From Turbulence Transition to Shell Buckling



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Emergent Complexity in Physical Systems

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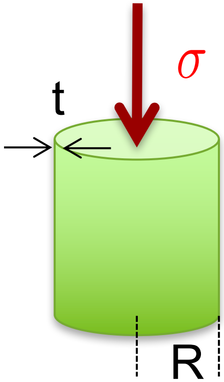
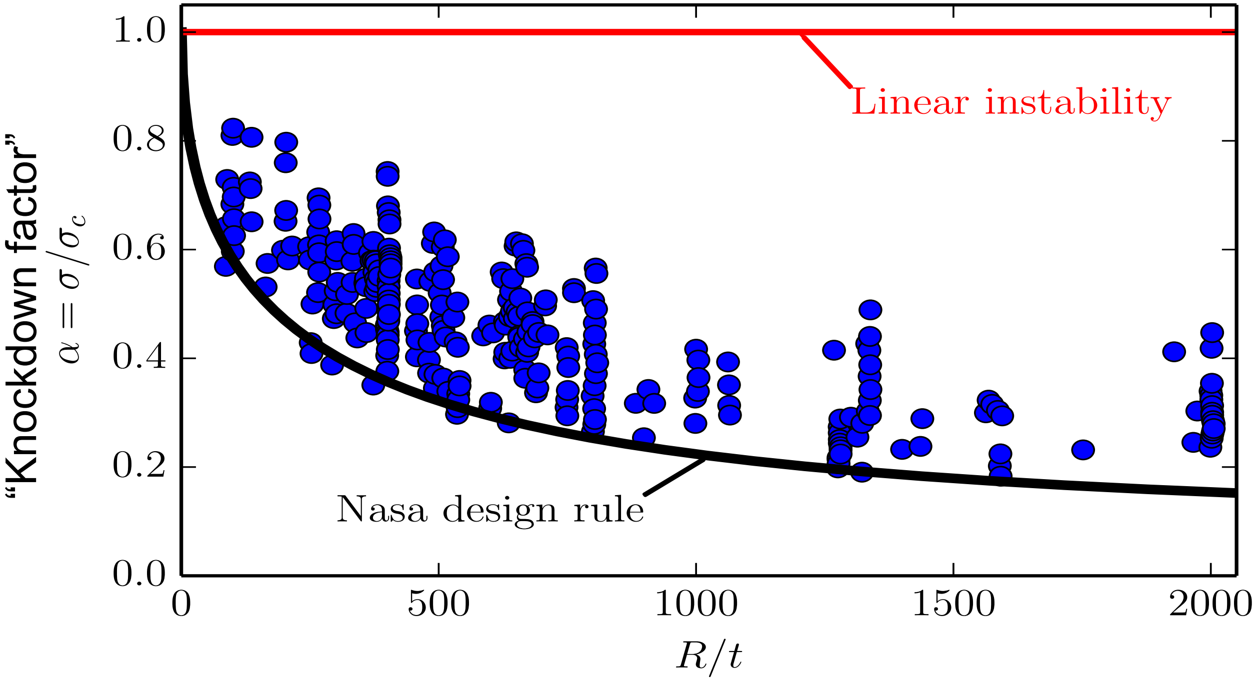
The (seemingly simple) question
How much load can a cylinder shell carry?

Axially loaded cylinder shell



$$\sigma_C = \frac{E}{3(1-\nu^2)}\frac{t}{R}$$

Buckling threshold (NASA)

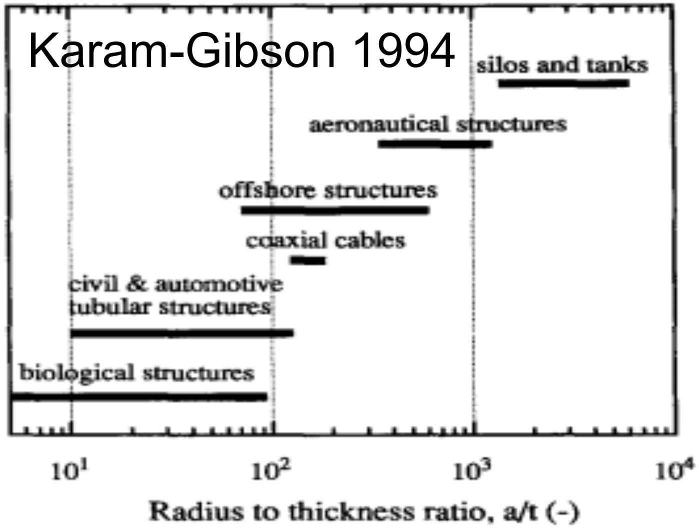


Observation: Failure of linear theory!
Nominally identical shells buckle at different load (‘stochastic’)
Thinner shells tend to buckle for lower load
Captured by empirical design rules (‘knockdown factors’)

Explanation: Extreme sensitivity to imperfections
(von Karman & Tsien 1939, Koiter 1945, …)

Thin shell stuctures

Structural regidity at minimal weight



Properties of curved shells

Exceptional load carrying capacity

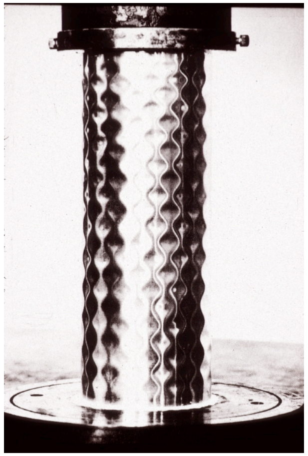
Challenging to predict buckling conditions

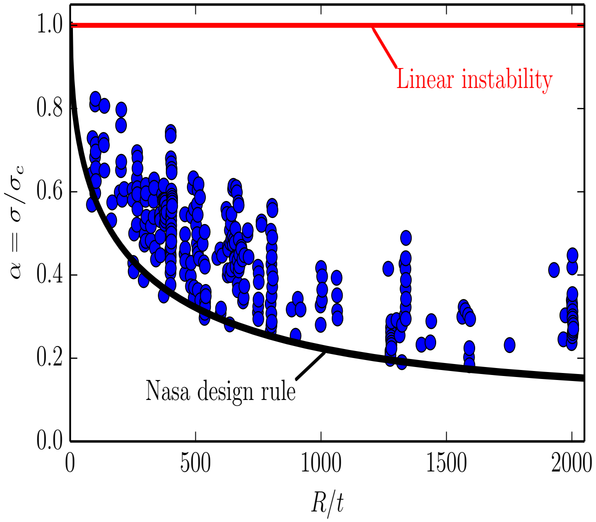
Reason: Imperfection sensitivity

Thin shell stuctures

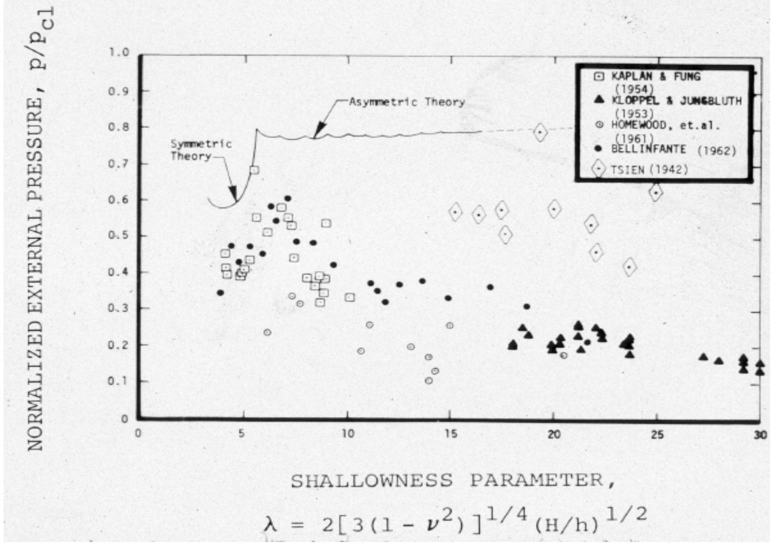
The two canonical examples

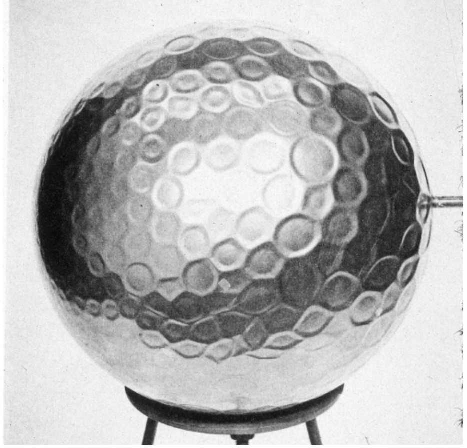
Axially loaded cylinder





Sphere under uniform pressure

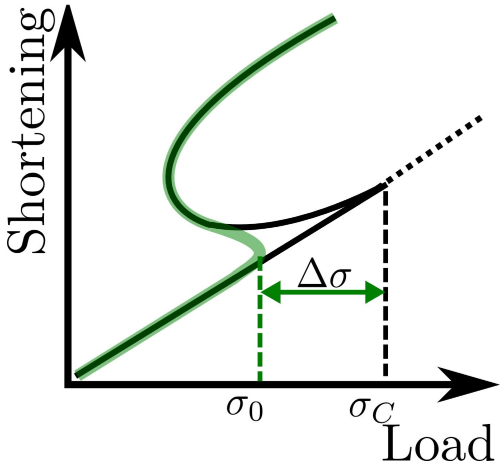




Shellbuckling.com

How to predict buckling loads?
The classical approach and its limitations

Classical approach: Imperfections modify the critical load

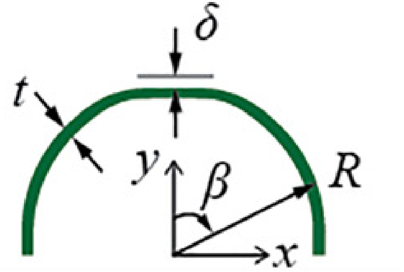


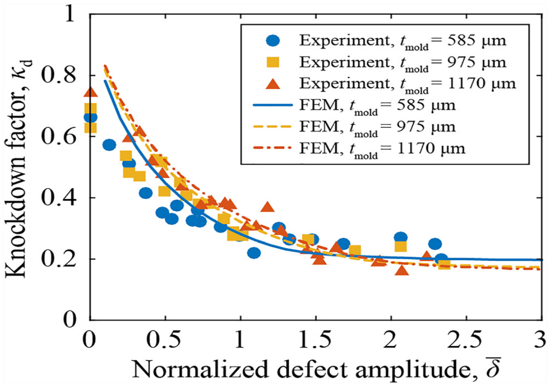
Scenario: *Linear* instability of the *imperfect* system

Given imperfections -> predict buckling load

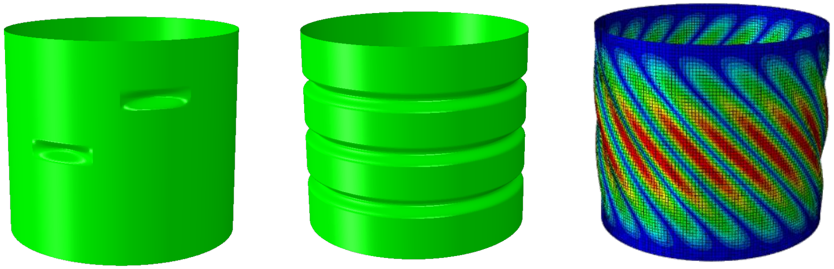
Example: Lee et al. 2016, Hutchinson 2016

Spherical cap





Problem: Imperfections typically unknown a priori



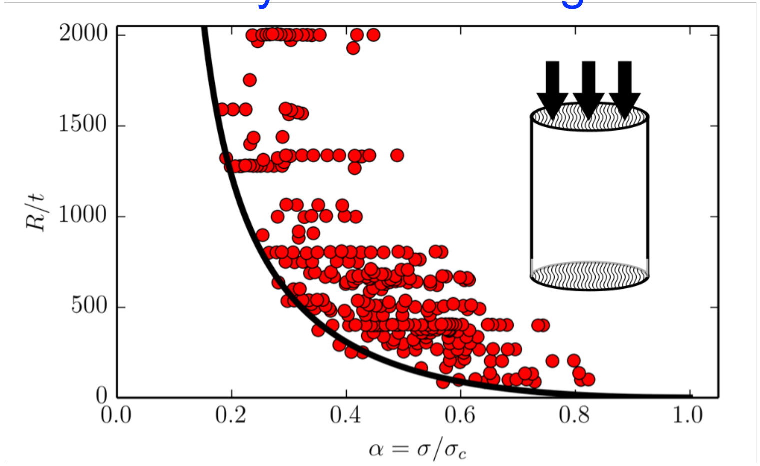
Credit: E. Lozano

Idea: Fully nonlinear dynamical systems approach?

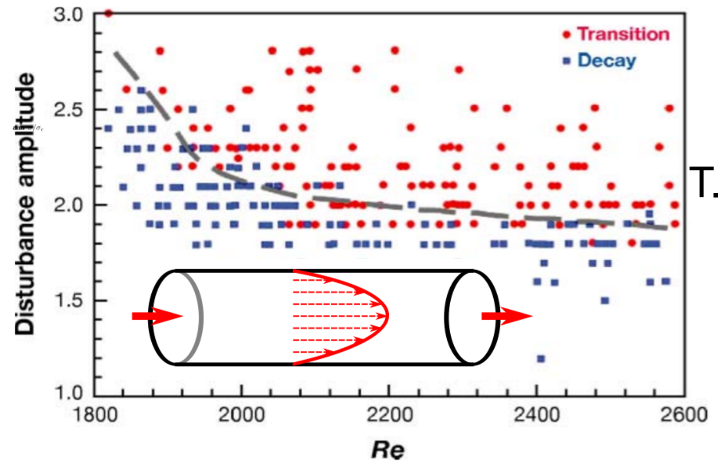
From turbulence transition to shell buckling

A new approach inspired by turbulence studies

Cylinder buckling



Pipe flow transition triggered by jet injection



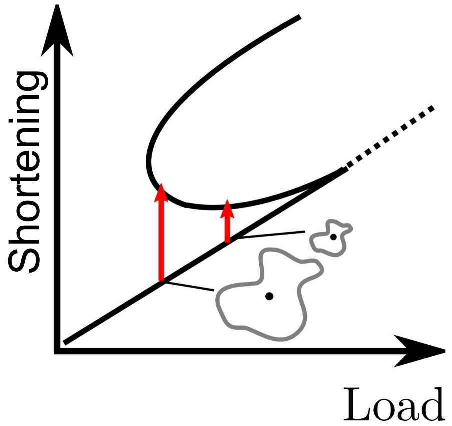
T. Mullin

Linearly stable!

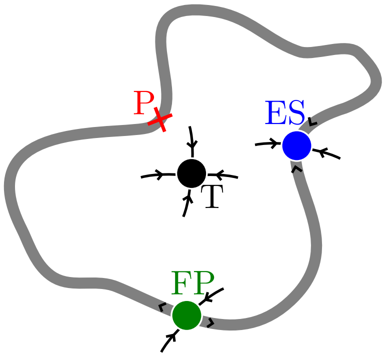
Similarity suggests paradigm shift:

Linear instability of the *imperfect* system

→ *Nonlinear* finite amplitude instability of the *perfect* system



Characterize basin of attraction as a function of load

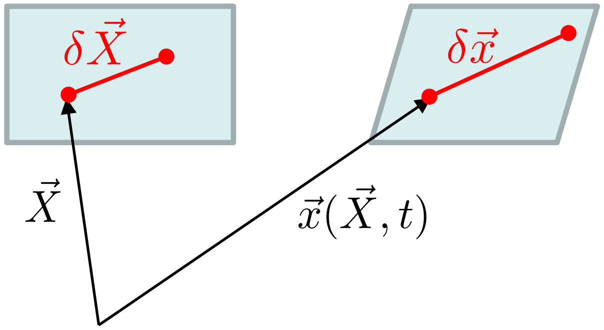


- ● : trivial (unbuckled) state
- ● : unstable equilibria
- ● : edge state

Geometric nonlinearity

Why is the problem intrinsically nonlinear?

Elastic response: deformation $\xrightarrow{\text{nonlinear}}$ strain $\longrightarrow$ stress



Displacement field $\vec{u}(\vec{X},t) = \vec{x} - \vec{X}$

Lagrangian coordinate

Strain: changed distance between material points

$$\delta\vec{x}^2 - \delta\vec{X}^2 = 2\epsilon_{ij}\delta X_i \delta X_j$$

With Green-Lagrange strain tensor

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right)$$

Nonlinear function of displacement field!!

Linear stress-strain relation

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\epsilon_{ij} + \frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}\right)$$

Isotropic Hookean material

Cauchy stress

Map back to reference configuration

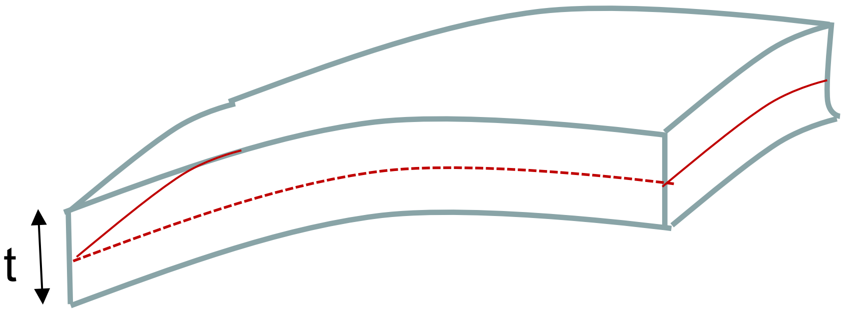
Force balance: $\cancel{\rho\partial_{tt}\vec{u}} + \gamma\partial_t\vec{u} = \rho\vec{g} + \nabla\cdot\tau$

overdamped

Piola-Kirchhoff stress tensor

Nonlinear dynamical system: $\partial_t\vec{u} = \mathcal{N}(\vec{u})$

Equations for a thin elastic shell
The Donnell-Mushtari-Vlassov theory



Deformation of the mid-surface

$$(x_1, x_2, h(x_1, x_2)) \rightarrow (x_1 + u_1, x_2 + u_2, h(x_1, x_2) + \underbrace{w}_{})$$

Transverse displacement

Asymptotic reduction from 3D: $\varepsilon = \frac{t}{L} << 1$

$w = \mathcal{O}(\varepsilon), u_\alpha = \mathcal{O}(\varepsilon^2)$

In-plane strain: $\epsilon_{\alpha\beta} = E_{\alpha\beta} + zK_{\alpha\beta}$

$$E_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) + \frac{1}{2}(h_{,\alpha}w_{,\beta} + h_{,\beta}w_{,\alpha}) + \frac{1}{2}w_{,\alpha}w_{,\beta},$$

$$K_{\alpha\beta} = -w_{,\alpha\beta}$$

- Use linear stress-strain relation
- Integrate stress over thickness: $N_{\alpha\beta} = \int_{-t/2}^{t/2} \tau_{\alpha\beta} dz$
- Introduce Airy stress potential $\phi_{,xx} = N_{yy}, \phi_{,yy} = N_{xx}$ and $\phi_{,xy} = -N_{xy}$
- Deviation from prebuckled base state $(w, \phi) = (w, \phi)_{tot} - (w^0, \phi^0)$

depends on axial load

Equations for a thin elastic shell

The Donnell-Mushtari-Vlassov theory

$w(x,y)$: normal displacement

X : axial coordinate

Y : azimuthal

Cylinder radius Axial load

$$w_{,\tau} = f(w) \equiv -\frac{1}{\delta}\left(D\Delta^2 w + \frac{1}{R}\phi_{,xx} - [w,\phi] + \lambda w_{,xx}\right)$$

$$\frac{1}{Et}\Delta^2\phi = \frac{1}{R}w_{,xx} - \frac{1}{2}[w,w]$$

Airy potential

DMV equations

$$[A,B] = A_{,xx}B_{,yy} + A_{,yy}B_{,xx} - 2A_{,xy}B_{,xy}$$

$$D = \frac{Et^3}{12(1-\nu^2)}$$ Bending stiffness

Critical load

$$\lambda_c = \frac{E}{3(1-\nu^2)}\frac{t^2}{R} = t\sigma_c$$

Clamped boundaries $w = w_{,x} = 0$ $\phi_{,x} = (\Delta\phi)_{,x} = 0$

Characteristics: Nonlinear & Nonlocal

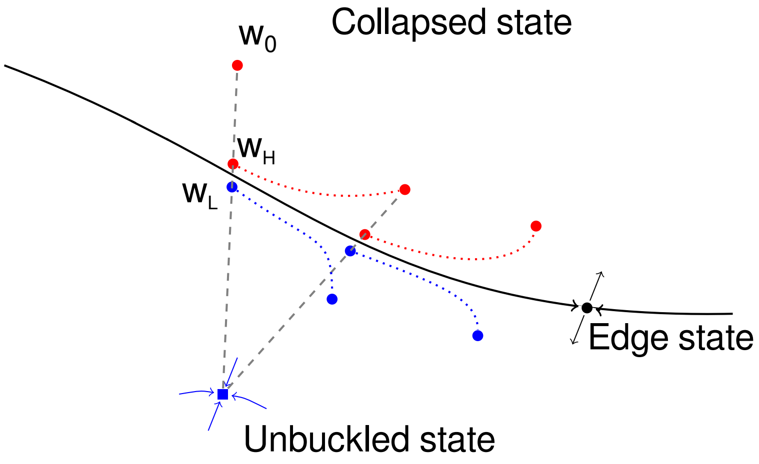
Numerical solution: Compact finite differences

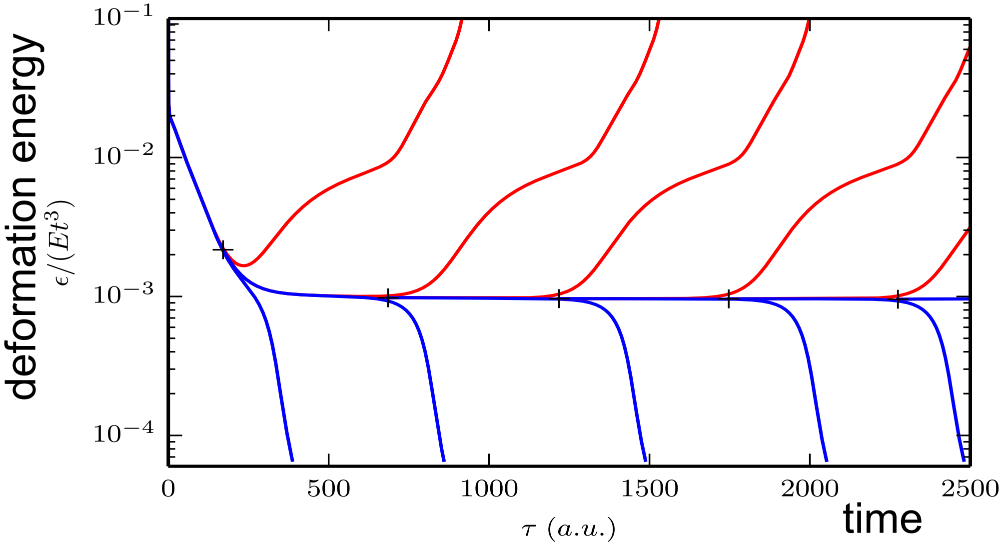
Aim: Construct nonlinear equilibria on the basin boundary of the unbuckled state

Dynamical systems methods for shells

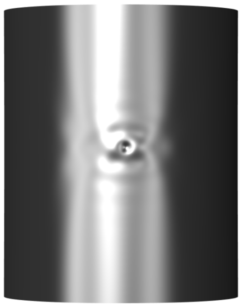
Unstable equilibria on the basin boundary

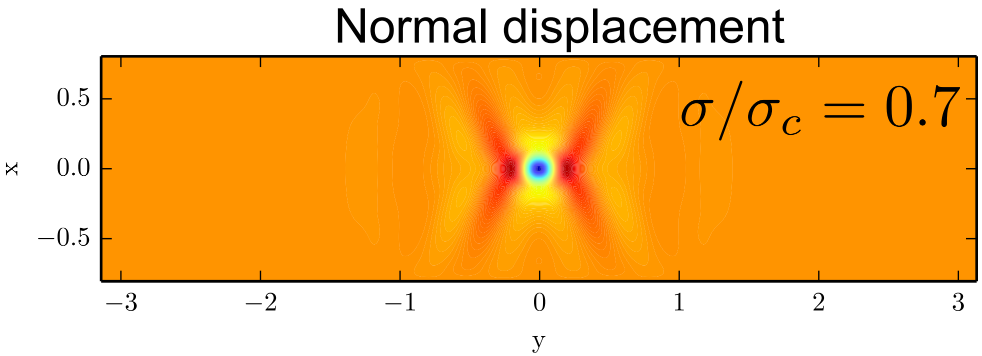
Edge tracking

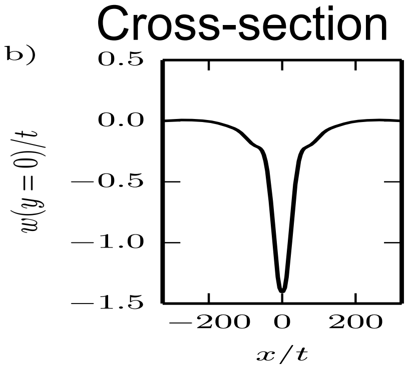




yields edge state equilibrium







Localized single dimple

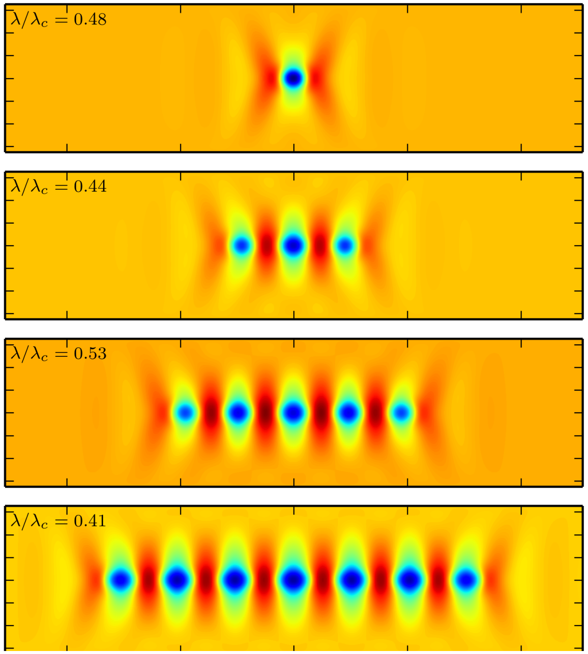
shallow

Yamaki parameters: R/t=405, L/R=1.6

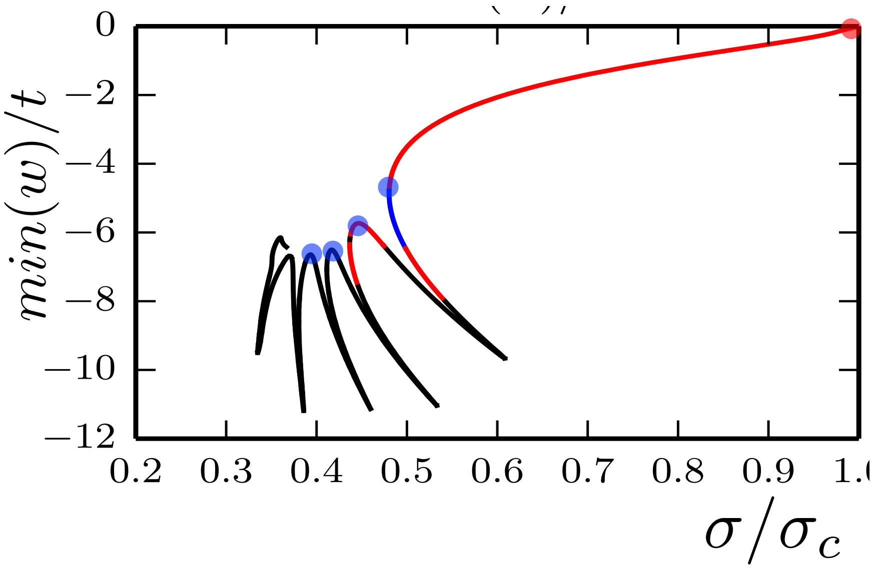
T. Kreilos & TMS, in prep.

Dynamical systems methods for shells
Unstable equilibria on the basin boundary

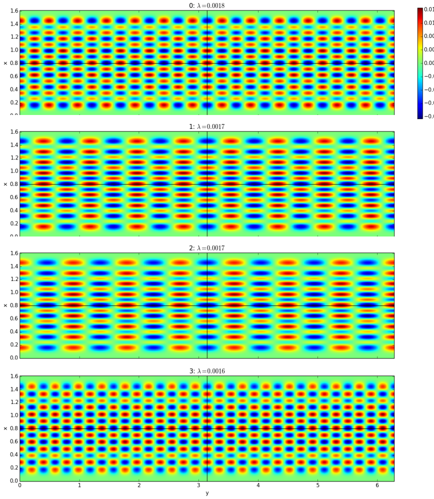
Snaking (azimuthally)



Continuation in axial load

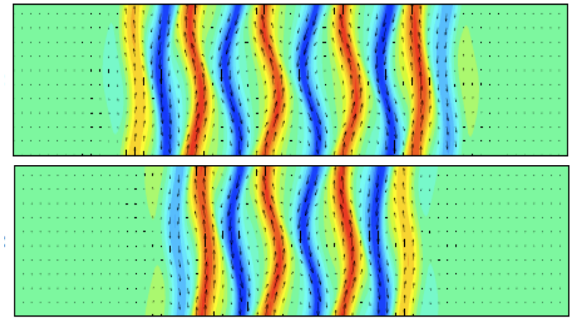


critical eigenmodes



Not localized

Compare: Plane Couette flow



Result: localized dimple patterns
low-dimensional unstable manifold
located on the stability boundary

Finite amplitude perturbations of a shell
Experiments with Shmuel Rubinstein, Harvard

Concept: Apply controlled perturbation to ‘perfect’ shell

Pipe flow transition (Mullin, Hof)

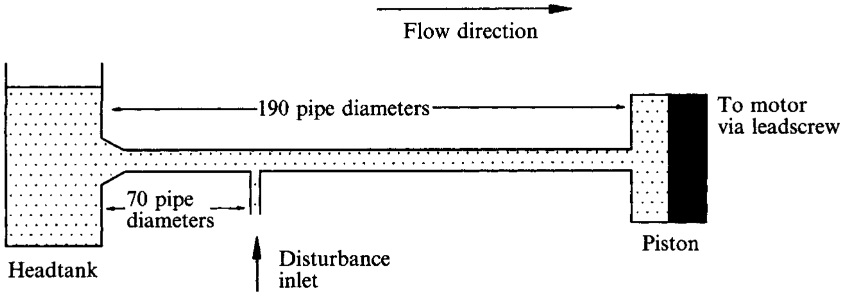
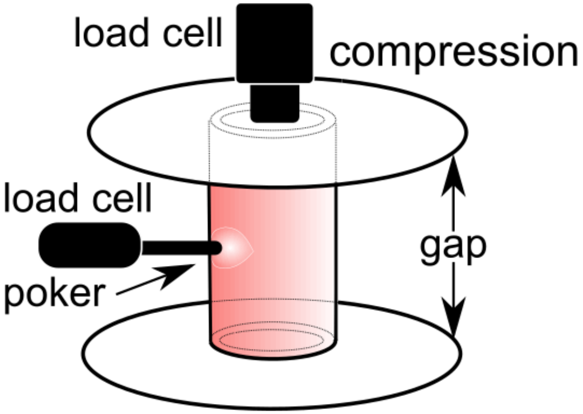


Figure 1. Schematic diagram of flow rig.

Dabyshire & Mullin 1994

Same for a shell

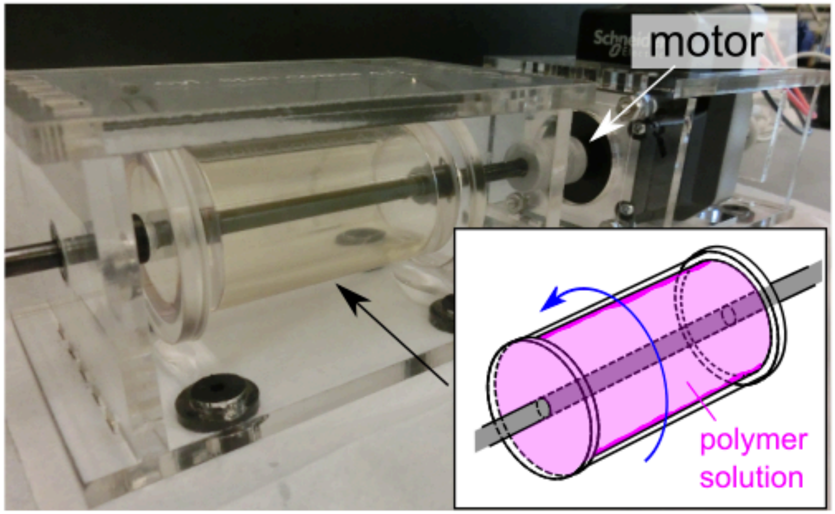


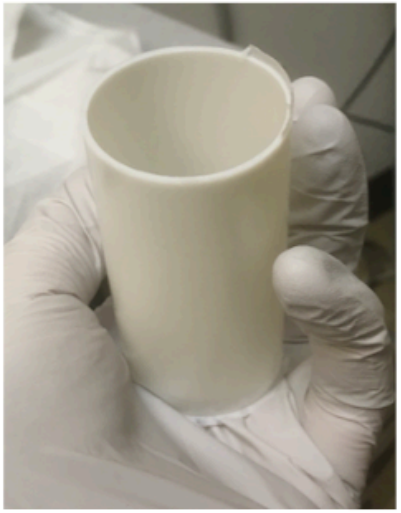
Needed:

1. Near ‘perfect’ shells
2. Biaxial testing machine – ‘shell poker’

Perturbing (almost) perfect shells

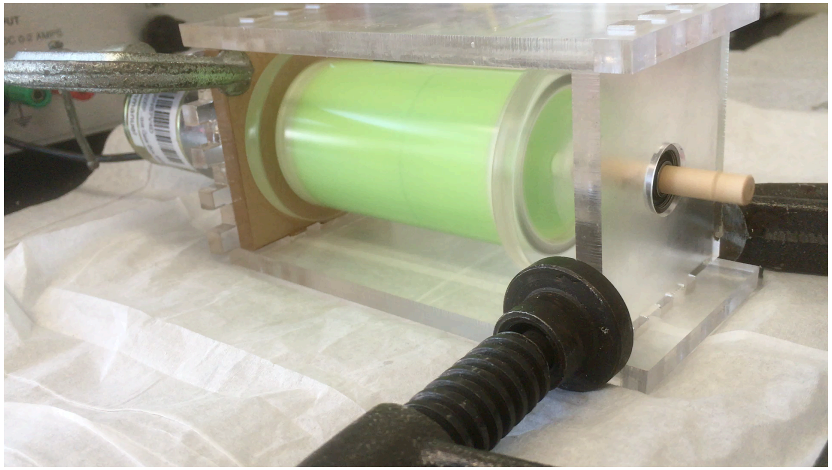
Polymer shells

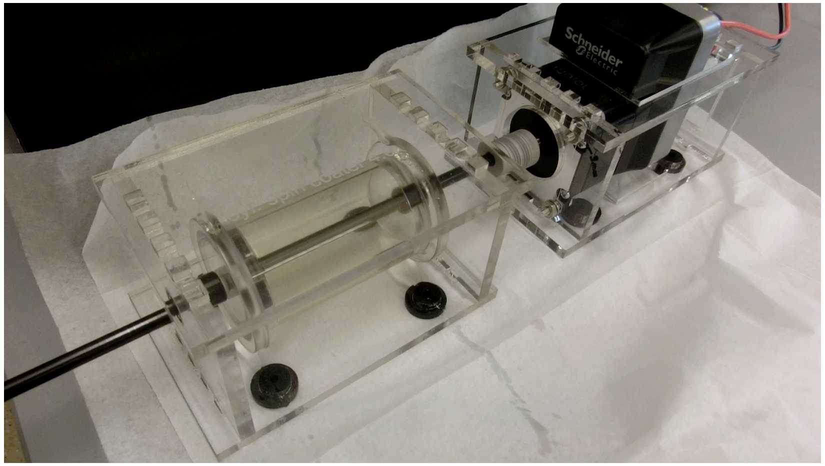




Spin coater (first iteration)

Spin coater (second iteration)





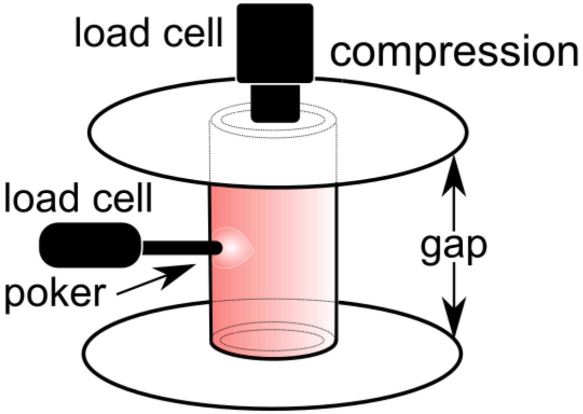
Advantage: No plastic deformation after buckling
Challenge: Making very thin shells

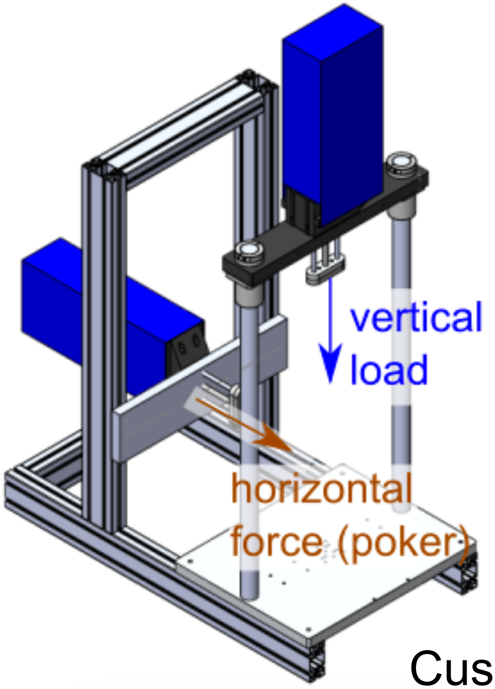
Perturbing (almost) perfect shells

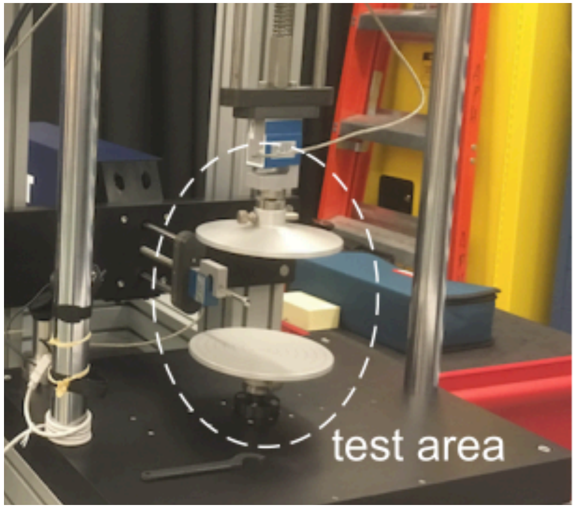
Aluminum cans

Perturbing (almost) perfect shells

Significance of unstable (edge) states

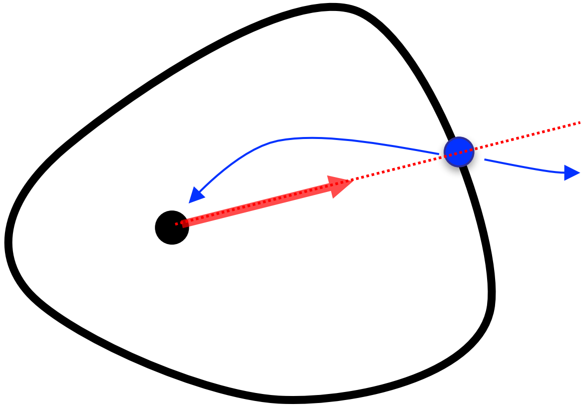




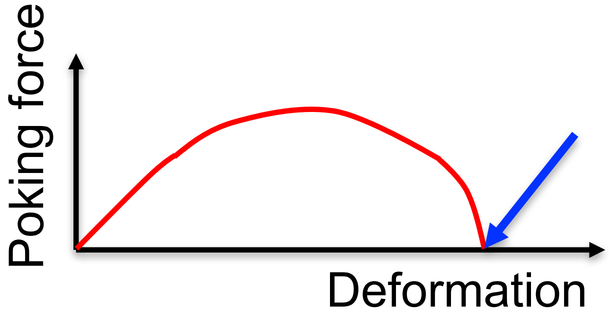


Custom made with ADMET company

Basin boundary at fixed load



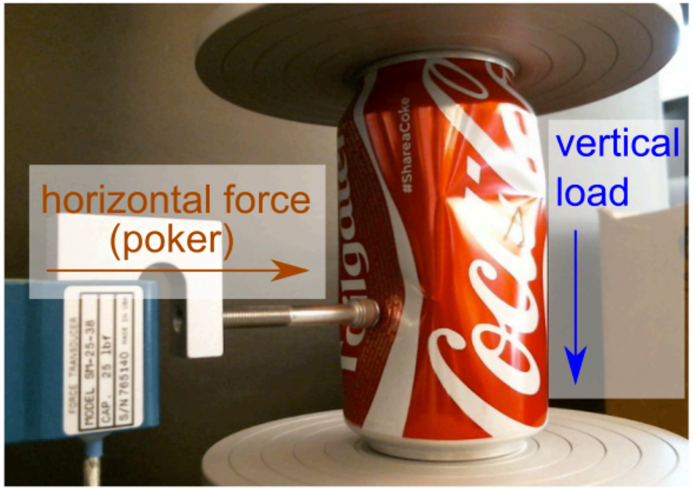
Control poker displacement and measure force



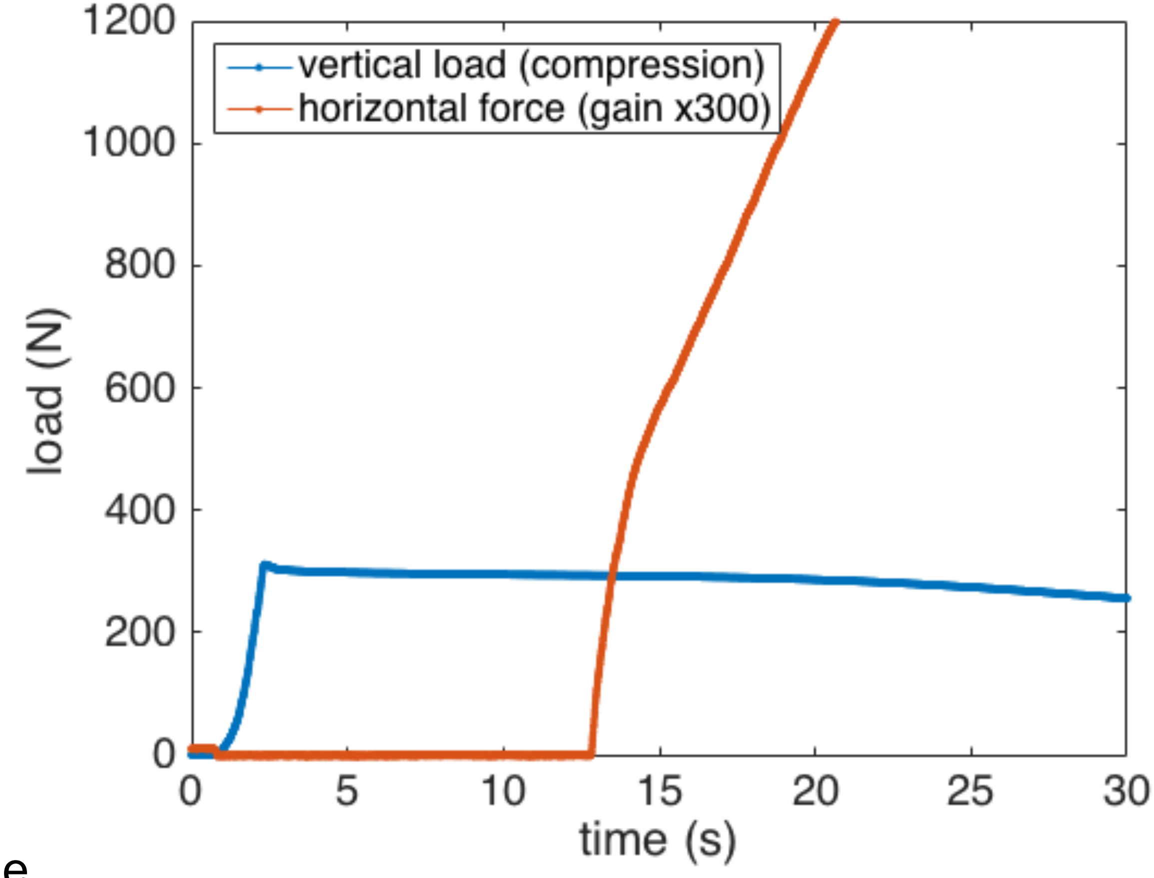
Expectation:
- reach edge state
- induce buckling

Poking experiment

First data (preliminary!!)



Load: 300 N



1: increase axial load to target value

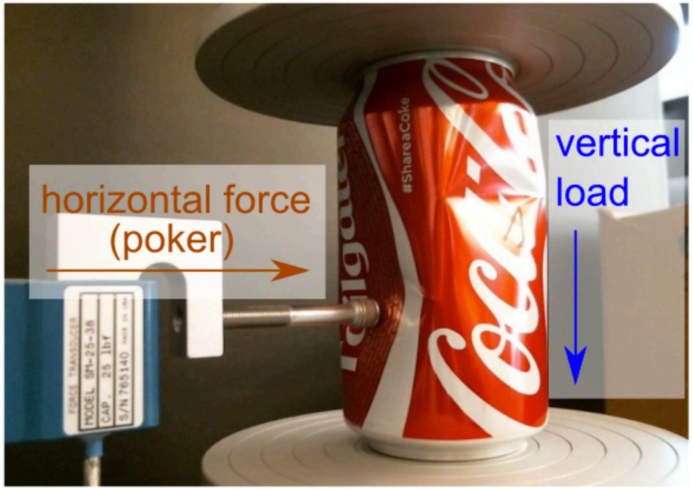
2. Advance lateral poker

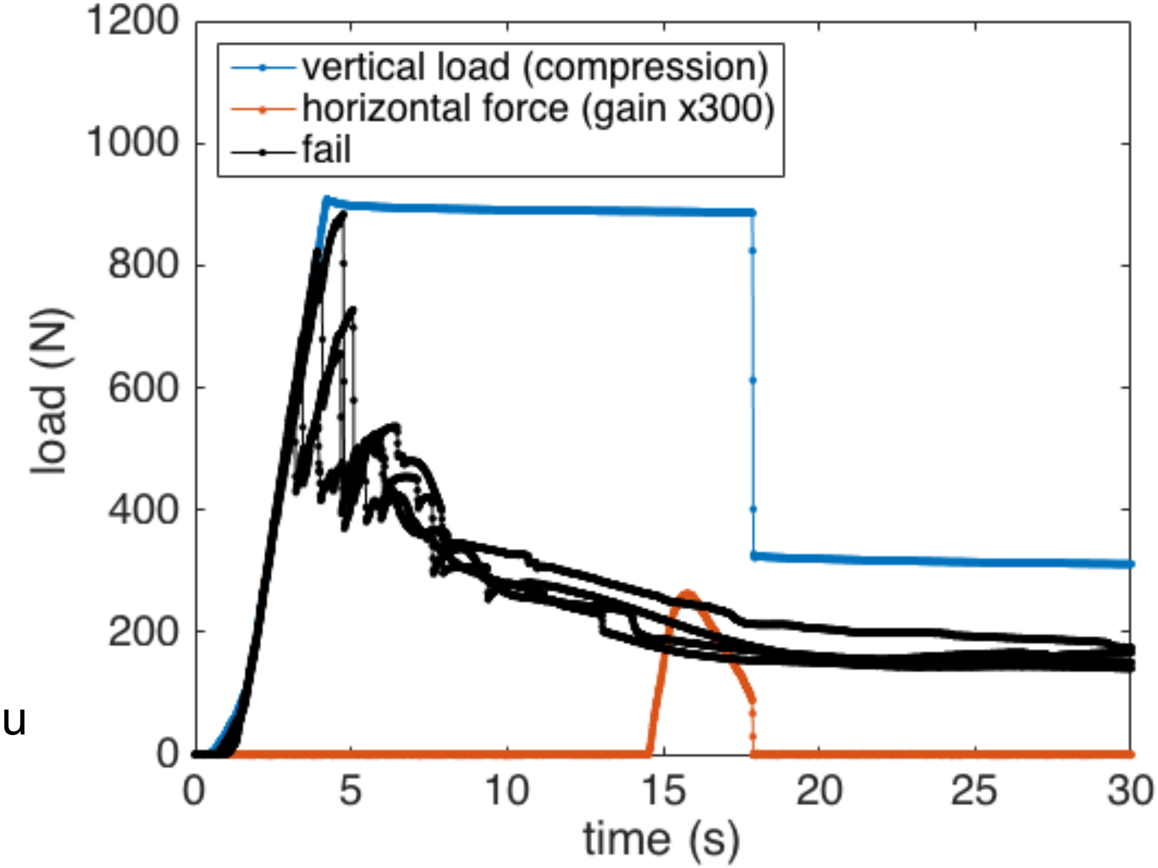
Constant speed 10mm/min

Poking experiment

First data (preliminary!!)

Load: 900 N





1: increase axial load to target valu
2. Advance lateral poker

Constant speed 10mm/min

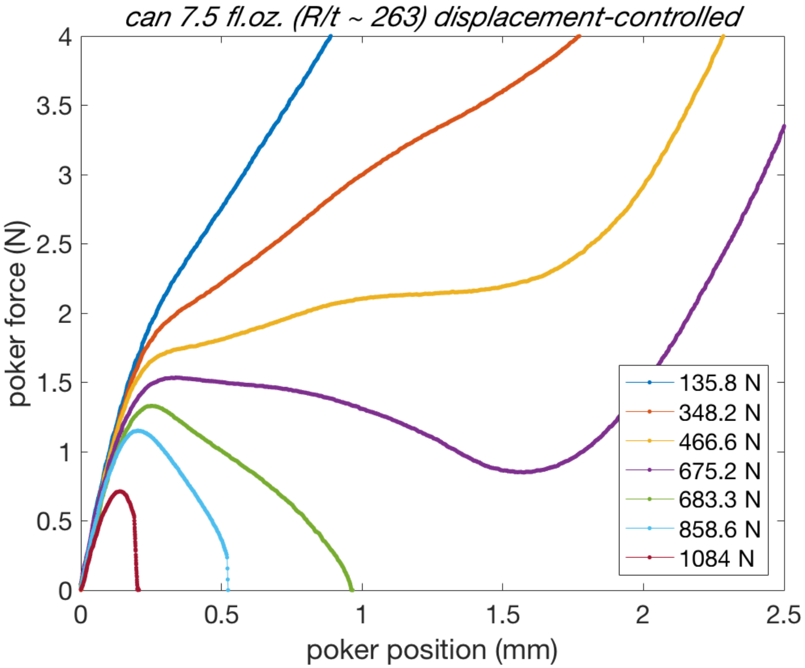
5 sec corresponds to 0.8 mm

Observation: buckling when edge state is reached
Interpretation: Confirms nonlinear finite amplitude instability triggers buckling

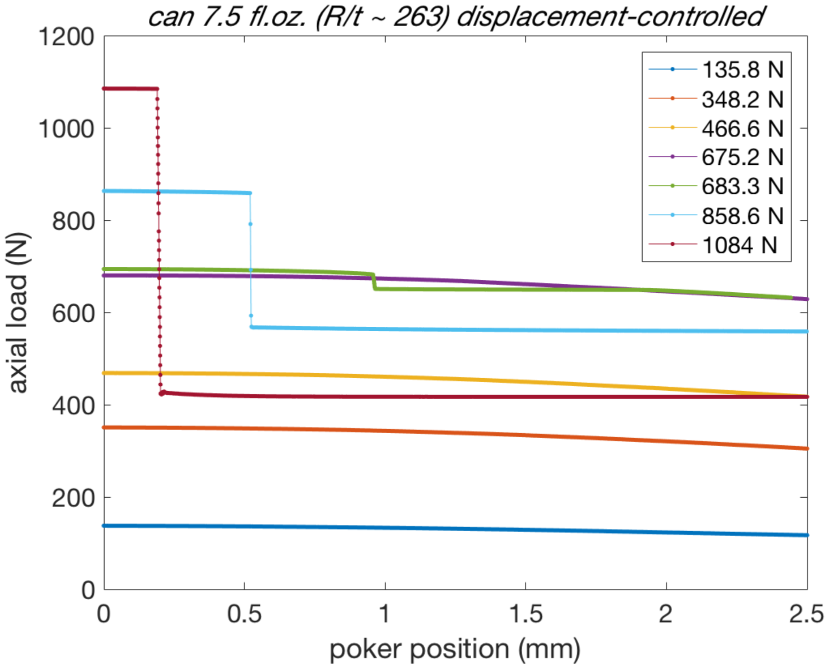
Poking experiment

Poking experiment
Quantifying critical perturbations

Poker force



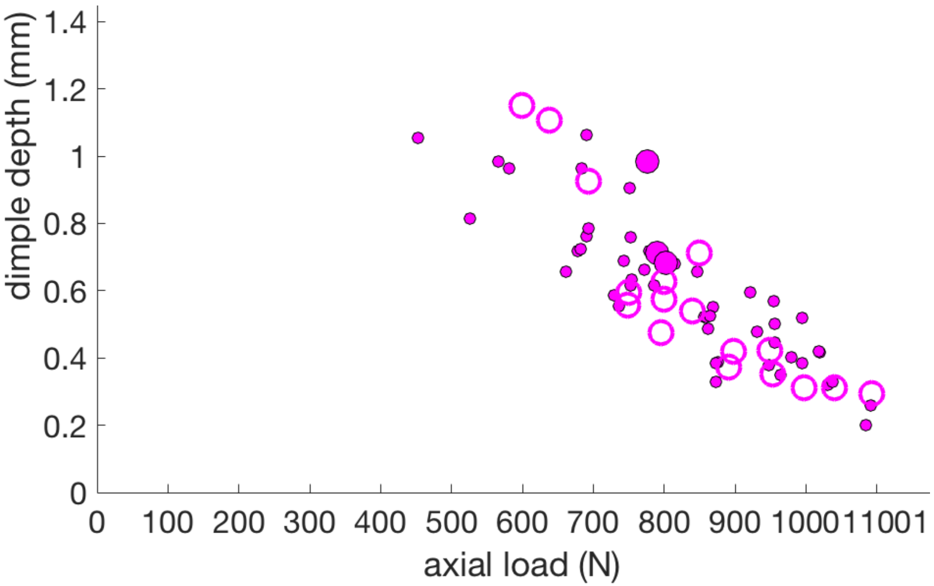
Axial force

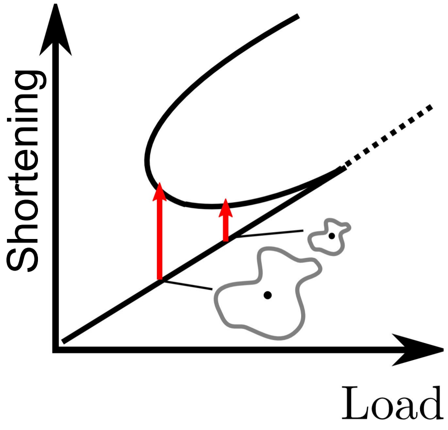


Poking experiment

Quantifying critical perturbations

Example: dimple depth when buckling is induced





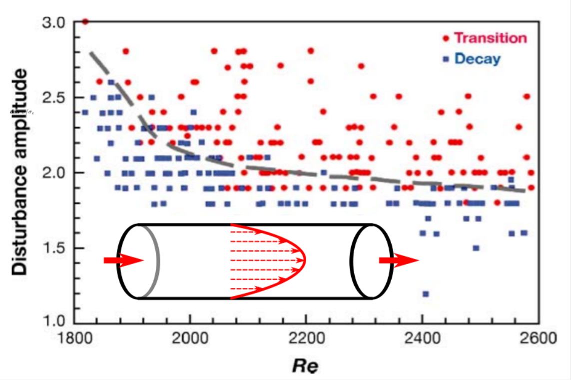
Interpretation:

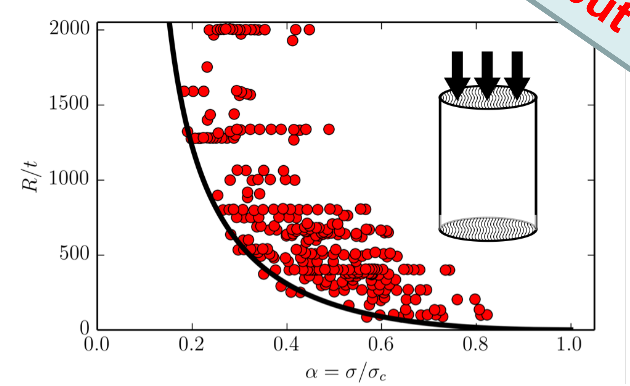
- basin of attraction shrinks with load
- basin vanishes at critical load

Summary and Outlook: Shell buckling

This is new, preliminary and highly speculative … but very exciting!!!!

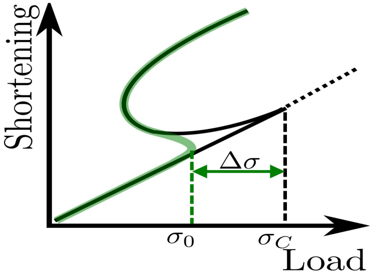
Similarity of turbulence transition and shell buckling

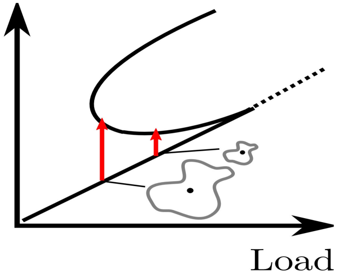




Suggests paradigm shift

Linear instability of the *imperfect* shell

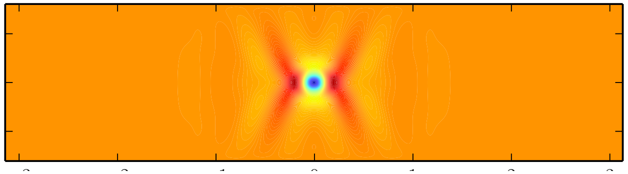




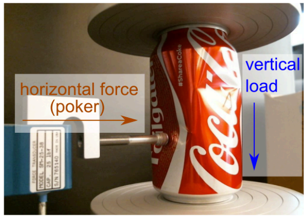
Nonlinear instability of the *perfect* shell

Approach appears supported by preliminary results

Theory: exact invariant states



Experiment: Critical perturbations



Plan / Dream: Predictive theory for buckling thresholds of thin shell structures