Koopman modes, Resolvent modes and a lid-driven cavity

Recurrent Flows, The Clockwork Behind Turbulence
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cavity / pipe DMD: **P Gomez, H Blackburn, M Rudman (Monash), B McKeon (Caltech)**

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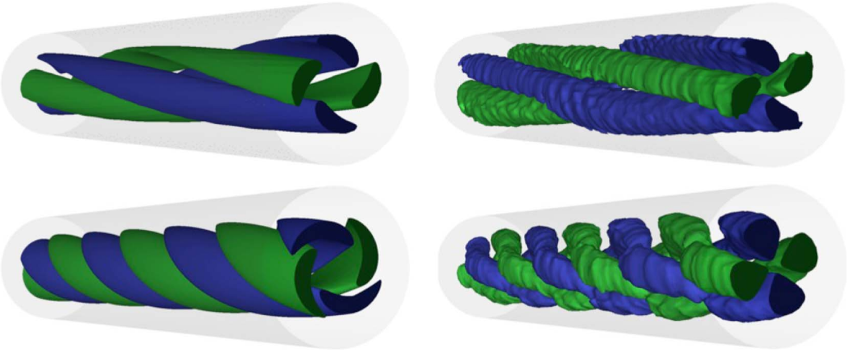
Why do DMD modes look like resolvent modes?



Gomez et al *PoF 2014*

"Paco's pipe projection puzzle"

$n = 2$, $\omega/2\pi = 0.1826,\ 0.5497$, isosurfaces $1/3u_{max}$.

Relation of RM to Koopman modes

Koopman modes are a general way of analysing nonlinear system dynamics, closely related to DMD*.

Why do resolvent modes look like DMD/Koopman modes?

Can we determine Koopman modes with limited data?

**Rowley* et al *JFM 2009; Schmid JFM 2010*

The Koopman operator

Suppose the dynamical system with state $\mathbf{u}$,

$$\partial_t \mathbf{u}(x,t) = \mathbf{f}\left(\mathbf{u}(x,t)\right).$$

Define a state transformation

$$S^\tau(\mathbf{u}(x,t)) = \mathbf{u}(x, t+\tau).$$

A Koopman operator U^τ is defined for each S^τ,

$$\mathrm{U}^\tau h(\mathbf{u}) = h(S^\tau \mathbf{u}).$$

The (family of) Koopman operators $\{U^\tau\}$ describes the dynamics of observable $h(\mathbf{u})$.

Spectrum of the Koopman operator

Consider the eigenvalue problem for U^τ

$$\mathrm{U}^\tau \varphi_n(\mathbf{u}) = \mu_n \varphi_n(\mathbf{u}).$$

Expanding some vector valued observable $\mathbf{g}(\mathbf{u})$ accordingly,

$$\mathbf{g}(\mathbf{u}) = \sum_n \varphi_n(\mathbf{u}) \mu_n \mathbf{g}_n.$$

Mezic Nonlin. Dyn. 2005, A. Rev. Fl. Mech. 2013; Rowley et al, *JFM 2009*

Koopman modes

For flow on a periodic orbit, decomposition in these eigenfunctions and Koopman modes reduces to a Fourier decomposition,

$$\mathbf{g}(x,t) = \mathbf{g}^*(x) + \sum_{n\neq 0} \mathbf{g}_n(x) e^{in\omega t}.$$

The Koopman mode $\mathbf{g}_n(x)$ is the projection of the field $\mathbf{g}$ on the subspace spanned by an eigenfunction of U^τ.

Note: temporal average emerges as $\mathbf{g}^*(x)$.

Koopman mode interpretation

Points to note:

- Selects frequencies present in system behaviour
- Chaotic system may have point and continuous spectrum
- Spatial structure is wrapped up in Koopman mode
- Periodic orbit gives Fourier decomposition
- …but any transformation (e.g. reflection, rotation) also possible

Generalising Koopman operator

We can define the Koopman modes where the Koopman operator does another transformation (spatial shift, reflection, etc)

Define a state transformation

$$S^{\chi}(\mathbf{u}(x,t)) = \mathbf{u}(x+\chi,t).$$

When periodic in x, U^{χ} gives

$$\mathbf{g}(x,t) = \bar{\mathbf{g}}(t) + \sum_{l\neq 0} \mathbf{g}_l(t) e^{il\alpha x}.$$

The spatial Koopman modes are $\mathbf{g}_l(t)$

Note: spatial average appears as $\bar{\mathbf{g}}(t)$.

Sharma, Mezic, McKeon, PRF, 2016

Combining Koopman operators

Combining the Koopman operators,

$$\mathrm{U}^{\chi}\mathrm{U}^{\tau}\mathbf{g}(\mathbf{u}) = \mu\mathbf{g}(\mathbf{u}).$$

gives

$$\mathbf{g}(x,t) = \sum_{n,m\in\mathbb{Z}} \mathbf{g}_{n,m} e^{i(n\omega t + m\alpha x)},$$

which is *an expansion in travelling waves*.

For turbulent attractor, same applies but continuous instead of point spectrum.

An expansion around the turbulent attractor I

Let $\mathbf{u}(t)$ be the state, and the Navier-Stokes equations be written

$$\dot{\mathbf{u}}(t) = \mathbf{f}\left(\mathbf{u}(t)\right). \tag{1}$$

In the "long run", decompose the state as

$$\mathbf{u}(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{i\omega t}\hat{\mathbf{u}}(\omega)d\omega.$$

Notice that the equation corresponding to $\omega = \mathbf{0}$ is the mean equation

$$0 = \int_{-\infty}^{\infty} \mathbf{f}(\mathbf{u})dt$$

with $\hat{\mathbf{u}}(0)$ the mean.

An expansion around the turbulent attractor II

The expansion of (1) about this mean (and subtracting this mean equation) is

$$\begin{aligned}\dot{\tilde{\mathbf{u}}}(t) &= \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\bar{\mathbf{u}}} \tilde{\mathbf{u}}(t) + \mathcal{O}(2) \\ &= \mathbf{L}\tilde{\mathbf{u}}(t) + \tilde{\mathbf{g}}(t)\end{aligned}$$

where $\tilde{\mathbf{g}}$ represents the second-order terms in the expansion of $\mathbf{f}$.

At a particular $\omega \neq \mathbf{0}$ we then have

$$\hat{\mathbf{u}}(\omega) = (\omega I - \mathbf{L})^{-1}\,\hat{\mathbf{g}}(\omega).$$

The second-order terms, rather than being truncated, act to excite the state.

A low-rank basis

The resolvent $\mathbf{H}(\omega) = (\omega I - \mathbf{L})^{-1}$ is well-approximated by a projection $\Pi(\omega)$,

$$\hat{\mathbf{u}}(\omega) = \mathbf{H}(\omega)\hat{\mathbf{g}}(\omega) \simeq \Pi(\omega)\hat{\mathbf{g}}(\omega).$$

The SVD gives the optimal N-dimensional basis on which the velocity field evolves, in sense that $\|\mathbf{H}(\omega) - \Pi_N(\omega)\|_F$ is minimised. [1]

The flow “lives” mostly in $\mathbf{\Pi}$.

The ‘error’ is $\hat{\mathbf{u}}^{\perp} = \Pi_N^{\perp}\hat{\mathbf{g}}$.

[1] McKeon & Sharma, JFM, 2010.

A basis for optimal projection; resolvent modes

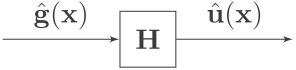
$$\mathbf{H}\hat{\mathbf{g}} = \sum_{m=1}^{\infty} \psi_m(\mathbf{x})\sigma_m \langle \phi_m^*(\mathbf{x}), \hat{\mathbf{g}}(\mathbf{x})\rangle$$

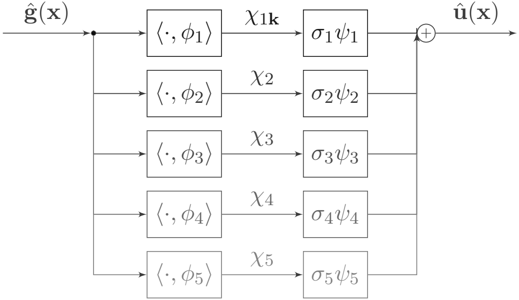
$$\langle \psi_m, \psi_{m'} \rangle = \delta_{m,m'}$$

$$\langle \phi_m, \phi_{m'} \rangle = \delta_{m,m'}$$

$$\sigma_1 \geq \sigma_2 \geq \dots$$

Each σ_m is a (real) gain, σ_1 is the maximum gain.
Velocity field response is $\psi_m(\mathbf{x})$.





Relation between Koopman and resolvent modes

Comparing starting point of RM derivation, we see that the resolvent relates the Koopman modes of $\mathbf{u}$ and $\mathbf{f}$

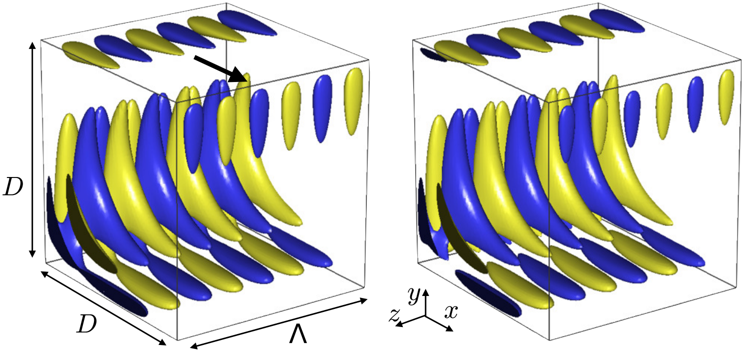
$$\mathbf{u}_m(y) = \mathcal{H}_m \mathbf{f}_m(y)$$

so the resolvent modes are (linear-)optimal basis to guess the Koopman modes.

Interpretation

- Resolvent modes a good 'first guess' of KM; for ordered expansion of KMs
- Koopman modes $\equiv$ RM modes $\equiv$ Fourier in homogeneous directions
- Where there are symmetries, they should be used *before* doing the DMD problem
- Expansion in RMs easily gives sensitivity of KMs to control / BC changes
- Meaning of spectrum of linear operator $\mathcal{H}_m(\mathbf{u_0})$ is clear

Application: 2D resolvent modes in a lid-driven cavity



- few, discrete frequencies
- complex geometry

With Gomez, Rudman, Blackburn; B McKeon, JFM 2016

Lid-driven cavity

- $Re = 1200$, $\Lambda = 0.945D$
- Nonlinear, low-dimensional behaviour
- Three dominant wavenumbers: $\beta = 0,\ 3,\ 6$
- Three dominant frequencies: $\omega = 0,\ 0.76,\ 1.52$
- spectral-*hp* 2D × Fourier (semtex)

Fourier transform velocity (translation invariant in time, z; neglect transients)

$$\mathbf{u} = \sum_{\beta,\omega} \mathbf{u}_{\beta,\omega}(x,y)\mathrm{e}^{i(\beta z - \omega t)}$$

Assume **time-space mean $\mathbf{u}_0$** to close.

Same for nonlinear term,

$$-\mathbf{u} \cdot \nabla \mathbf{u} = \sum_{\beta,\omega} \mathbf{f}_{\beta,\omega}(x,y)\ \mathrm{e}^{i(\beta z - \omega t)}$$

Fluctuations:

$$\mathbf{u}_{\beta,\omega} = (i\omega - \mathcal{L}_\beta)^{-1}\, \mathbf{f}_{\beta,\omega}.$$

Mean:

$$0 = \mathbf{f}_0 - \mathbf{u}_0 \cdot \nabla \mathbf{u}_0 + \frac{1}{Re} \nabla^2 \mathbf{u}_0.$$

Take SVD of transfer function,

$$(i\omega - \mathcal{L}_\beta)^{-1} = \sum_m \psi_{\beta,\omega,m}\ \sigma_{\beta,\omega,m}\ \phi^*_{\beta,\omega,m}$$

Gives gain-optimal basis to represent $\mathbf{u}$ and $\mathbf{f}$, scalar coefficient c,

$$\mathbf{u}_{\beta,\omega}(x,y) = \sum_m \psi_{\beta,\omega,m}(x,y)\ c_{\beta,\omega,m}$$

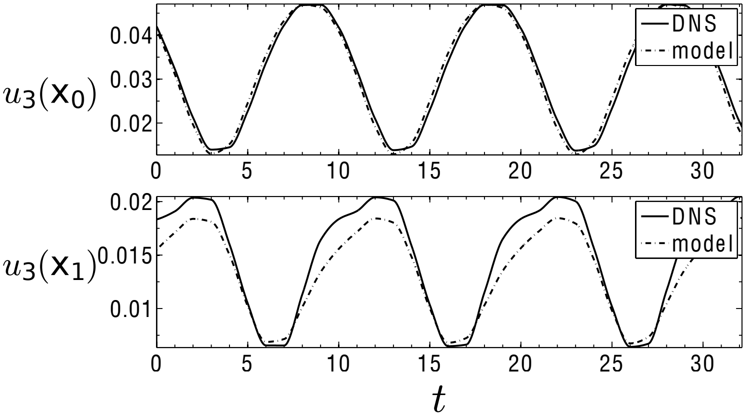
$$\mathbf{f}_{\beta,\omega}(x,y) = \sum_m \phi_{\beta,\omega,m}(x,y)\ c_{\beta,\omega,m}\ /\ \sigma_{\beta,\omega,m}$$

Estimating mode coefficients from probe signal

Focus on $\beta = 3$, fit $m = 1$ coefficients at three frequencies.

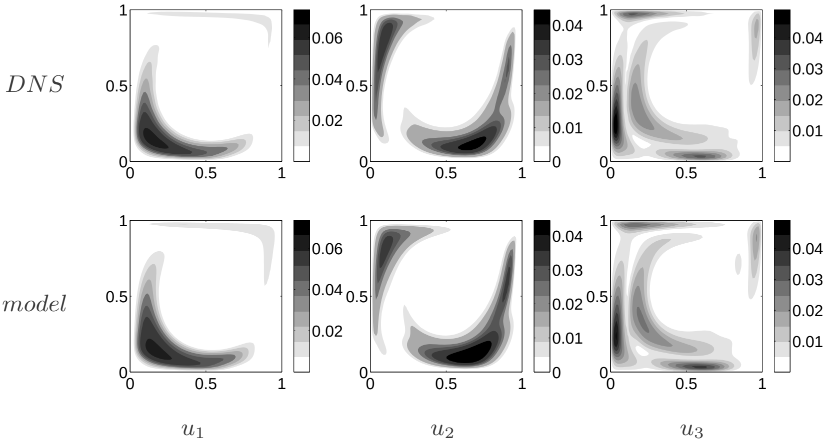
$$C_\beta = \Psi_\beta^+(\mathbf{x}_p) U_\beta(\mathbf{x}_p)$$

Reconstructed signals



LSQ fit with single probe at $\mathbf{x}_0 = (0.1, 0.1, 0)$;
reconstruction at $\mathbf{x}_0$ and $\mathbf{x}_1 = (0.82, 0.95, 0)$

Reconstructed field (RMS fluctuations)



About 20% model error

isosurfaces at 30% max $w_{\beta=3}$.

Conclusions (cavity)

Limitations:

- Needs limited data to fix amplitudes, phases, dominant ω, β, mean

Benefits:

- Meaning of mean flow in linear operators is now clear
- Approximates whole flow from probed points + mean
- Modes are orthogonal (unlike eigenmodes)

Conclusions

- DMD modes look like RM because RM look like Koopman modes
- Resolvent modes offer (linear-) optimal basis to expand Koopman modes of velocity
- For homogeneous directions, it's just Fourier modes
- Global coherence allows fixing of coefficients from point data
- Can use to predict response of nonlinear system to control