathcal{K}^\dagger(\alpha) + \beta C'(u) = 0$$

$$\mathcal{K}(u) = 0$$

$$C(u) = c_0$$

where $\mathrm{d}J(u; v) = \langle J'(u), v \rangle_X$, $\mathrm{d}C(u; v) = \langle C'(u), v \rangle_X$

Lagrange multipliers:

$$\alpha : \Omega \to \mathbb{R}, \quad \beta \in \mathbb{R}$$

Application to N-S

Governing equations:

$$\partial_t u = -u \cdot \nabla u - \nabla p + \nu \Delta u + f$$

$$\nabla \cdot u = 0$$

$$u(\cdot, t_0) = u_0$$

Observable:

$$I(u) = \frac{1}{|\Omega|} \int_\Omega u \cdot f dx$$

Application to N-S

Function space:

$$u \in X = L^2(\Omega)$$

Corresponding functional:

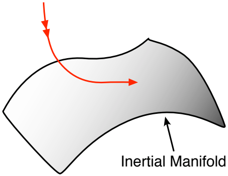
$$\begin{aligned} J(u) &\triangleq \mathrm{d}I(u; \mathcal{N}(u)) \\ &= \int_\Omega \big[u \cdot (u \cdot \nabla f) + \nu\, u \cdot (\Delta f) \big]\, \mathrm{d}x \end{aligned}$$

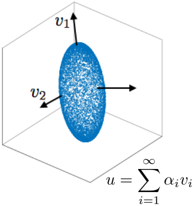
Optimization problem:

$$\sup_{u \in X} J(u)$$

$$\text{subject to } \begin{cases} \nabla \cdot u = 0 \\ C(u) = c_0 \end{cases}$$

Constraint





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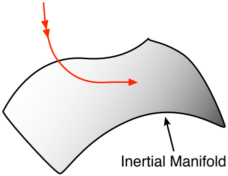

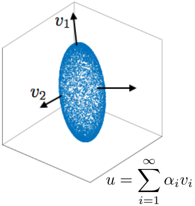

$$C(u) := \frac{1}{|\Omega|} \int_\Omega \frac{|A(u)|^2}{2}\, \mathrm{d}x$$

Energy	$A = \mathrm{id}$	$C(u) = E(u)$
Enstrophy	$A = \nabla$	$C(u) = \frac{1}{2\nu} D(u)$
POD	$A(v_i) = \sigma_i v_i$	$C(u) = \frac{1}{2\|\Omega\|} \sum_{i=1}^{\infty} \sigma_i^2 \alpha_i^2$

Euler–Lagrange Equations for N-S

$$\left(\nabla f + \nabla f^\top\right) u + \nu \Delta f - \nabla \alpha - \beta \Delta u = 0$$

$$\nabla \cdot u = 0$$

$$\frac{1}{|\Omega|} \int_\Omega \frac{|\nabla u|^2}{2} dx = c_0$$

Solve for

$$u : \Omega \to \mathbb{R}^d, \quad \alpha : \Omega \to \mathbb{R}, \quad \beta \in \mathbb{R}$$

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Kolmogorov forcing:

$$f(x, y) = \sin(k_f y) e_1$$

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Exact solutions:

$$u_\pm = \pm \frac{2\sqrt{c_0}}{k_f} \sin(k_f y) e_1, \quad \alpha_\pm = \pm \sqrt{c_0} \int \sin(2k_f y) \mathrm{d}y, \quad \beta_\pm = \pm \frac{\nu k_f}{2\sqrt{c_0}}$$

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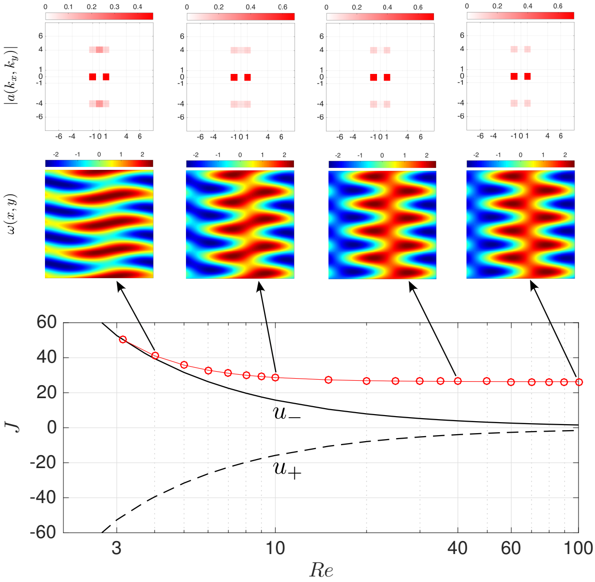
Approximate solutions by Newton–GMRES-hook method

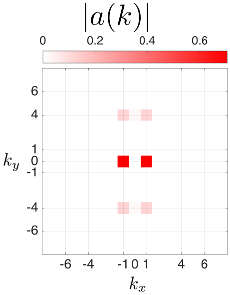
Y. Saad, and M. H. Schultz, SIAM J. Sci. Stat. Comput. (7), 1986, J. E. Dennis and R. B. Schnabel, SIAM 1996

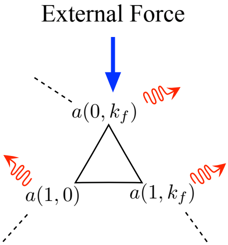
Optimal solutions, $k_f = 4$



M. Farazmand and T. Sapsis, A variational method for probing extreme events in turbulent dynamical systems, preprint

Critical energy transfers

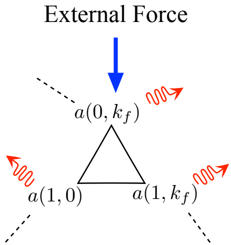




$$u(x) = \sum_k \frac{a(k)}{|k|} \begin{pmatrix} k_y \\ -k_x \end{pmatrix} e^{\hat{i} k \cdot x}, \qquad k = (k_x, k_y)$$

Critical energy transfers

$Re = 40, \quad k_f = 4$



Mean flow energy $= |a(0,k_f,t)|$

$$u(x,t) = \sum_k \frac{a(k,t)}{|k|} \begin{pmatrix} k_y \\ -k_x \end{pmatrix} e^{\hat{i}k\cdot x}, \qquad k = (k_x, k_y)$$

Predicting the bursts

To be predicted

$$\gamma(t) = \max_{\tau \in [t+t_i, t+t_f]} D(\tau)$$

W. Cousins and T. Sapsis, Physica D (280-281), 2014

Predicting the bursts

To be predicted

$$\gamma(t) = \max_{\tau \in [t+t_i, t+t_f]} D(\tau)$$

Predictor

$$\lambda(t) := |a(1,0,t)|$$

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Conditional statistics

$$p_{\gamma|\lambda} = \frac{p_{\gamma,\lambda}}{p_\lambda}$$

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Probability of future extreme energy dissipation

$$P_{ee}(\lambda_0) := \mathcal{P}\left(\gamma(t) > \gamma_e | \lambda(t) = \lambda_0\right) = \int_{\gamma_e}^{\infty} p(\gamma' | \lambda_0) \, \mathrm{d}\gamma'$$

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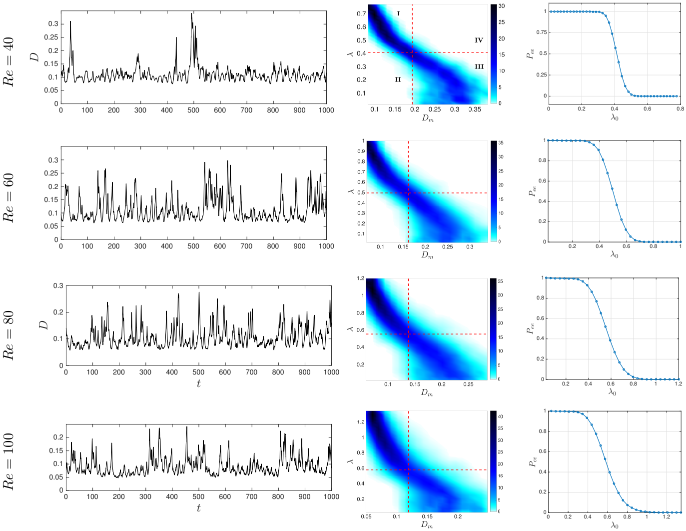
Small values of $\lambda \Rightarrow$ Future extreme event (almost surely)

W. Cousins and T. Sapsis, Physica D (280-281), 2014

Predicting the bursts



M. Farazmand and T. Sapsis, A variational method for probing extreme events in turbulent dynamical systems, preprint

Conclusions

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- Future work:
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M. Farazmand, An adjoint-based approach for finding invariant solutions of Navier–Stokes equations, J. Fluid Mech. (795), 2016

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