Energy dissipation caused by boundary layer instability at vanishing viscosity

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Workshop on Recurrent Flows: the Clockwork behind Turbulence
KITP, UC Santa Barbara, January 17th 2017

1. Revisiting d'Alembert's paradox

Dissipative properties of vortices in wall-bounded 2D turbulent flows

Comparison between Navier-Stokes, Euler and Prandtl solutions

Jean Le Rond d'Alembert
(1717-1783)

Leonhard Euler
(1707-1783)

Prize of mathematics for 1750

On 16 May 1748 Euler, president of the Prussian Academy of Sciences, read the problem he proposed for the Prize of Mathematics to be given in 1750 :

'Deduce from new principles, as simple as possible,
a theory to explain the resistance
exerted on a body moving in a fluid,
as a function of the body's velocity, shape and mass,
and of the fluid's density and compressibility'.

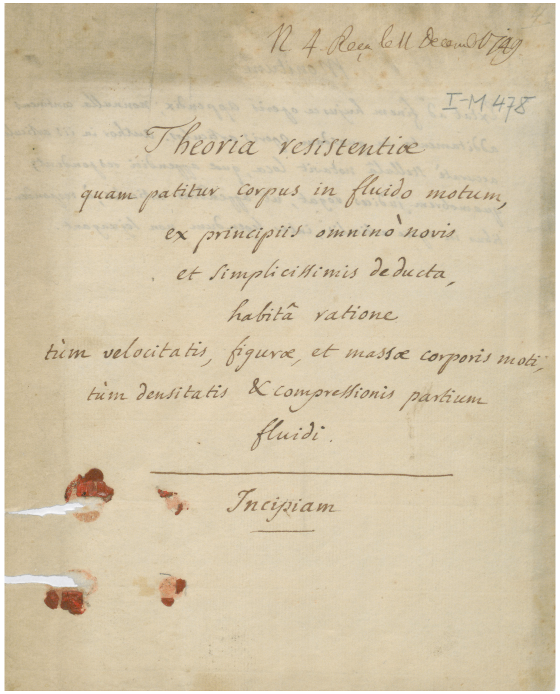
Six mathematicians, including d'Alembert, sent a manuscript, but Euler was not satisfied with them and decided to postpone the prize to 1752.

***Grimberg**, D'Alembert et les équations aux dérivées partielles en hydrodynamique, Thèse de Doctorat, Université de Paris VII, 1998*

D’Alembert’s manuscript

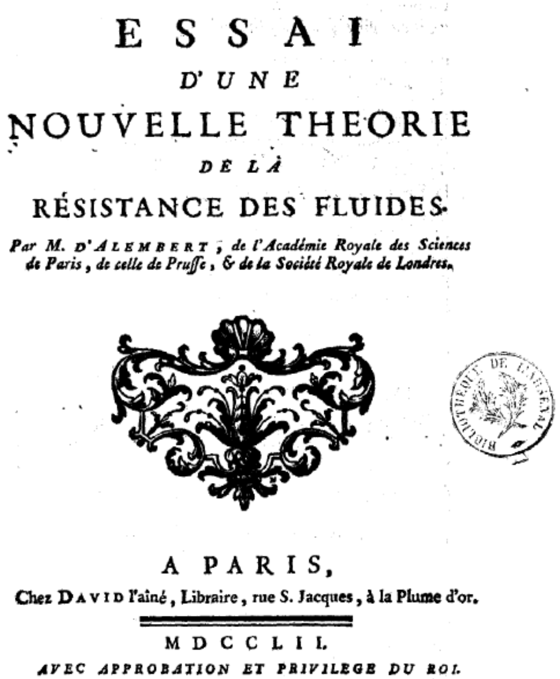
D’Alembert was upset and took back his manuscript of 1749.
He translated it into French and published it in 1752 under the title
’Essai d’une nouvelle théorie de la résistance des fluides’.

1749



Theoria resistentiæ
quam patitur corpus in fluido motum,
ex principiis omnino novis
et simplicissimis deducta,
habitâ ratione
tum velocitatis, figuræ, et massæ corporis moti,
tum densitatis & compressionis partium
fluidi.

Incipiam



ESSAI
D’UNE
NOUVELLE THEORIE
DE LA
RÉSISTANCE DES FLUIDES.
Par M. D’ALEMBERT, de l’Académie Royale des Sciences de Paris, de celle de Prusse, & de la Société Royale de Londres.

A PARIS,
Chez DAVID l’aîné, Libraire, rue S. Jacques, à la Plume d’or.
MDCCLII.
AVEC APPROBATION ET PRIVILEGE DU ROI.

1752

The prize was finally given in 1752 to Jacobo Adami,
a friend of Euler, and published by the Prussian Academy.

D’Alembert’s paradox

Euler, in his treaty *‘New principles of gunnery’* of 1745, had noticed that potential flows exert no drag on moving bodies :

’The boat would be slowed down at the prow as much as it would be pushed at the poop.’

In 1749, while working on the Berlin Academy Prize, d’Alembert was also conscious of this problem and wrote :

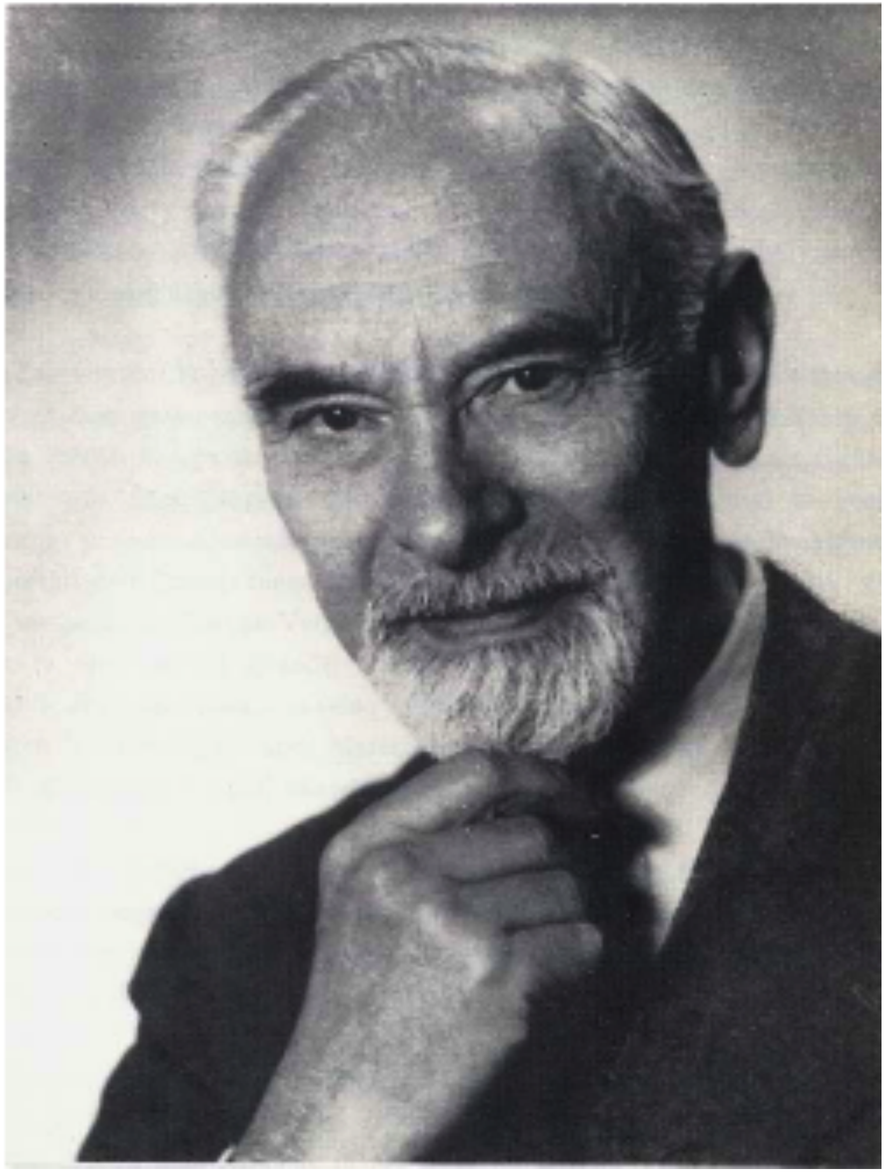
’It seems to me that the theory, developed in all possible rigor, gives, at least in several cases, a strictly vanishing resistance, a singular paradox which I leave to future geometers to elucidate.’

Darrigol, *World of flows: a history of hydrodynamics from Bernoulli to Prandtl, Oxford university Press, 2005*

Adhémar Jean-Claude Barré de Saint-Venant (1797-1886)

George Stokes (1819-1903)

Ludwig Prandtl
(1875-1953)

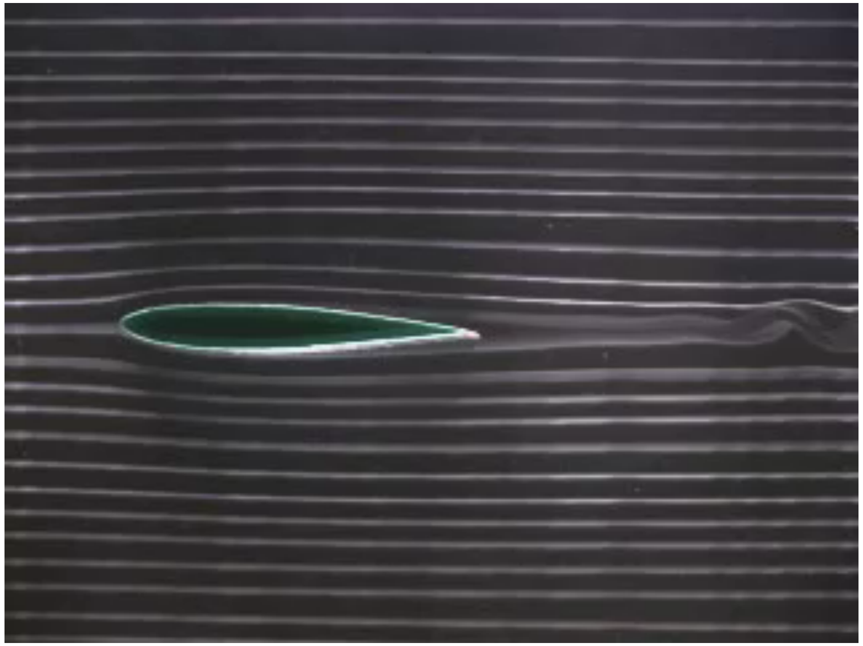


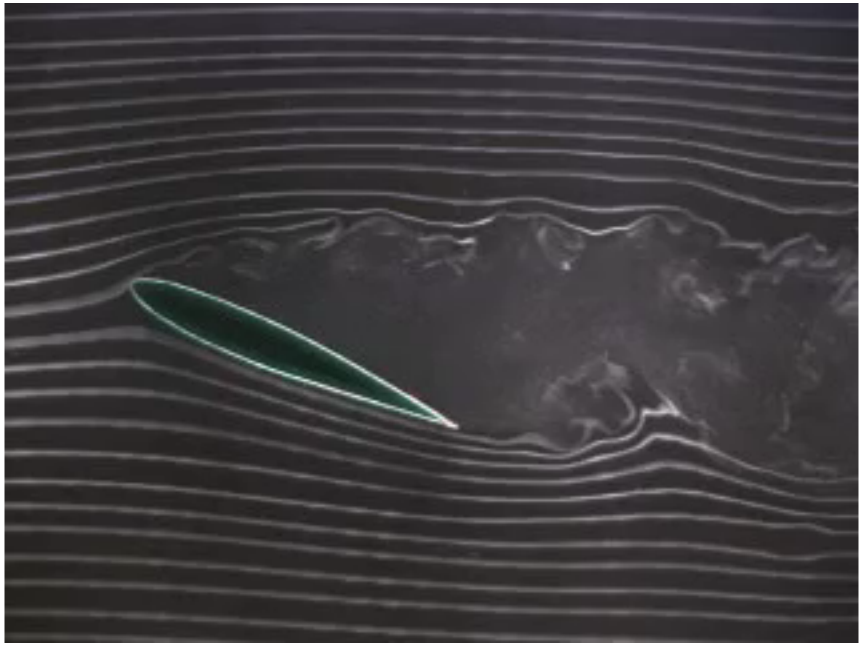
Theodore von Karman
(1881-1963)

Limit of validity for Prandtl's theory

- Prandtl was aware that his approach is only valid if the boundary layer remains attached to the wall, *i.e.*, away from separation points and wakes.
- Separated flow regions and wakes, where boundary layer detaches, should be included in an *adhoc* fashion

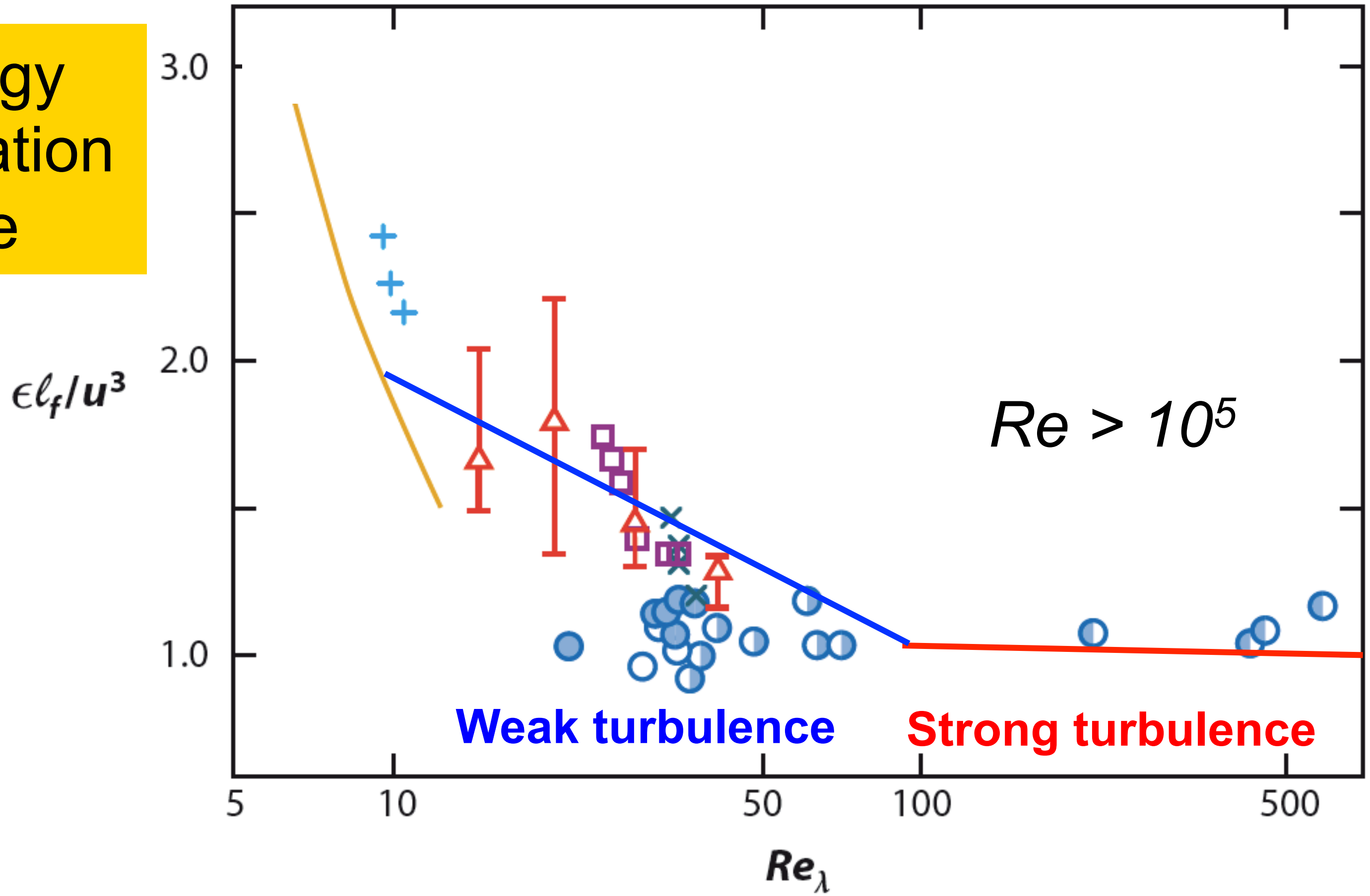
since Prandtl's theory doesn't predict their behavior.





Dissipation rate: laboratory experiments

Energy dissipation rate



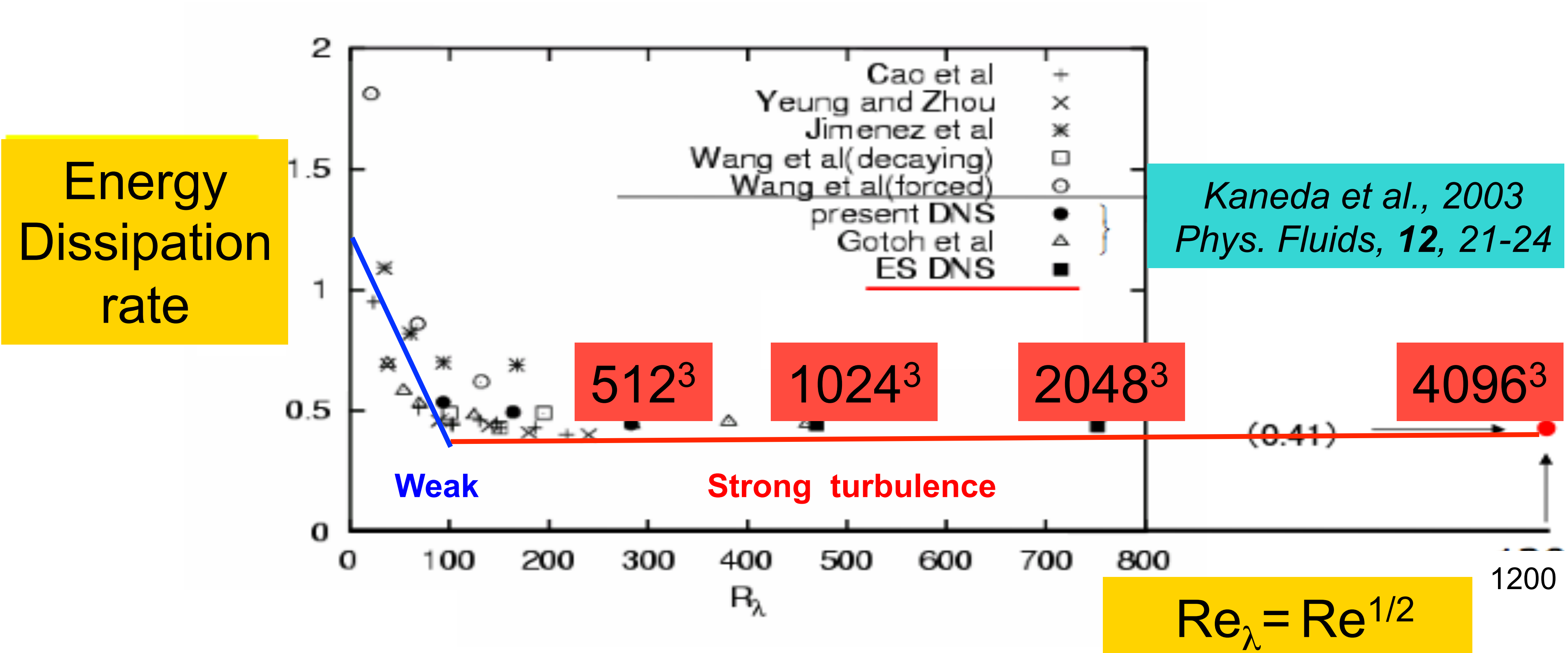
Log ($Re_\lambda = Re^{1/2}$)

Sreenivasan,
Phys. Fluids, ***27****, 1984*

Dissipation rate: numerical experiments

Normalized energy dissipation α → ?

$\alpha = \epsilon L/u'^3$ as $\nu \to 0$, or $Re \to \infty$



Both laboratory experiments and numerical experiments show that the dissipation rate becomes independent of the fluid viscosity for $Re \to \infty$

Dissipation of energy in the inviscid limit

What happens when $\nu \to 0$?

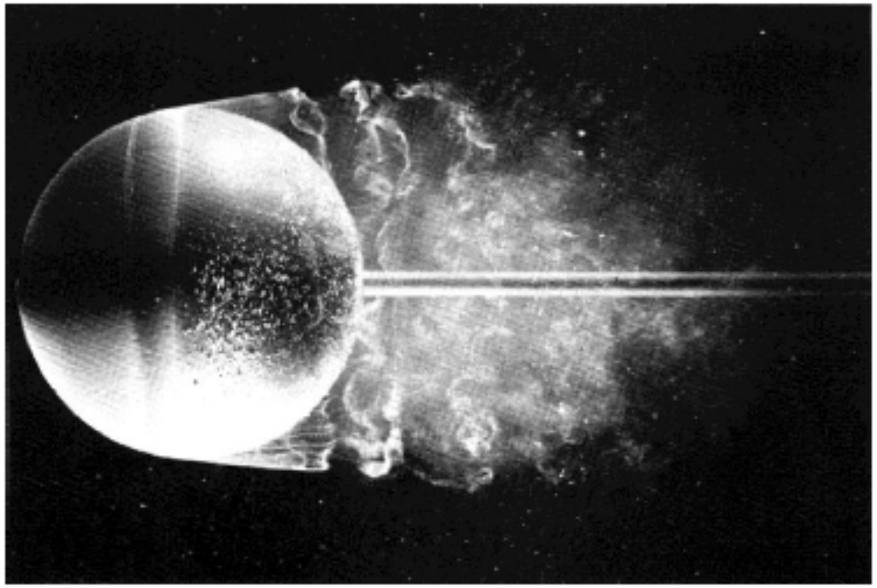
- In an incompressible flow ($\rho = 1$)

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int \frac{\mathbf{u}^2}{2} = -\nu \int \omega^2 = -2\nu Z$$

- To dissipate energy, vorticity needs to be created and/or amplified, in such a way that $Z \sim \nu^{-1}$.

Possible vorticity distributions:
$\omega \sim \nu^{-1/2}$ over $O(1)$ area,
$\omega \sim \nu^{-1}$ over $O(\nu)$ area.

E energy, Z enstrophy,
ν fluid kinematic viscosity
ω flow vorticity.



What is the inviscid limit of Navier-Stokes?

Navier-Stokes equations

with no-slip boundary conditions :

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u} \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{u}_{|\partial\Omega} = \mathbf{0}, \qquad \mathbf{u}(0,\cdot) = \mathbf{v} \end{cases}$$

with Reynolds number Re = UL/ν

Euler equations

with slip boundary conditions :

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{u}_{|\partial\Omega} \cdot \mathbf{n} = \mathbf{0}, \qquad \mathbf{u}(0,\cdot) = \mathbf{v} \end{cases}$$

solution

$\mathbf{u}_{\mathrm{Re}}(t,\mathbf{x})$ $\xrightarrow[\mathrm{Re} \to +\infty]{\nu \to 0}$?

?

solution

$\mathbf{u}(t,\mathbf{x})$ $\quad \nu = 0$, $\mathrm{Re} = +\infty$

1984: Kato's theorem

The Navier-Stokes solution converge towards the Euler solution, if and only if, the energy dissipation vanishes

$$\Delta E_{\mathrm{Re}}(0,T) = \mathrm{Re}^{-1} \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}\mathbf{x} \left| \nabla \mathbf{u}(t,\mathbf{x}) \right|^2 \underset{\mathrm{Re}\to\infty}{\longrightarrow} 0, \quad \nu \to 0$$

and, if and only if, this happens in a boundary layer of thickness proportional to Re^{-1}

Kato, 1984, Remarks on zero viscosity limit for non stationary Navier-Stokes flows with boundary, MSRI Berkeley

~~$\delta x \propto \mathrm{Re}^{-\frac{1}{2}}$~~

$$\delta x \propto \mathrm{Re}^{-1}$$

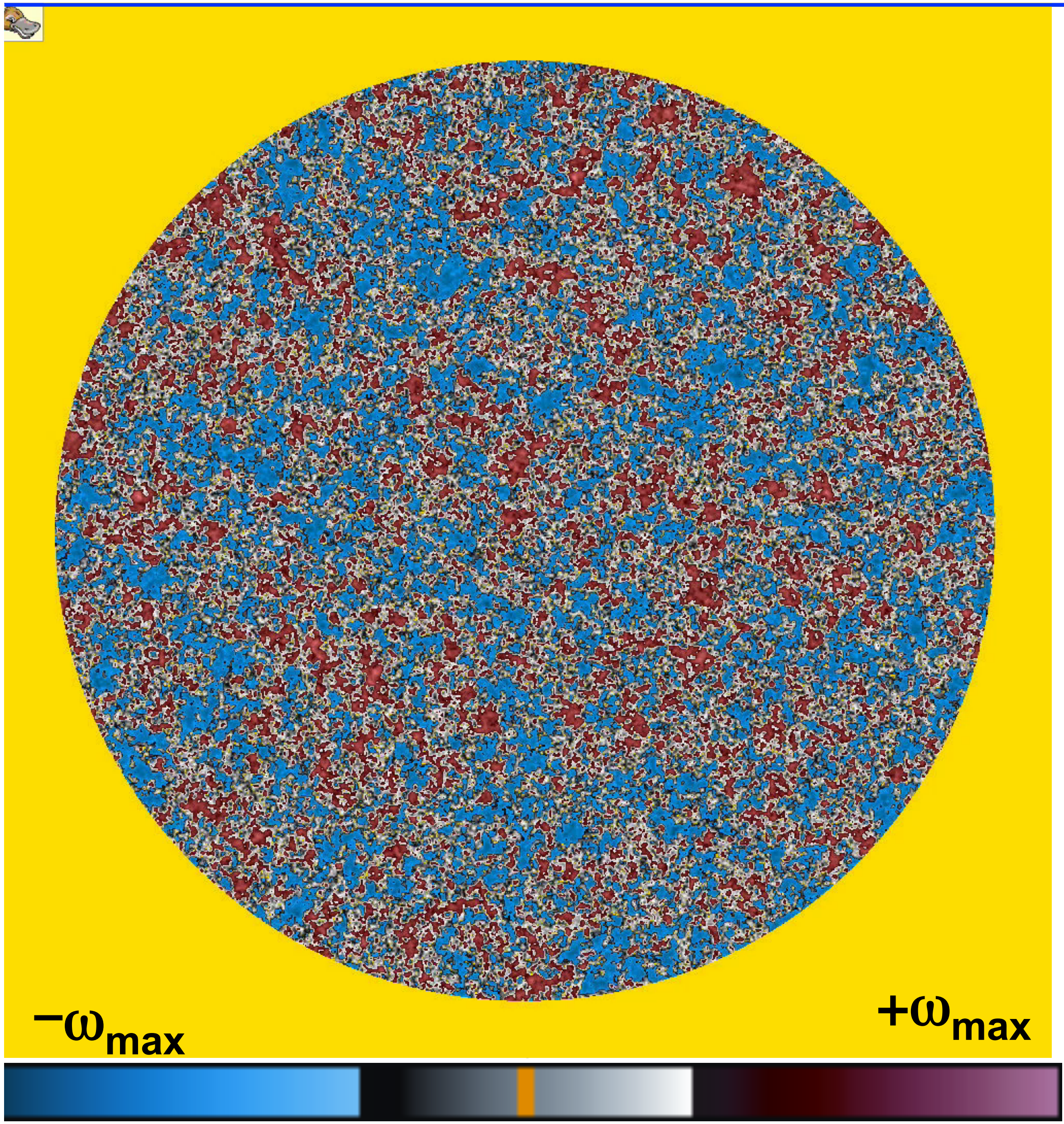
This requires using smaller resolution to compute very high Reynolds flows

Revisiting d'Alembert's paradox

2. Dissipative properties of vortices in wall-bounded 2D turbulent flows

Comparison between Navier-Stokes, Euler and Prandtl solutions

Wall-bounded 2D turbulent flow



DNS
Resolution
$N=1024^2$

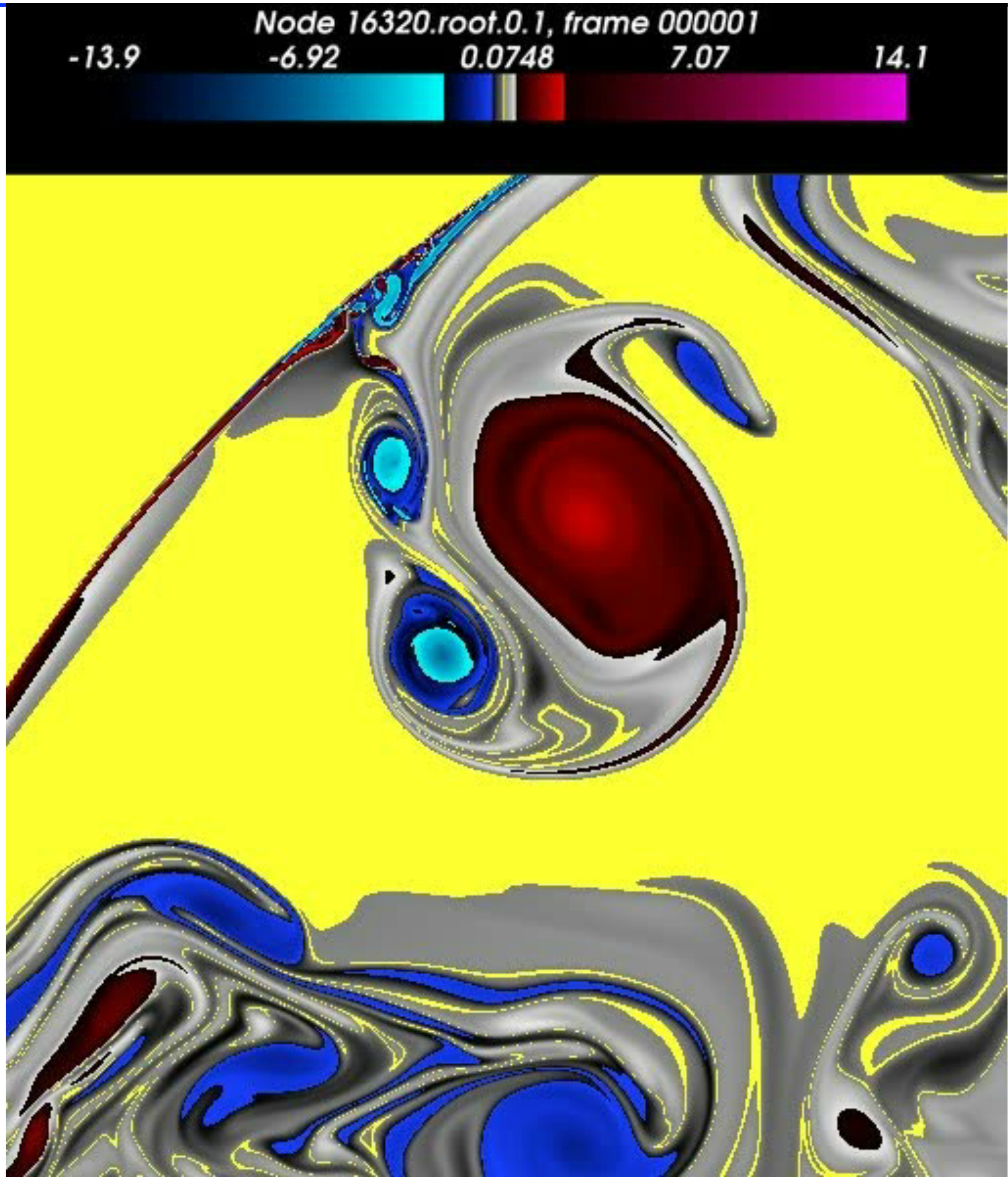
Random initial conditions

Pseudo-spectral method with volume penalization

*K. Schneider and M. F., Phys. Rev. Lett., **95**, 244502 (2005)*

DNS of 2D flow in a cylindrical domain

Resolution
N=8192^2



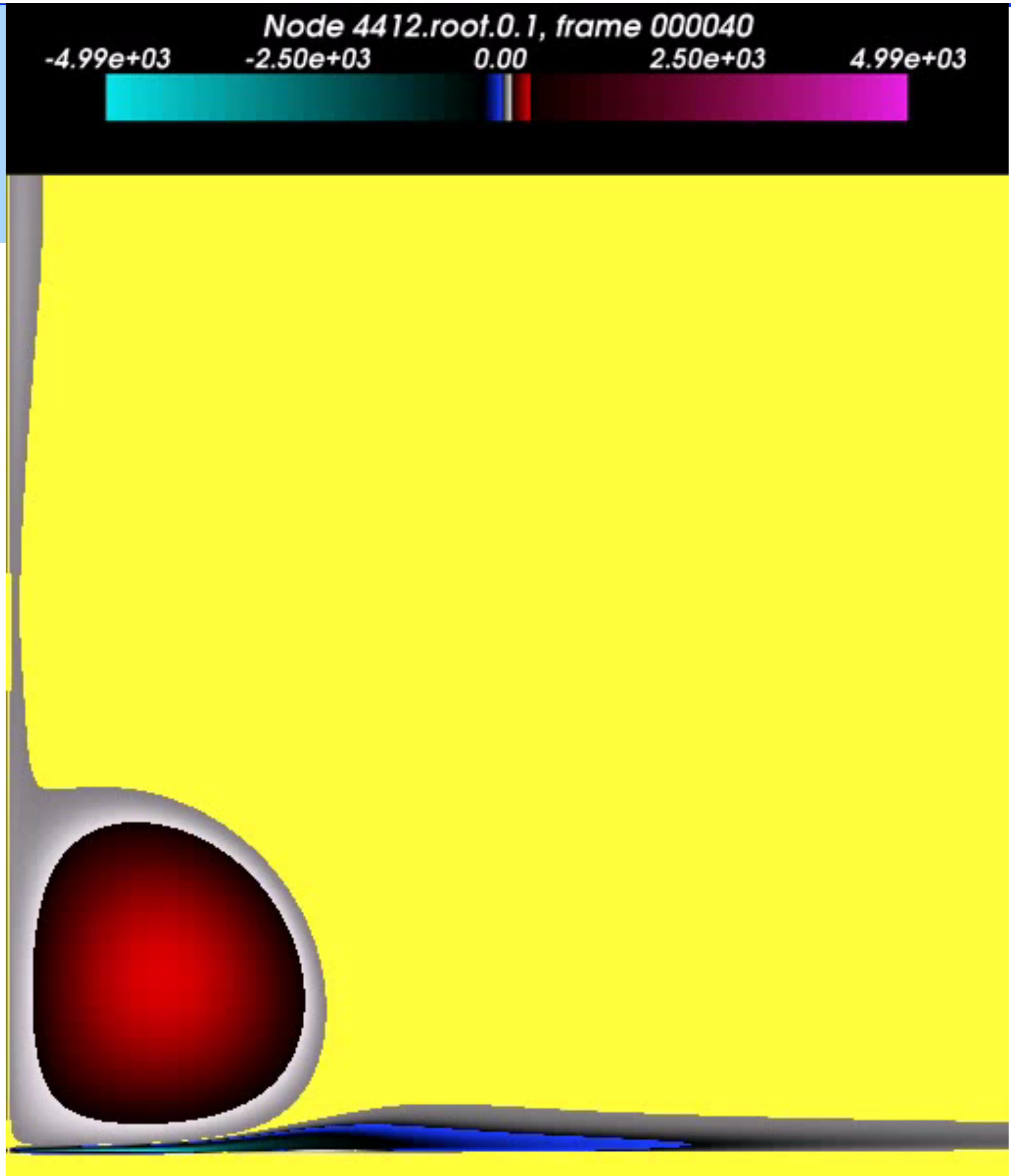
Time evolution of vorticity at the wall computed on IBM Blue-Gene, IDRIS, 2010

(100 Tflops)

Nguyen van yen, M. F. and Schneider, 2010

Vortex dipole onto a wall at Re=8000

DNS
Resolution
$N=8192^2$

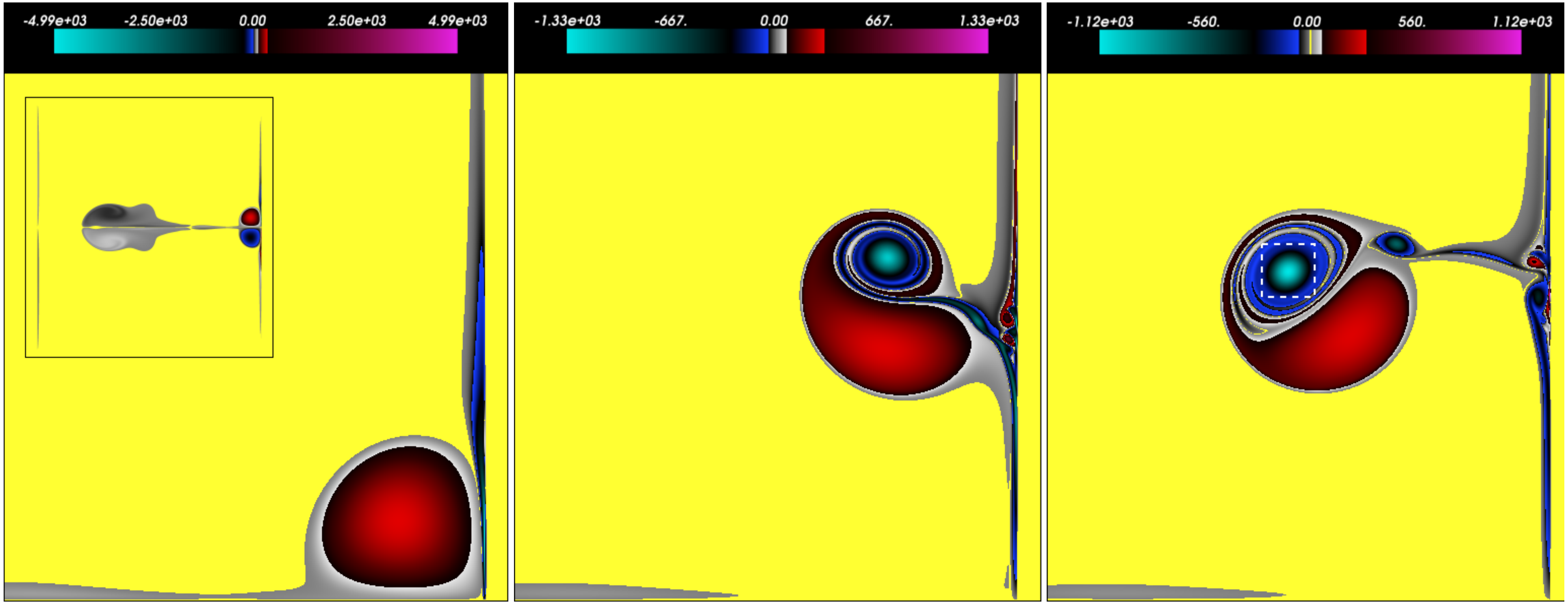


*Nguyen van yen, M. F. and Schneider, 2011 PRL, **106**(18)*

Vortex dipole onto a wall at Re=16000

Resolution
$N=16384^2$

*Nguyen van yen, M. F. and Schneider, 2011, PRL, **106**(18)*

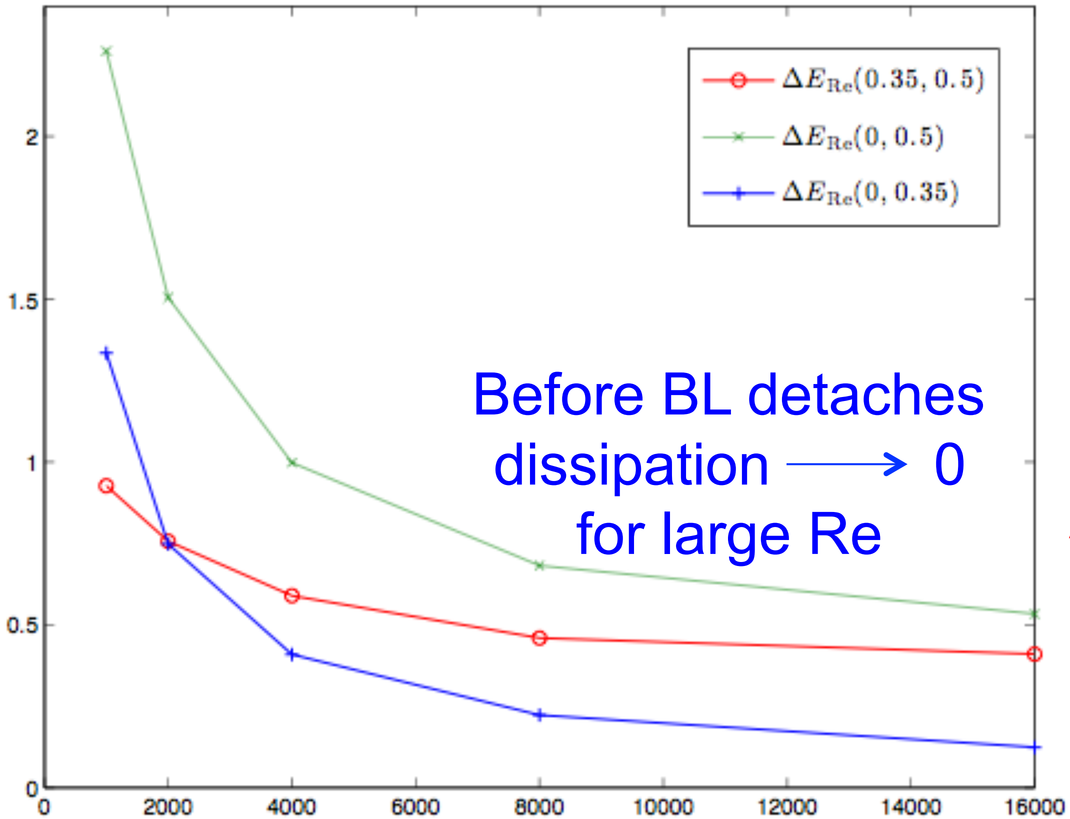


t=0.3 *t=0.4* *t=0.5*

Energy dissipation

Energy dissipated before boundary layer detaches
and after, for increasing Reynolds numbers

ΔE



Before BL detaches
dissipation ⟶ 0
for large Re

After BL detaches
dissipation ⟶ C
for large Re

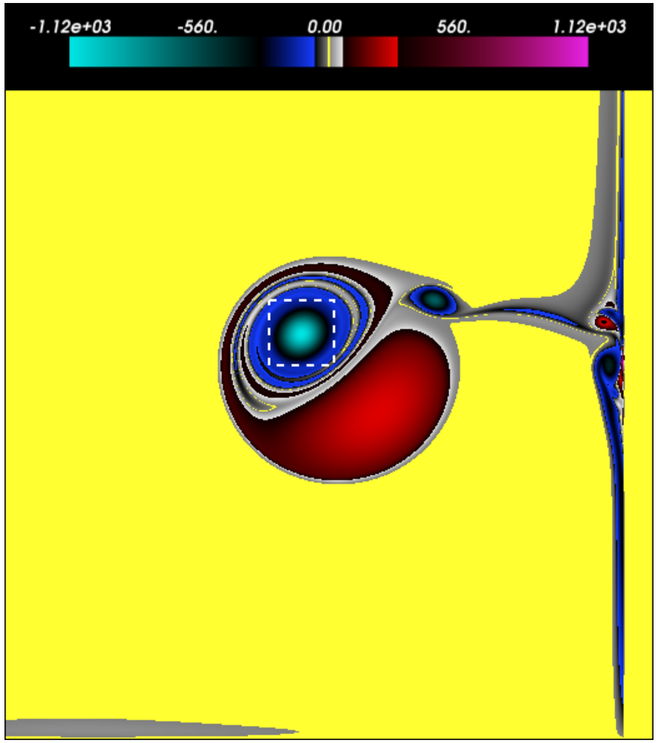
Re

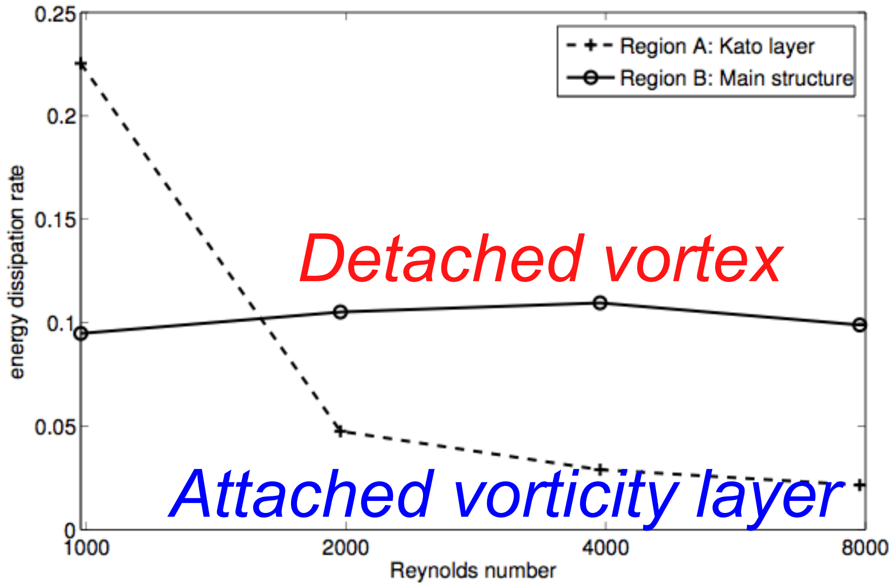
Re=1000

Re=16000

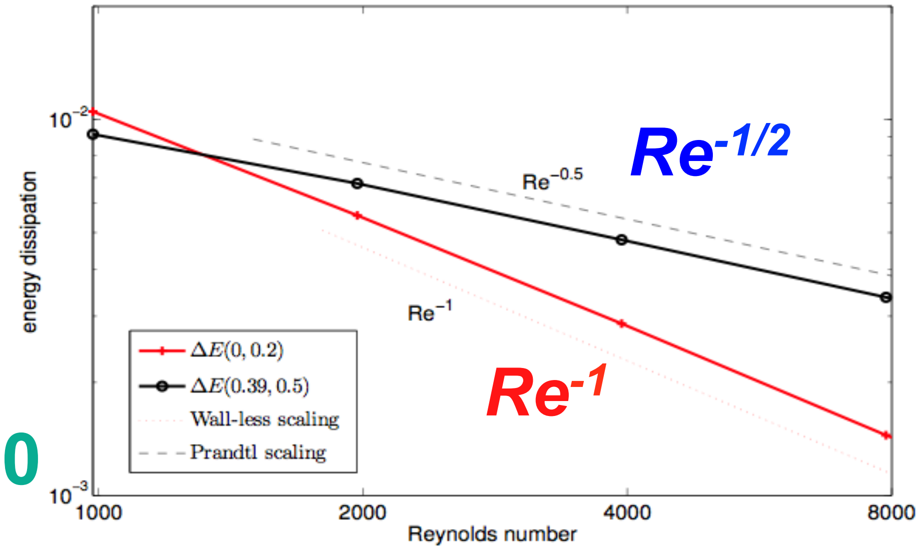
Dissipative structures

*Nguyen van yen, M. F. and Schneider, 2011, PRL, **106**(18)*





Energy dissipation rate



Energy dissipation

Re=1000

Re=8000

Revisiting d'Alembert's paradox

Dissipative properties of vortices in wall-bounded 2D turbulent flows

3. Comparison between Navier-Stokes, Euler and Prandtl solutions

Part II: Energy dissipation caused by boundary layer instability at vanishing viscosity

Comparison between Navier-Stokes Euler and Prandtl solutions

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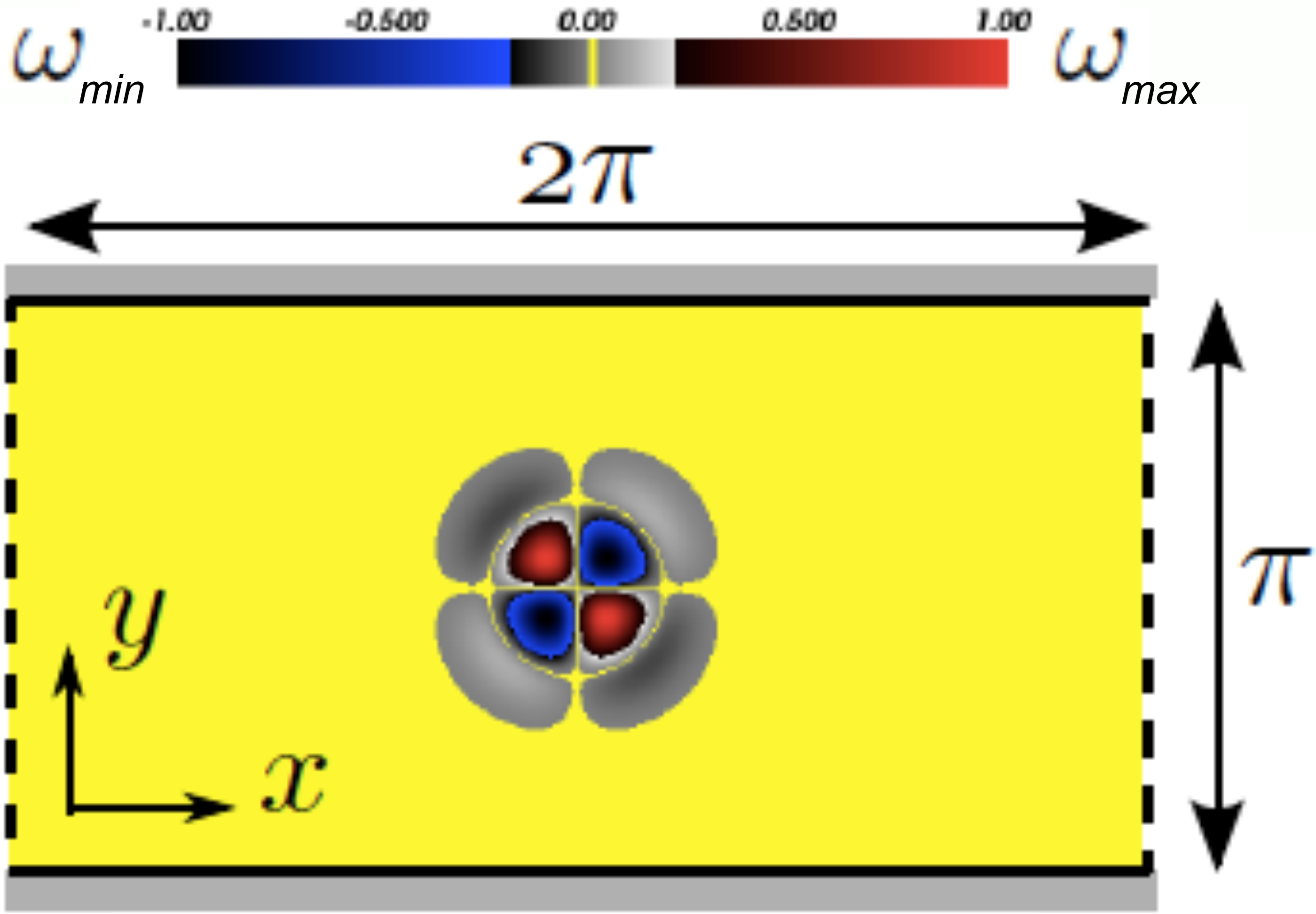
Initial flow

Vorticity field (quadrupole)



$$\psi_i(x,y) = Axy \exp\left(-\frac{(x-x_0)^2+(y-y_0)^2}{2s^2}\right)$$

2d Euler equations

$$\partial_t \omega_E + \nabla \cdot (\mathbf{u}_E \omega_E) = 0$$

$$\nabla \cdot \mathbf{u}_E = 0$$

$$\nabla \times \mathbf{u}_E = \omega_E$$

with the streamfunction $\Delta \psi_E = \omega_E$

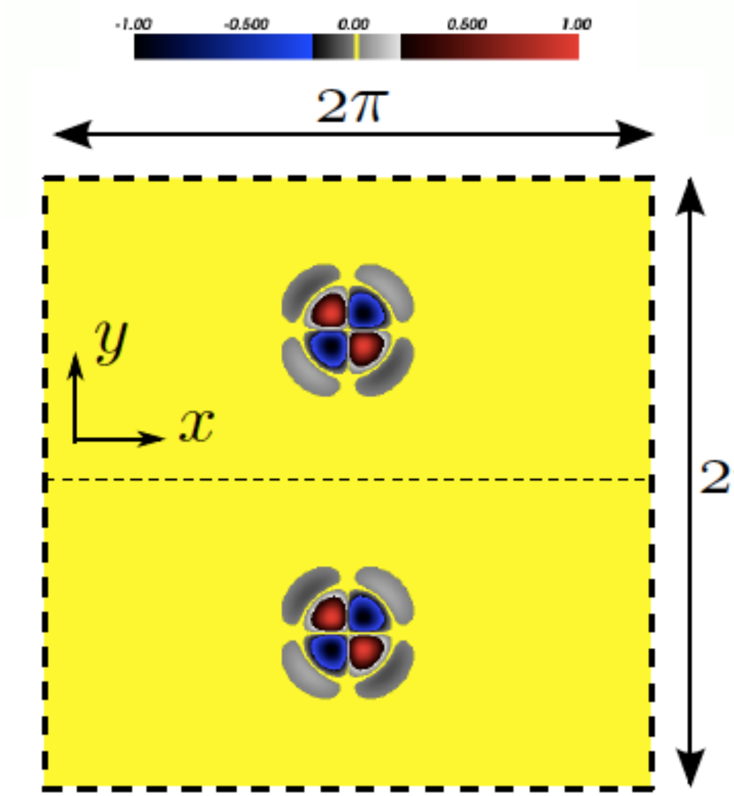
and Dirichlet boundary conditions

$$\psi_E(x, 0, t) = \psi_E(x, \pi, t) = 0$$

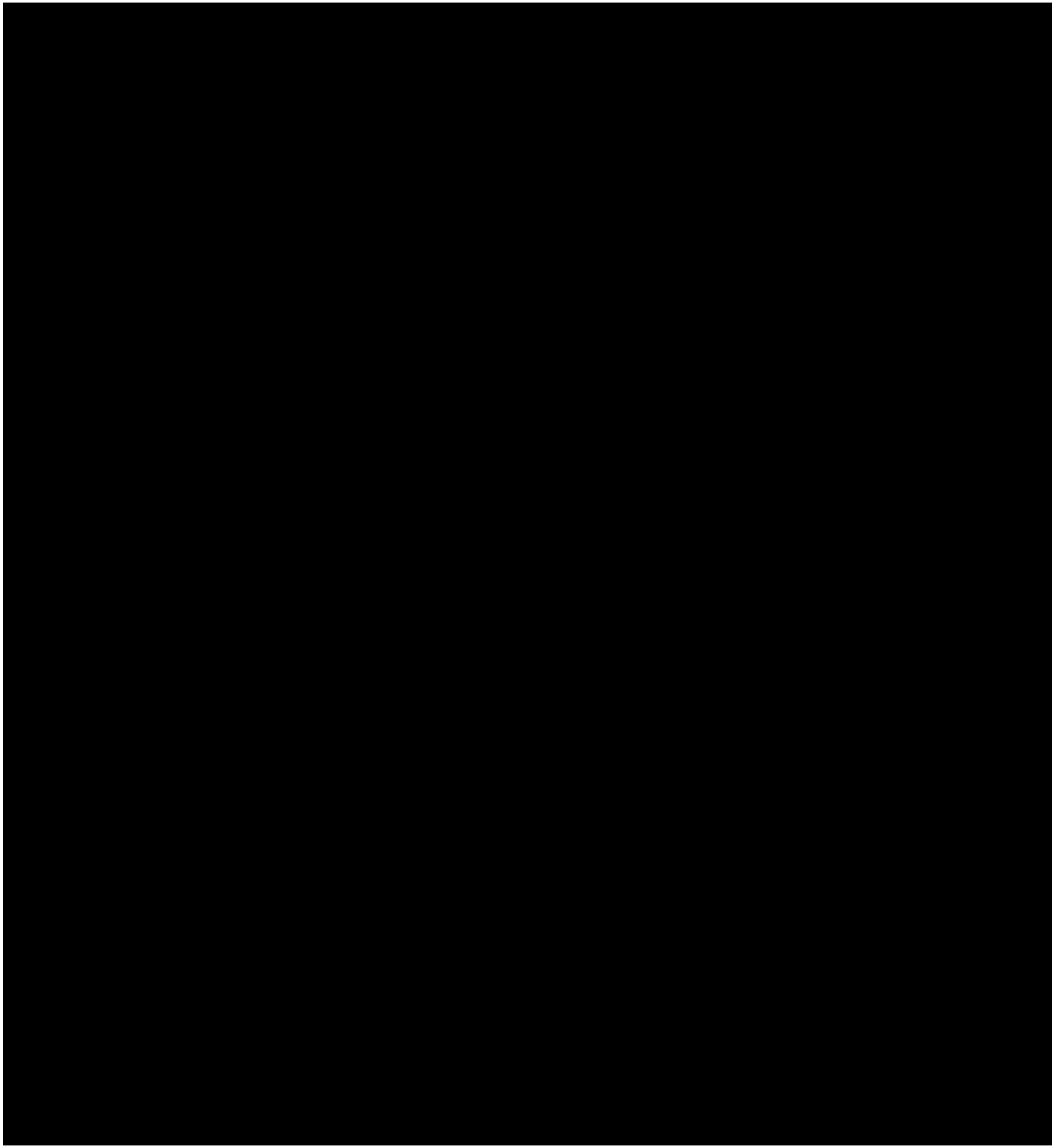
plus initial conditions

$$\omega_E(x, y, t = 0) = \omega_E^i(x, y)$$

Euler solver



- Use mirror symmetry around $y = 0$ to impose boundary condition.
- Spatial discretization: Fourier pseudo-spectral with hyperdissipation, $k_{max} = 682$.
- Time discretization: third order low storage Runge-Kutta, with exponential propagator for the viscous term.



insect_rigid_turb.avi

Prandtl equations

$$\partial_t \omega_P + \boldsymbol{\nabla}\cdot(\mathbf{u}_P \omega_P) = \partial^2_{y_P} \omega_P$$

$$\omega_P(x, y_P, 0) = 0$$

$$\psi_P(x, y_P, t) = \int_0^{y_P} \mathrm{d}y'_P \int_0^{y'_P} \mathrm{d}y''_P \omega_P(x, y''_P, t)$$

$$\partial_{y_P} \omega_P(x, 0, t) = -\partial_x p_E(x, 0, t),$$

where p_E is the pressure calculated from ω_E

Prandtl solver

- Artifical boundary condition $\partial_{y_P}\omega_P = 0$ at $y_P = 64$
- Spatial discretization: Fourier in x, compact FD, 5th order in y
- Time discretization: low storage third order Runge-Kutta
- Neumann boundary condition

$$\partial_{y_P}\omega_P = -\partial_x p_E \quad \text{at} \quad y_P = 0$$

where p_E is the pressure calculated from ω_E

- To close the system we impose

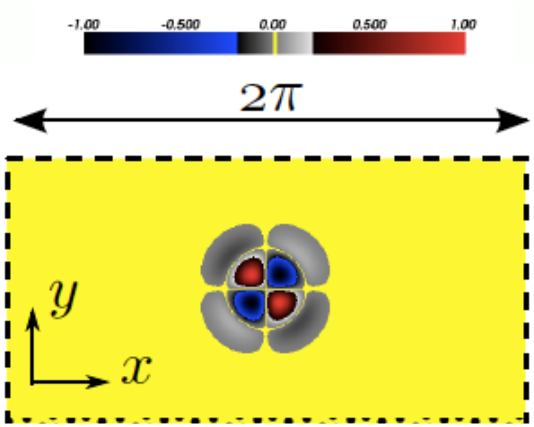
$$\partial^2_{y_P}\omega_P = 0 \quad \text{at} \quad y_P = 64$$

which is consistent with the rapid decay of ω_P

Navier-Stokes solver

1. **Fourier/compact finite differences (5th order)**

   Similar to the one for the Prandtl equations, except that non-uniform grids are used in the *y* direction. Two linear integral constraints are applied on ω to satisfy the no-slip boundary conditions in *y*. Integrating factor for the viscous term and 3rd order Runge-Kutta for the advection term.



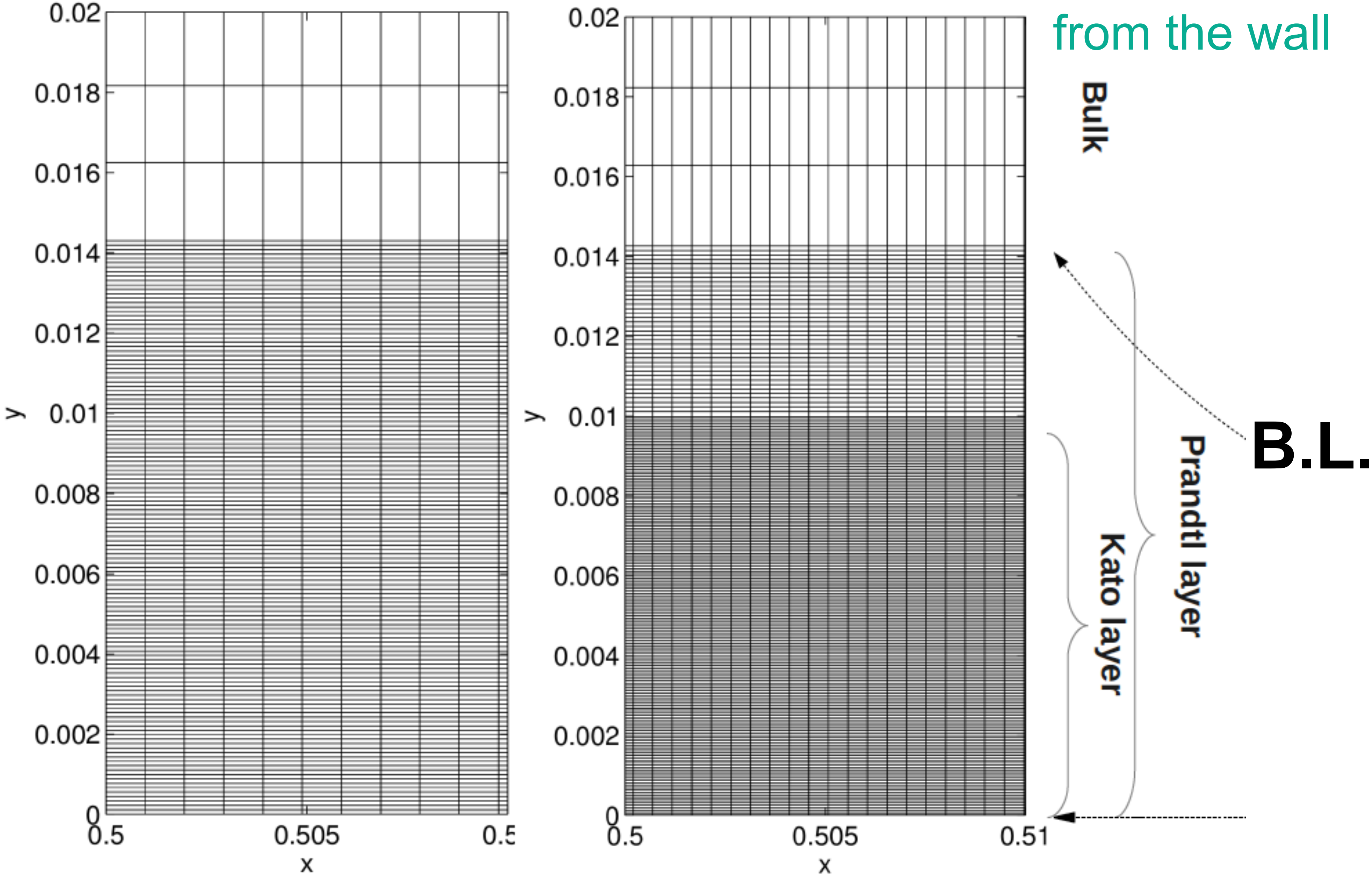
Resolution:

N_x = 1024

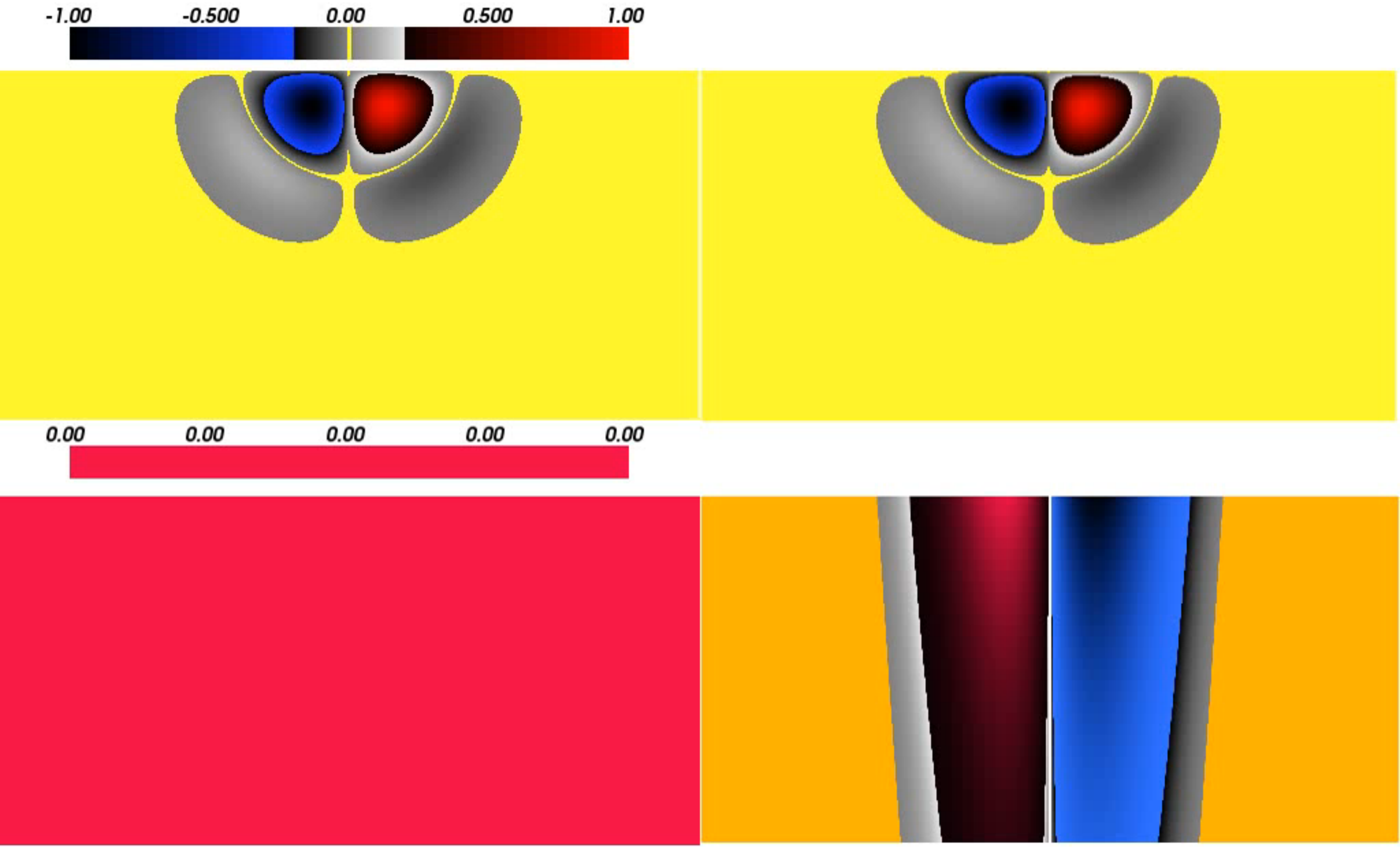
N_y = 385 - 449

Computational grid

Before t=54 After t=54 when B.L. detaches from the wall



Euler +Prandtl   Navier-Stokes



Prandtl's singularity

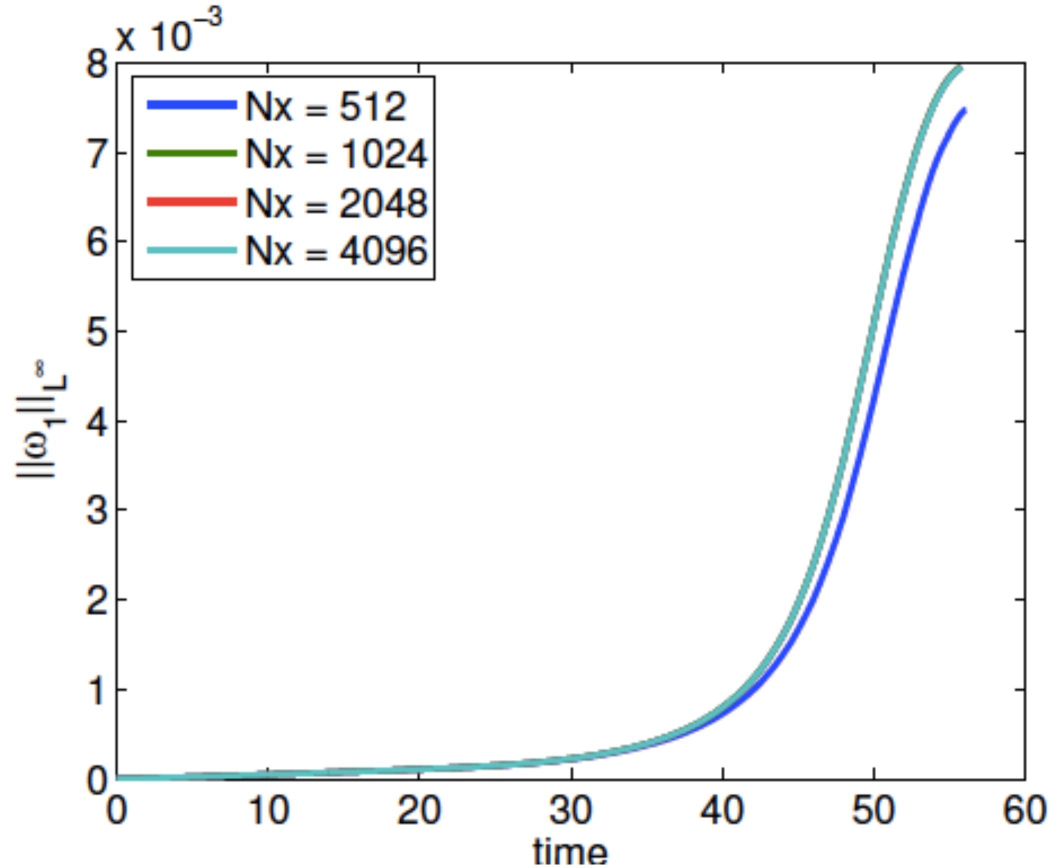
Prandtl equation has well-known finite time singularity

- $|\partial_x \omega_1|$ and $u_{1,y}$ blows up,
- ω_1 remains bounded.

L. L. van Dommelen and S. F. Shen., 1980
J. Comp. Phys., ***38****(2)*

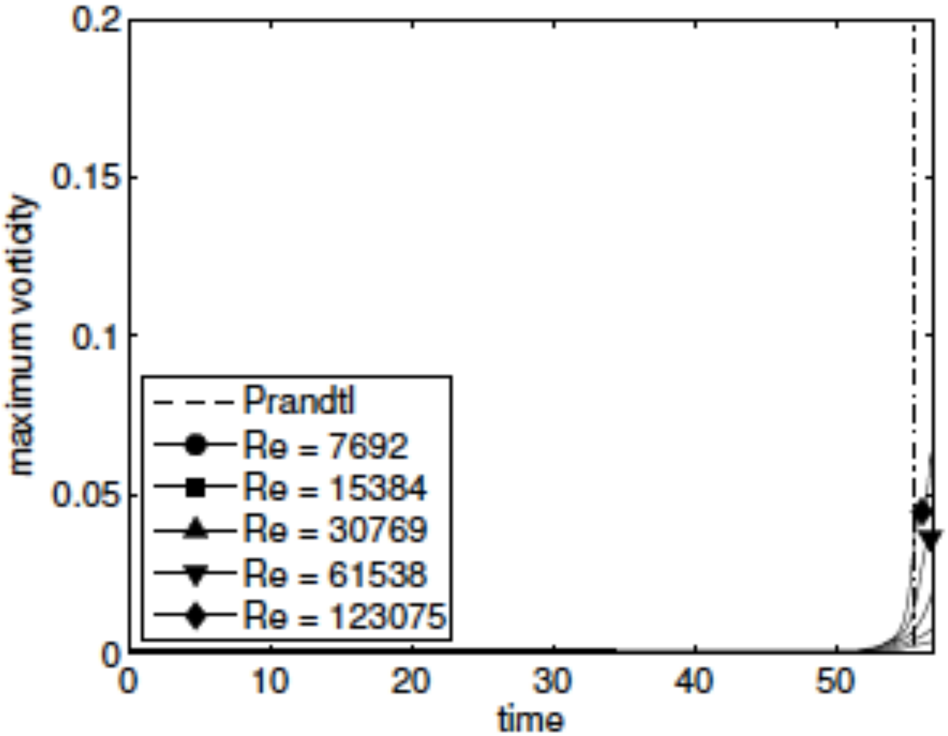
This is observed in our computations as expected,

for $t \to t_D \simeq 55.8$

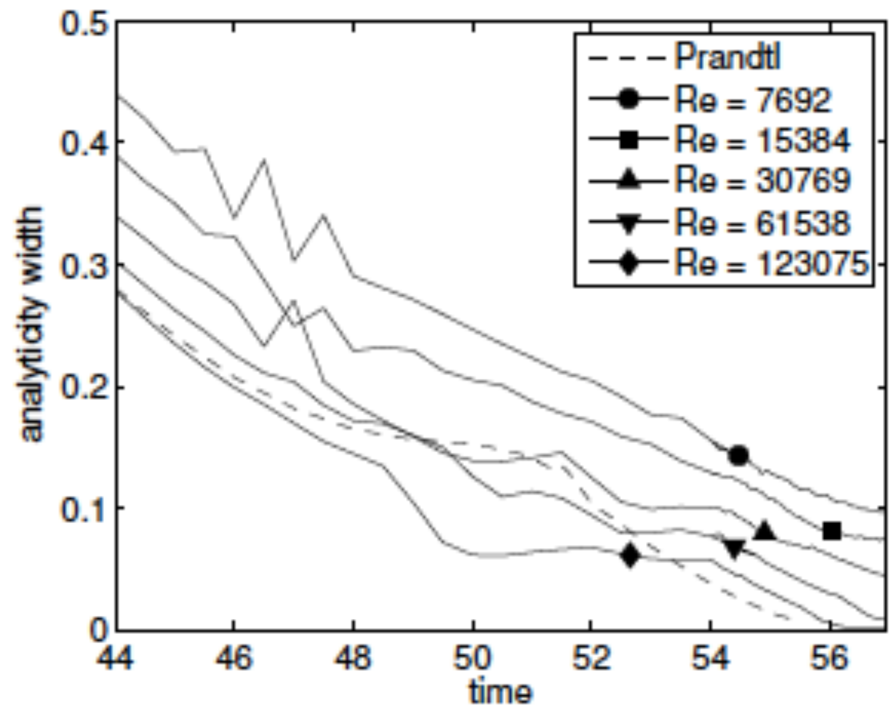


Prandtl solution's blow-up

According to Kato's theorem, and since ω_1 remains bounded uniformly until t_D, we expect that $\mathbf{u}_\nu \xrightarrow[\nu \to 0]{L^2} \mathbf{u}_0$ uniformly on $[0, t_D]$.



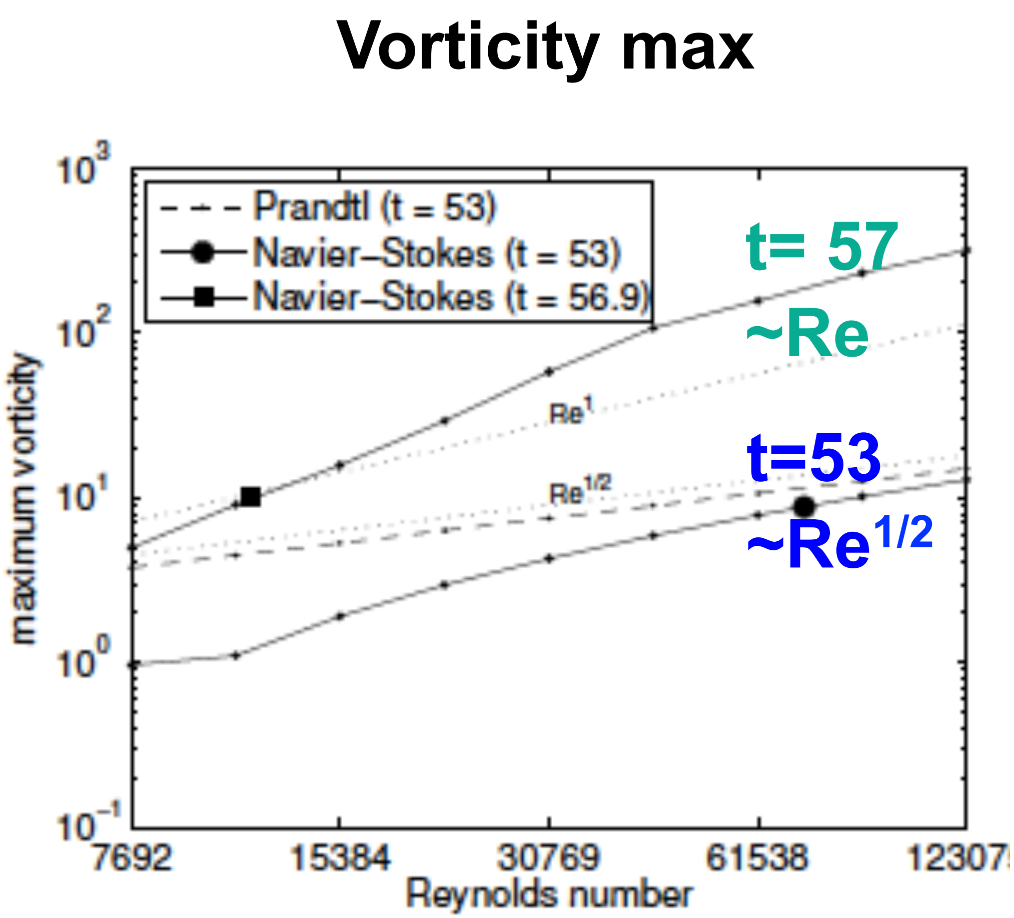
Evolution of vorticity max

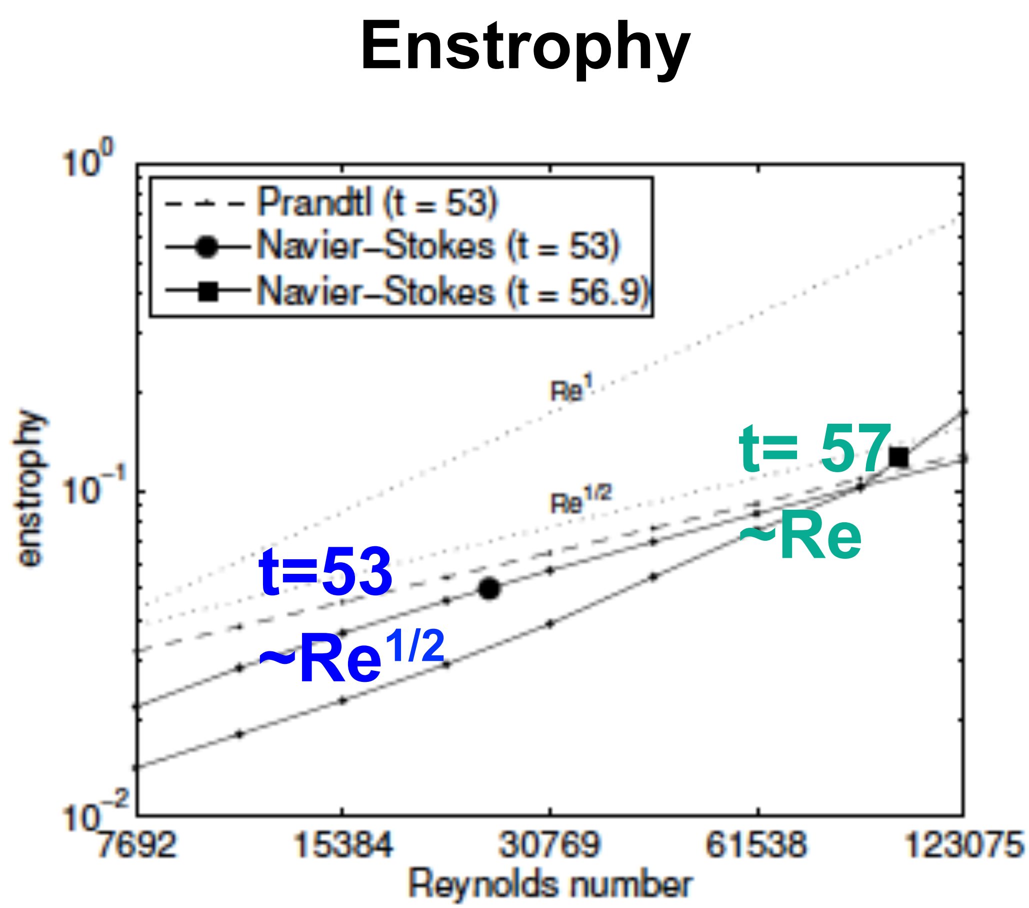


Evolution of analyticity strip

Show convergence!

What happens after the singularity?

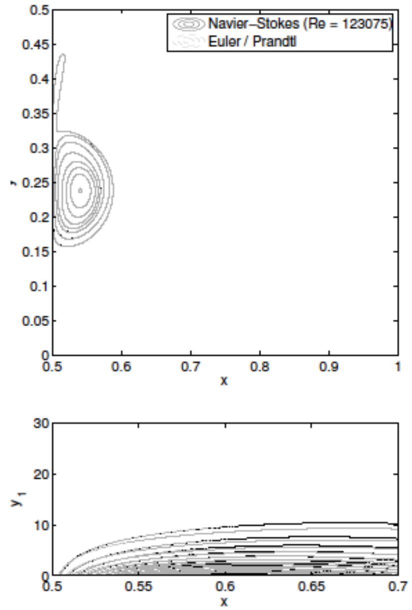




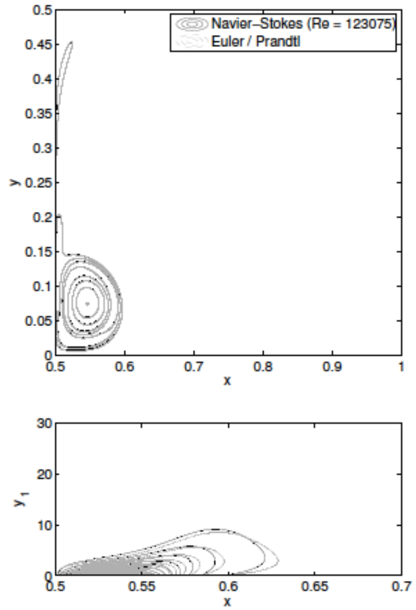
We observe Prandtl's scaling in $Re^{1/2}$ before $t_D \sim 55.8$ and Kato's scaling in Re after

Comparison Prandtl and Navier-Stokes

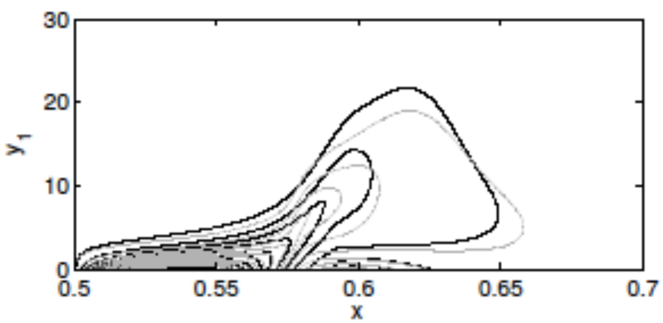
t=30



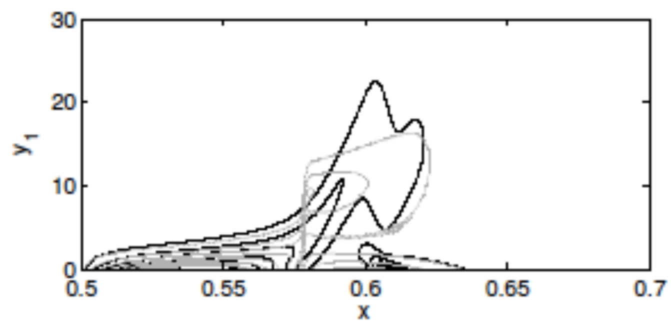
t=50



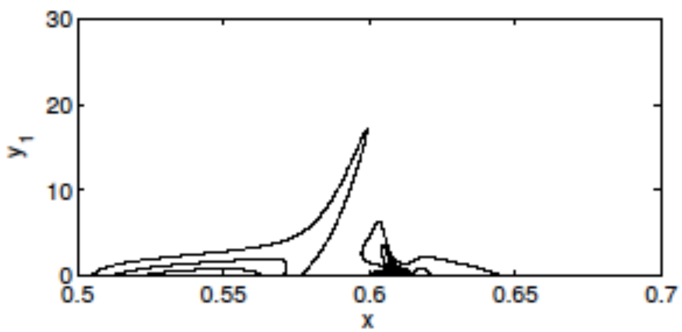
t=54



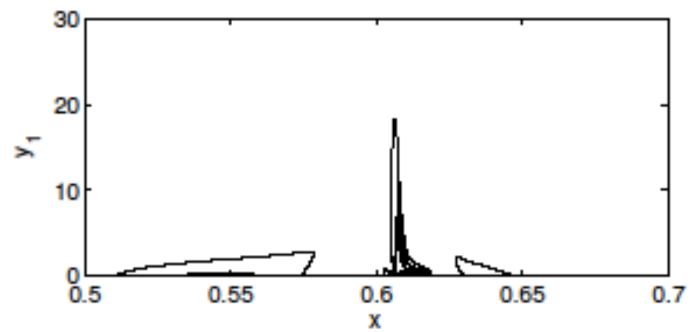
t=55.3



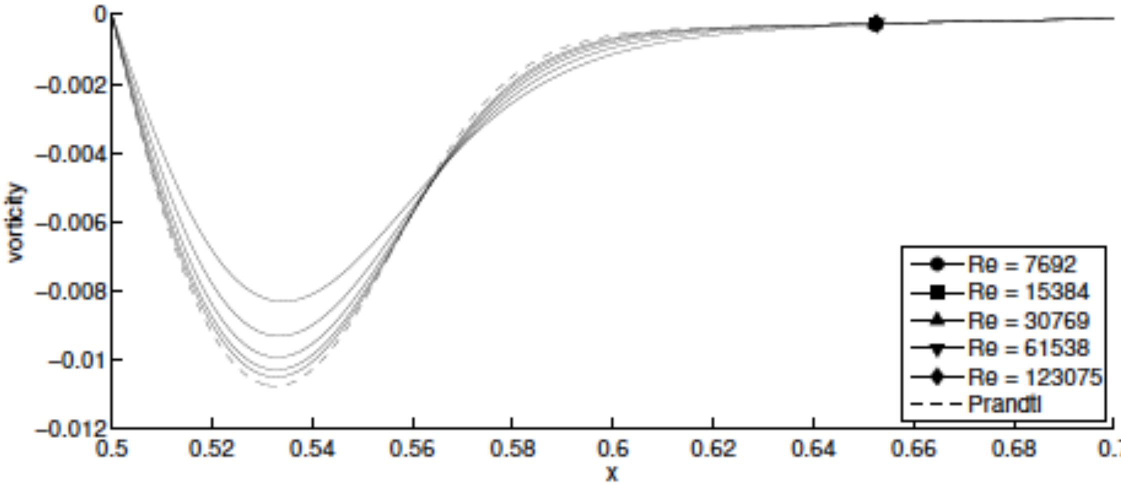
t=56



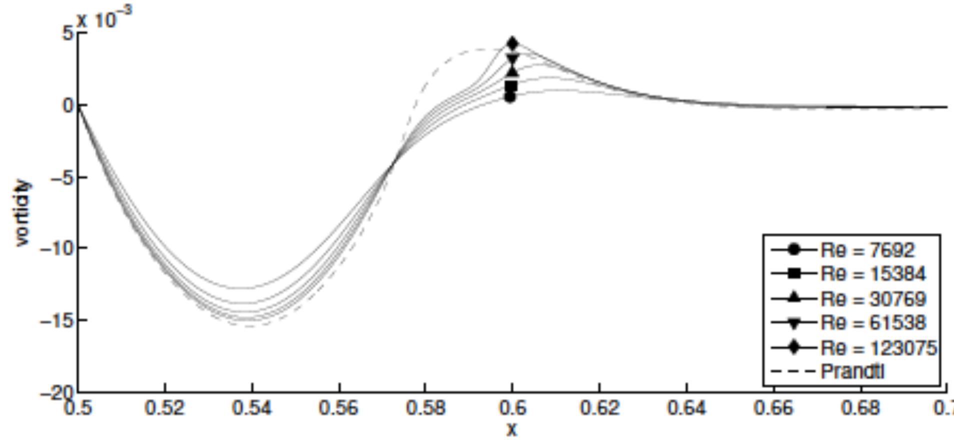
t=56.9



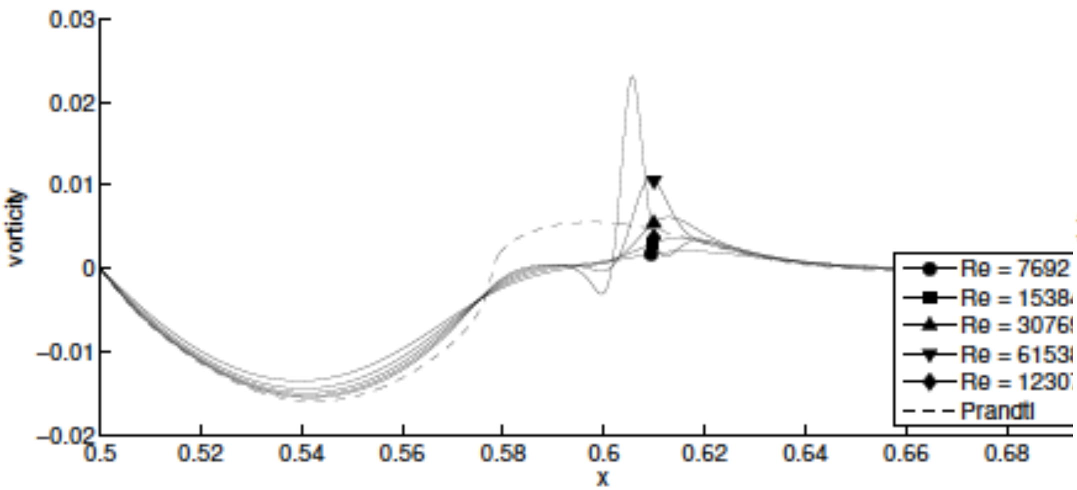
t=50



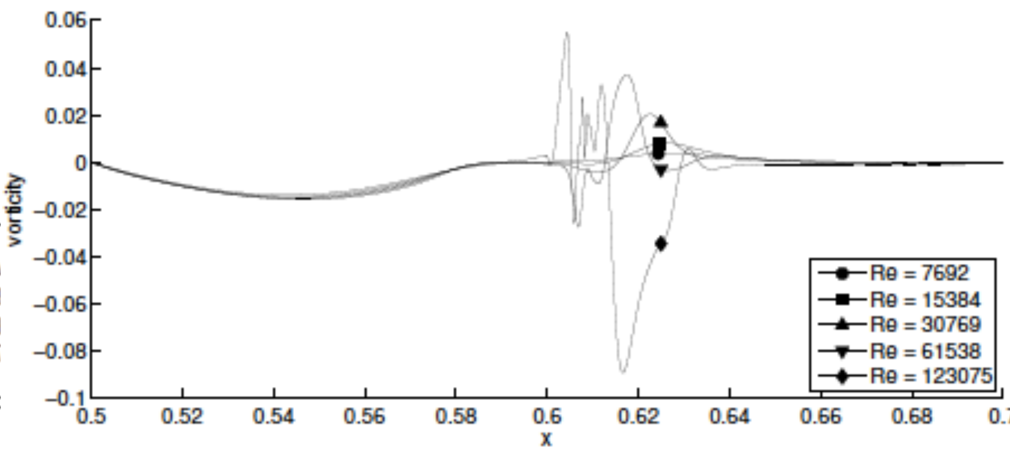
t=54



t=55.3



t=57.5



Conclusion

The production of dissipative structures
is the key feature of boundary layer (BL) detachment
at vanishing viscosity limit of incompressible flows.

The viscous Prandtl solution becomes singular at t_D.

The viscous Navier-Stokes solution converges uniformly
to the inviscid Euler solution for $t<t_D$,
and ceases to converge for $t>t_D$.

The detachment process involves spatial scales
in different directions, and not only parallel to the wall,
that are as fine as Re^{-1}.

Conclusion

The Navier-Stokes boundary layer detachment dynamics are very different from the dynamics of the finite time singularity developing in Prandtl's equation with:

- non locality in the parallel direction,
- formation of small scales scaling at least as Re^{-1}, in different directions and not only in the direction parallel to the wall,
- pressure plays an essential role in the detachment process.

*R. Nguyen van yen, M. Farge and K. Schneider, 2011 Phys. Rev. Lett., **106**(18), 184502*

R. Nguyen van yen, M. Waidman, R. Klein, M.Farge and K. Schneider, 2016 Preprint

Open questions

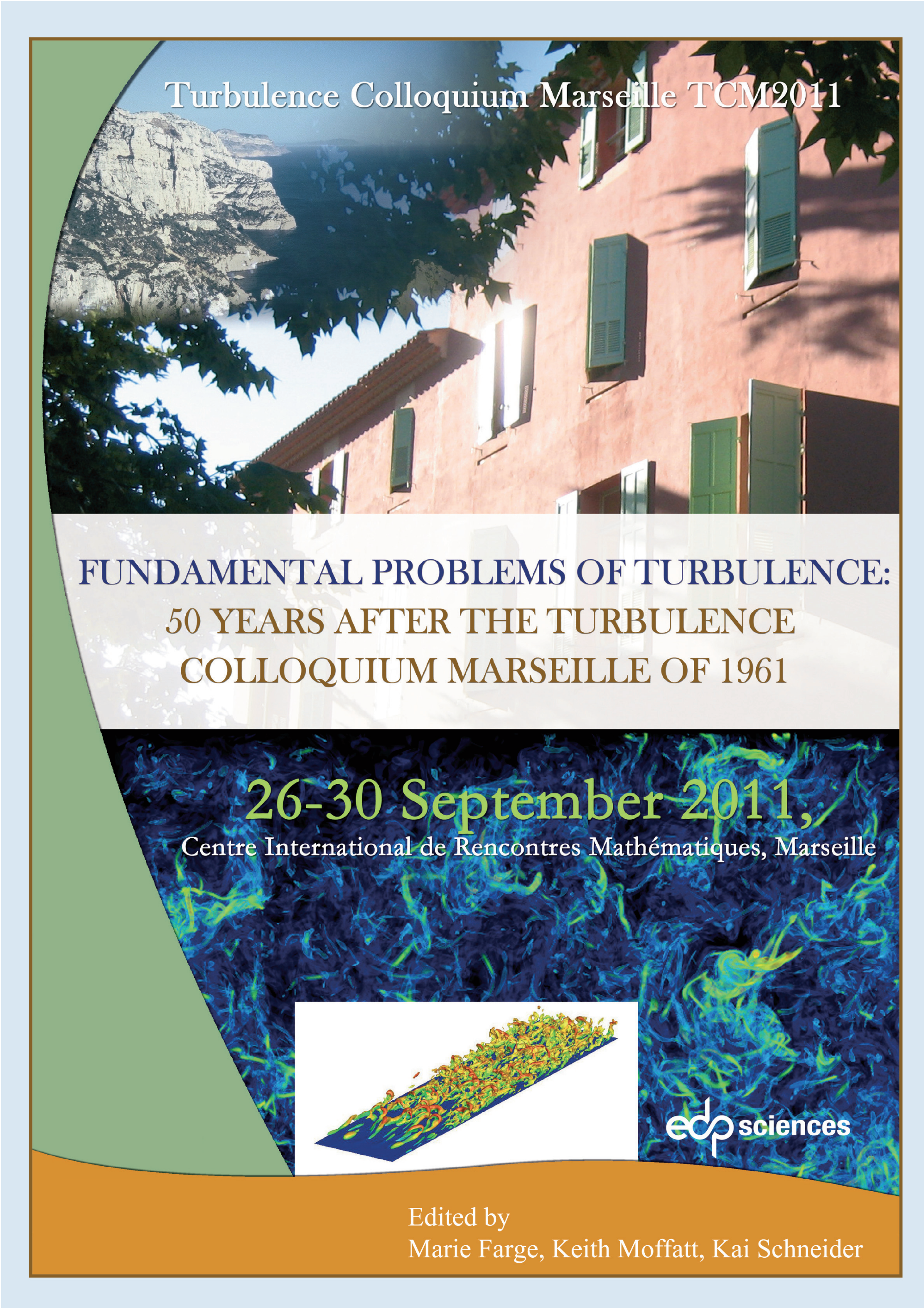
Numerical results suggest that a new asymptotic description of the flow beyond the breakdown of the Prandtl regime is possible, and studying it might help to understand the observed scalings.

Here are few open questions related to this:

- Would Navier-Stokes solution loose smoothness after t_D?
- Would it converge to a weak singular dissipative solution of Euler's equation, as suggested by Leray in 1934, analogously to dissipative shocks produced in Burgers solution?
- How can such a weak solution be approximated numerically?

J. Leray, 1934
Sur le mouvement d'un fluide visqueux,
Acta Mathematica, ***63***

C. de Lellis and L. Székzlyhidi, 2010
Archives Rational Mechanics and Analysis,
195*(1), 221-260*



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