Disordered systems and turbulence

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Outline

- Burgers, Cole-Hopf and disordered systems

- Methods and results for disordered elastic systems (replica, large D, Functional RG)

- Conjecture for decaying Burgers in D>1

- Freezing transition in decaying Burgers

- FRG for Navier Stokes

Decaying Burgers

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = \nu \nabla^2 \vec{v}$$

$$(\vec{v} \cdot \vec{\nabla})\vec{v} = \frac{1}{2}\vec{\nabla}\vec{v}^2$$

initial value: $\vec{v}(\vec{r}, t = 0) = \vec{\nabla} V(\vec{r})$ random function $\vec{r} \in R^D$

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$$\vec{v}(\vec{r}, t) = \vec{\nabla}\hat{V}(\vec{r}, t) \qquad \hat{V}(\vec{r}, t = 0) = V(\vec{r})$$

$$H(\vec{u}) = \frac{(\vec{u} - \vec{r})^2}{2t} + V(\vec{u})$$

energy function

$$e^{-\frac{1}{2\nu}\hat{V}(\vec{r},t)} = \int d^D\vec{u}\; e^{-\frac{1}{2\nu}H(\vec{u})}$$

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energy function

$$e^{-\frac{1}{2\nu}\hat{V}(\vec{r},t)} = \int d^D\vec{u}\; e^{-\frac{1}{2\nu}H(\vec{u})} \equiv Z$$

$$T \equiv 2\nu$$

$$\vec{v}(\vec{r}, t) = \frac{1}{t}\langle \vec{r} - \vec{u} \rangle_Z$$

inviscid limit $\nu \to 0^+$ zero temperature

$D = 1$

$$\hat{V}(r,t) = \min_u [\frac{(u-r)^2}{2t} + V(u)]$$

inviscid limit $\nu \to 0^+$ zero temperature

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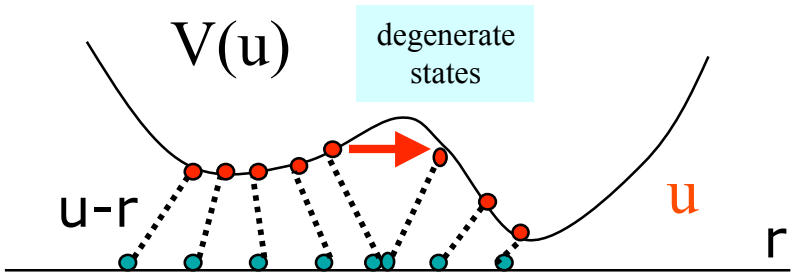
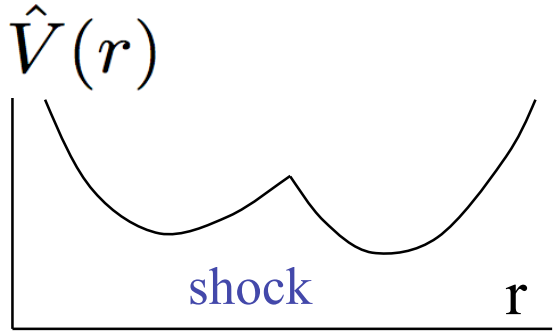
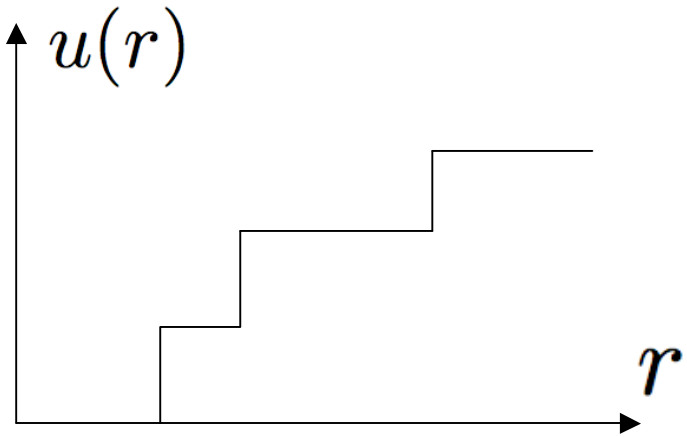
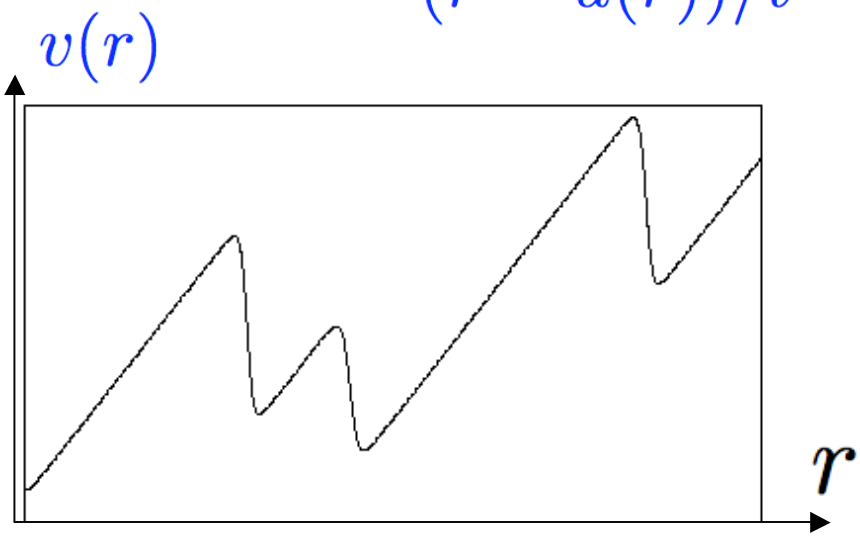




$v(r) = \hat{V}'(r)$ jumps

$= (r - u(r))/t$





Decaying statistical turbulence

scale $\ell(t) \sim t^{\zeta/2}$ $\quad r \gg \ell(t)$ infrared

$\ell_\nu(t) \ll r \ll \ell(t)$ inertial range

scaling of velocity $< v(r,t)v(0,t) >= \frac{\ell(t)^2}{t^2}\tilde{\Delta}(\frac{r}{\ell(t)})$

energy cascade

$$\frac{\ell_\nu(t)}{\ell(t)} \sim \tilde{T} \sim \tilde{\nu} \sim \nu t^{-\theta/2}$$

$$\lim_{\nu\to 0} \nu\overline{(\nabla v)^2} \neq 0$$

Systems with quenched disorder

- Electrons in random potentials, localization

Glasses, many metastable states: shocks

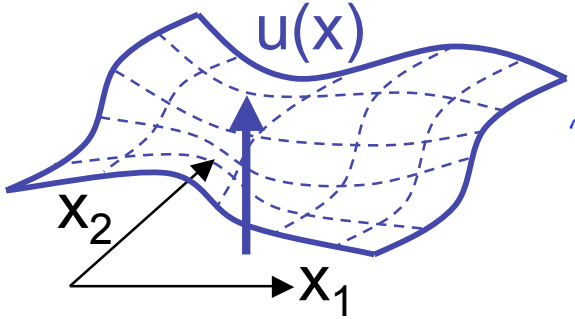
- Spin glasses $H = -\sum_{ij} J_{ij} S_i S_j$
- Disordered elastic systems

$$u \to u(x)$$

$$V(u) \to V(x, u)$$

Elastic manifolds in random potential

- domain wall in higher dimension



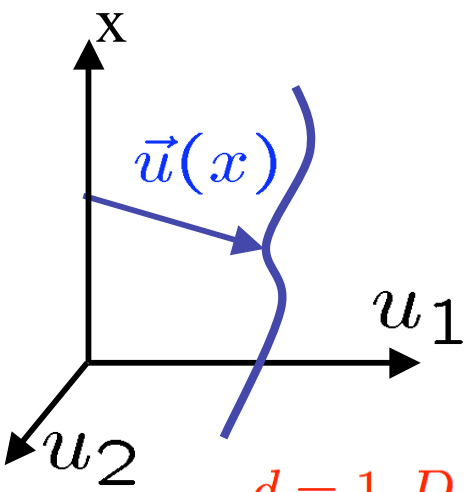
interface

$u(x_1, x_2)$

$d = 2, D = 1$

$\vec{u}(\vec{x})$ $\in R^D$, $\in \mathbb{R}^d$

- directed polymer



$d = 1, D = 2$

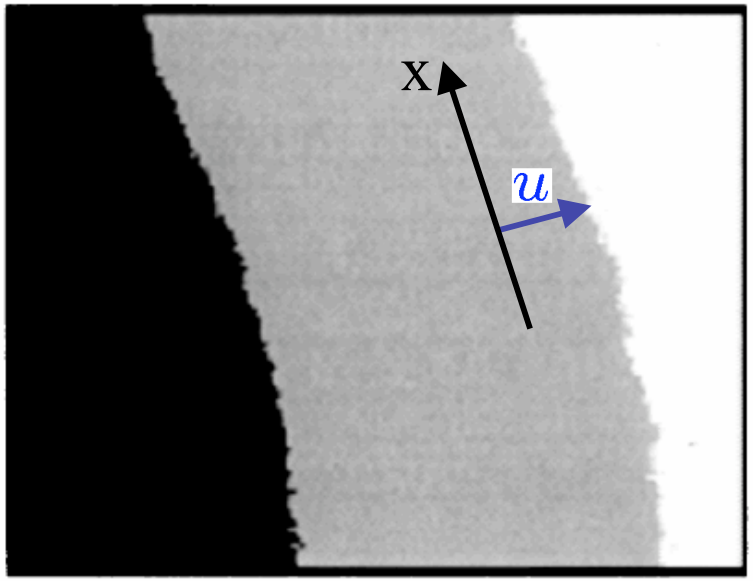
$$H = \int d^d x \quad \frac{c}{2}(\nabla u)^2 + V(x, u(x))$$

$$\overline{\langle (u(x) - u(0))^2 \rangle} \sim |x|^{2\zeta}$$

critical object

Magnetic interface

Lemerle, Ferre, Chappert,
Mathe, Giamarchi PLD, PRL 98



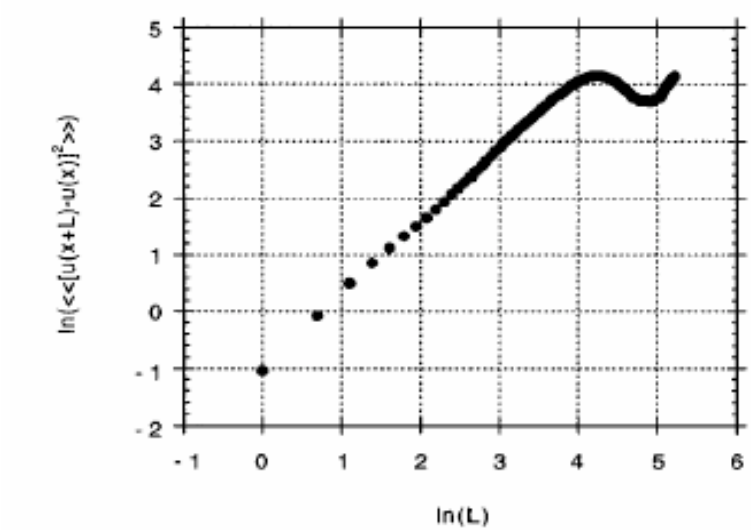
Ising magnetic film Co

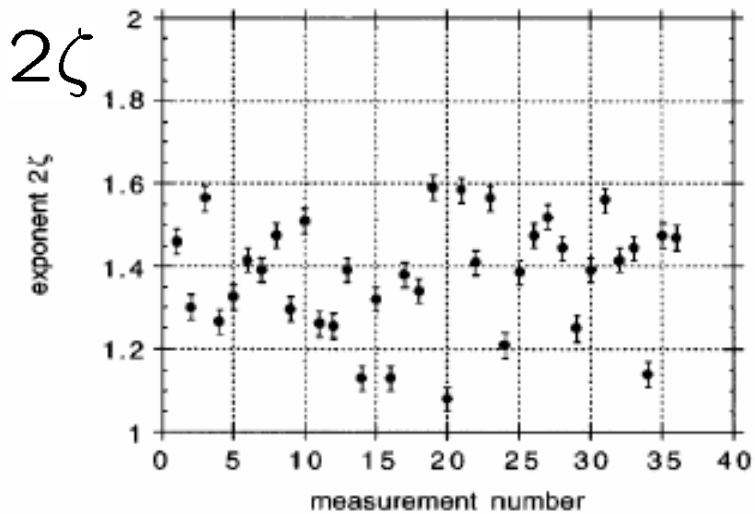
D=1+1 interface

short range disorder

$$\overline{(u(x)-u(0))^2} \sim |x|^{2\zeta}$$

thermally equilibrated:
minimum energy configuration





Shocks for elastic manifolds

$$H[u] = \int d^d x[(\nabla u)^2 + V(x, u(x)) + \frac{m^2}{2}(u(x) - r)^2]$$

$$e^{-\frac{1}{T}\hat{V}(r)} = \int Due^{-\frac{1}{T}H[u]}$$

$$m^2 = 1/t$$

$$x_m = 1/m$$

$$u \to u(x) \qquad V(u) \to V(x, u)$$

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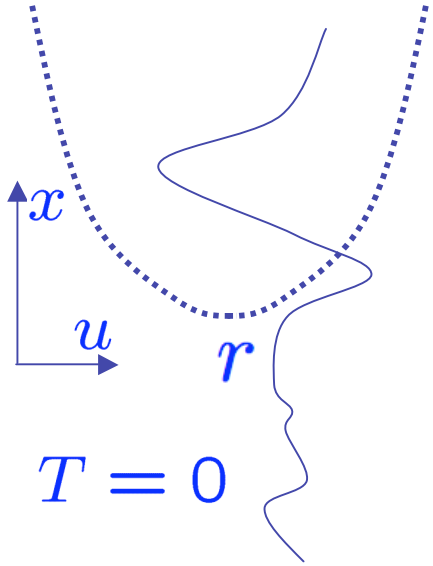

$T = 0$

minimum energy configuration $u_{min}(x; r)$

$$u(r) = L^{-d} \int_{x \in L^d} u_{min}(x; r)$$

$r - u(r)$ *exhibits shocks*

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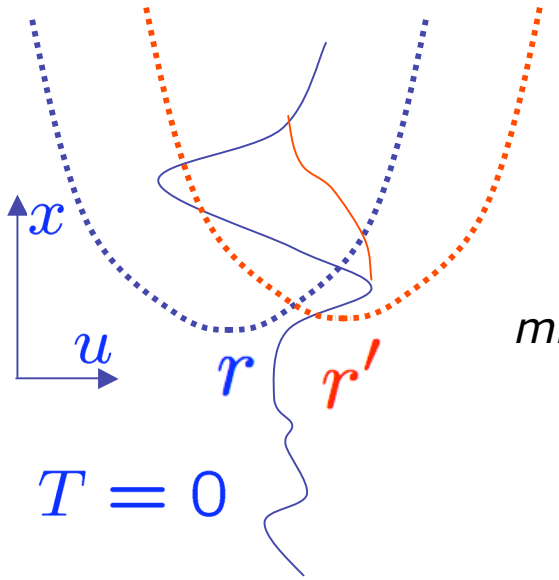

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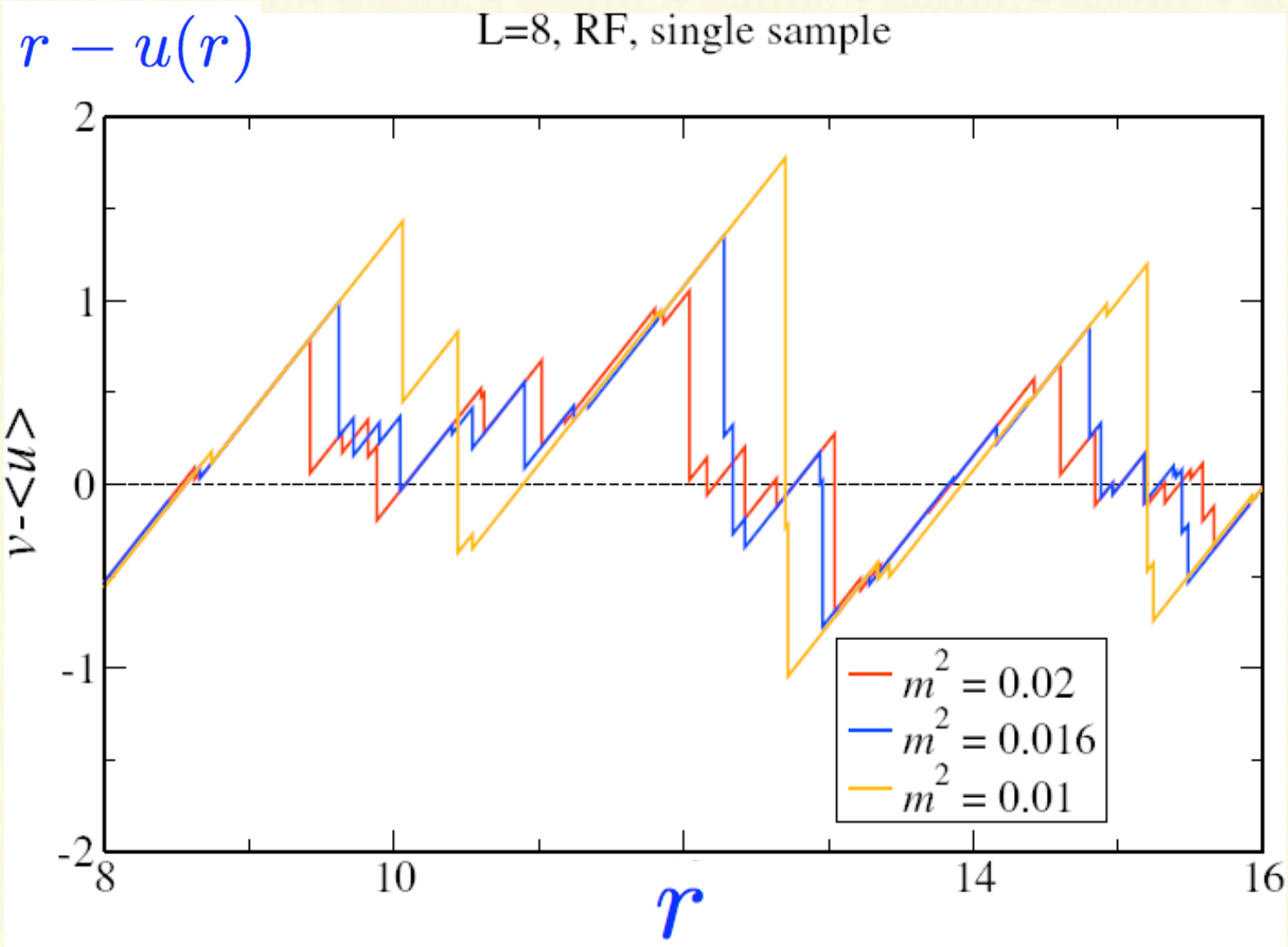
with Alan Middleton, U. Syracuse

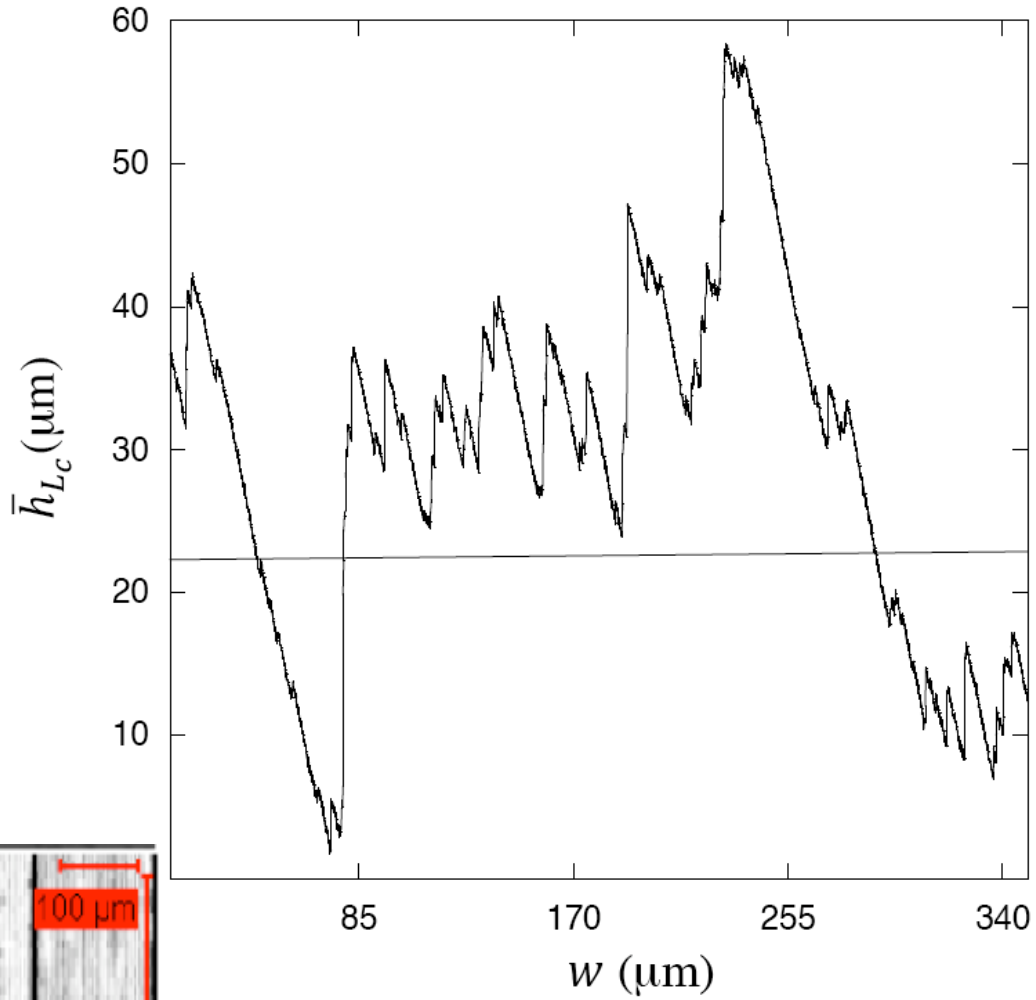
Sequence of m^2 in a single sample

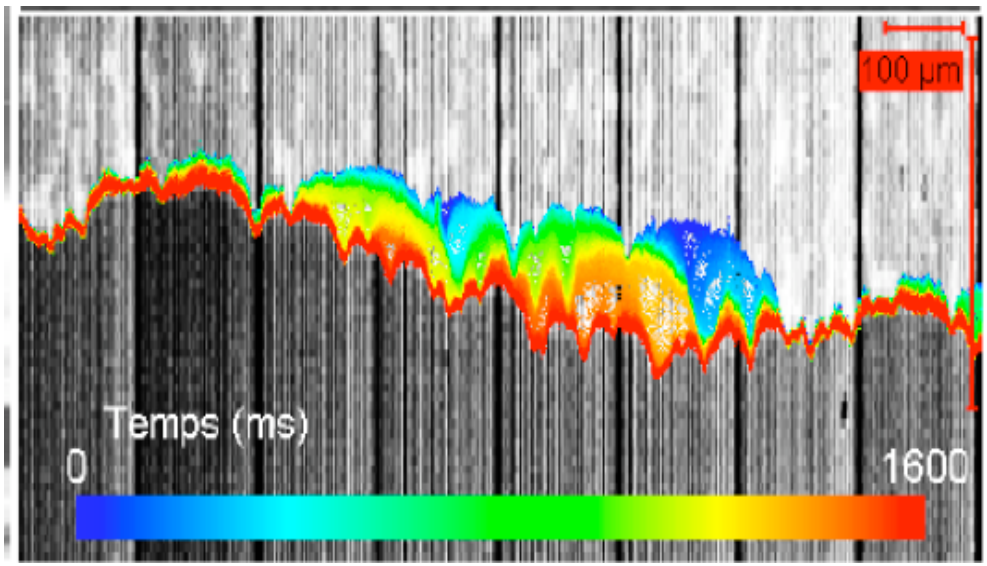


$$u(w) - w$$

$$w = vt$$

u(w) = center of mass of the contact line





summary

$$\overline{v(r)v(r')} = \overline{\hat{V}'(r)\hat{V}'(r')} = t^{-2}\overline{(r-u(r))(r'-u(r'))}$$

$$\overline{v(r)v(r')} = L^d \Delta(r)$$

general d elastic manifold

d=0 decaying Burgers

$$\overline{v_i(\vec{r})v_j(\vec{r}')} = L^d \Delta_{ij}(\vec{r}-\vec{r}')$$

Forced Burgers $\longleftrightarrow$ Elastic line (d=1)

$$\partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v} = \nu \nabla^2 \vec{v} + \vec{f}(\vec{u}, t) \qquad \vec{\nabla} \equiv \nabla_{\vec{u}}$$

$$\vec{v}(\vec{u}, t) = -\vec{\nabla} h \qquad \vec{f}(\vec{u}, t) = -\vec{\nabla} V(u, t)$$

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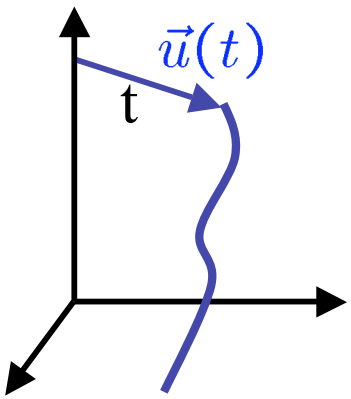

free energy $\quad h = -T \ln Z \qquad$ KPZ growth

$$Z(\vec{u}, t) = \sum_{paths, \vec{u}(t) = \vec{u}} e^{-\sum_{t'=0}^{t} V(t', \vec{u}(t'))/T}$$

Burgers time t $\longleftrightarrow$ length of line

forcing $\longleftrightarrow$ disorder

develop shocks $\longleftrightarrow$ metastable states

viscosity ν $\longleftrightarrow$ temperature

summary

- decaying Burgers in D dimension — initial condition

$\updownarrow$ $\updownarrow$

elastic manifold internal dimension d=0 (a point) moving in D dimension — V(u)

- forced Burgers in D dimension — forcing

$\updownarrow$ $\updownarrow$

elastic manifold internal dimension d=1 (a line) moving in D dimension — V(u,t)

Methods and results for disordered sytems

replica method

$$t^2\overline{\vec{v}^2} = \overline{\langle \vec{u}^2 \rangle} = \lim_{n\to 0} \int \prod_{a=1}^{n} d\vec{u}_a \vec{u}_1^2 \overline{e^{-\frac{1}{2\nu}\sum_a \frac{\vec{u}_a^2}{2t} + V(\vec{u}_a)}}$$

$$\overline{\langle .. \rangle} = \frac{\overline{\int d\vec{u}..}}{Z_V}$$

disorder average -> replica interaction

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disorder average -> replica interaction

•1) infinite D limit:

• Forced Burgers (directed line d=1 in disorder)
Bouchaud,Mezard,Parisi,1995

• Decaying Burgers (d=0) PLD, Mueller,Wiese,2010
distribution of shock sizes

measure=gaussian in replica + replica symmetry breaking saddle point

⟷ measure=superposition of gaussians centered on random metastable states

cells/walls=shocks

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•2) around d=4 any D: perturbative (functional) RG

why does it become perturbative?

$$(\nabla u)^2$$

as internal dimension d increases $\longrightarrow$ elasticity stronger

disorder/elasticity weaker

$$d > d_{uc} = 4$$

weak disorder does nothing

naïve perturb. th. works

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$$d < 4$$

expansion in $\epsilon = 4 - d$

$$\tilde{\Delta}(\vec{r}) = \epsilon \tilde{\Delta}_1(\vec{r}) + \epsilon^2 \tilde{\Delta}_2(\vec{r}) + ..$$

FRG equation

$$\Delta(r) = t^{\frac{d-4}{2}+\zeta}\tilde{\Delta}(r/t^{\zeta/2})$$

$$\epsilon = 4 - d$$

$$2t\partial_t\tilde{\Delta}(r) = (\epsilon - 2\zeta)\tilde{\Delta}(r) + \zeta r\tilde{\Delta}'(r)$$

similar any D

$$-m\partial_m = 2t\partial_t$$

$$-\frac{1}{2}\frac{d^2}{dr^2}(\tilde{\Delta}(r) - \tilde{\Delta}(0))^2$$

$$+\frac{1}{2}\frac{d^2}{dr^2}(\tilde{\Delta}'^2 - \tilde{\Delta}'(0)^2)(\tilde{\Delta} - \tilde{\Delta}(0)) + O(\tilde{\Delta}^4)$$

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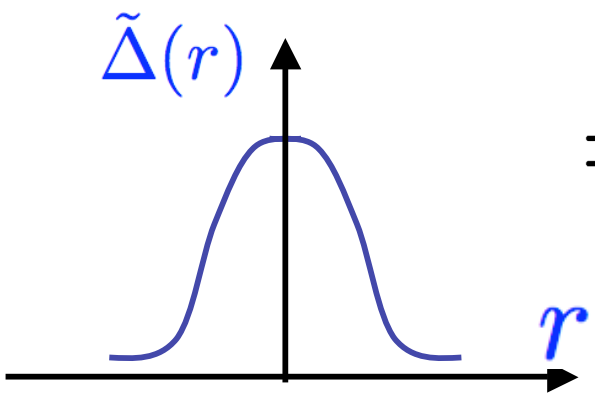

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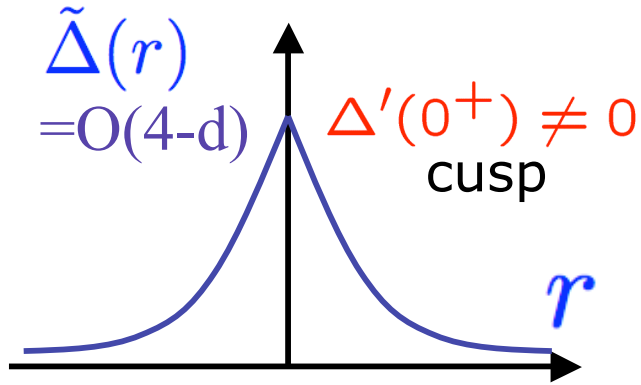

$$\zeta = \zeta_1\epsilon + ..$$

$$\zeta_{RF} = \epsilon/3$$

$$\zeta_{RB} = 0.208\epsilon$$

Unique solution each Univ class (B.C)

Inertial range $r \ll \ell(t) \sim t^{\zeta/2}$ $r = x^{\zeta} \sim m^{-\zeta} \sim t^{\frac{\zeta}{2}}$

$\tilde{\Delta}(r) \approx -\Delta'(0^+)|r|$ $\overline{(v(r) - v(0))^2} \sim |r|^{\zeta_2}$ $\zeta_2 = 1$

shocks

$E(t) = \frac{1}{2}\overline{v^2} = \frac{1}{2}\Delta(0)$ energy conservation (smooth flow)

$$2t\partial_t \Delta(0) = 0$$

dis. systems= dimensional reduction

Inertial range $r \ll \ell(t) \sim t^{\zeta/2}$ $\qquad r = x^{\zeta} \sim m^{-\zeta} \sim t^{\frac{\zeta}{2}}$

$$\tilde{\Delta}(r) \approx -\Delta'(0^+)|r| \qquad \overline{(v(r)-v(0))^2} \sim |r|^{\zeta_2} \qquad \zeta_2 = 1$$

shocks

$$E(t) = \frac{1}{2}\overline{v^2} = \frac{1}{2}\Delta(0)$$

energy conservation (smooth flow)

$$2t\partial_t \Delta(0) = 0$$

cusp (shocks) $\longrightarrow$ $2t\partial_t \Delta(0) \sim -\Delta'(0^+)^2$

$$E(t) = \frac{1}{2} t^{\frac{d-4}{2}+\zeta} \tilde{\Delta}(0) \qquad (4-d-2\zeta)\tilde{\Delta}(0) = -\tilde{\Delta}'(0^+)^2 + ..$$

$$= \nu t^{-\frac{\theta}{2}} \tilde{\Delta}''(0)$$

$$\theta = d - 2 + 2\zeta$$

Energy cascade

matched at dissipative scale

$$\lim_{\nu \to 0} \nu \overline{(\nabla v)^2} \neq 0$$

distribution of shock sizes

size $S = v(r^-) - v(r^+)$ $\qquad tv = r - u(r)$

$$\overline{(v(r) - v(0))^p} \sim \overline{S^p}\, r + O(r^2)$$

FRG yields $d = 4$ $\qquad P(S) \sim S^{-3/2} e^{-S/(4S_m)}$

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$$d = 4 - \epsilon$$

$$P(S) = \frac{< S >}{2\sqrt{\pi}} S_m^{\tau-2} A S^{-\tau} \exp(C(\frac{S}{S_m})^{1/2} - B(\frac{S}{4S_m})^{\delta})$$

$\tau = \frac{3}{2} - \frac{\epsilon - \zeta}{8}$ $\qquad \delta = 1 + \frac{\epsilon - \zeta}{4}$ $\qquad S_m = c\, t^{\frac{d-2+\zeta}{2}}$

$$\tau_{\text{conj}} = 2 - \frac{2}{d + \zeta}$$

Can this method make predictions directly for d=0 i.e. decaying Burgers ?

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YES !

Shock statistics in D>1 Burgers

$$\delta\vec{v}(\vec{r}_1,\vec{r}_2) = \vec{v}(\vec{r}_1,0) - \vec{v}(\vec{r}_2,0)$$

$$\frac{1}{2}\overline{\delta v_i(\vec{r}_0,\vec{r}_0+\vec{r})\delta v_j(\vec{r}_0,\vec{r}_0+\vec{r})} = \frac{B}{2}|\vec{r}|(\delta_{ij}+\hat{r}_i\hat{r}_j)$$

generalization of Brownian initial velocity in D=1

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generalization of Brownian initial velocity in D=1

$$2t\partial_t\tilde{\Delta}'(r) = (\epsilon-\zeta)\tilde{\Delta}(r) + \zeta\tilde{\Delta}''(r)$$
$$-3\tilde{\Delta}'\tilde{\Delta}'' - \tilde{\Delta}'''(\tilde{\Delta}-\tilde{\Delta}(0)) + O(\tilde{\Delta}^3)$$

$$\Delta'_{t=0}(r) = B$$

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$$-3\tilde{\Delta}'\tilde{\Delta}'' - \tilde{\Delta}'''(\tilde{\Delta}-\tilde{\Delta}(0)) + O(\tilde{\Delta}^3)$$

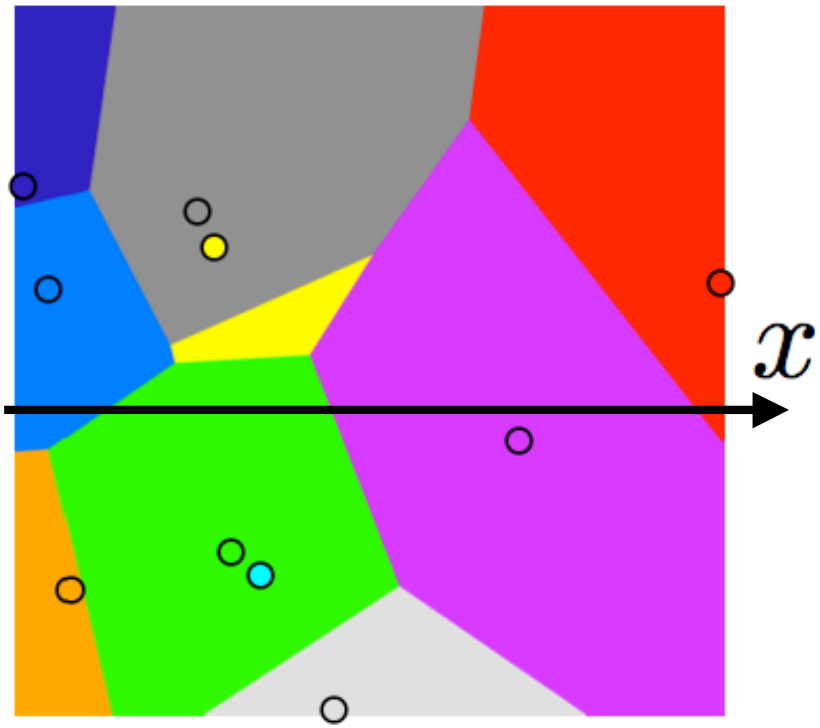
$$\Delta'_{t=0}(r) = B$$

$$\zeta = 4-d$$

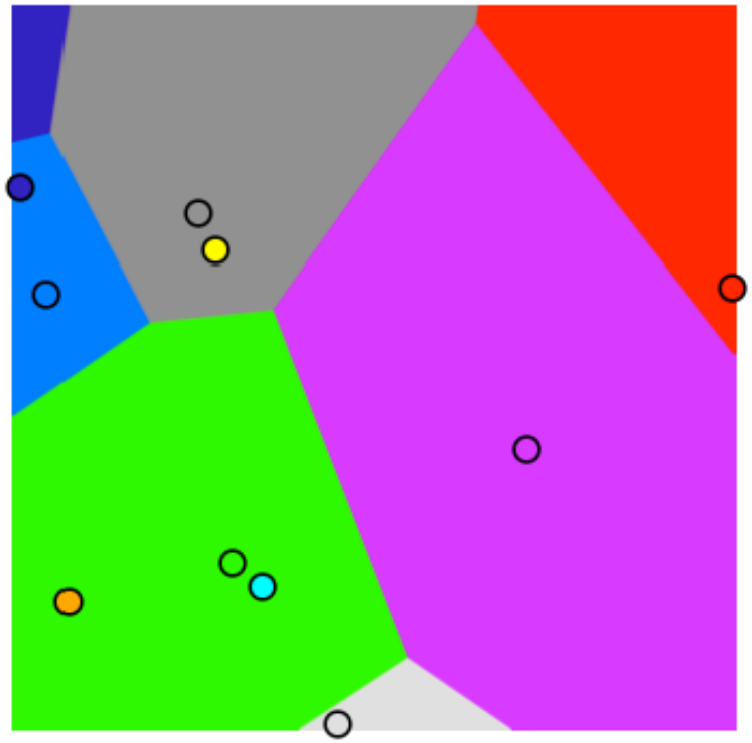
$$\longrightarrow \partial_t\tilde{\Delta}(r) = 0$$

$$\ell(t) \sim t^2$$

Conjecture: TRUE (i) any order
(ii) any D



$t = 0.035$



$t = 0.1$

Consequences of conjecture+ numerical checks

$$\overline{e^{-\vec{\lambda}\cdot[\vec{v}(x\vec{e}_1,t)-\vec{v}(0,t)]}} = e^{x[Z_t(\vec{\lambda})-\lambda_x]}$$

$$\tilde{Z}(\vec{\lambda}) = \int \mathrm{d}\vec{s}\left(e^{\vec{\lambda}\cdot\vec{s}}-1\right)p(\vec{s})$$

shocks along a line are uncorrelated !

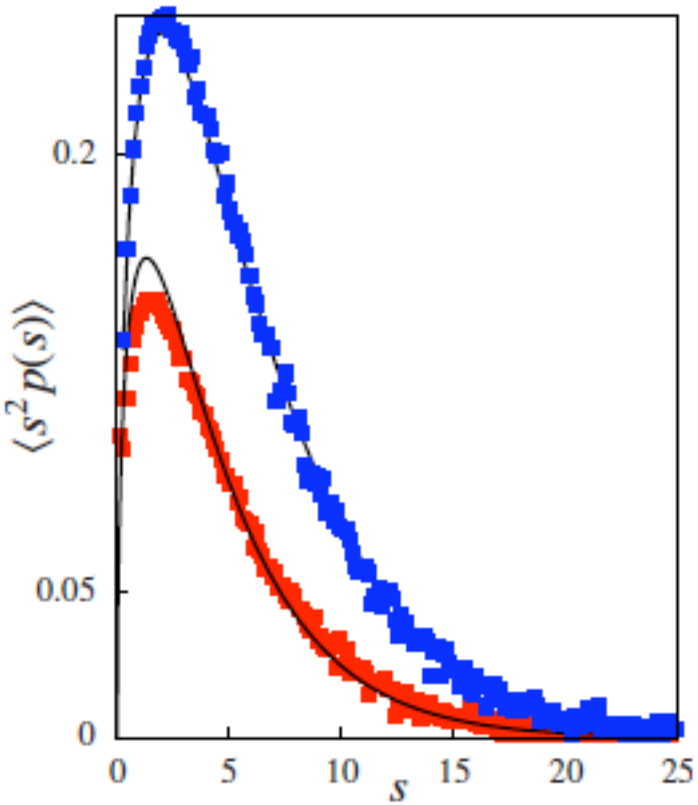
$$p_1(s) = \frac{1}{2\sqrt{\pi}s^{3/2}}e^{-s/4} \qquad p_1(s_x) := \int \mathrm{d}s_\perp p(s_x, s_\perp)$$

D=1 proved by Bertouin

$$p_2(s_\perp) = \int \mathrm{d}s_x\, p(s_x, s_\perp)$$

$$\lambda(\theta) = \sin\theta \frac{\sqrt{5-\cos(4\theta)}+2}{\left[1-\cos(2\theta)+\sqrt{5-\cos(4\theta)}\right]^2}$$

$$\tilde{Z}_2(\theta) = \frac{\cos\theta}{2}\frac{\sqrt{5-\cos(4\theta)}-2}{1-\cos(2\theta)+\sqrt{5-\cos(4\theta)}}\,.$$



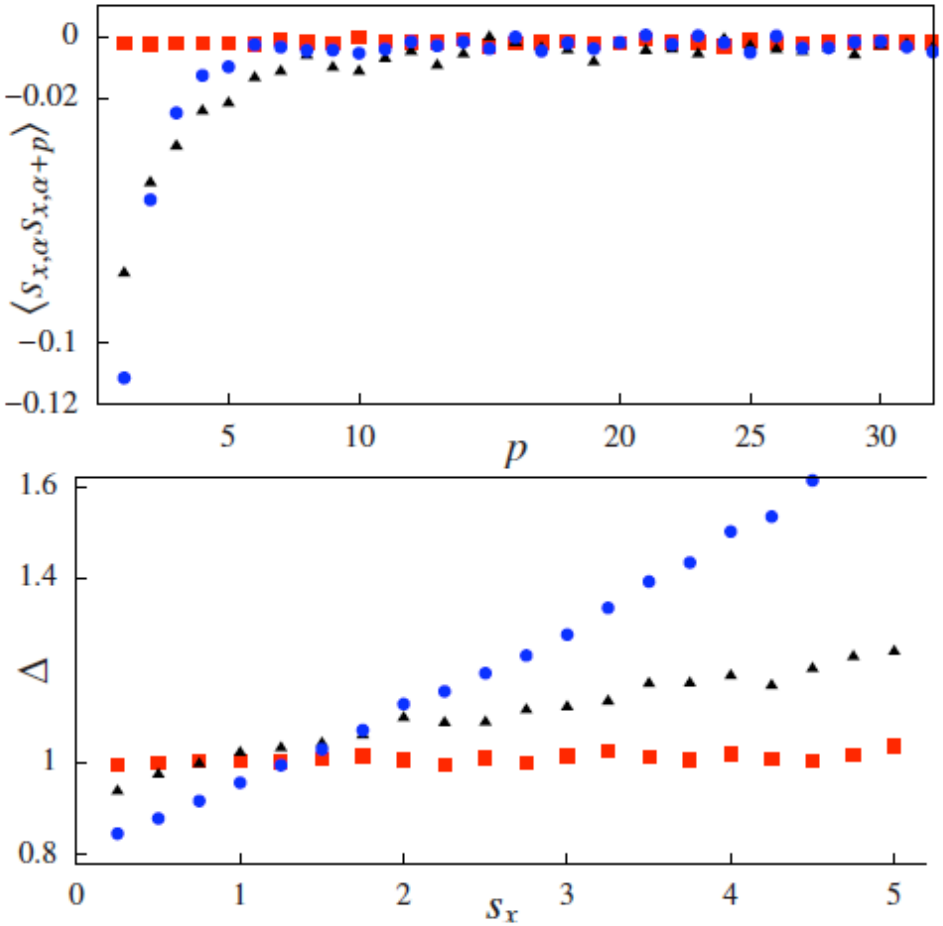


FIG. 4: Color Online. Circles corresponds to $H = 1/2$, Triangles to $H = 1$ and Squares to $H = 3/2$. Top: Connected size correlations of subsequent shocks. Correlation decay is slower for $H = 1$, for $H = 3/2$ no correlation has been detected. Bottom: Normalized shock distance.

(Decaying incompressible) Navier Stokes..?..

$$\partial_t v^\alpha_{k,t} = -\nu k^2 v^\alpha_{k,t} - \frac{1}{2} P_{\alpha;\beta\gamma}(k) \sum_{p+q=k} v^\beta_{q,t} v^\gamma_{p,t}$$

$$P_{\alpha;\beta\gamma}(k) = i k^\alpha \delta_{\beta\gamma}$$ Burgers

$$P_{\alpha;\beta\gamma}(k) = i k^\beta P^T_{\alpha\gamma}(k) + i k^\gamma P^T_{\alpha\beta}(k)$$ NS $k^\gamma v^\gamma(k,t) = 0$

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no Cole-Hopf.. loop expansion

$$\langle v^\alpha_{kt} v^\beta_{k't} \rangle = \delta_{k,k'} \Delta_{t,\alpha\beta}(k)$$

$$\partial_t \Delta = t\Delta * \Delta + t^3 \Delta * \Delta * \Delta + ..$$
$$C^{(3)} = t\Delta * \Delta + t^3 \Delta * \Delta * \Delta + .. \quad , \quad C^{(4)} = \Delta * \Delta + t^2 \Delta * \Delta * \Delta + ..$$
$$C^{(5)} = t\Delta * \Delta * \Delta + .. \quad , \quad C^{(6)} = \Delta * \Delta * \Delta + ..$$

renormalized small time expansion..

One loop FRG

N = D

$$\Delta_{t,\alpha\beta}(k) = t^{\zeta-2+N\frac{\zeta}{2}}\tilde{\Delta}_{t,\alpha\beta}(kt^{\zeta/2})$$

$$\Delta_{\alpha\beta}(k) = P^T_{\alpha\beta}(k)\Delta(k)$$

$$E(k) \sim k^{N-1}\Delta(k)$$

$$t\partial_t\tilde{\Delta}(k) = (2-\zeta-N\frac{\zeta}{2}-\frac{\zeta}{2}k\cdot\partial_k)\tilde{\Delta}(k)$$

$$+\frac{2}{N-1}\sum_q \tilde{b}_{k,k-q,q}(\tilde{\Delta}(q)\tilde{\Delta}(k-q)-\tilde{\Delta}(q)\tilde{\Delta}(k))$$

$$\tilde{b}_{k,k-q,q} = \frac{k^2q^2-(k\cdot q)^2}{k^2q^2(k-q)^2}\left\{(k^2-q^2)\left[(k-q)^2-q^2\right]+(N-2)k^2(k-q)^2\right\}$$

cusp or no cusp ?

$$\langle (v(r) - v(0))^2 \rangle \sim |r|^{\zeta_2} \qquad \tilde{\Delta}(k) \sim k^{-(N+\zeta_2)}$$

$$C = \frac{-\sqrt{\pi}}{4(4\pi)^{N/2}} \Big(\frac{2^{\zeta_2}((N-2)\zeta_2 - (N-1))(N+\zeta_2)\Gamma(-\frac{\zeta_2}{2})\Gamma(\frac{N}{2}+\zeta_2)}{\Gamma(\frac{3-\zeta_2}{2})\Gamma(\frac{2+N+\zeta_2}{2})^2}$$

$$- \frac{4\sqrt{\pi}N\Gamma(\frac{N}{2})}{\sin(\frac{\pi\zeta_2}{2})\Gamma(\frac{4+N-\zeta_2}{2})\Gamma(\frac{N+\zeta_2}{2})} \Big) = 0$$

$$N = 3 \quad \longrightarrow \quad \zeta_2 = 1$$

Kolmogorov $\zeta_2 = 2/3$

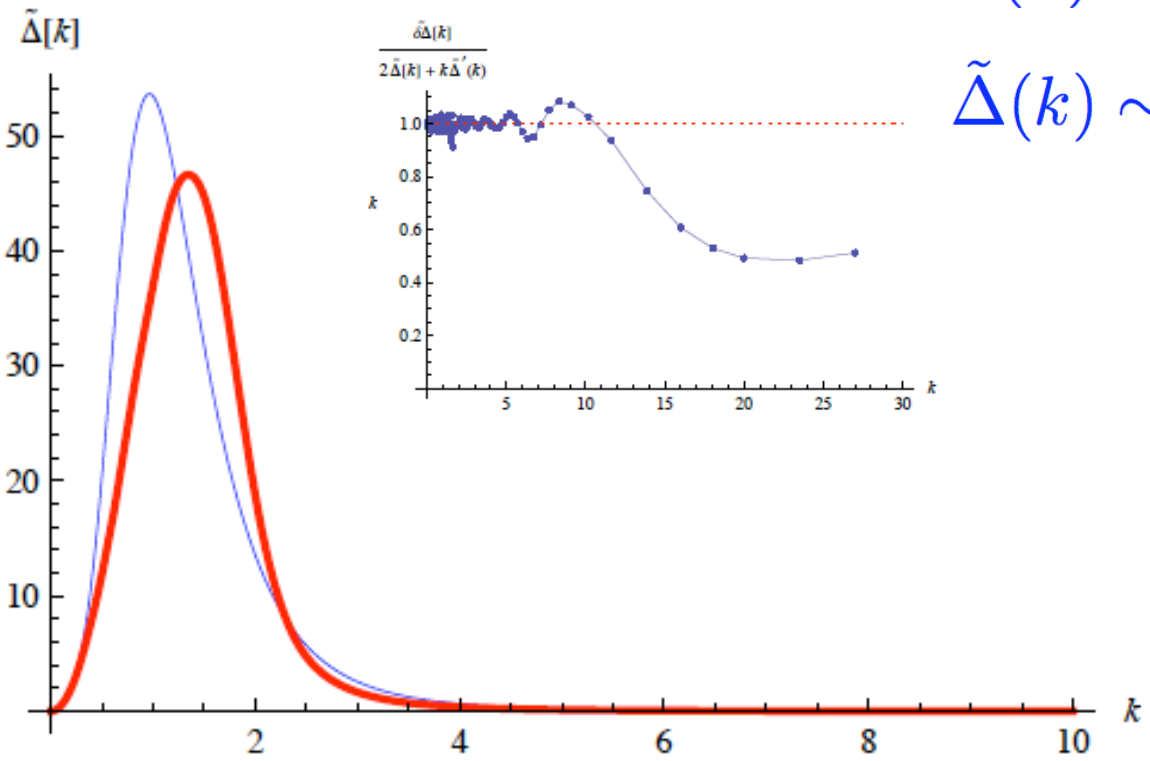
$$\Delta_{\alpha\beta}(k) = P^T_{\alpha\beta}(k)\Delta(k)$$

N=2

$$\tilde{\Delta}(k) \sim k^2 \qquad k \ll 1$$

$$\tilde{\Delta}(k) \sim \frac{1}{k^4} \qquad k \gg 1$$

$$E(k) \sim k\Delta(k)$$



large N

becomes similar to Burgers..

conclusion

•Burgers generalized to manifolds d - similar physics

energy cascade
inertial range
shocks

•around d=4 loop expansion/FRG/truncation becomes controled

compute e.g. energy decay, shock size distributions

•conjecture for a solution Burgers D>1 generalization of Brownian IC

• NS: same approach gives $\zeta_2 = 1$ at one loop D=3

cascades etc..

Why ? Higher loop?

Interesting to learn about situation where truncations become controled..

Frg equation for forced burgers.. In progress