El Niño as a Topological Insulator: A Surprising Connection Between Geophysical Fluid Dynamics and Quantum Physics

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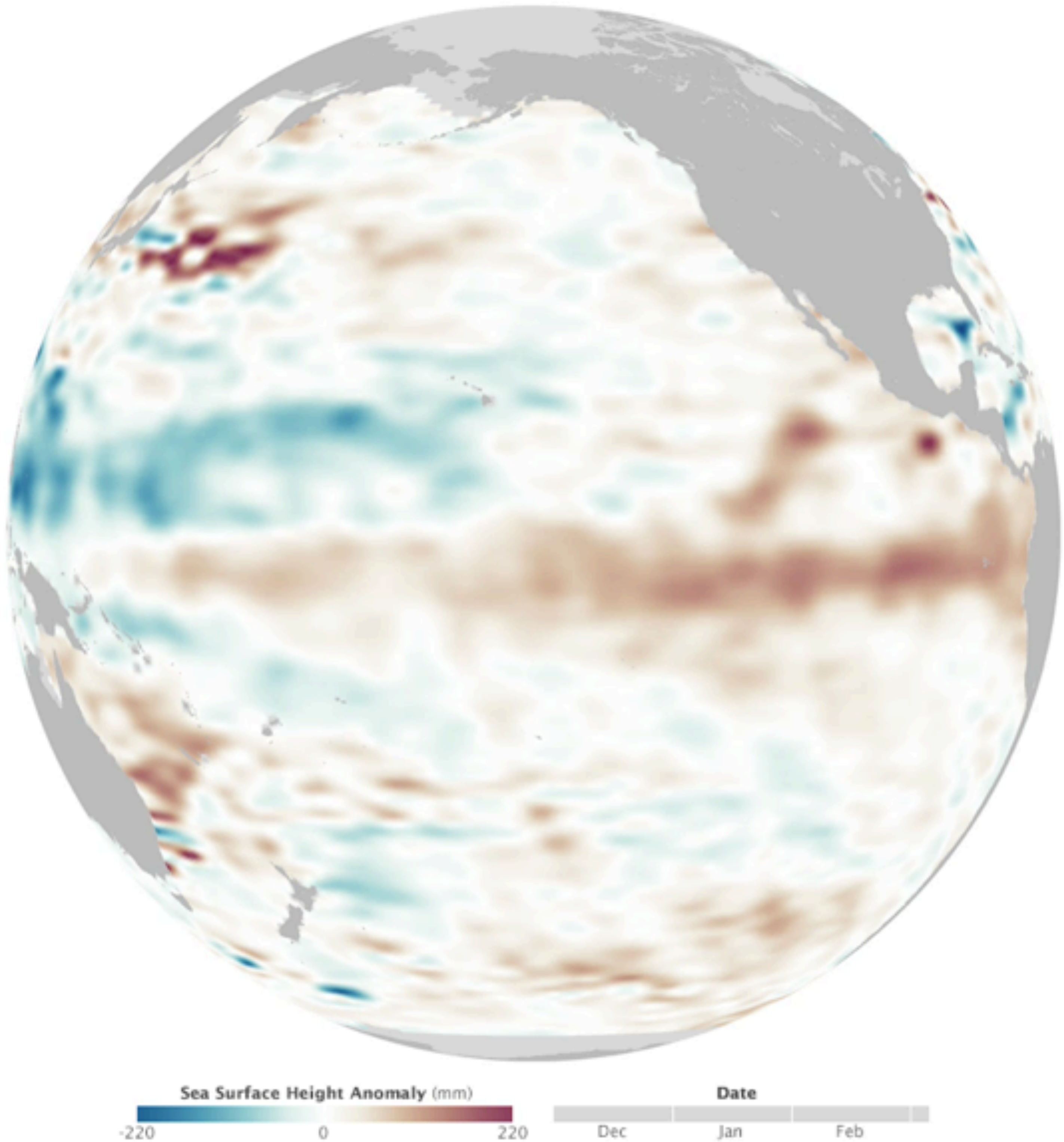
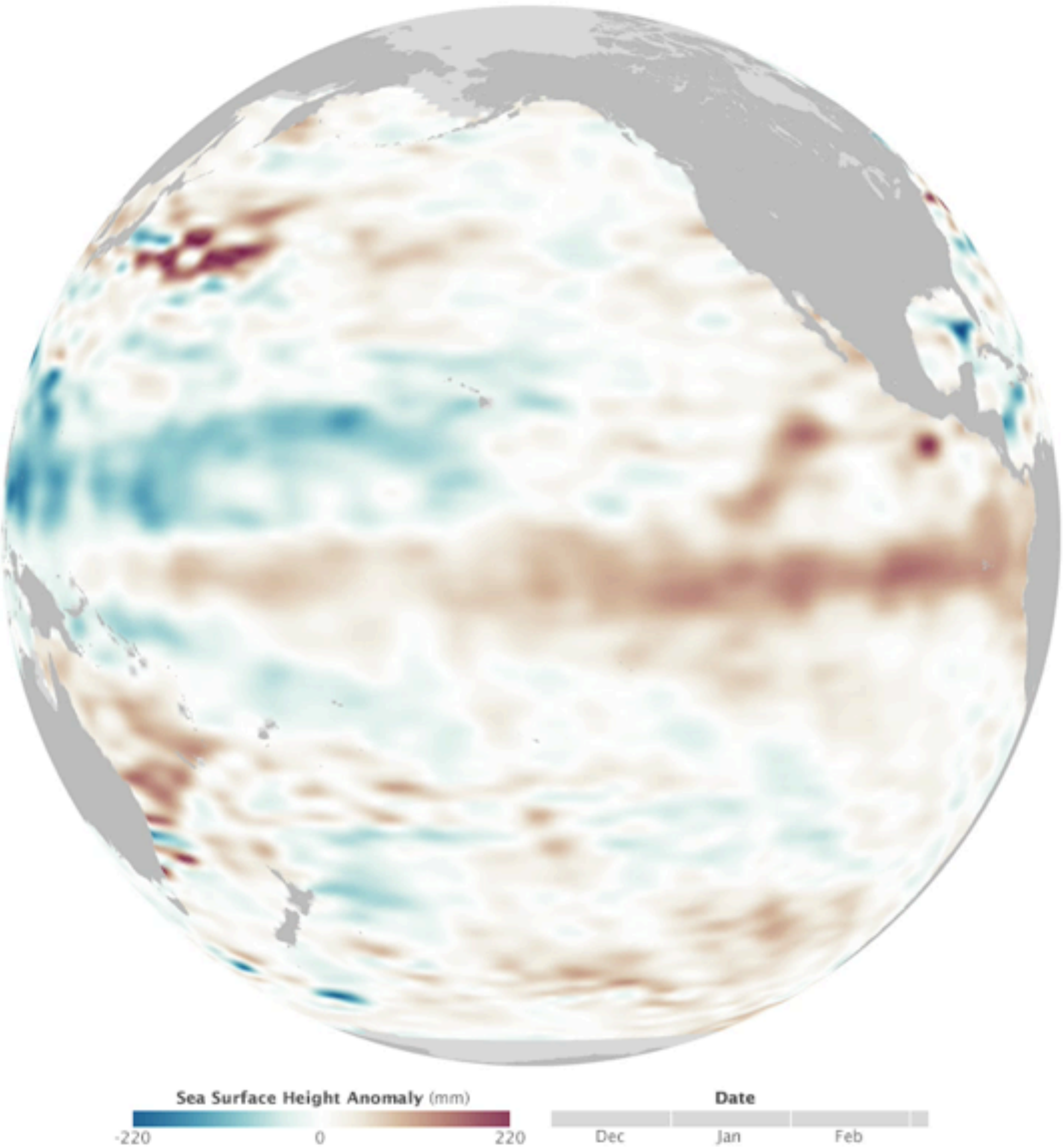
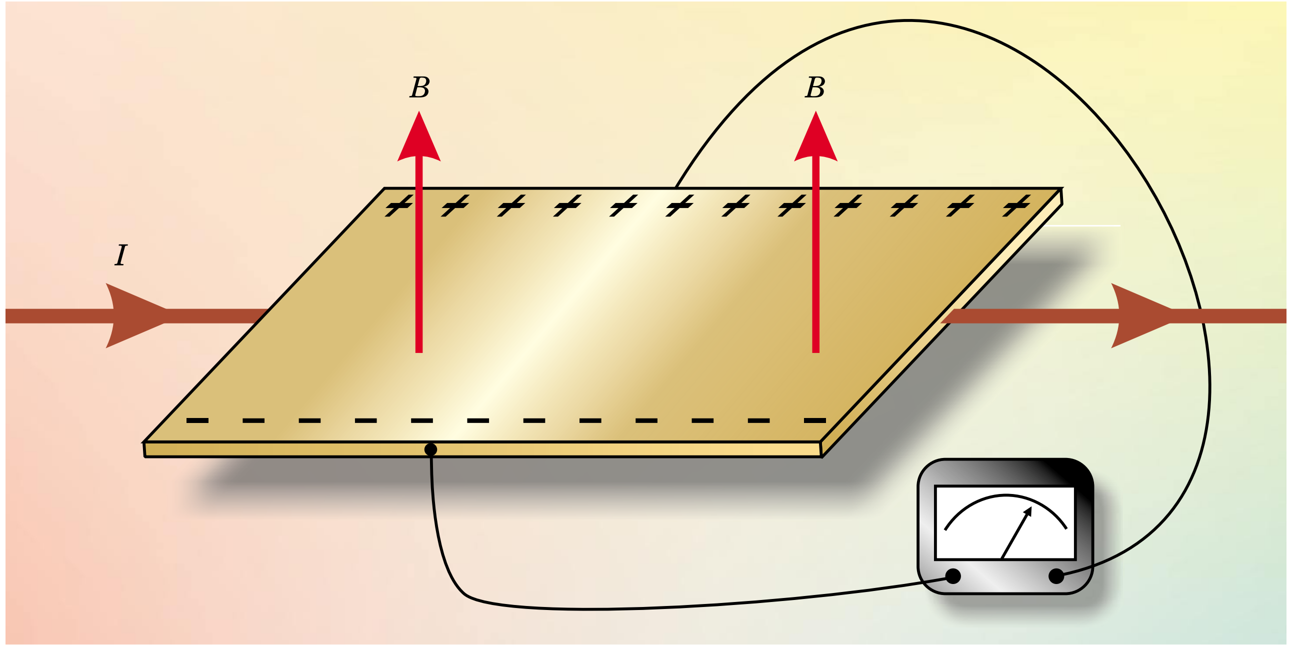
El Niño 2009 - 2010



El Niño 2009 - 2010



Integer Quantum Hall Effect



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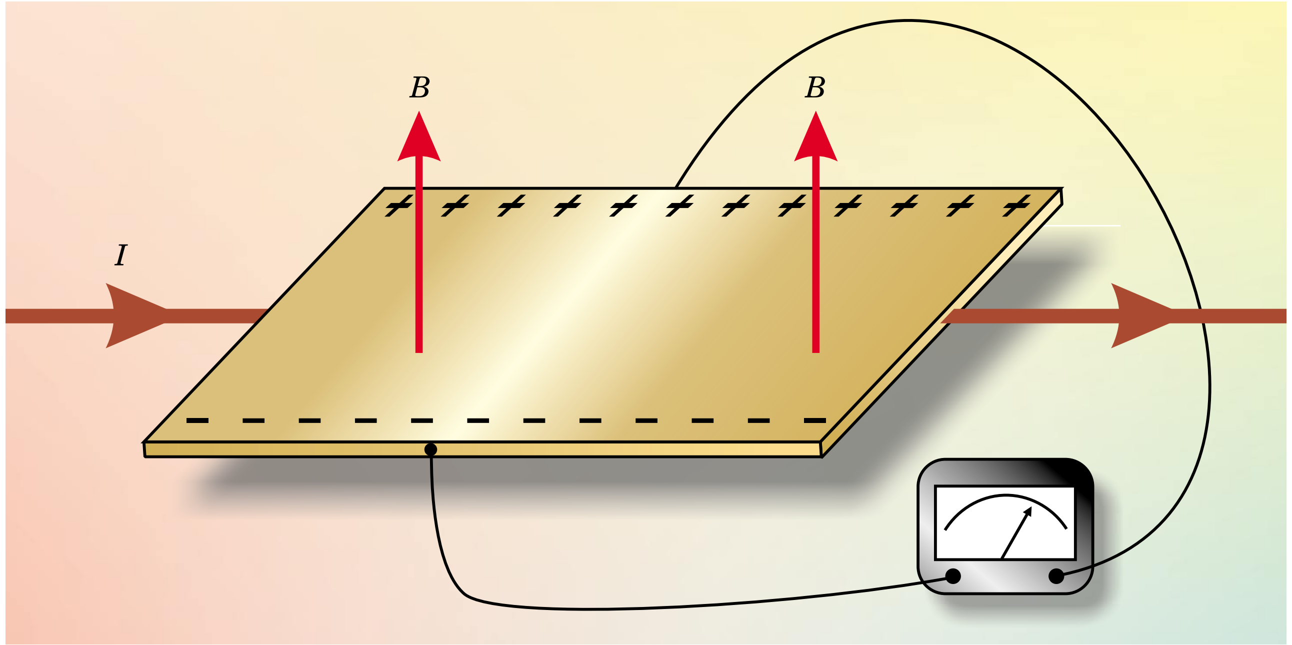


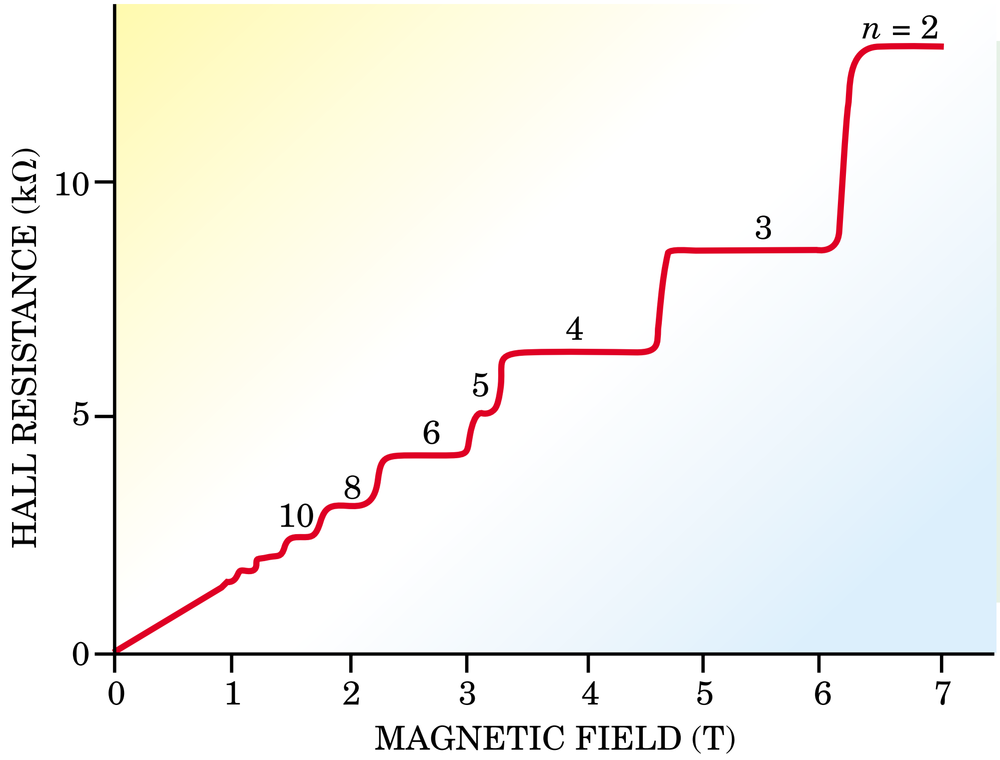
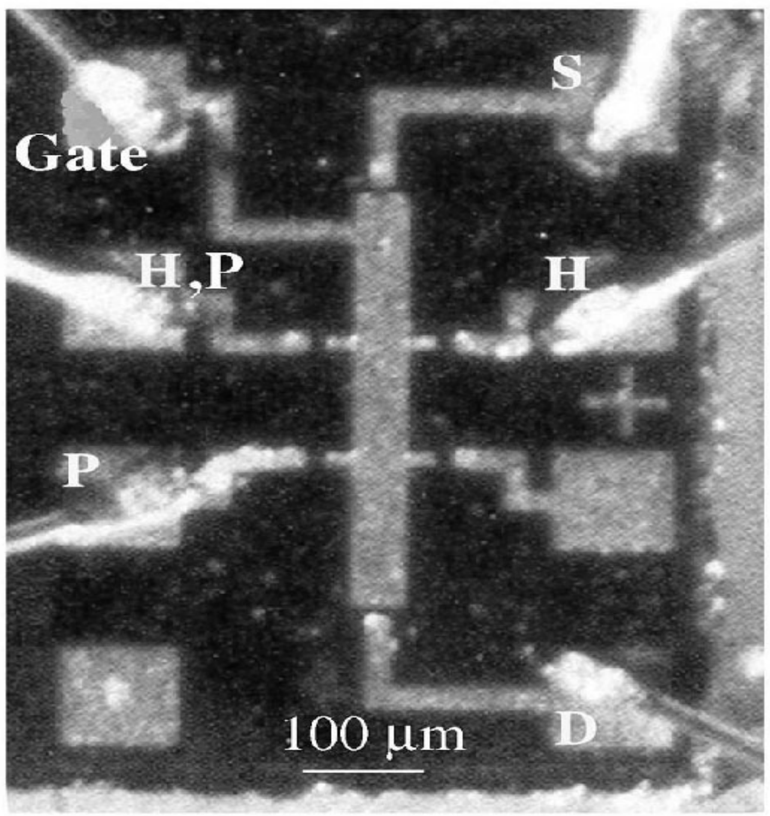


Figure 2. The integer quantum Hall effect. Plotting the Hall resistance (essentially the reciprocal of the Hall conductance) of a low-temperature two-dimensional electron gas against the strength of the imposed magnetic field normal to the gas plane, one finds a stairlike quantized sequence of Hall conductances very precisely equal to ne^2/h, where n is the integer that characterizes each plateau. The natural unit of resistance defined by this effect is about 26 kΩ. (Adapted from M. Paalanen, D. Tsui, A. Gossard, *Phys. Rev. B.* **25**, 5566 [1982].)

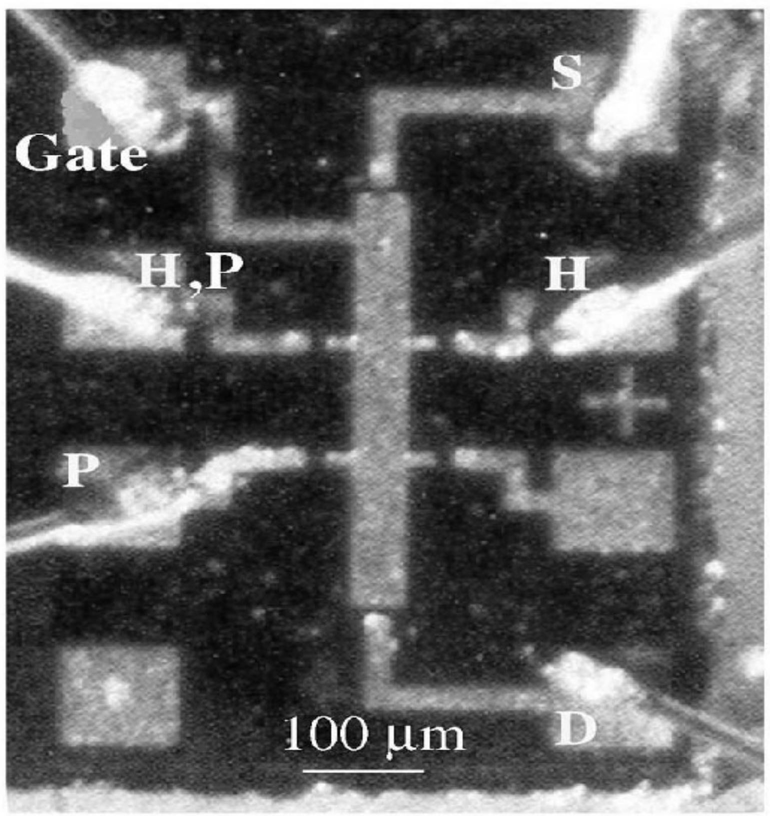
Topological Protection

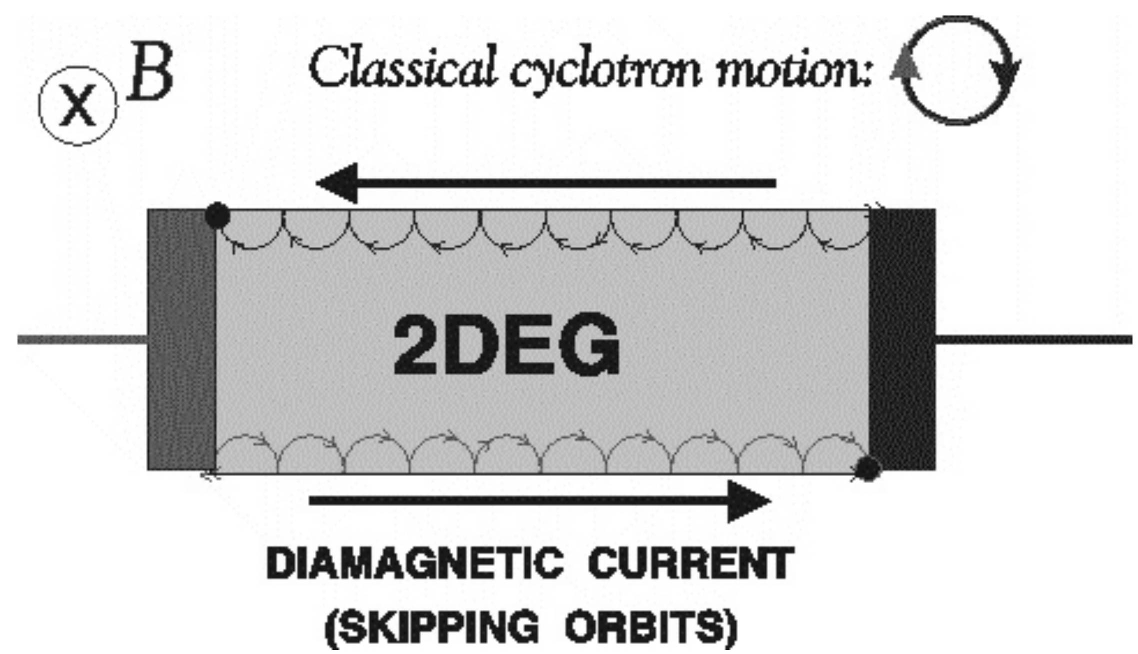
(Laughlin 1981 & Thouless 1983)



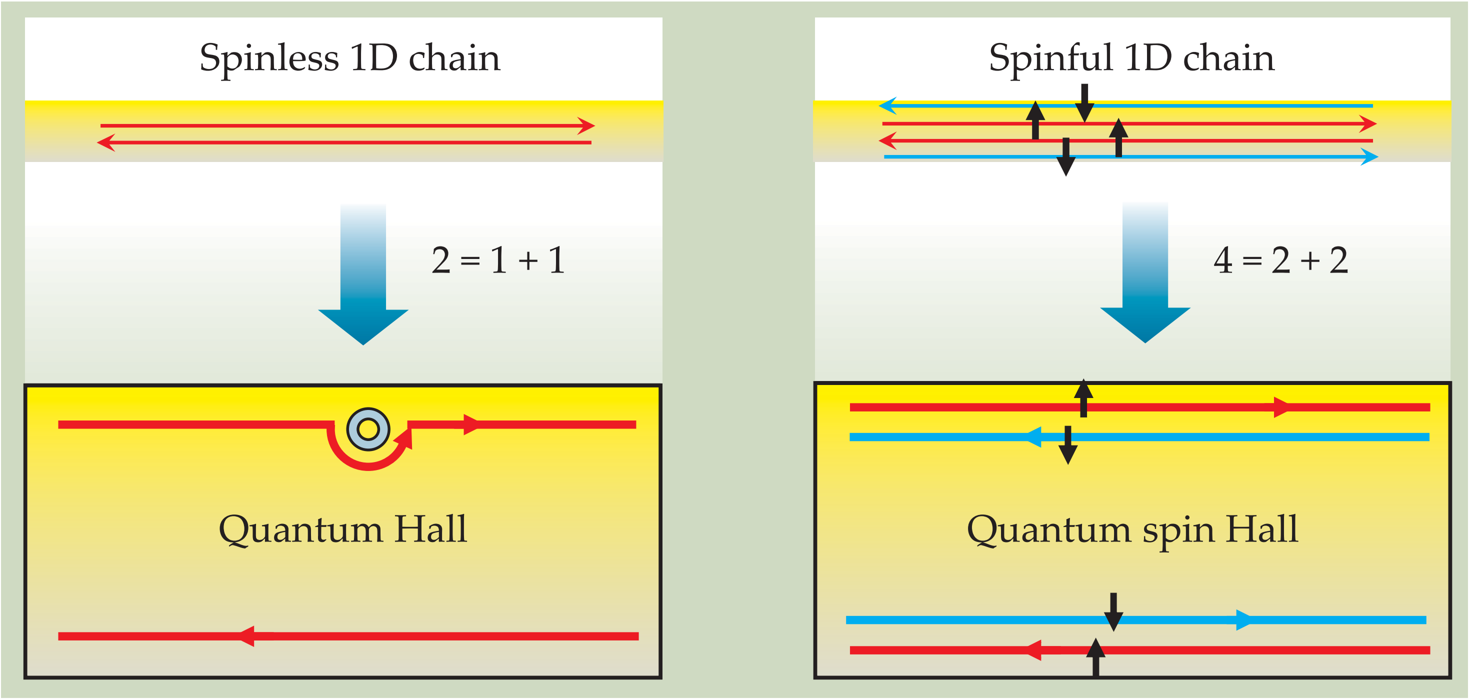
Topological Protection

(Laughlin 1981 & Thouless 1983)



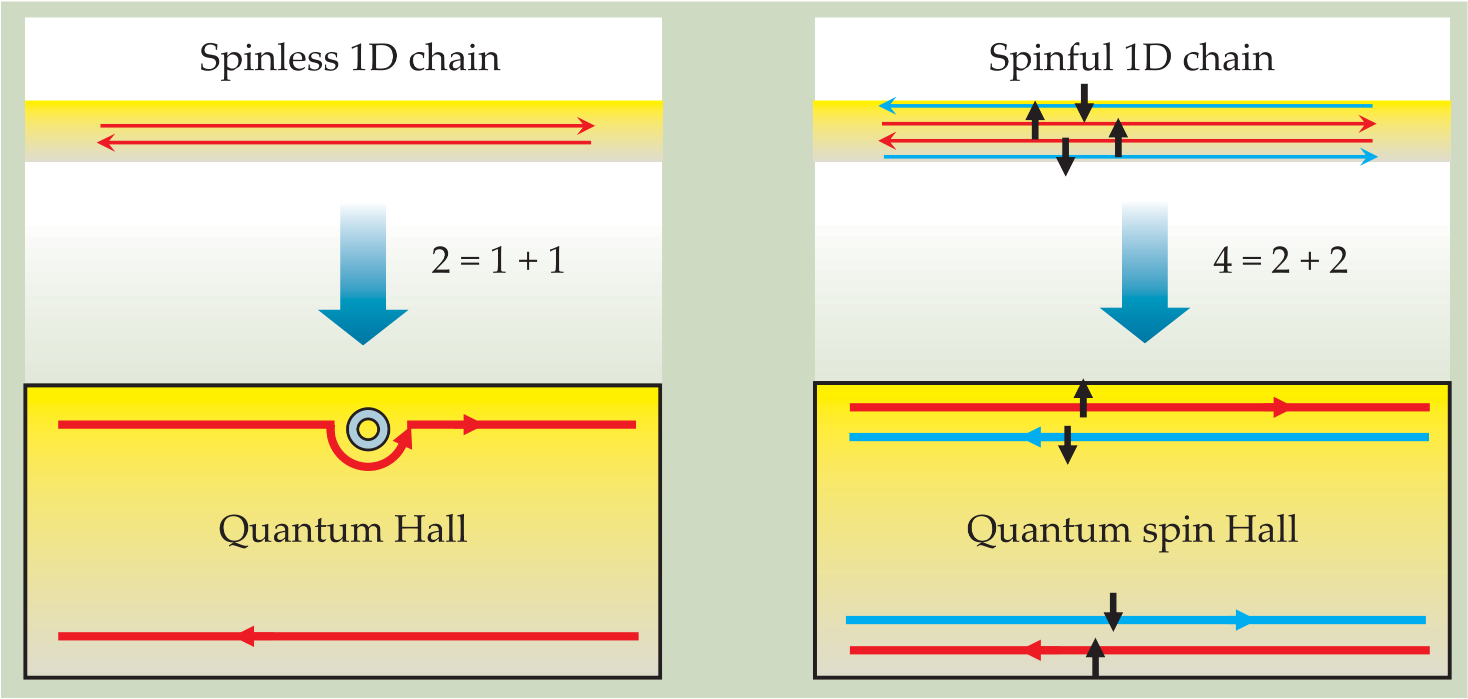


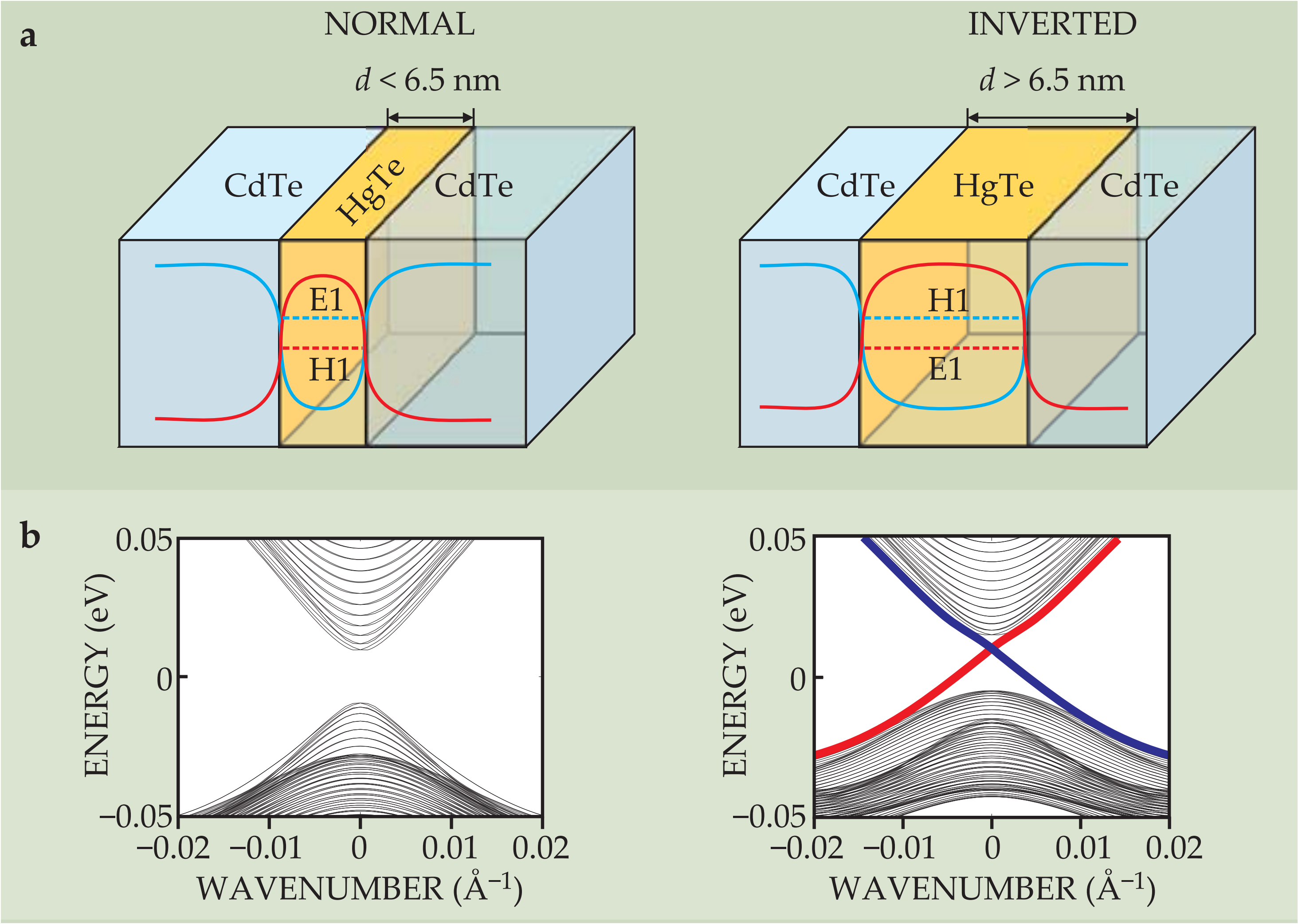
Topological Insulators



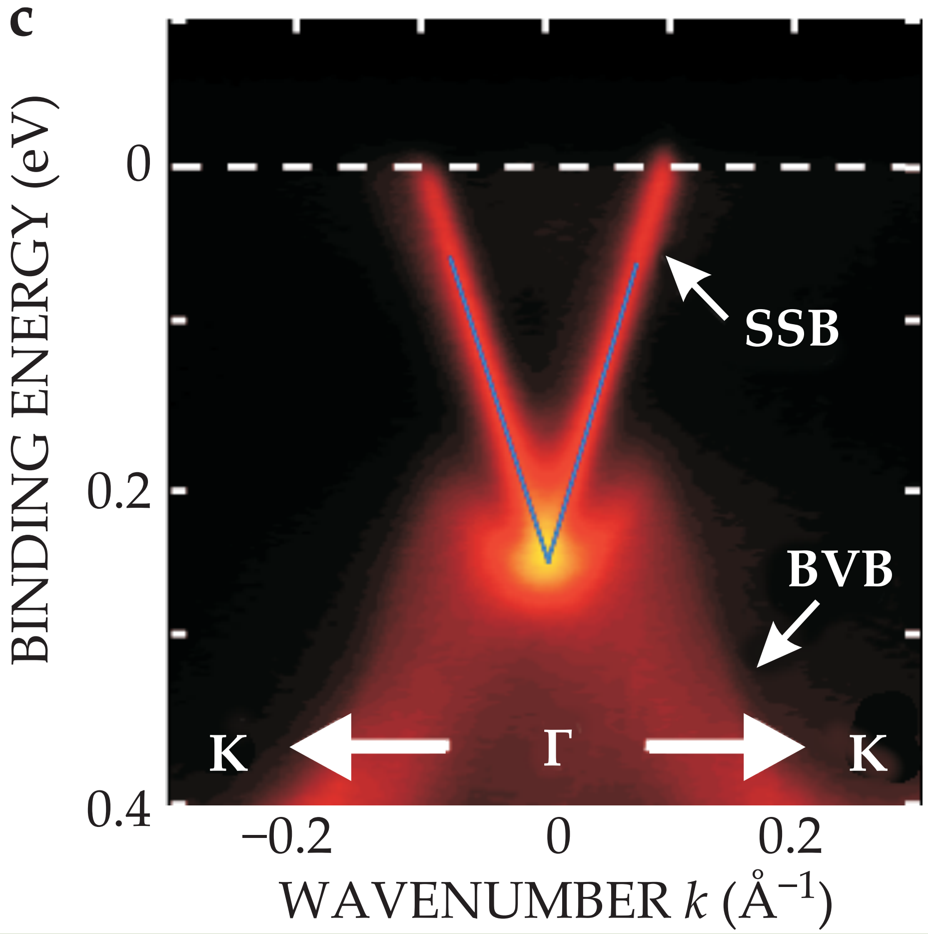
Qi and Zhang
(2010).

Topological Insulators





Qi and Zhang (2010).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BINDING ENERGY (eV)
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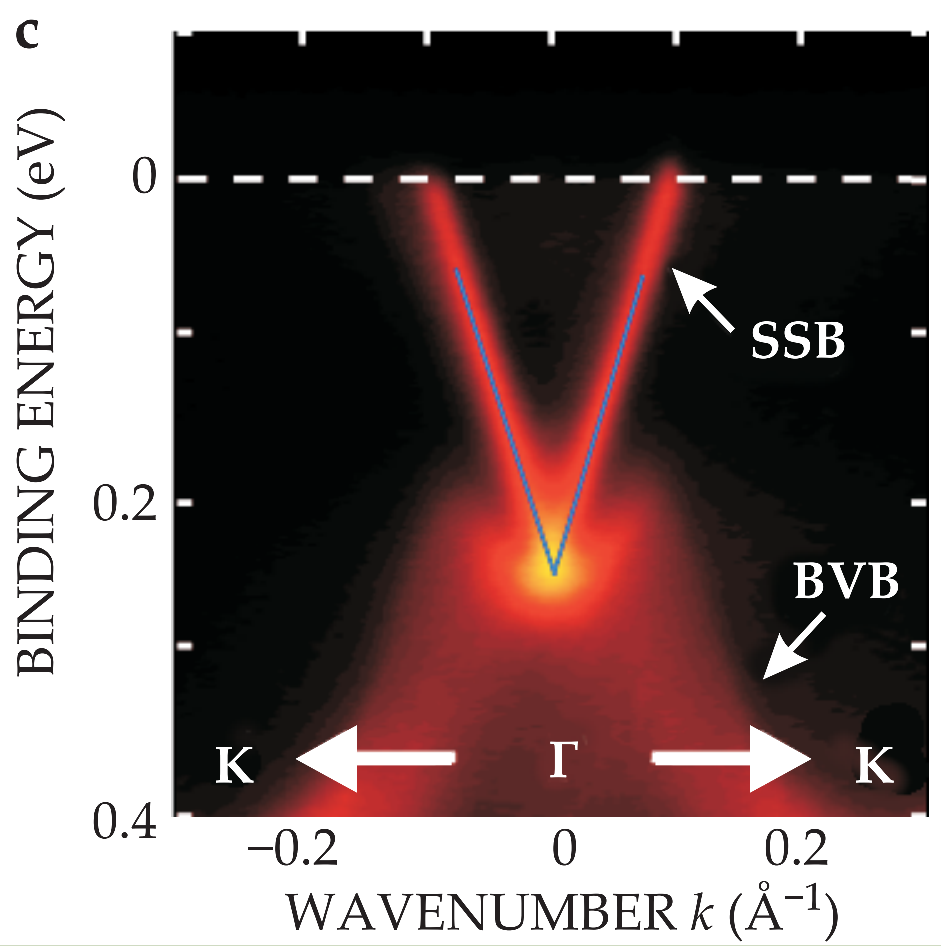
0.2
0.4
SSB
BVB
K
Γ
K
−0.2
0
0.2
WAVENUMBER k (Å−1)


c
BINDING ENERGY (eV)
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0.2
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K
Γ
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0
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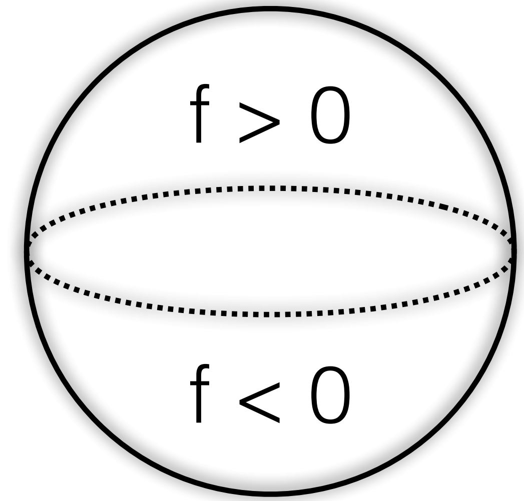
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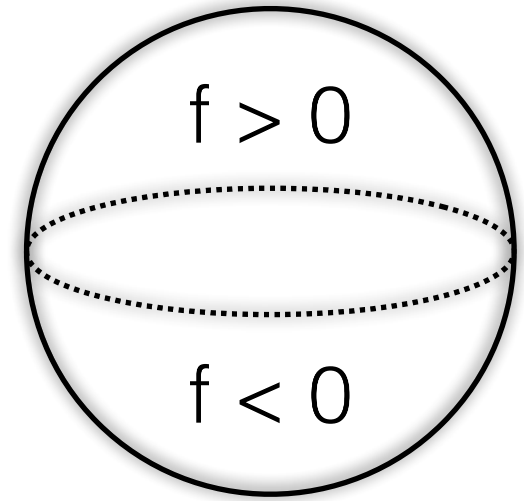
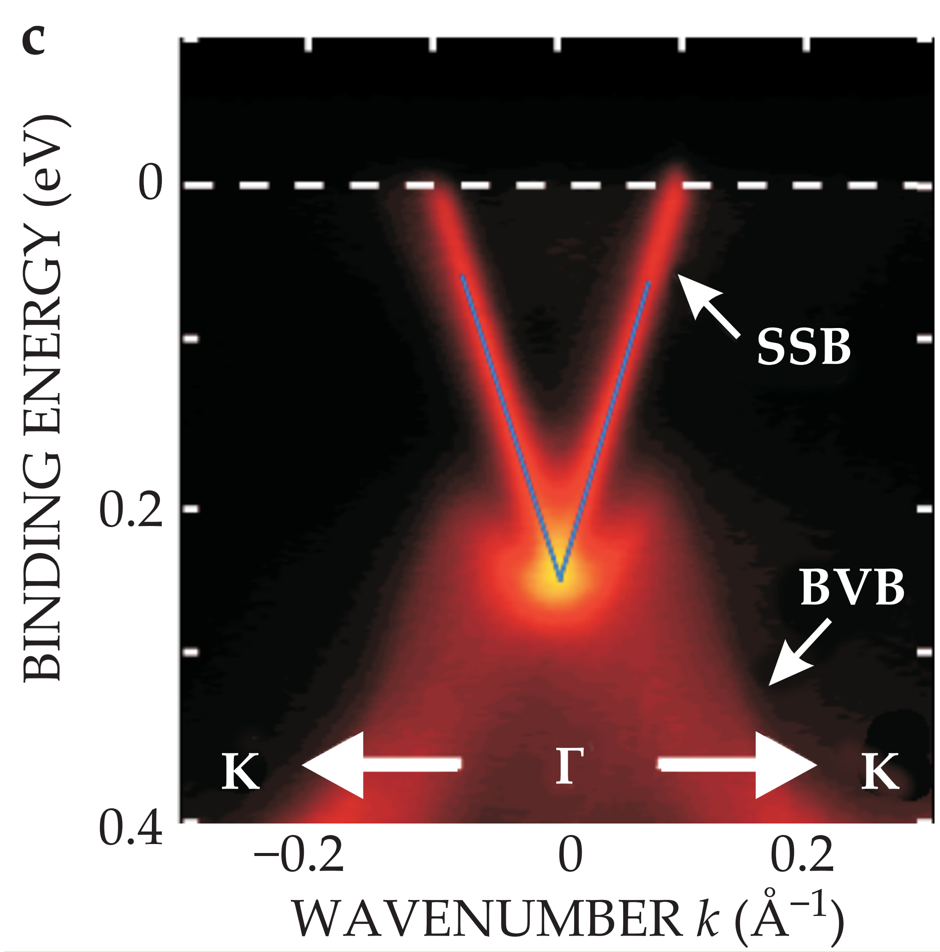
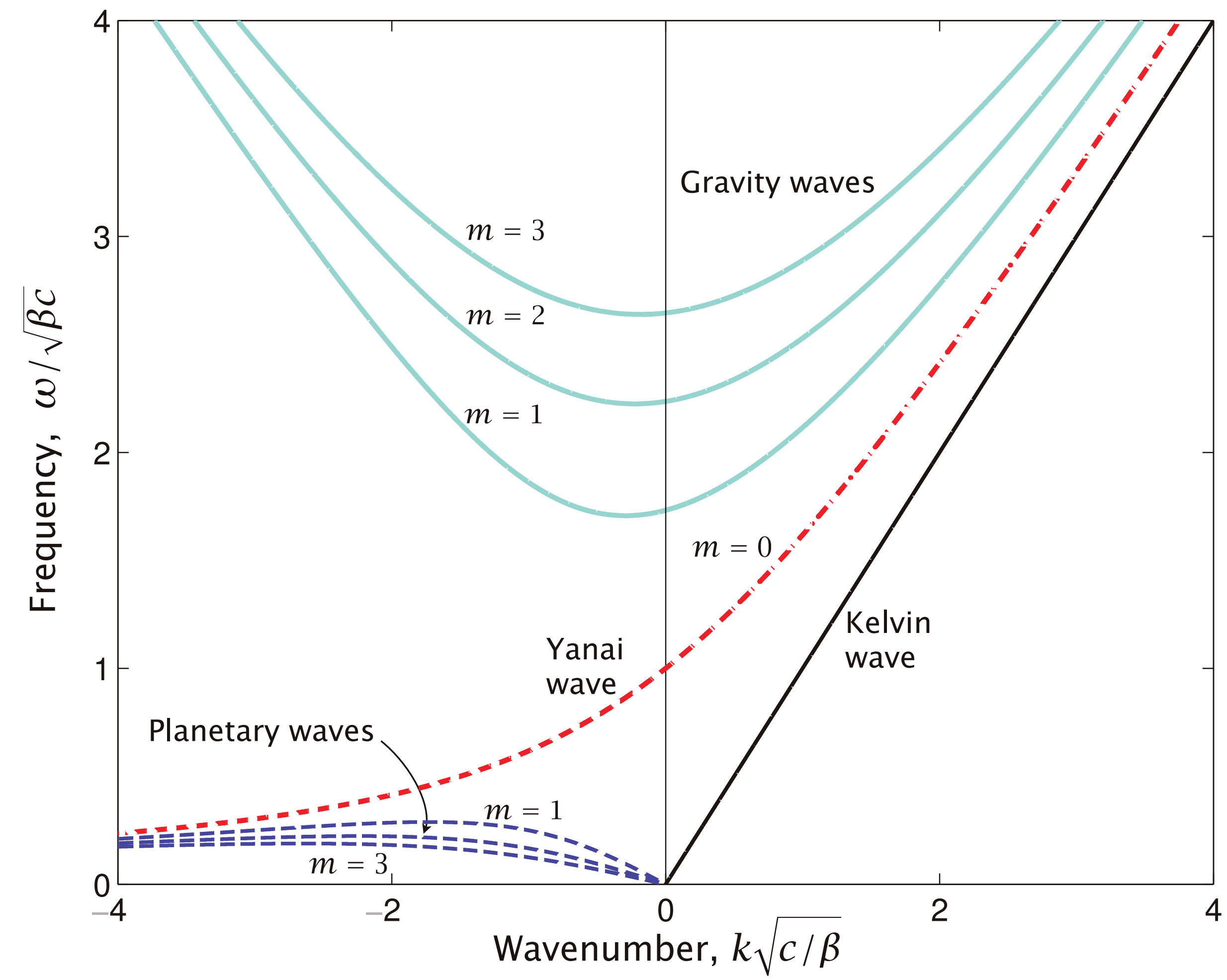
f < 0



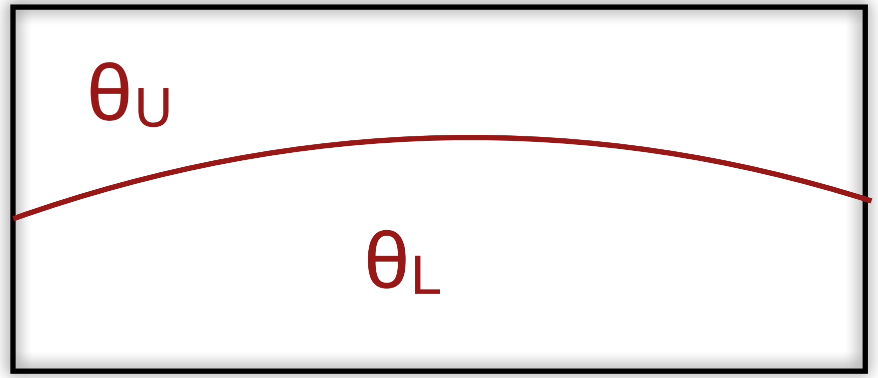


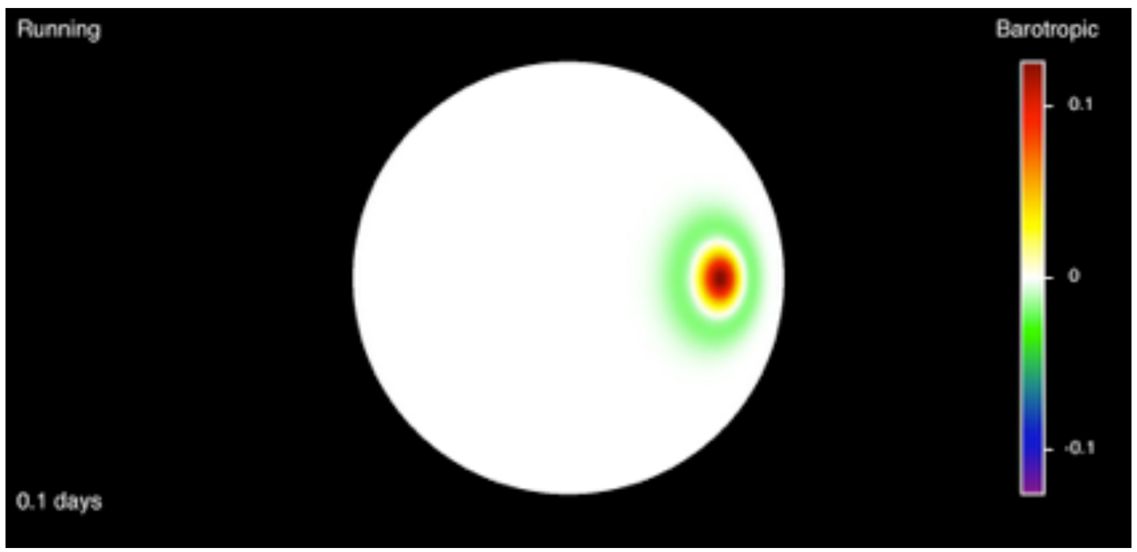


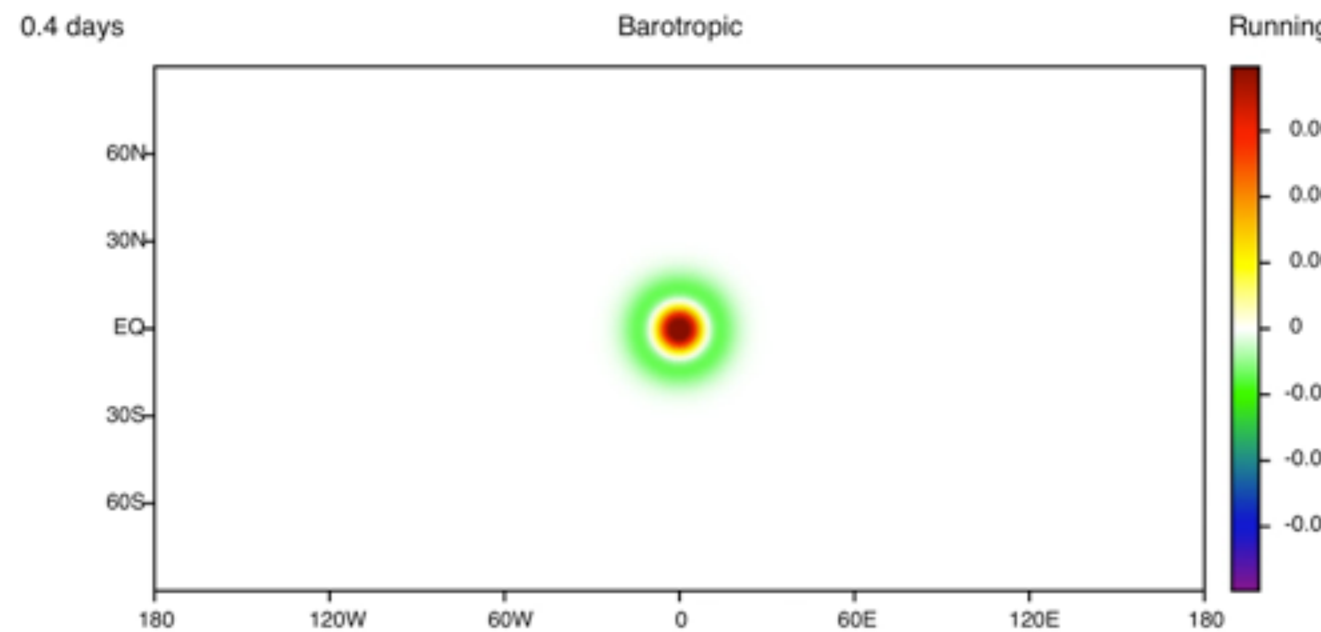
Matsuno 1966 and Longuet-Higgins 1968.

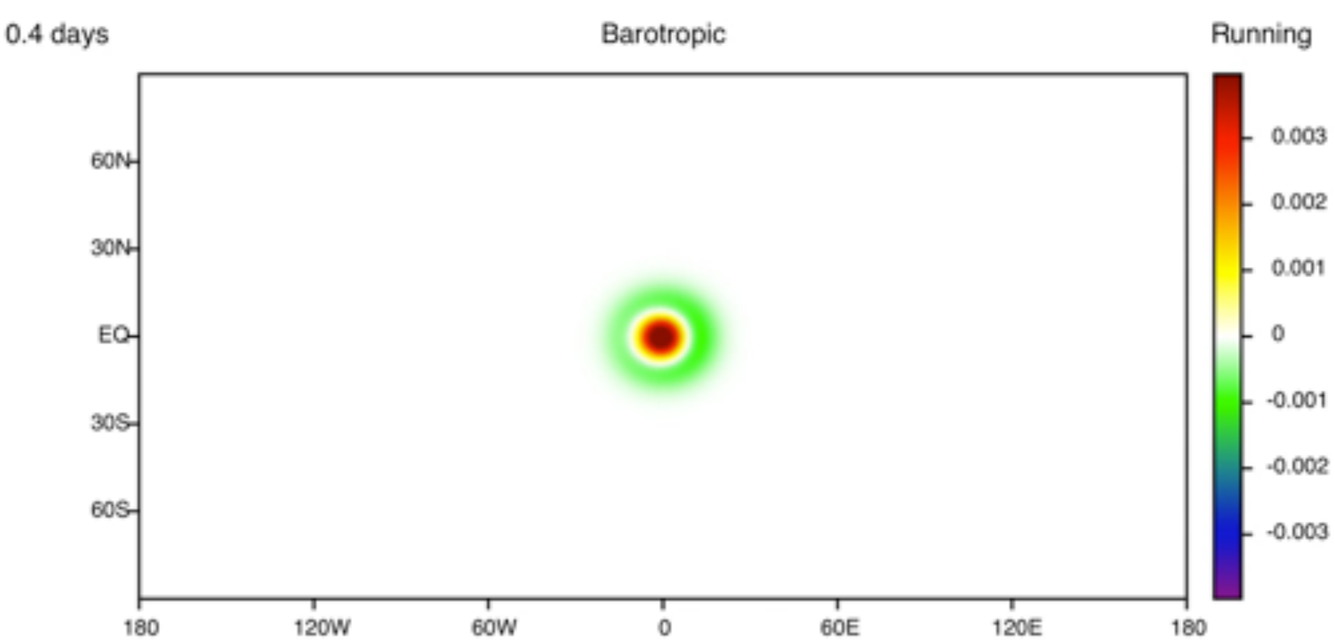
[Geoff Vallis, *Atmospheric and Oceanic Fluid Dynamics* (notes for a 2nd ed.)]

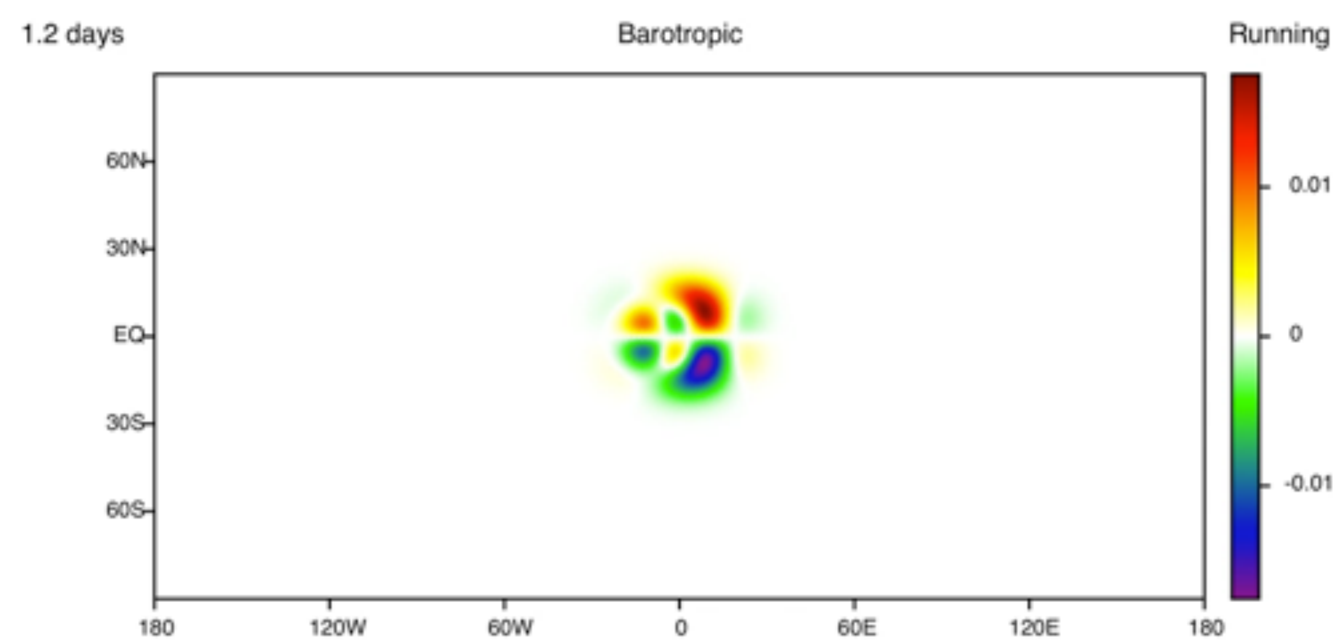
2 Layer Model: Vertical Mean Temperature











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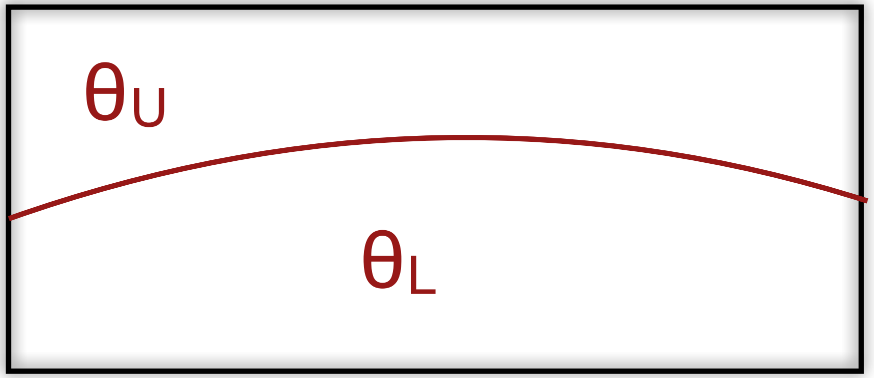

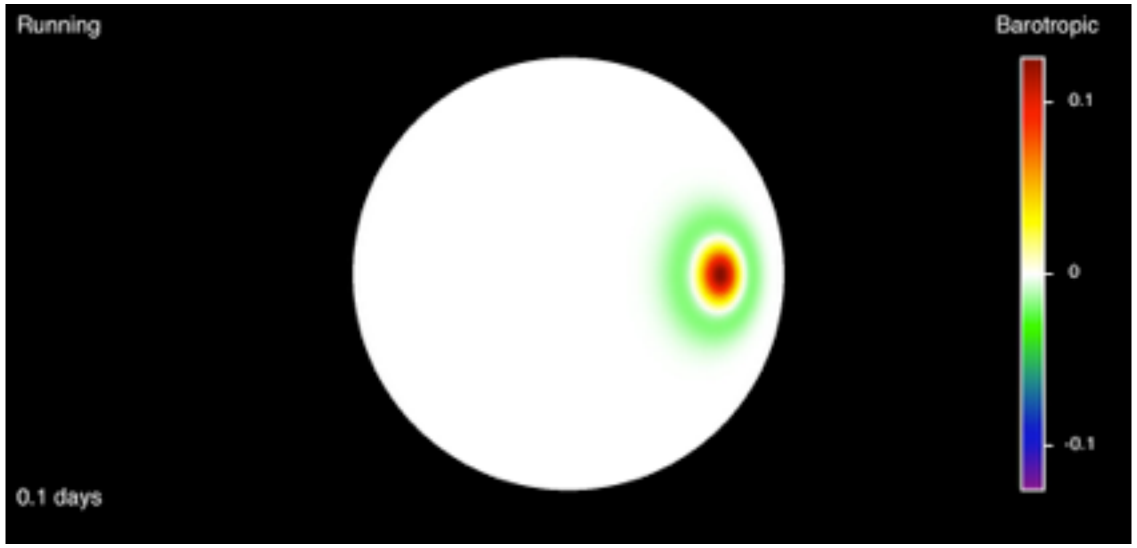

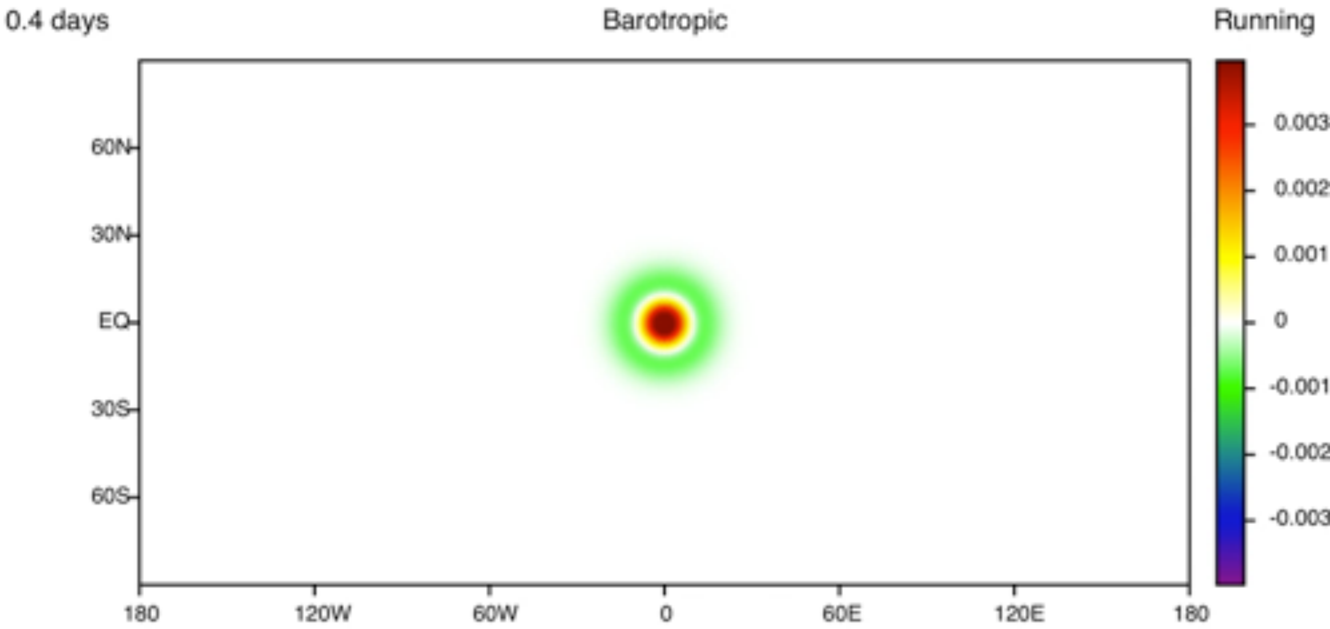

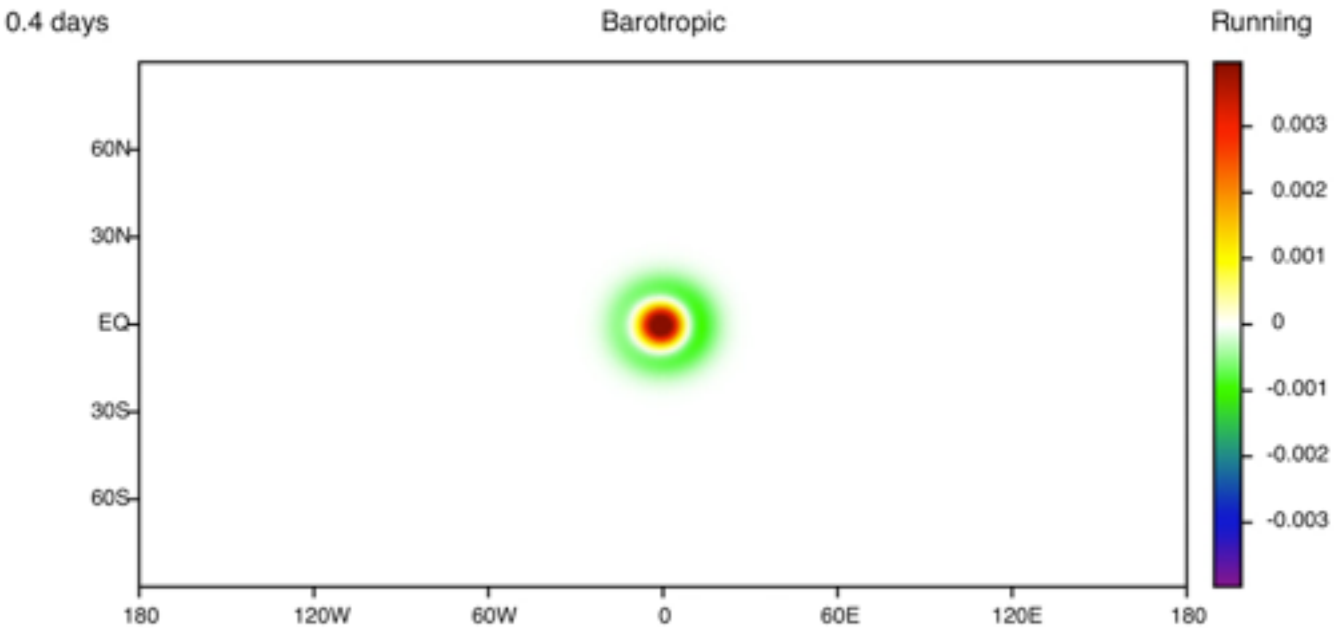

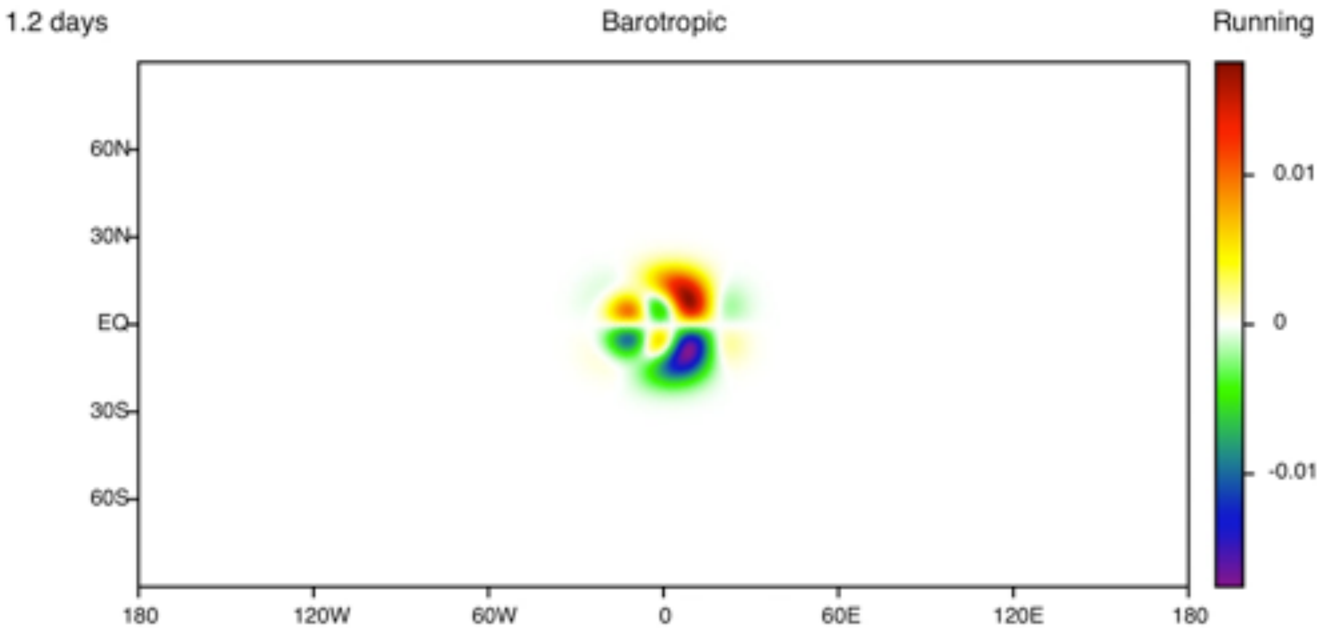

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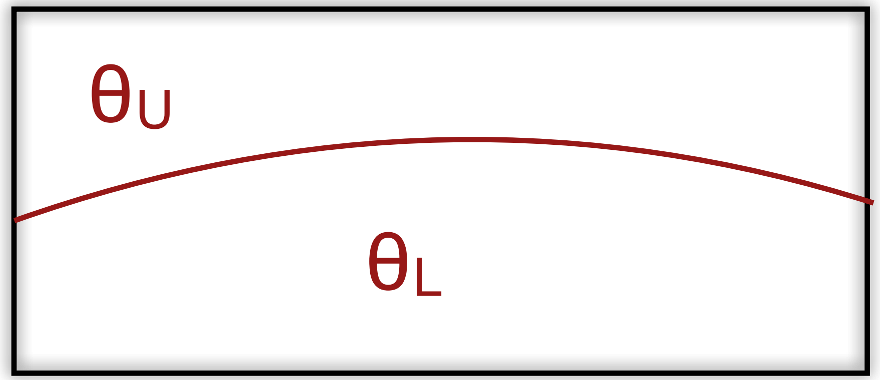

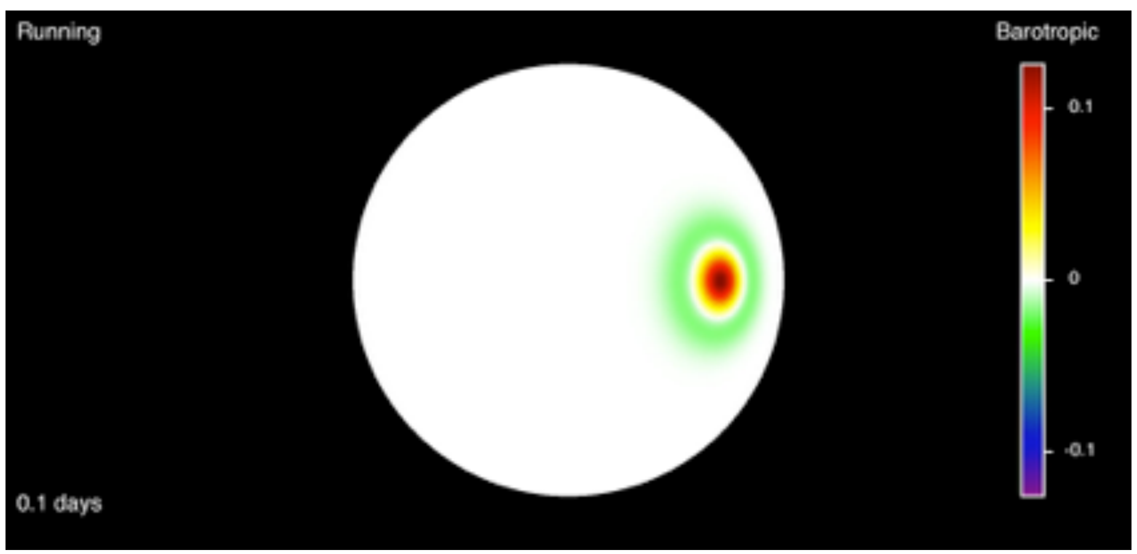

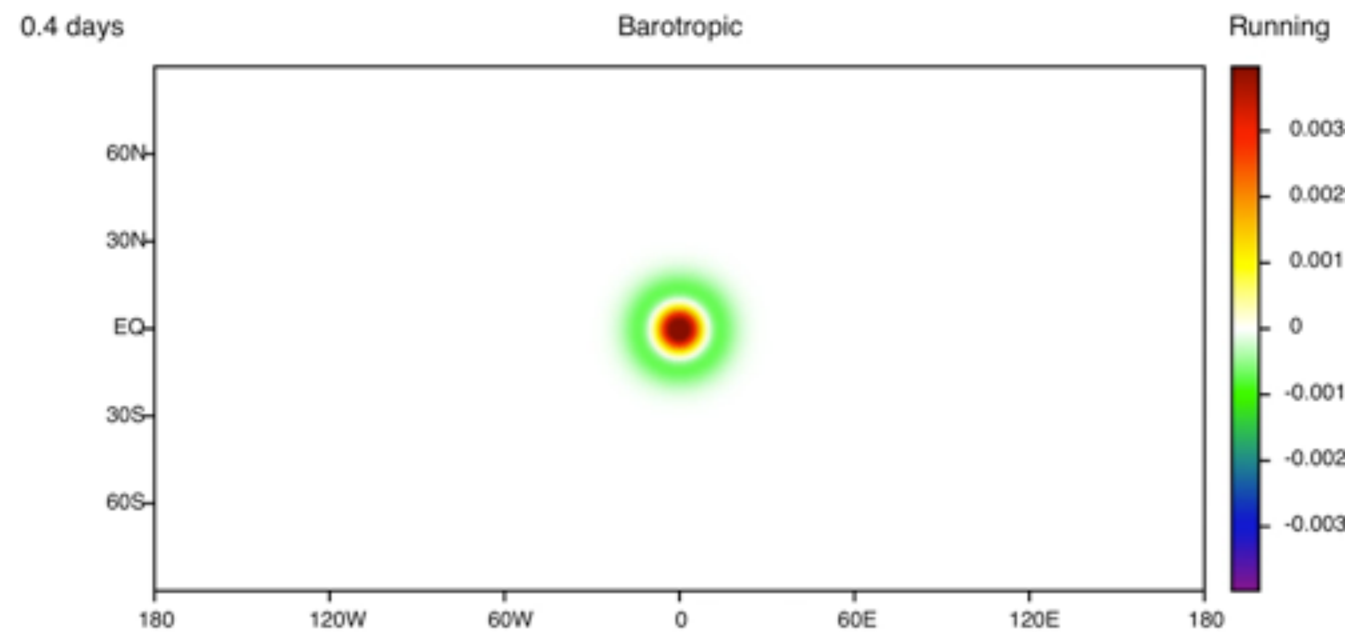

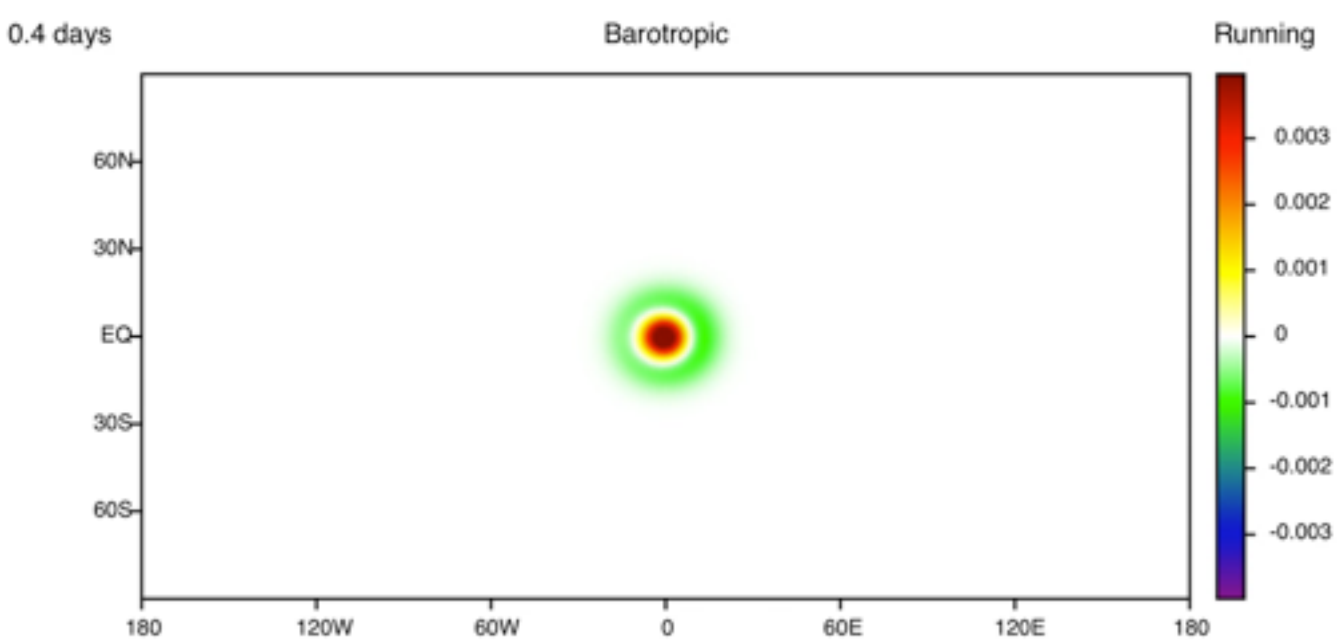

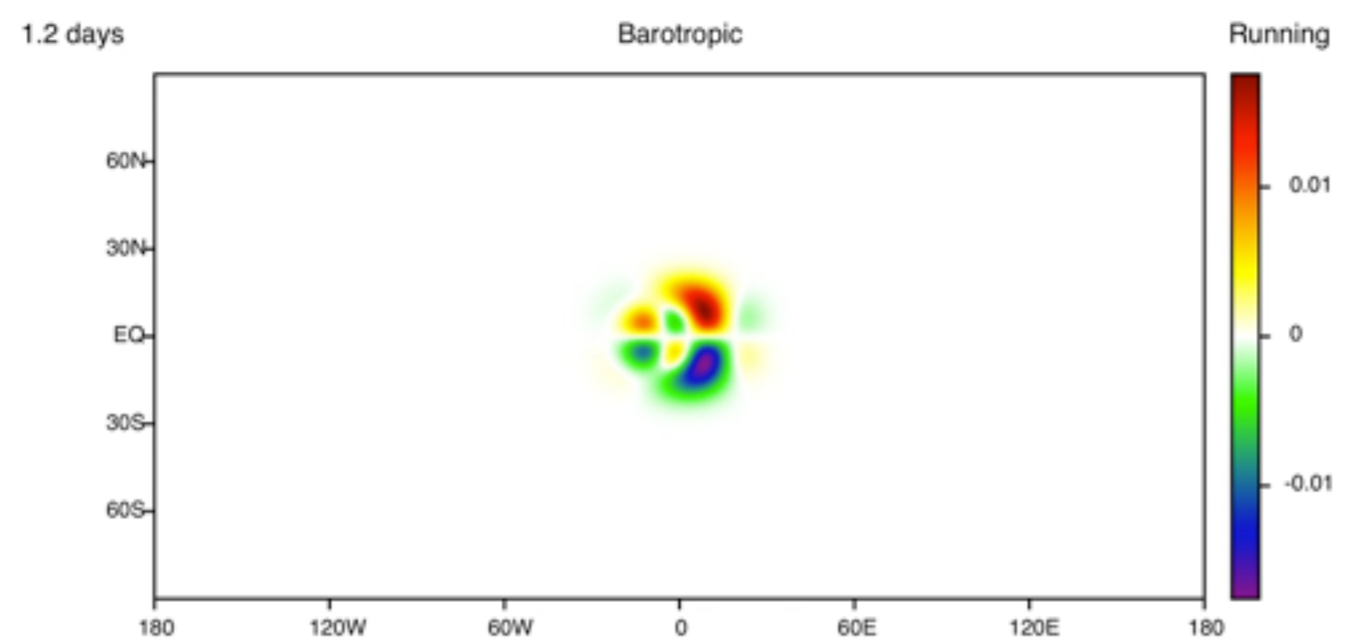

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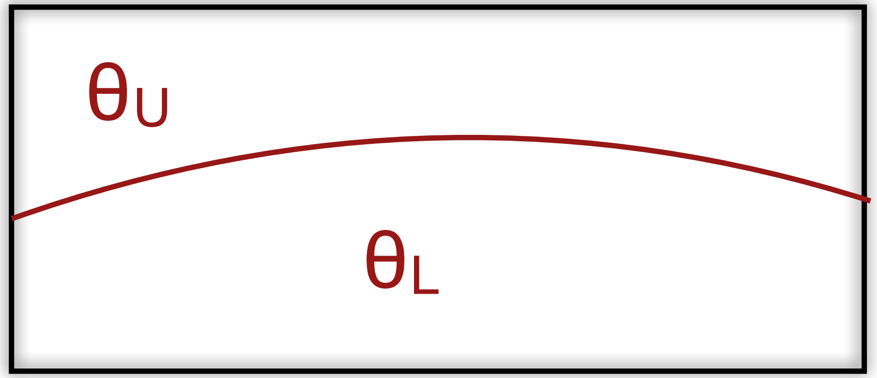

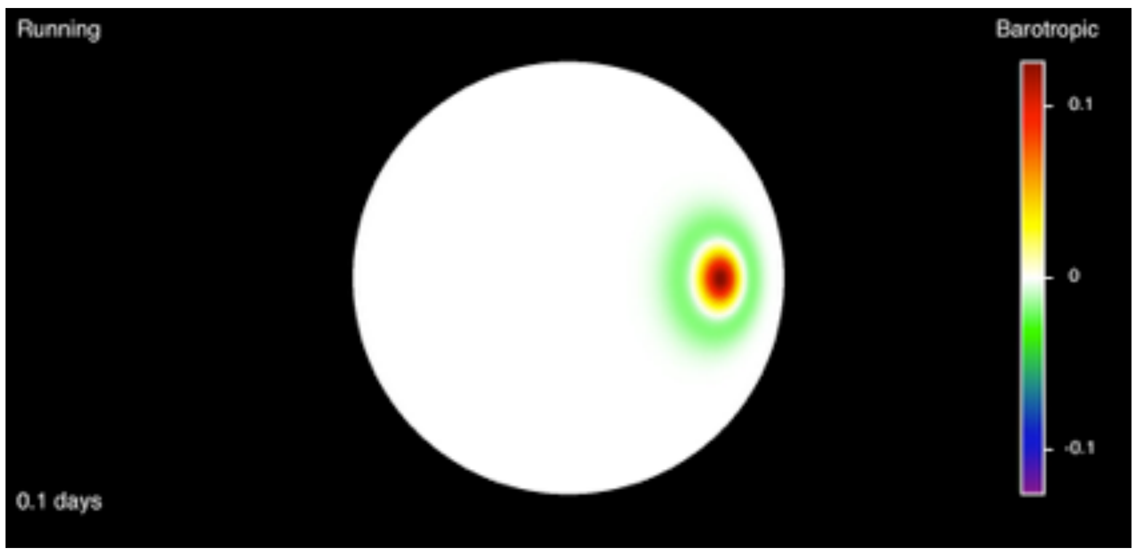

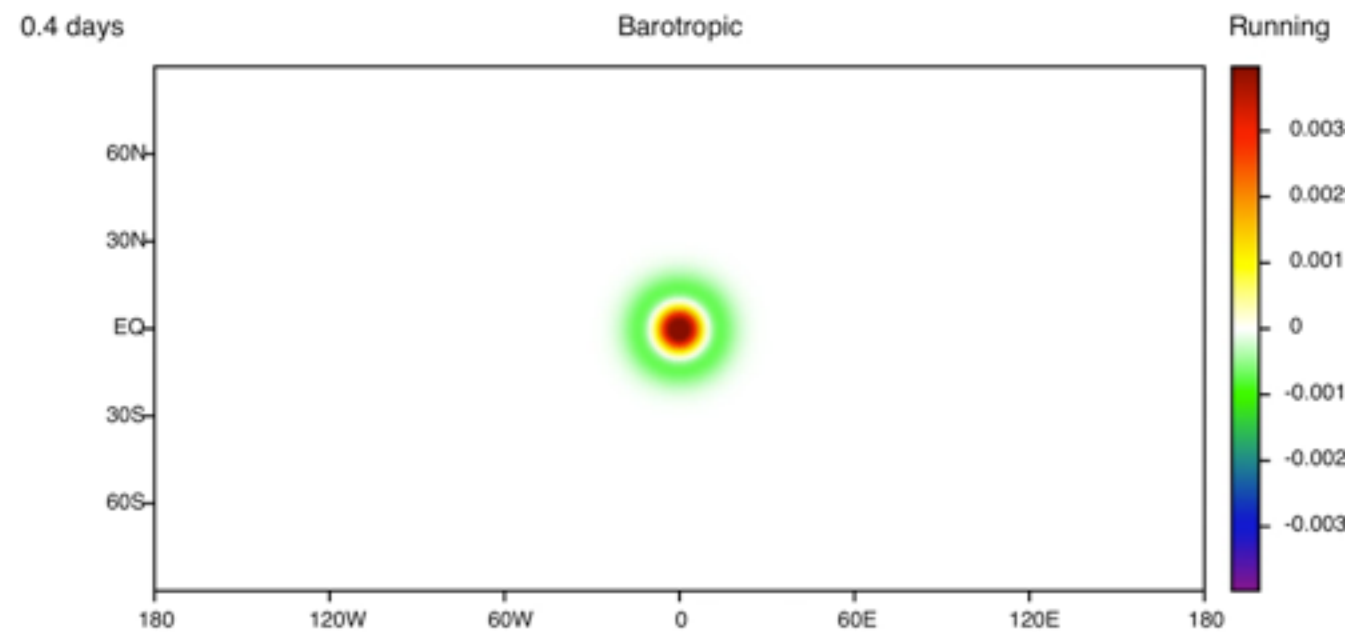

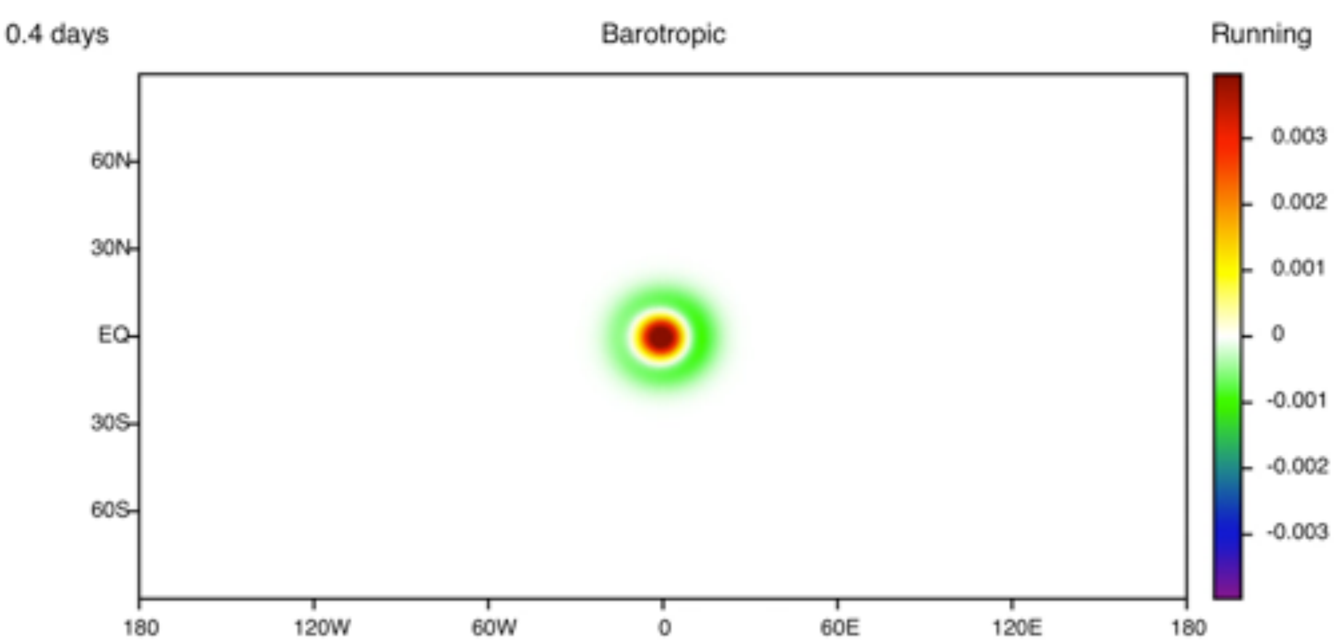

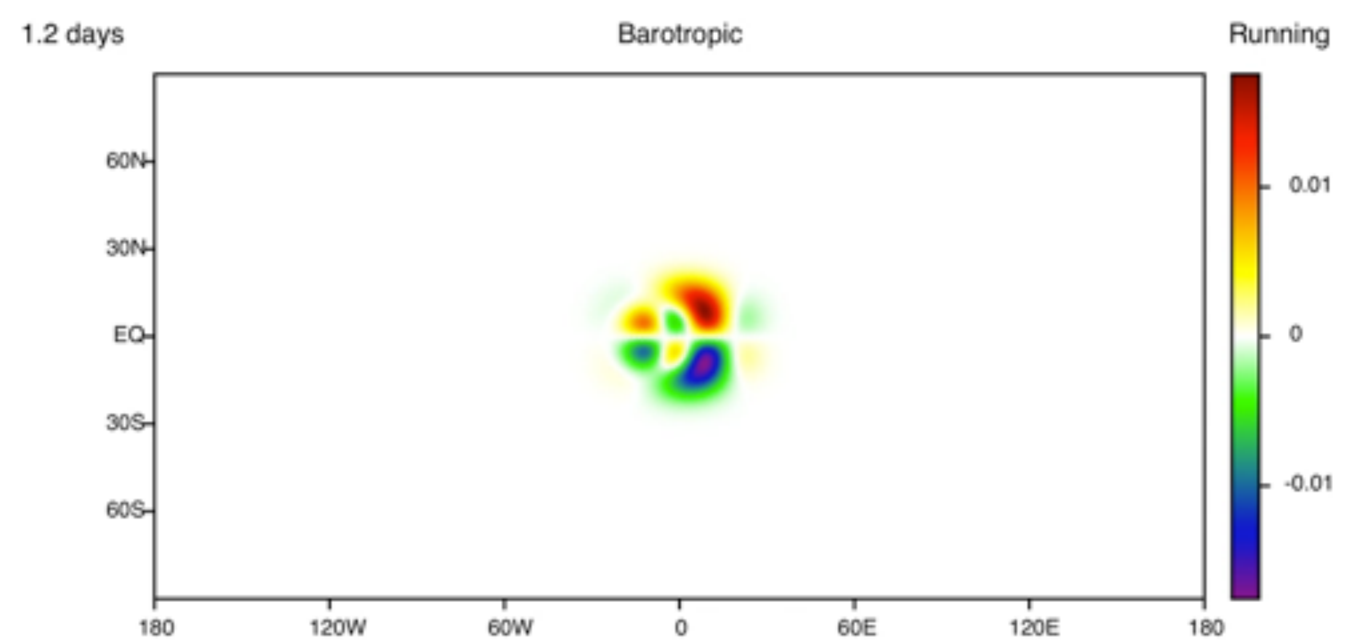

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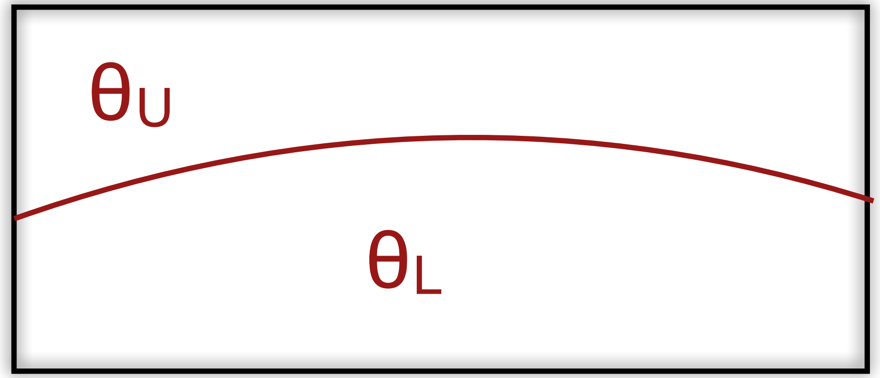
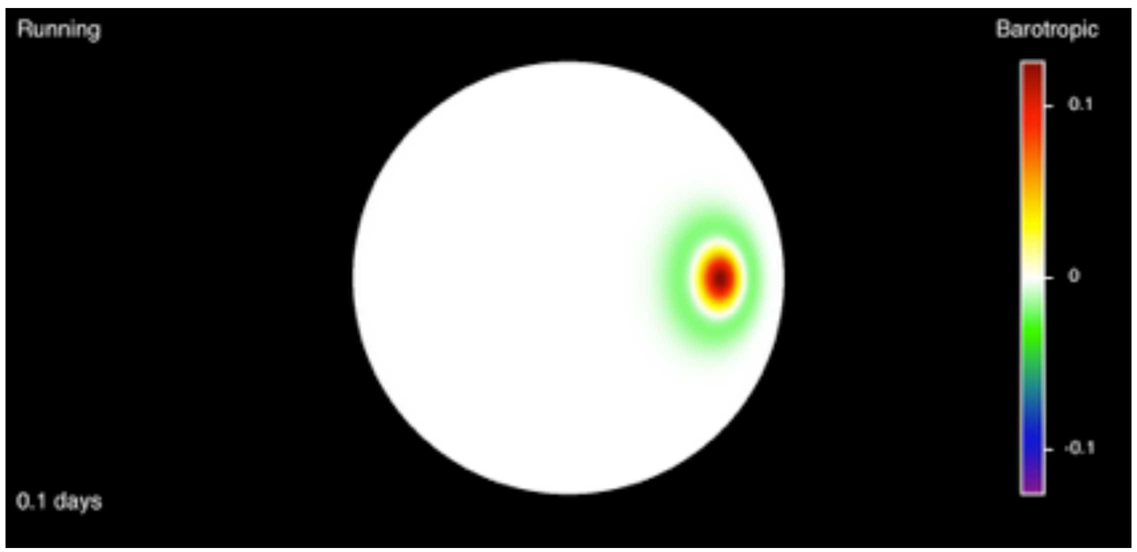
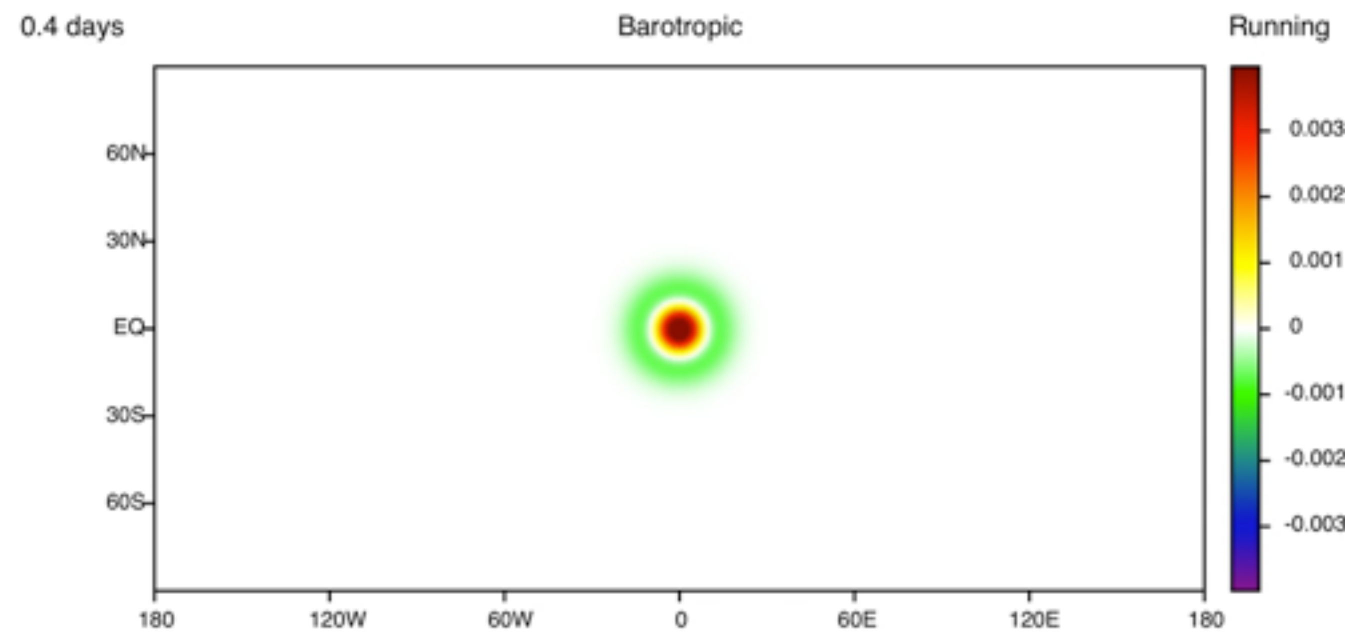
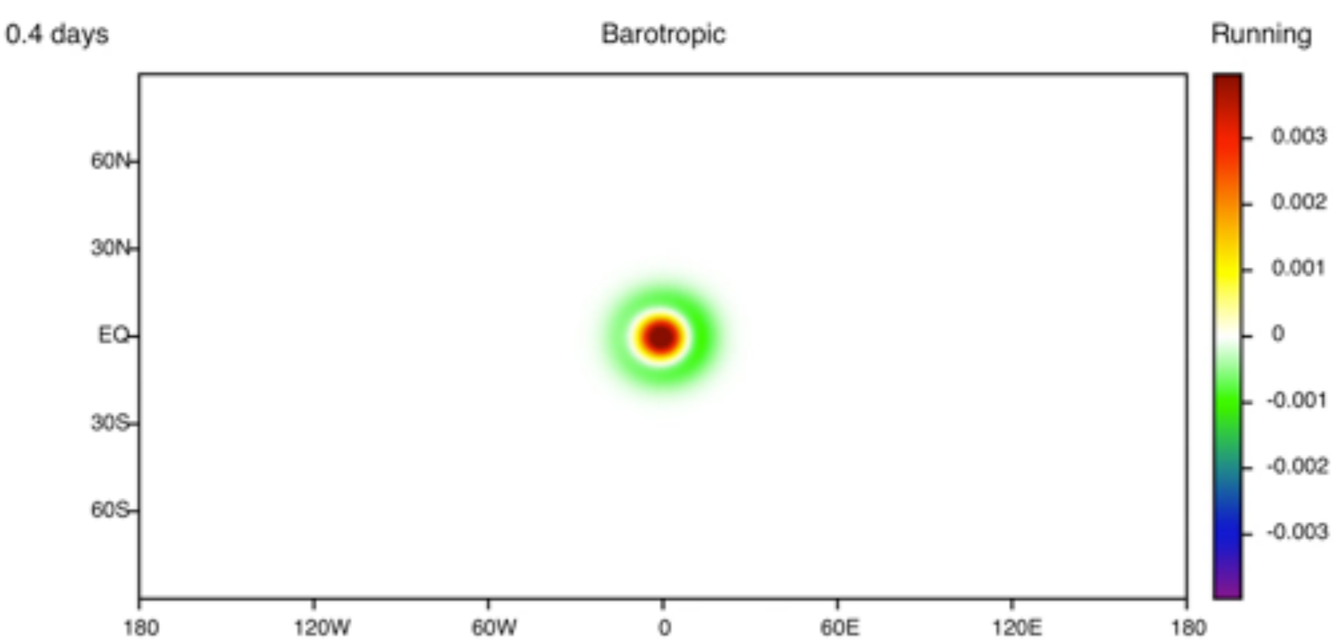
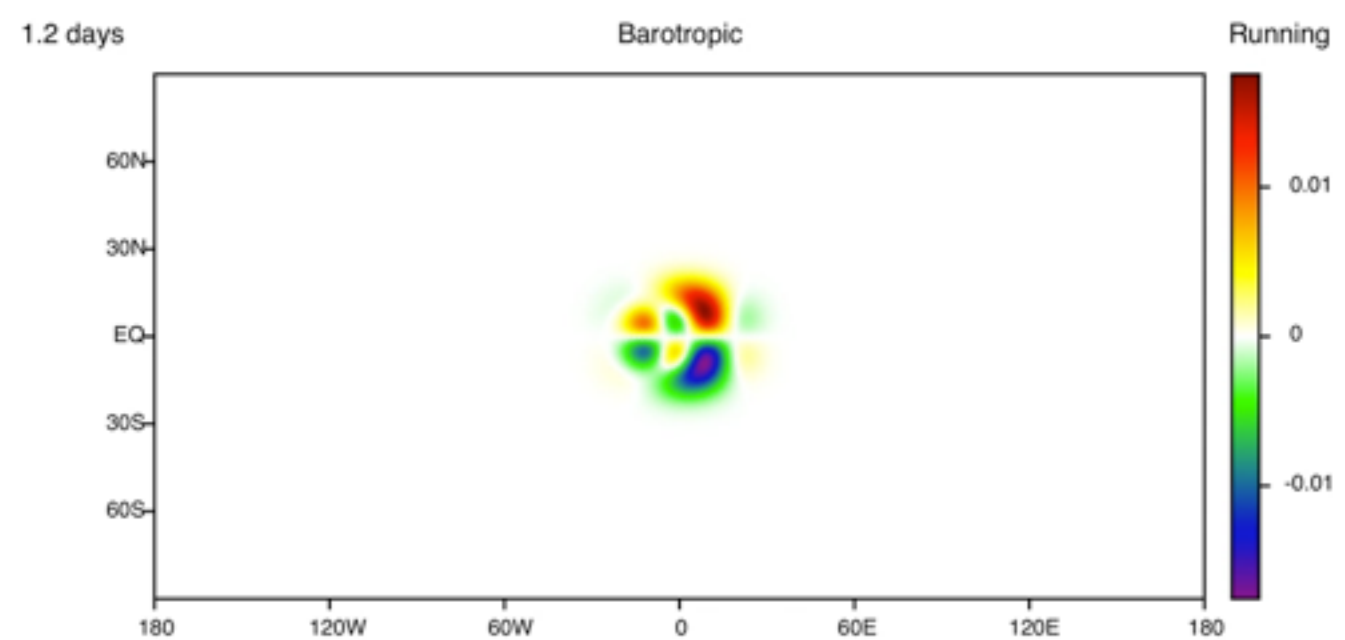










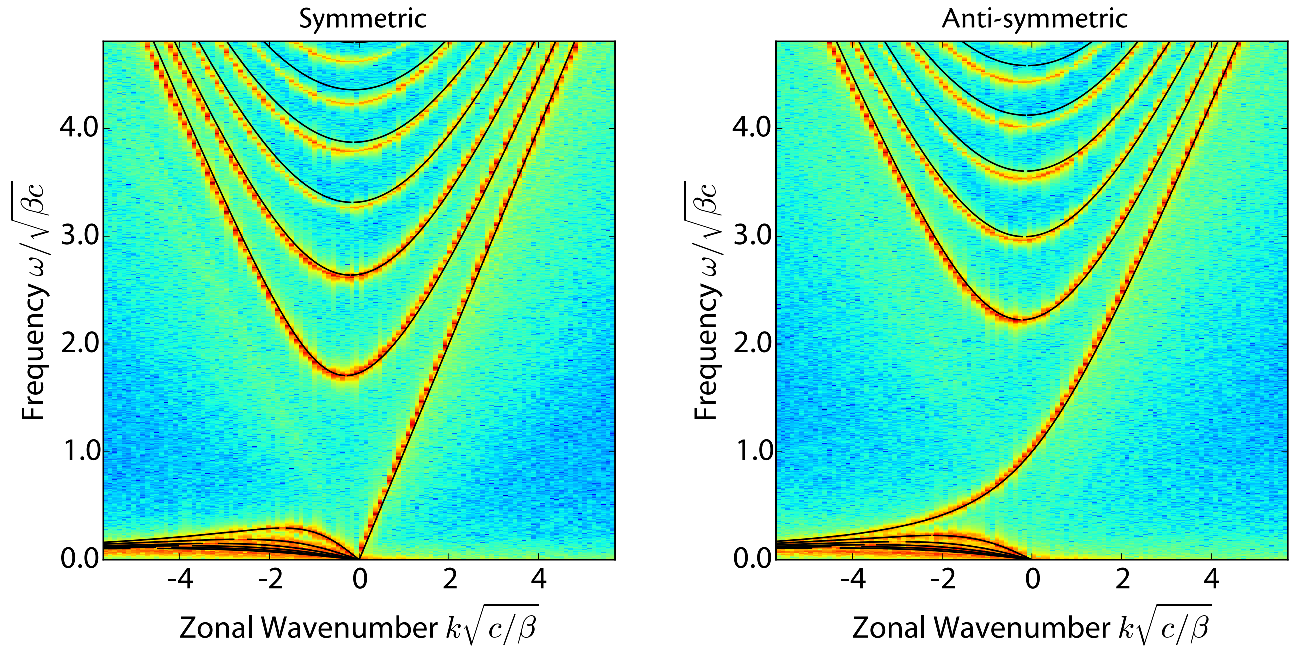
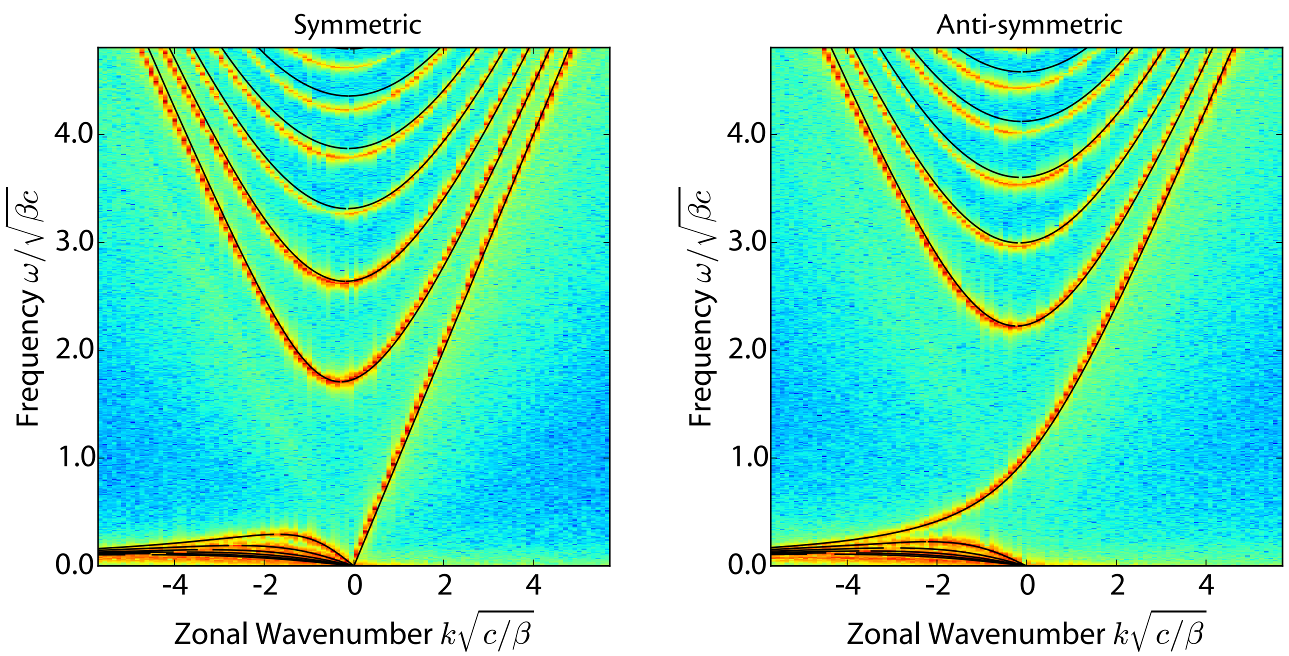


Fig. 3.7 Power spectrum from a numerical simulation of the shallow water equations (colour shading, with red the most intense), with the analytic dispersion relation for equatorial Rossby and gravity waves overlaid (solid black lines, as in Fig. 3.6). The left panel shows the symmetric component, obtained by adding Northern and Southern Hemispheres and with only the odd values of m plotted analytically, and the the right panel plots the antisymmetric component and the even values of m.



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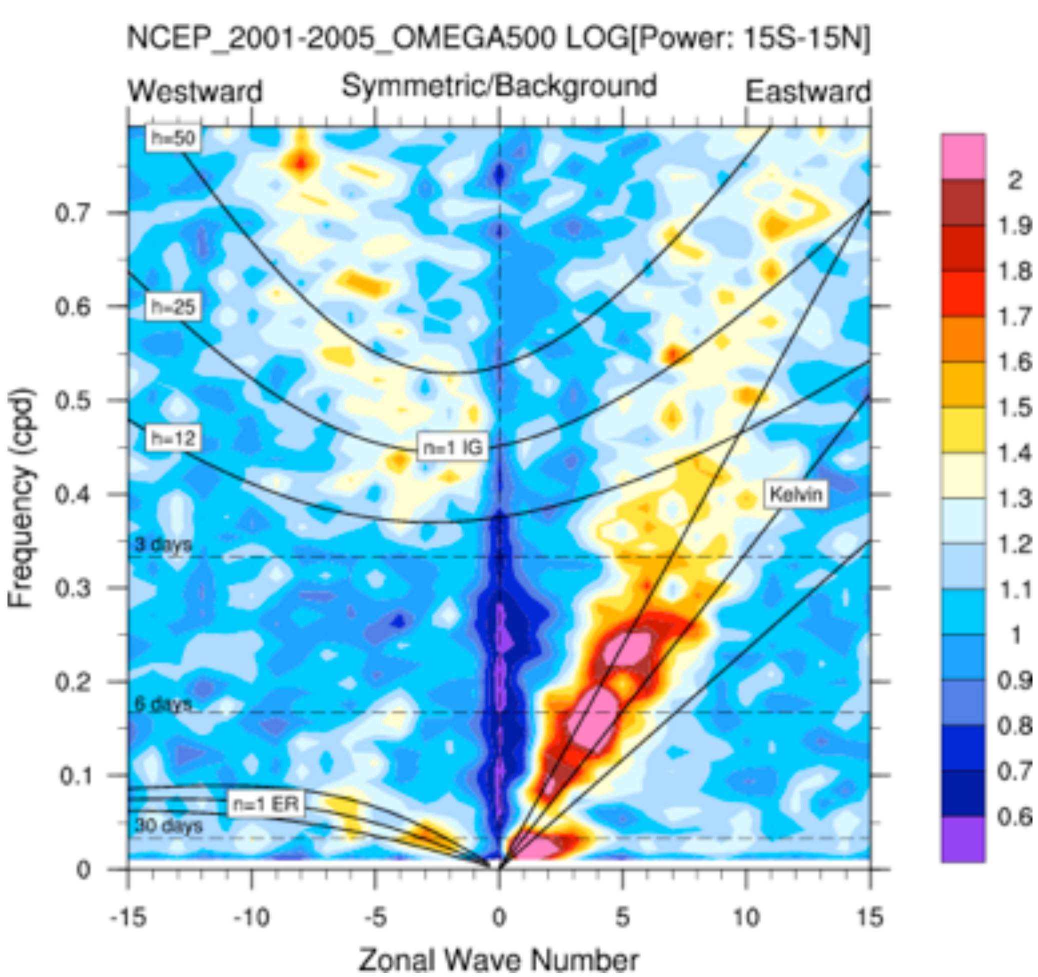

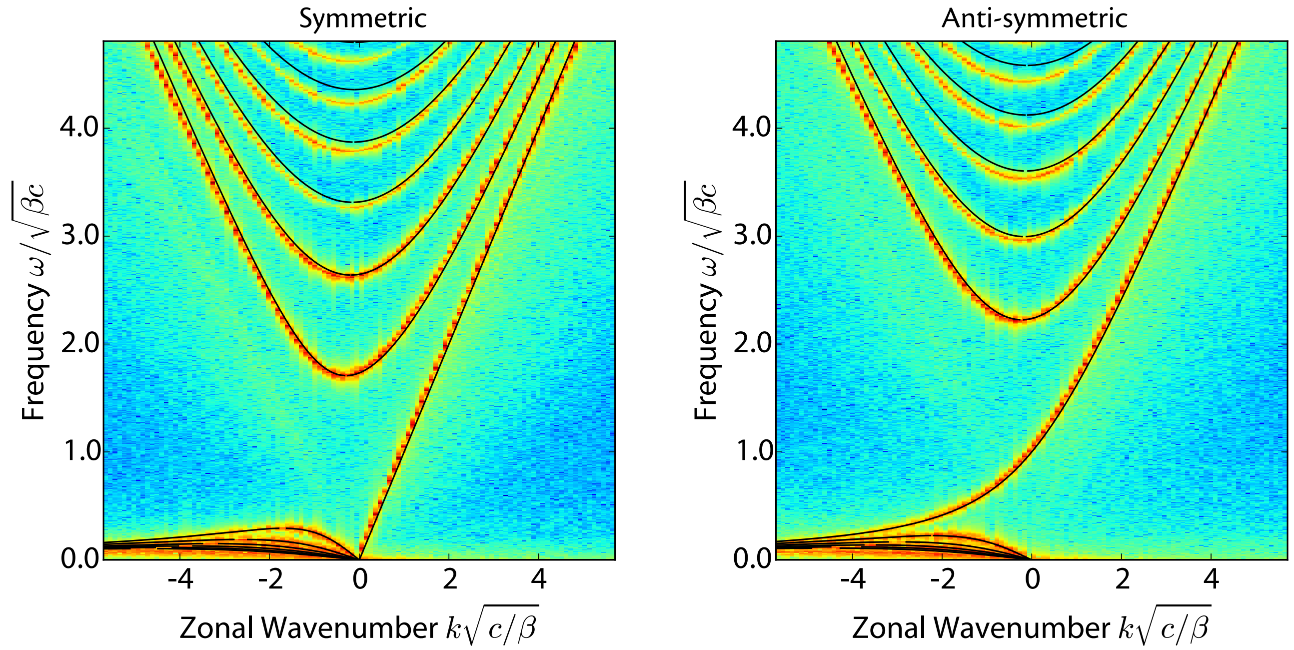

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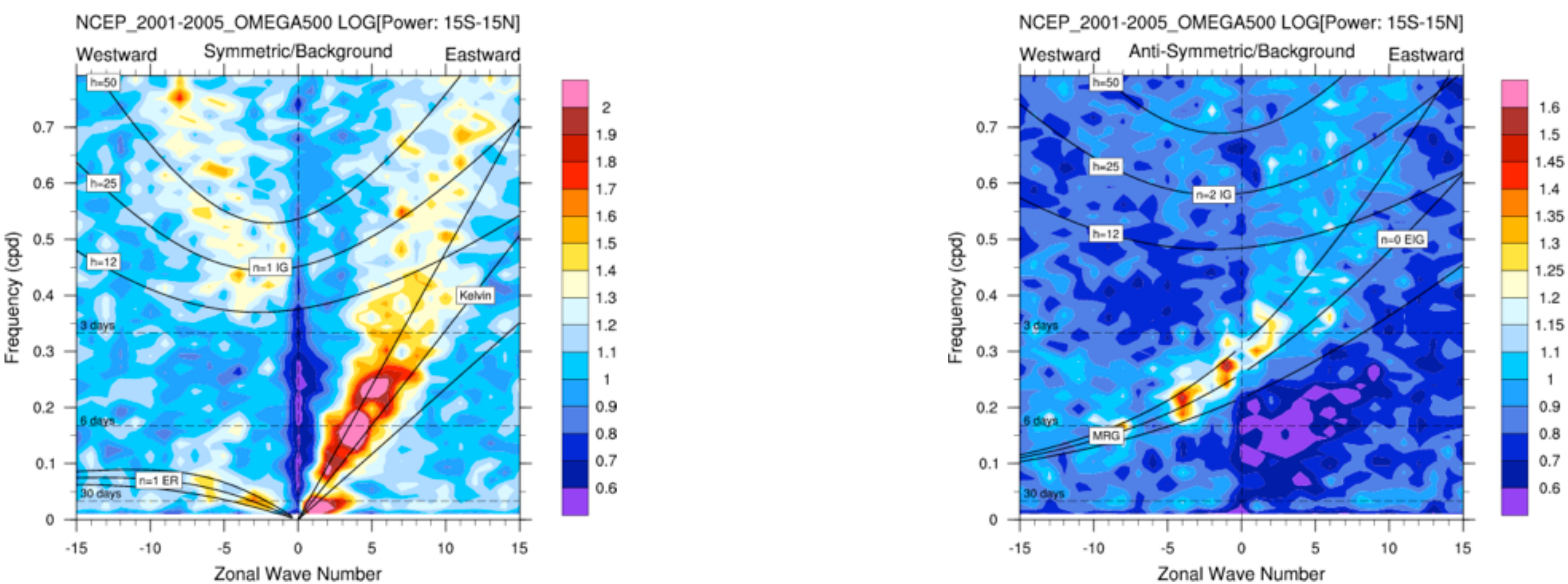
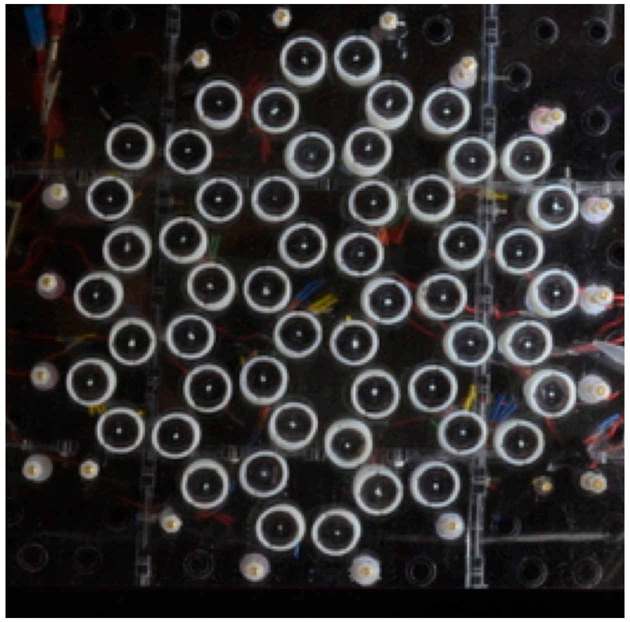
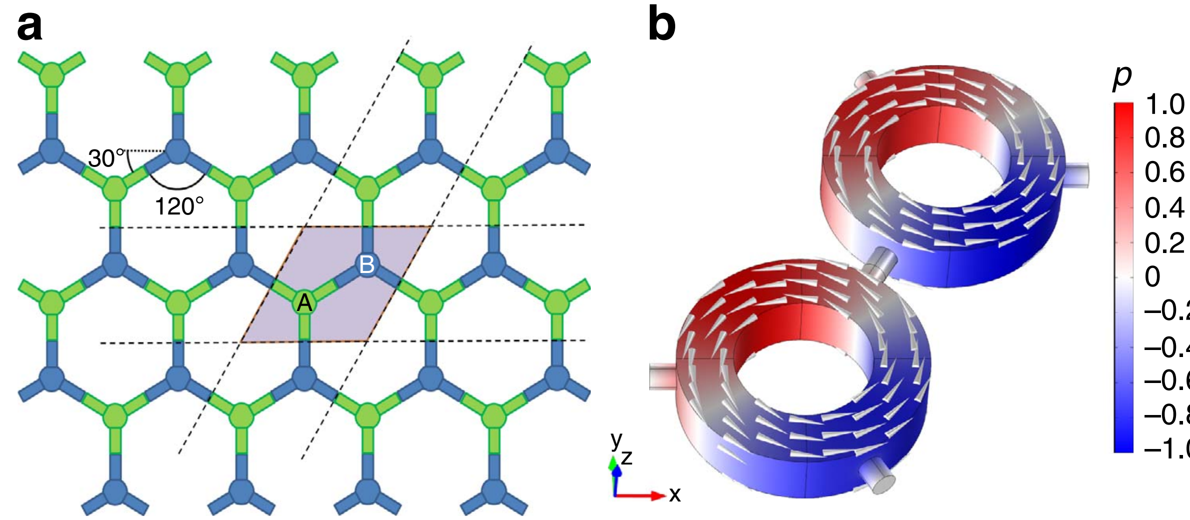


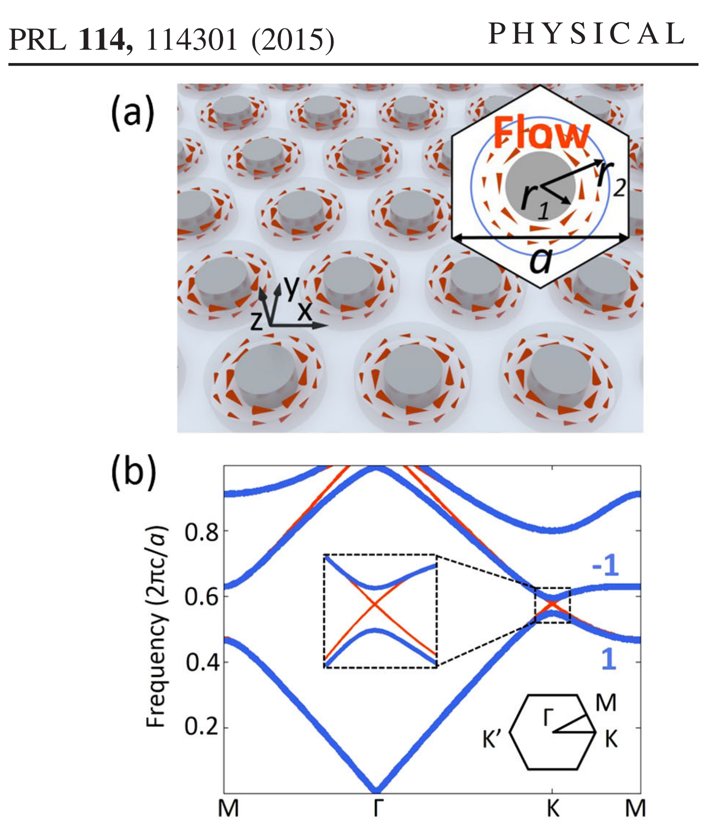
Other Topologically Protected Classical Systems



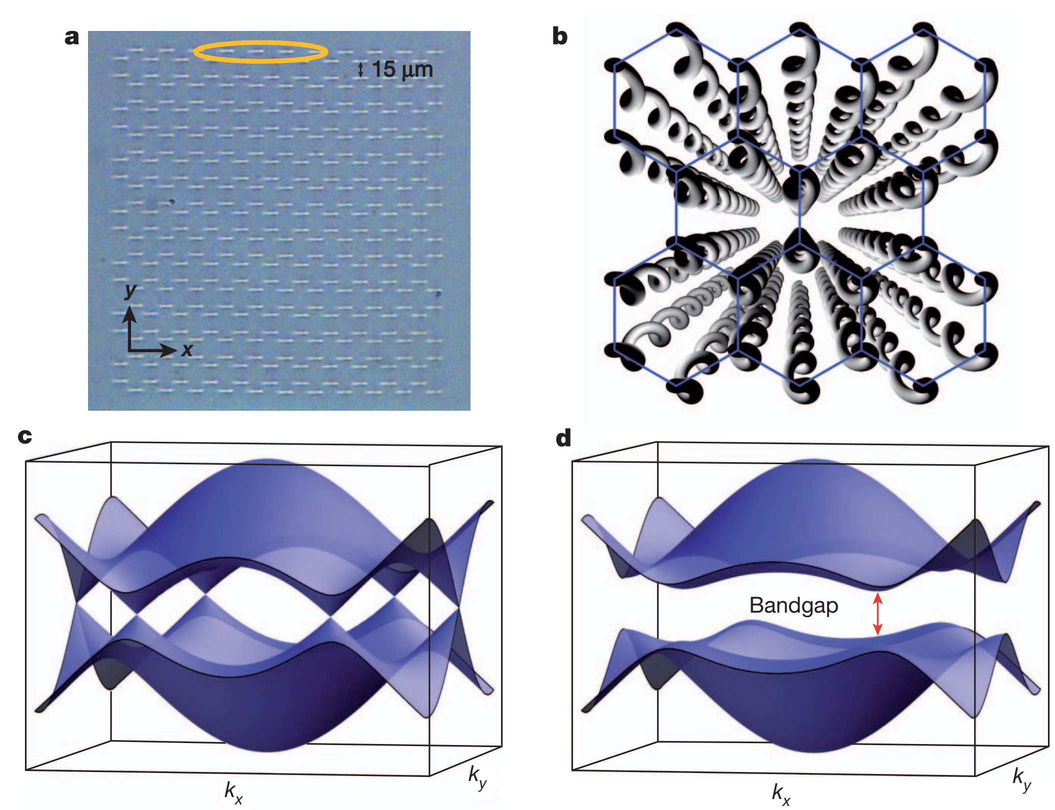
Nash *et al.* PNAS (2015)



Khanikaev *et al.* Nature Communications (2015)



Yang *et al.* PRL (2015)



Rechtsman *et al.* Nature (2013)

Shallow Water Equations on Equatorial **f**-plane and Torus

$$
\begin{aligned}
\partial_t h + \nabla (h\mathbf{u}) &= 0 \\
\partial_t u + \mathbf{u}\cdot\nabla u &= -g\partial_x h + fv \\
\partial_t v + \mathbf{u}\cdot\nabla v &= -g\partial_y h - fu
\end{aligned}
$$

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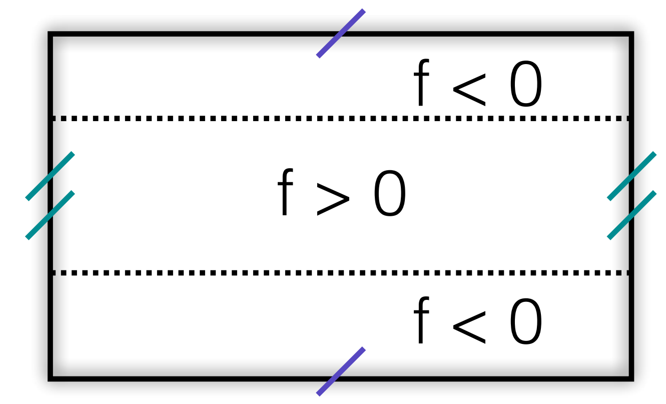

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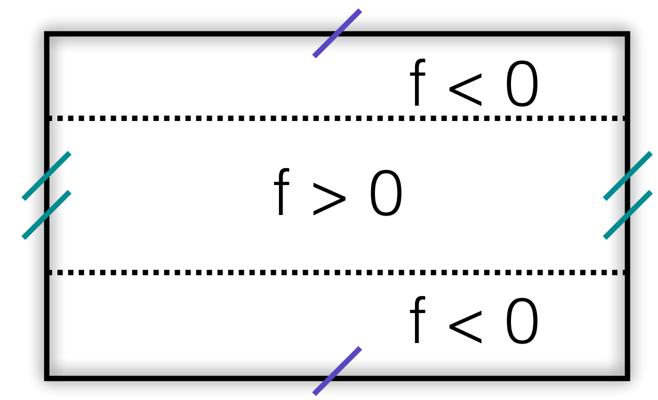

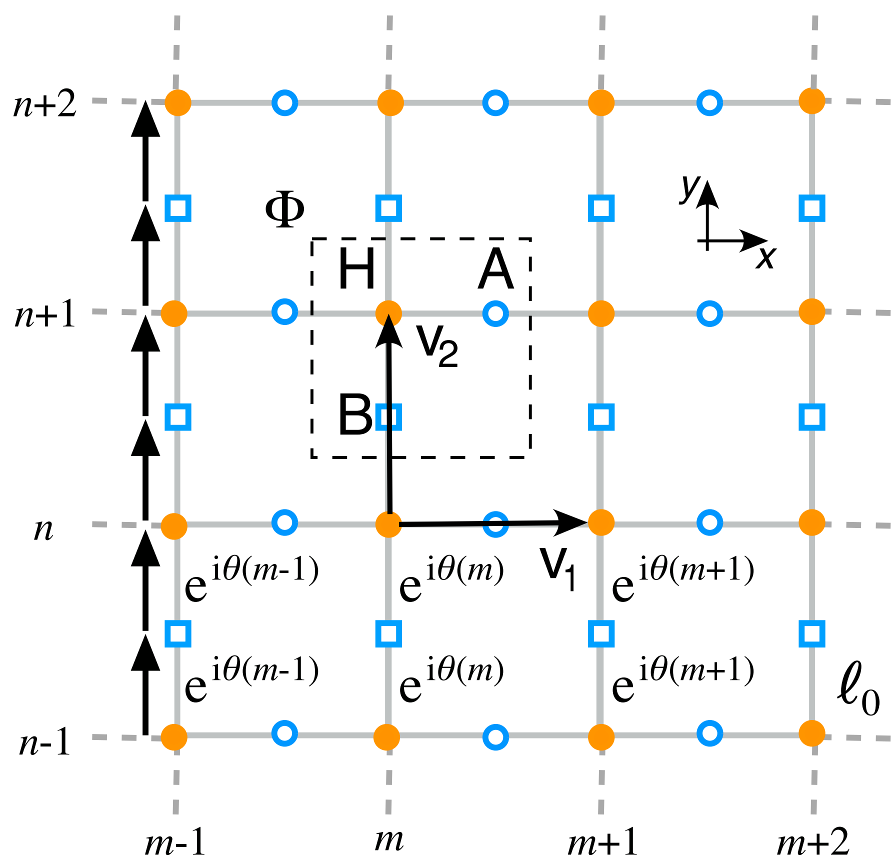

Lieb lattice
(arXiv:1101.4500v1)

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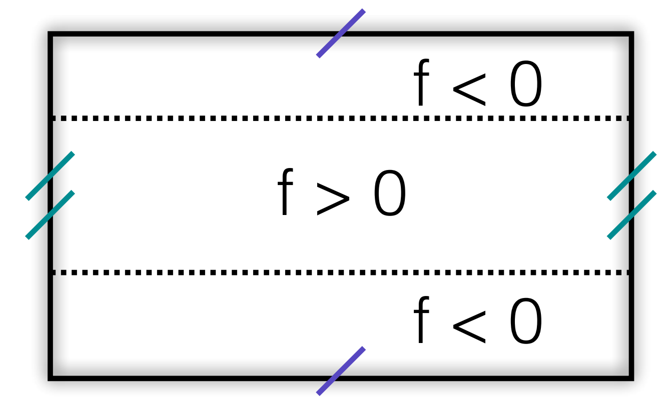

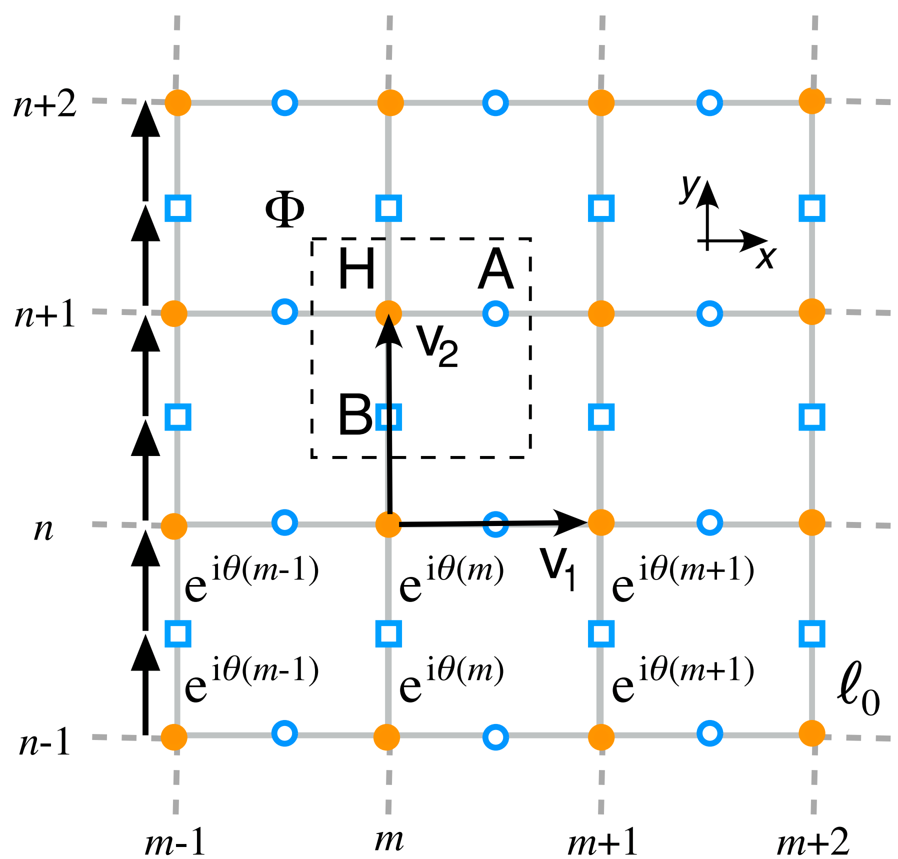

Lieb lattice

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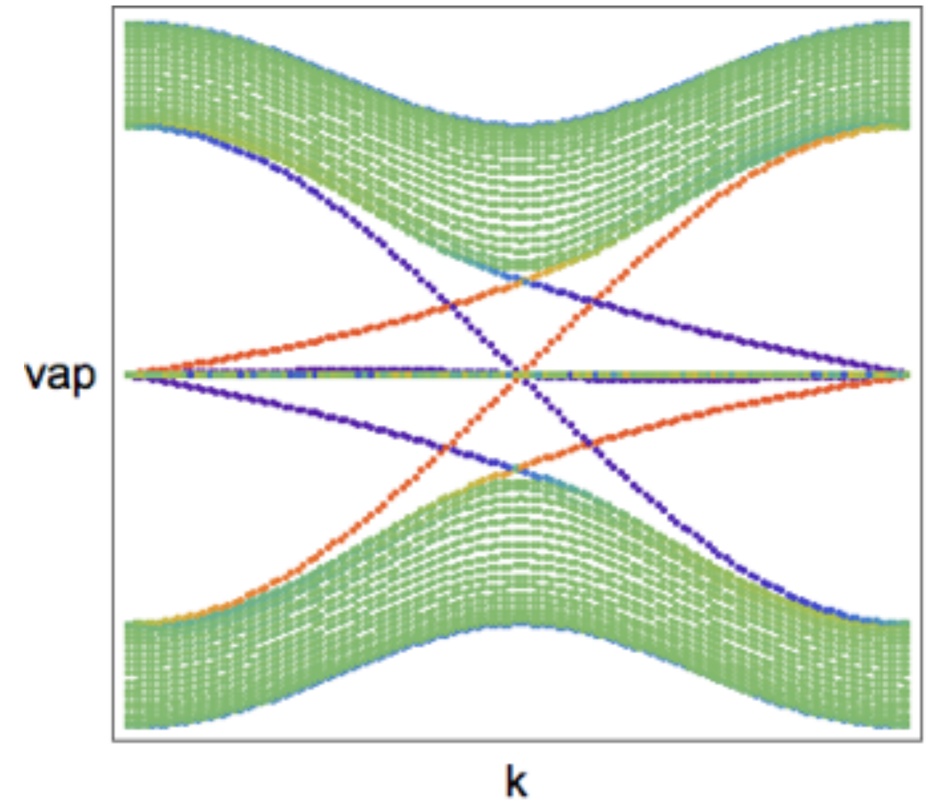

Chern Number & Bulk-Edge Correspondence

$$\omega \begin{pmatrix} \hat{\eta} \\ \hat{u} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} 0 & ck_x & ck_y \\ ck_x & 0 & -if \\ ck_y & if & 0 \end{pmatrix} \begin{pmatrix} \hat{\eta} \\ \hat{u} \\ \hat{v} \end{pmatrix}$$

Chern Number & Bulk-Edge Correspondence

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$$\Psi_+(f, k_x, k_y) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{ck}{\sqrt{c^2k^2+f^2}} \\ \frac{k_x}{k} - i\frac{fk_y}{k\sqrt{c^2k^2+f^2}} \\ \frac{k_y}{k} + i\frac{fk_x}{k\sqrt{c^2k^2+f^2}} \end{pmatrix}$$

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$$k_x = \cos\varphi \cos\theta$$

$$k_y = \sin\varphi \cos\theta$$

$$f = \sin\theta$$

Chern Number & Bulk-Edge Correspondence

$$\omega \begin{pmatrix} \hat{\eta} \\ \hat{u} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} 0 & ck_x & ck_y \\ ck_x & 0 & -if \\ ck_y & if & 0 \end{pmatrix} \begin{pmatrix} \hat{\eta} \\ \hat{u} \\ \hat{v} \end{pmatrix}$$

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$$k_x = \cos\varphi \cos\theta$$
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$$f = \sin\theta$$

$$\Psi_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} \sin\theta \\ \cos\varphi - i\cos\theta\sin\varphi \\ \sin\varphi + i\cos\theta\cos\varphi \end{pmatrix}$$

Chern Number & Bulk-Edge Correspondence

$$\omega \begin{pmatrix} \hat{\eta} \\ \hat{u} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} 0 & ck_x & ck_y \\ ck_x & 0 & -if \\ ck_y & if & 0 \end{pmatrix} \begin{pmatrix} \hat{\eta} \\ \hat{u} \\ \hat{v} \end{pmatrix}$$

$$\Psi_+(f, k_x, k_y) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{ck}{\sqrt{c^2k^2+f^2}} \\ \frac{k_x}{k} - i\frac{fk_y}{k\sqrt{c^2k^2+f^2}} \\ \frac{k_y}{k} + i\frac{fk_x}{k\sqrt{c^2k^2+f^2}} \end{pmatrix}$$

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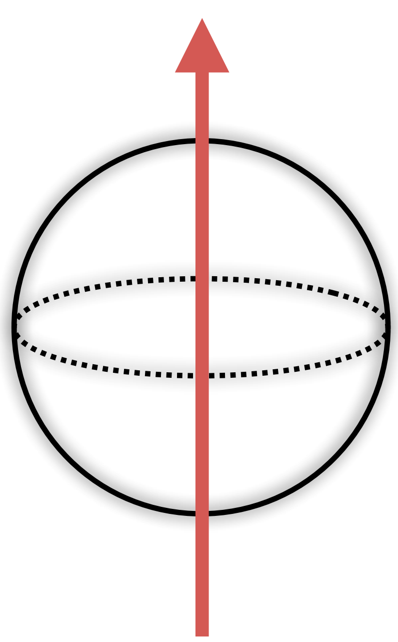

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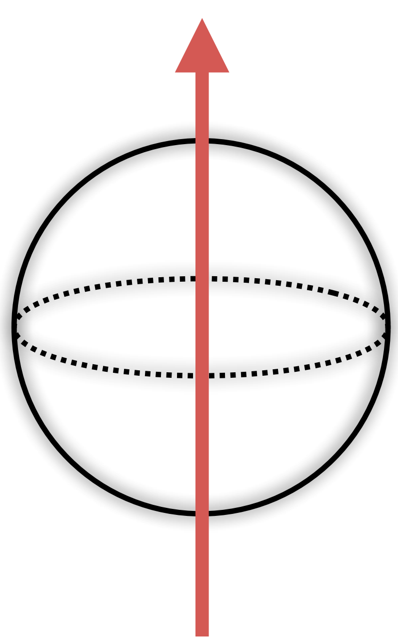

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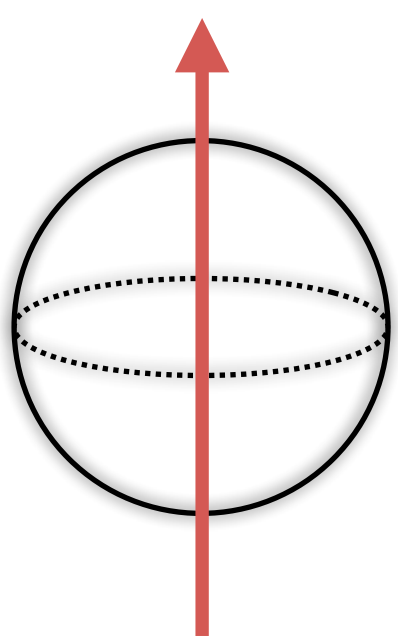

$$\mathbf{A}_+^N = \mathbf{A}_+^S + 2\nabla_s\varphi$$

$$\Delta\mathcal{C}_+ = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \, (\mathbf{A}_+^N - \mathbf{A}_+^S)\cdot\hat{e}_\varphi$$

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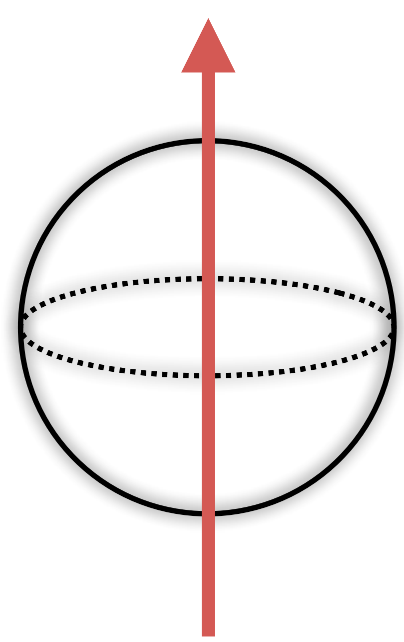

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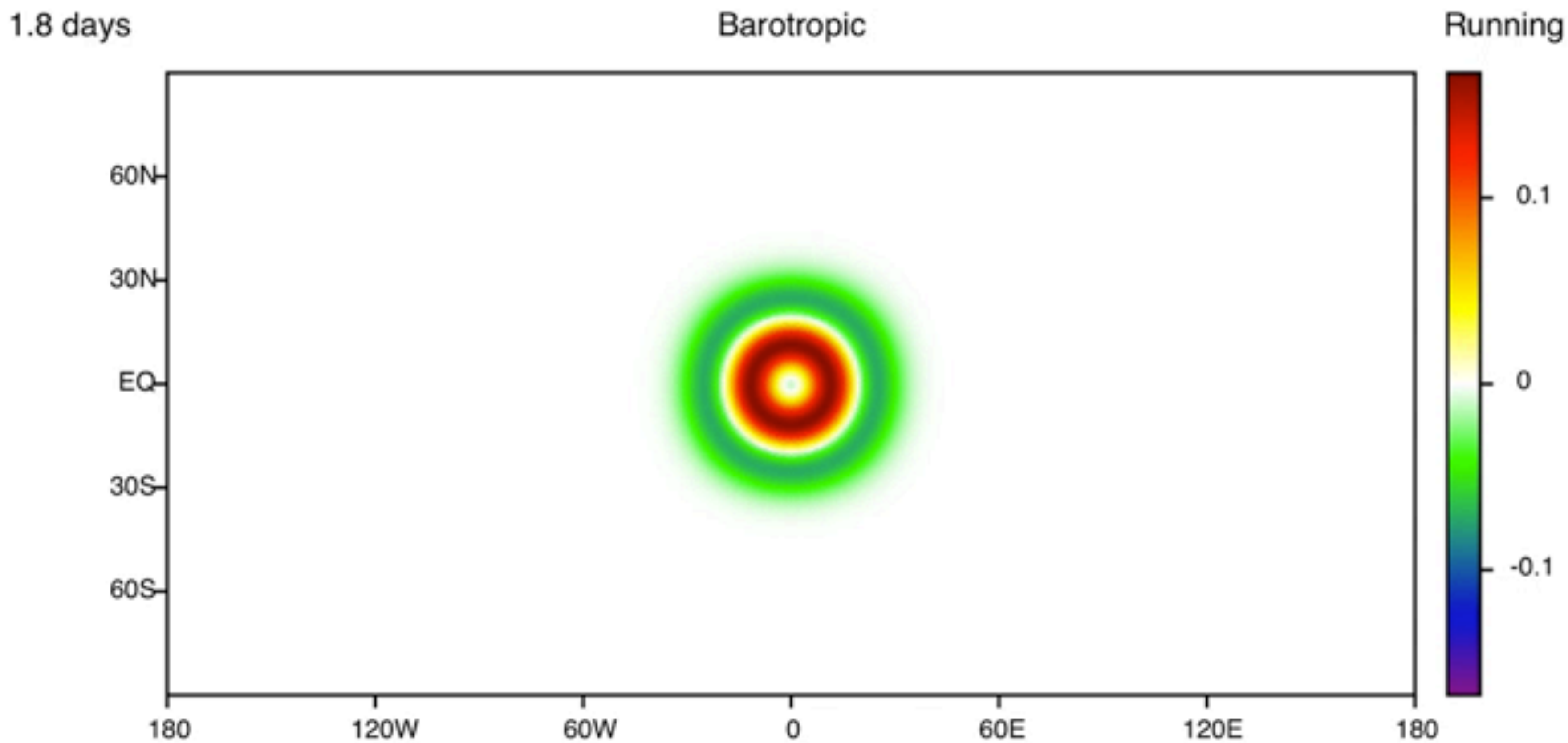
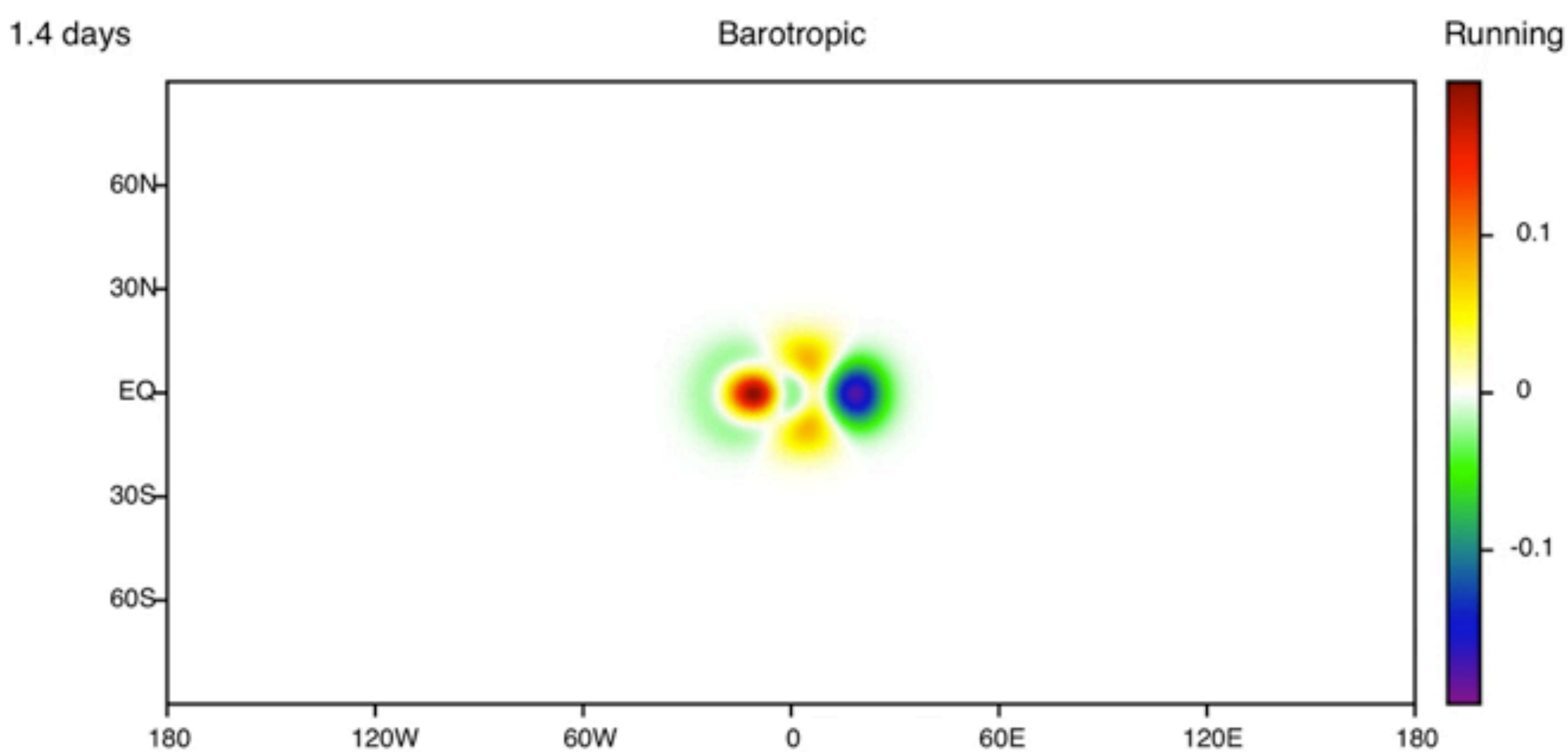
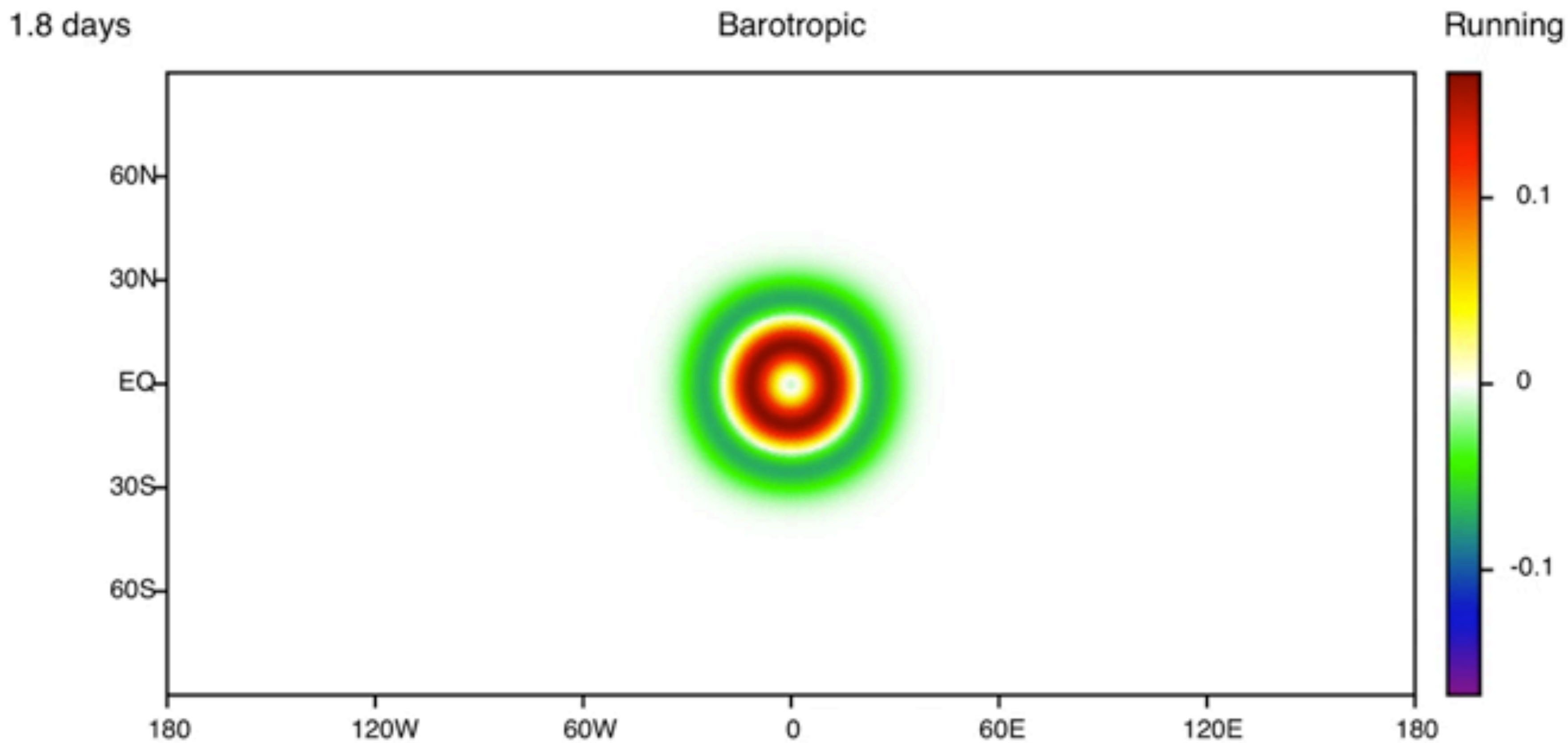
$$\begin{aligned} \Delta\mathcal{C}_+ &= \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, (\mathbf{A}_+^N - \mathbf{A}_+^S)\cdot\hat{e}_\varphi \\ &= 2 \end{aligned}$$

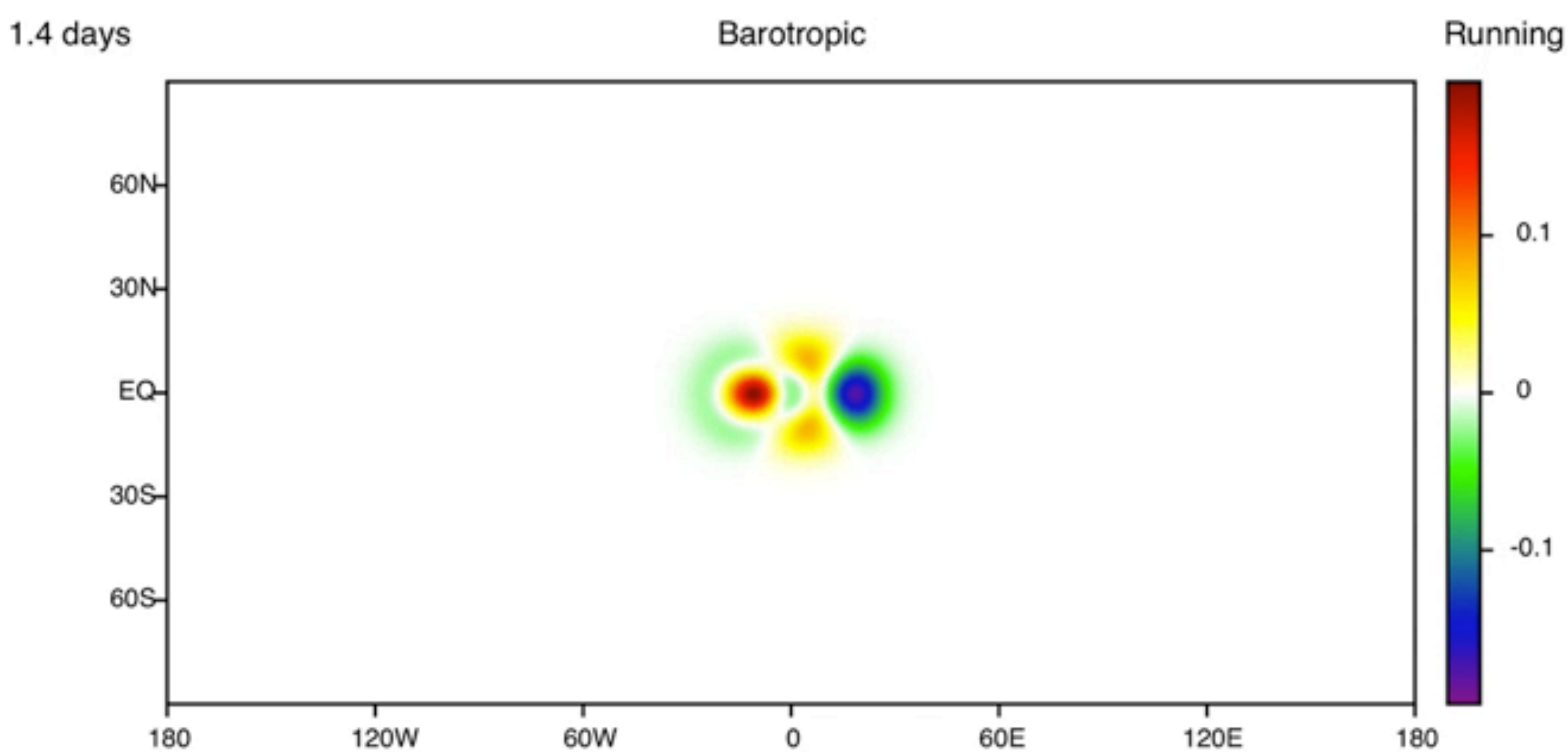
Protection from Obstacles



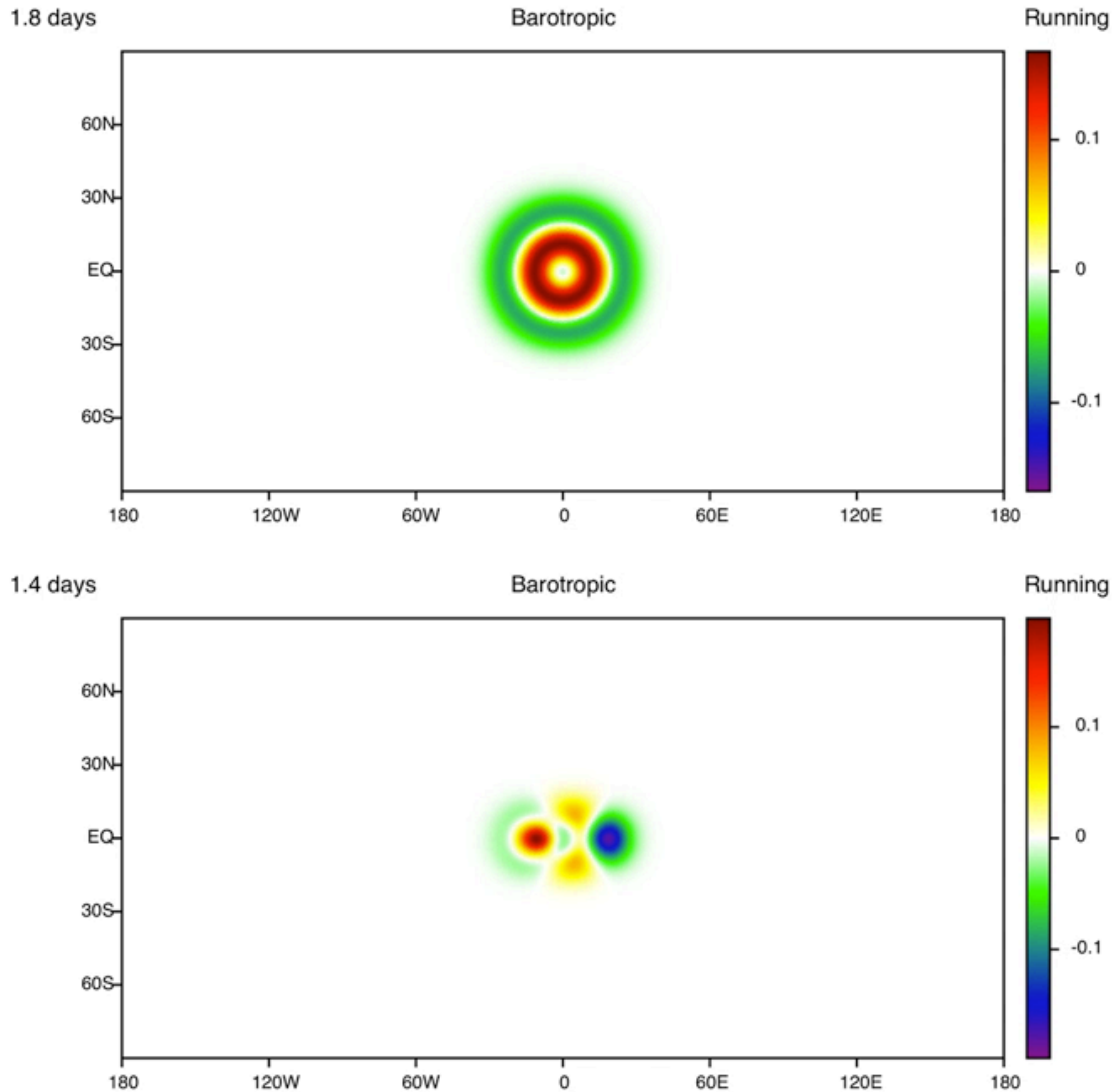


Protection from Obstacles





Protection from Obstacles



Other Topologically Protected Fluid Waves

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- Edge modes (eg. Kelvin waves at ocean basin boundaries)

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- Edge modes (eg. Kelvin waves at ocean basin boundaries)
- Magneto-Rossby waves (slow & fast)
- Fluids with mean flows (breaking time-reversal symmetry)?
- Relationship to exact coherent states?

Protection from Interactions / Nonlinearities?

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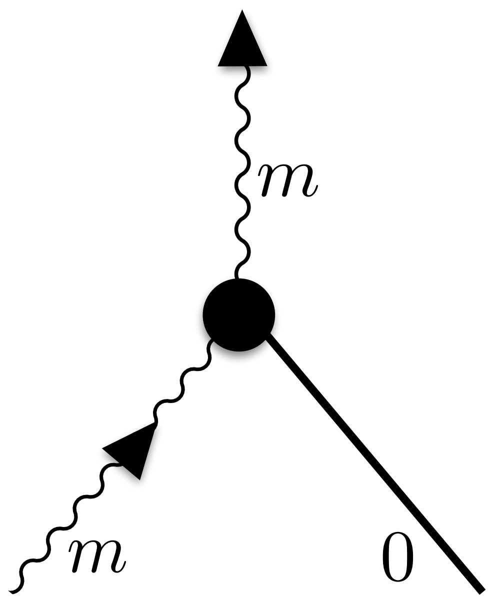
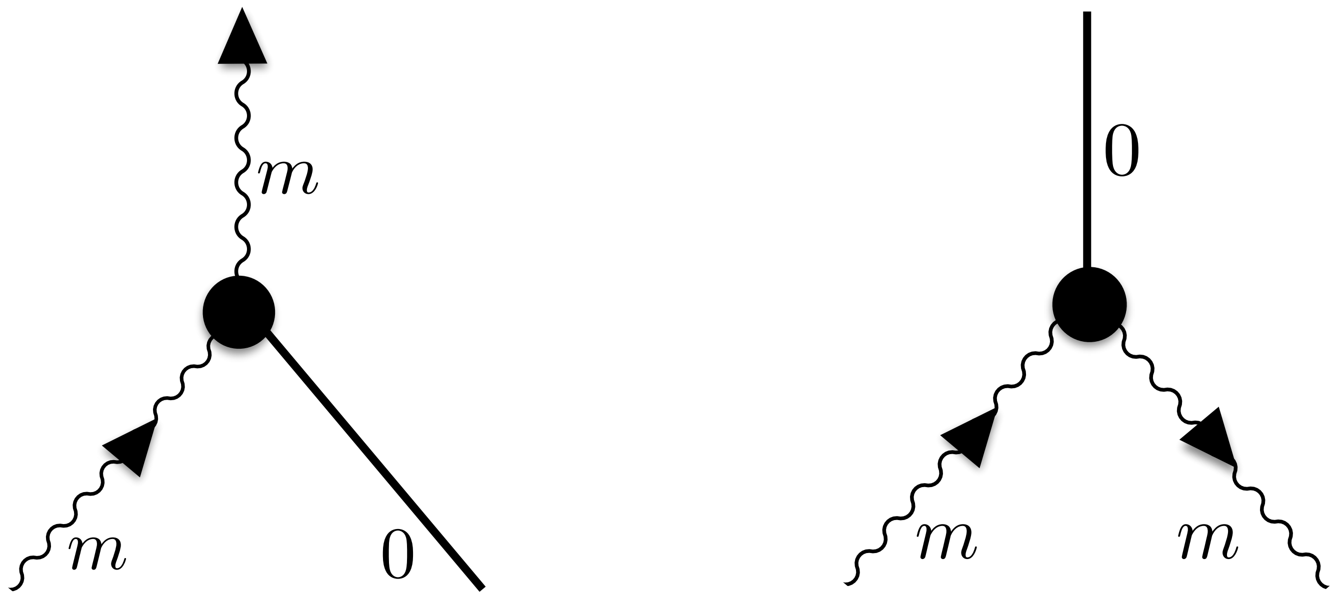
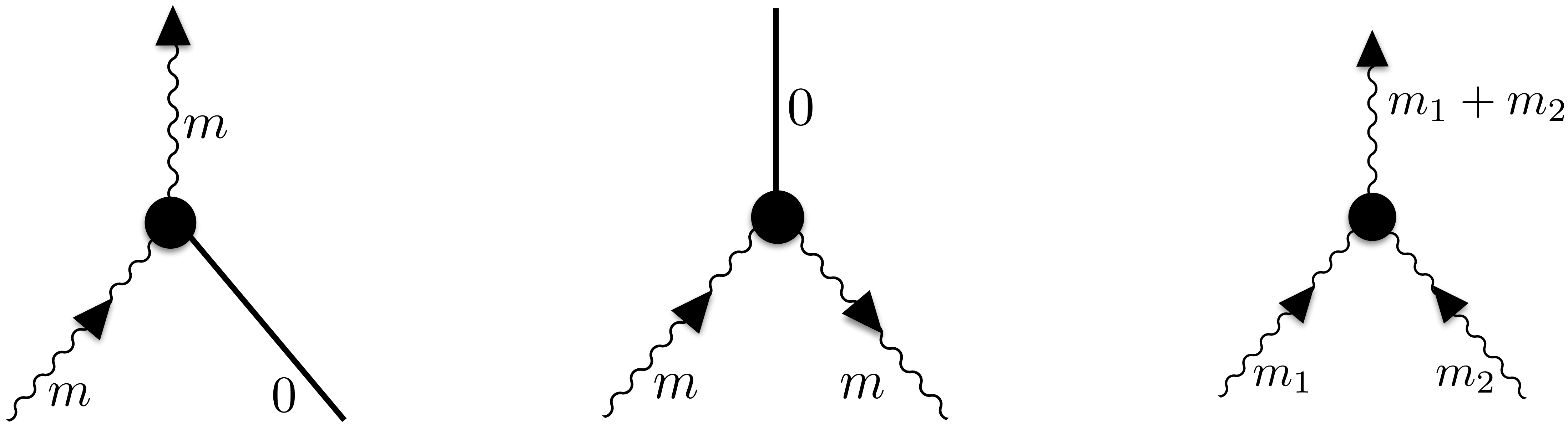


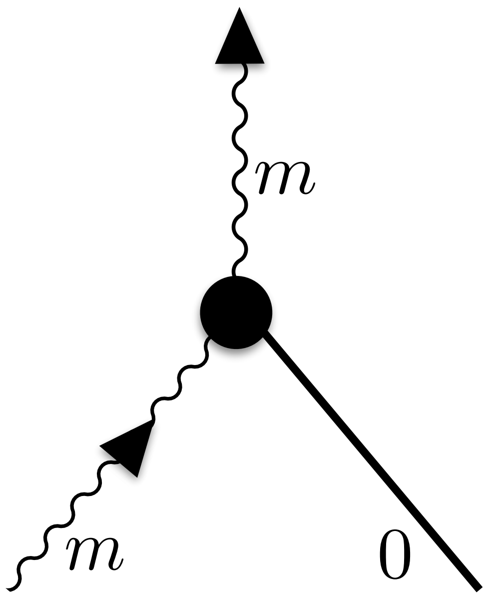
Protection from Interactions / Nonlinearities?

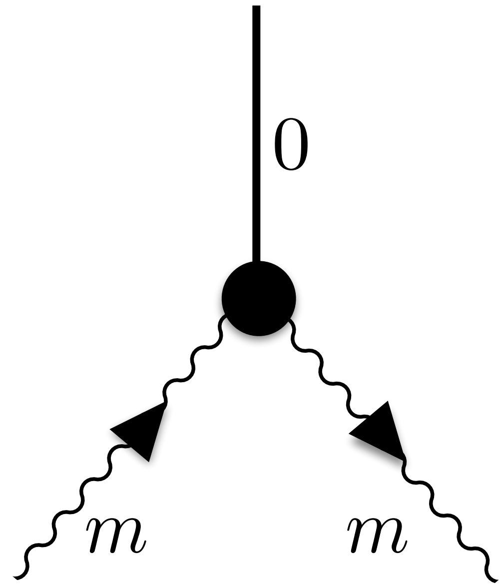


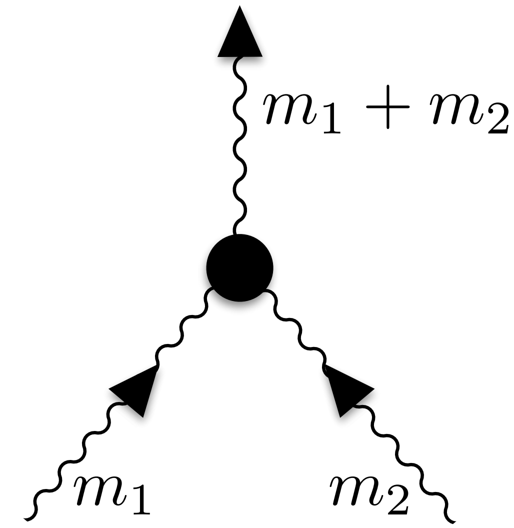
Protection from Interactions / Nonlinearities?



Protection from Interactions / Nonlinearities?







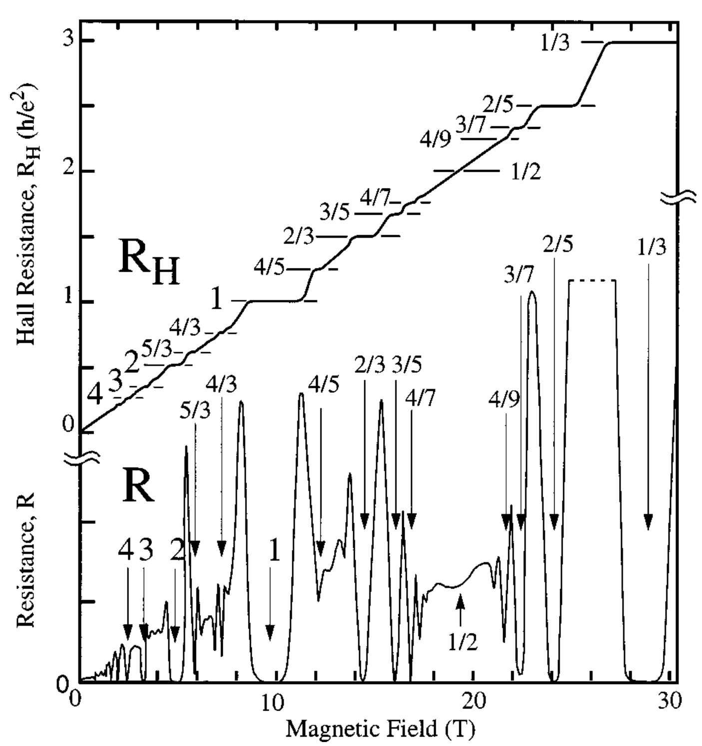
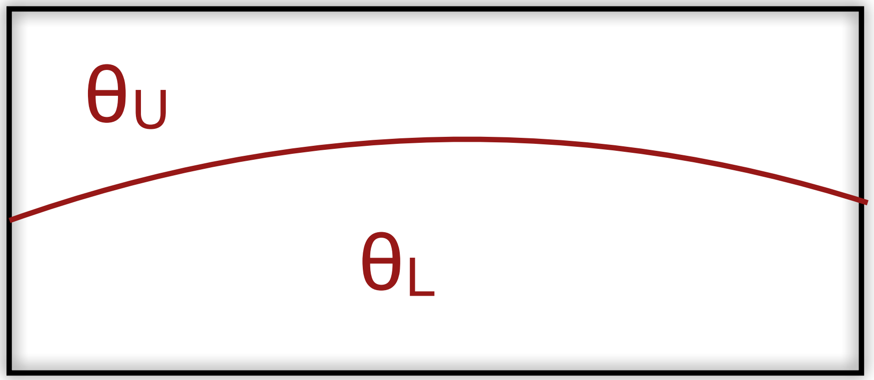


FIG. 18. The FQHE as it appears today in ultrahigh-mobility modulation-doped GaAs/AlGaAs 2DESs. Many fractions are visible. The most prominent sequence, $\nu = p/(2p \pm 1)$, converges toward $\nu = 1/2$ and is discussed in the text.

Fractional Quantum Hall Effect

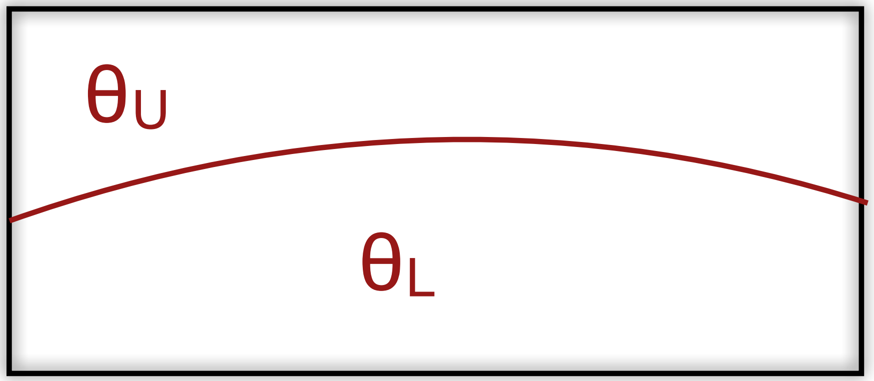
Thank You

2 Layer Primitive Equations



$$\vec{v} = \hat{r} \times \vec{\nabla}\psi + \vec{\nabla}\chi,$$

2 Layer Primitive Equations



$$\vec{v} = \hat{r} \times \vec{\nabla}\psi + \vec{\nabla}\chi,$$

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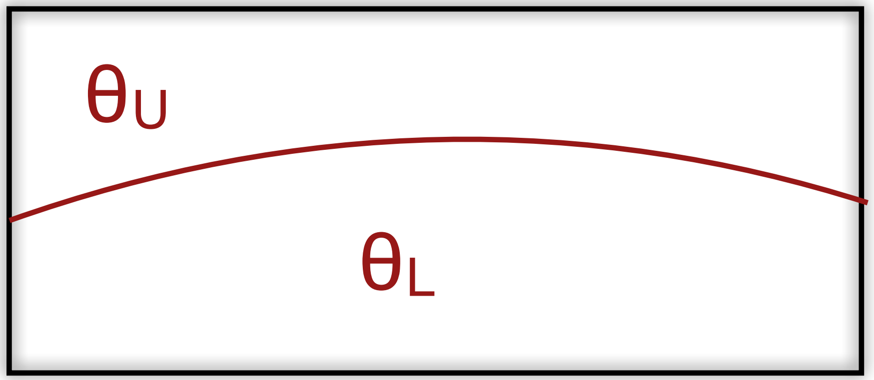

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$$F[A,B] \equiv \vec{\nabla} \cdot (A\vec{\nabla}B),$$

$$\dot{\bar{q}} = J[\bar{q},\bar{\psi}] + J[\hat{q},\hat{\psi}] - F[\hat{q},\hat{\chi}] - J[\hat{\delta},\hat{\chi}] - F[\hat{\delta},\hat{\psi}],$$

$$\dot{\hat{q}} = J[\hat{q},\bar{\psi}] + J[\bar{q},\hat{\psi}] - F[\bar{q},\hat{\chi}],$$

$$\dot{\hat{\delta}} = J[\bar{q},\hat{\chi}] + F[\hat{q},\bar{\psi}] + F[\bar{q},\hat{\psi}] - \nabla^2(\hat{K} + C_p B\bar{\theta}),$$

$$\dot{\bar{\theta}} = J[\bar{\theta},\bar{\psi}] + J[\hat{\theta},\hat{\psi}] - F(\hat{\theta},\hat{\chi}), \text{ and}$$

$$\dot{\hat{\theta}} = J[\hat{\theta},\bar{\psi}] + J[\bar{\theta},\hat{\psi}] - F(\bar{\theta},\hat{\chi}) + \bar{\theta}\hat{\delta}.$$