Power-law decay of the energy spectrum in linearized perturbed systems

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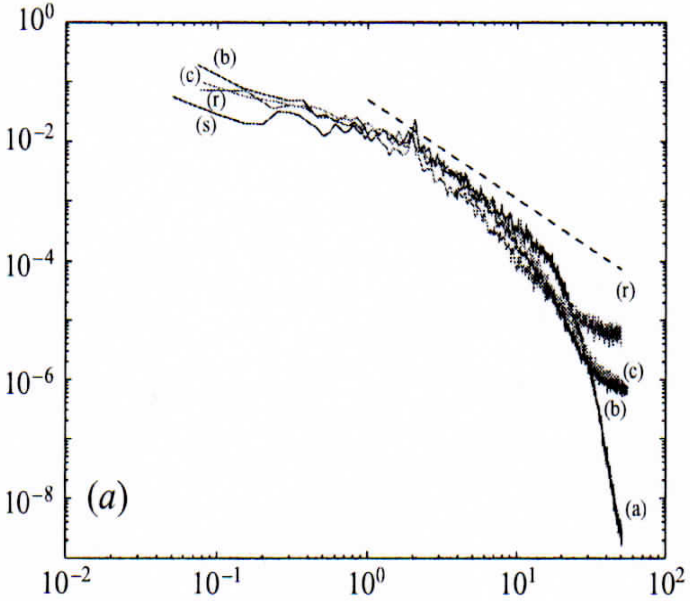

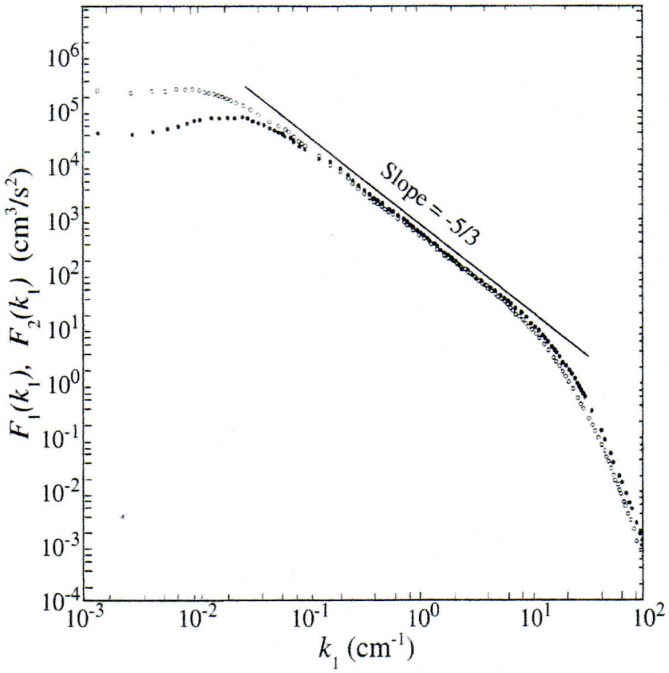

(left) *Evangelinos & Karniadakis, JFM 1999*. (right) *Champagne, JFM 1978*.

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- The perturbative evolution is ruled out by the **initial-value problem** associated to the Navier-Stokes linearized formulation.

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  - The difference is large $\Rightarrow$ quantitative measure of the nonlinear interaction in spectral terms;
  - The difference is small $\Rightarrow$ higher degree of universality on the value of the exponent of the inertial range, not necessarily associated to the nonlinear interaction.

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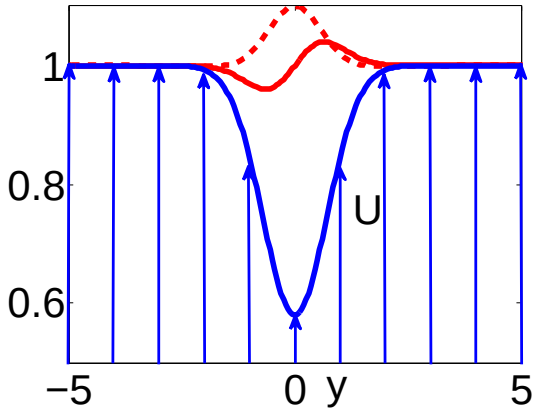

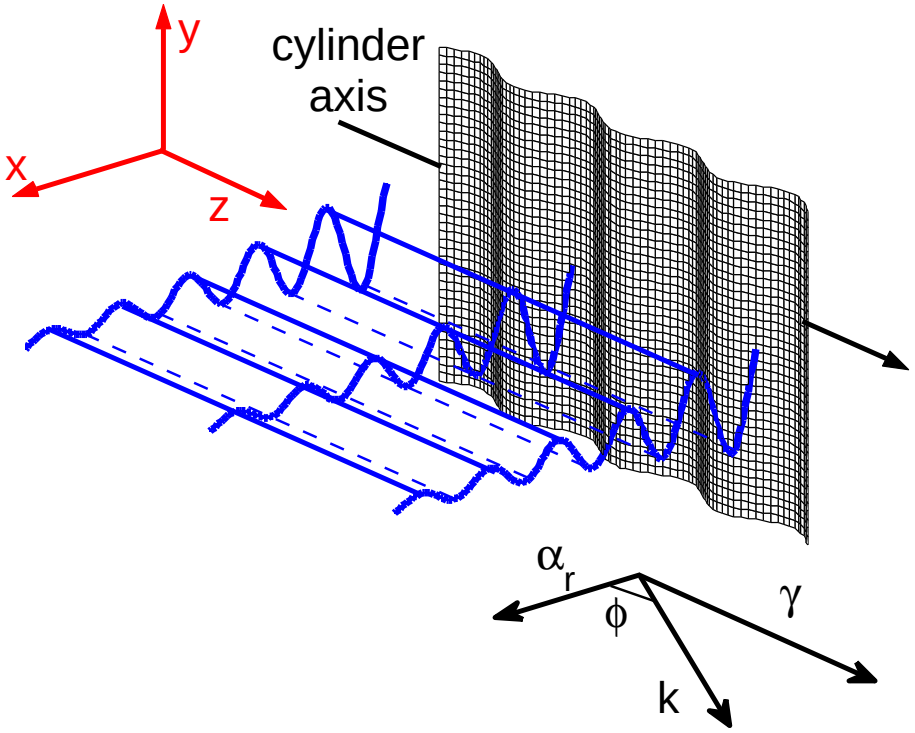

Perturbative equations

- Perturbative linearized system:

$$\begin{aligned}
\frac{\partial^2 \hat{v}}{\partial y^2} &- (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i)\hat{v} = \hat{\Gamma} \\
\frac{\partial \hat{\Gamma}}{\partial t} &= (i\alpha_r - \alpha_i)(\frac{d^2 U}{dy^2}\hat{v} - U\hat{\Gamma}) + \frac{1}{Re}[\frac{\partial^2 \hat{\Gamma}}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i)\hat{\Gamma}] \\
\frac{\partial \hat{\omega}_y}{\partial t} &= -(i\alpha_r - \alpha_i)U\hat{\omega}_y - i\gamma\frac{dU}{dy}\hat{v} + \frac{1}{Re}[\frac{\partial^2 \hat{\omega}_y}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i)\hat{\omega}_y]
\end{aligned}$$

  The transversal velocity and vorticity components are $\hat{v}$ and $\hat{\omega}_y$ respectively, $\hat{\Gamma}$ is defined as $\widetilde{\Gamma} = \partial_x\widetilde{\omega}_z - \partial_z\widetilde{\omega}_x$.
- Initial conditions:
  - $\hat{\omega}_y(0, y) = 0$;
  - $\hat{v}(0, y) = e^{-y^2}\sin(y)$ or $\hat{v}(0, y) = e^{-y^2}\cos(y)$;
- Boundary conditions: $(\hat{u}, \hat{v}, \hat{w}) \to 0$ as $y \to \infty$.

Perturbation energy

- Kinetic energy density e:

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\begin{aligned}
e(t;\alpha,\gamma) &= \int_{-y_d}^{+y_d} (|\hat{u}|^2 + |\hat{v}|^2 + |\hat{w}|^2) dy \\
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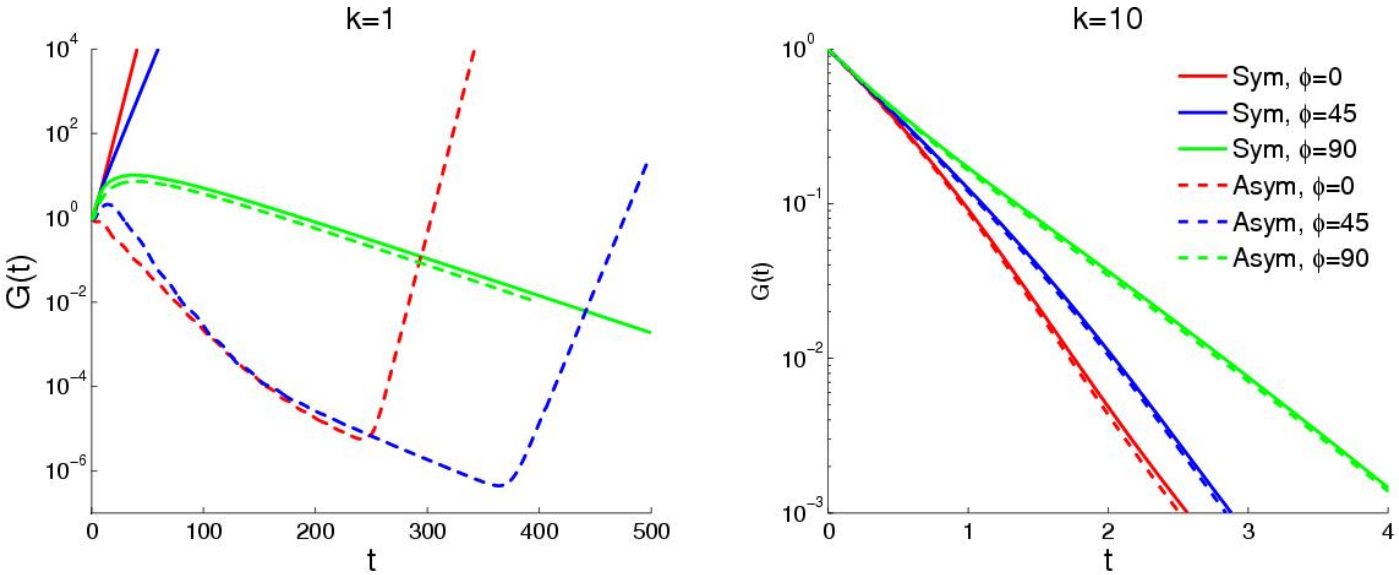
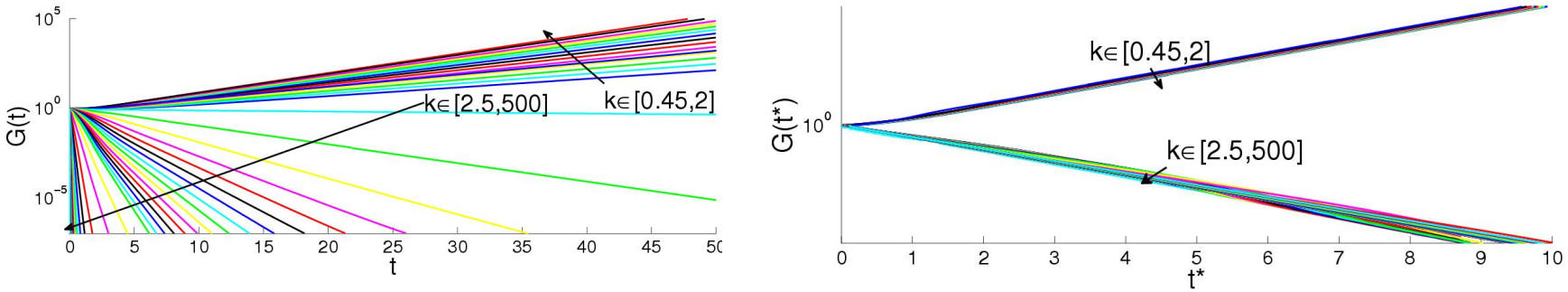


The amplification factor G, obtained at $Re = 100$, $x_0 = 10$, with different initial condition and obliquity angle for a long (on the left) and a short (on the right) waves

Time normalization

- We have different temporal scales associated to the different perturbation wavelengths $\Rightarrow$ A continuous instantaneous normalization can be used by defining as $t* = \frac{t}{\tau_G}, \qquad \tau_G = \frac{G(t)}{\left|\frac{dG(t)}{dt}\right|}$



The amplification factor G, obtained at $Re = 100$, $x_0 = 10$, with symmetric initial condition, $\phi = 0$ as a function of t (on the left) and of t* (on the right)

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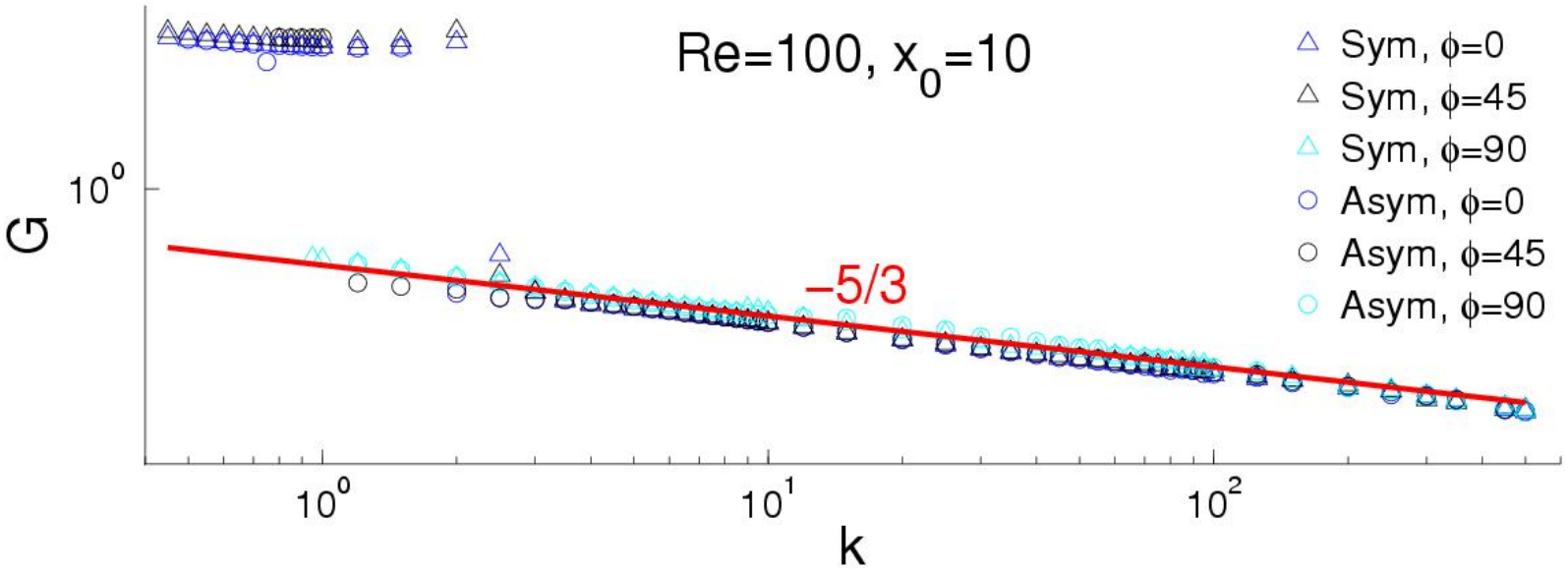
High Reynolds number and intermediate wake configuration



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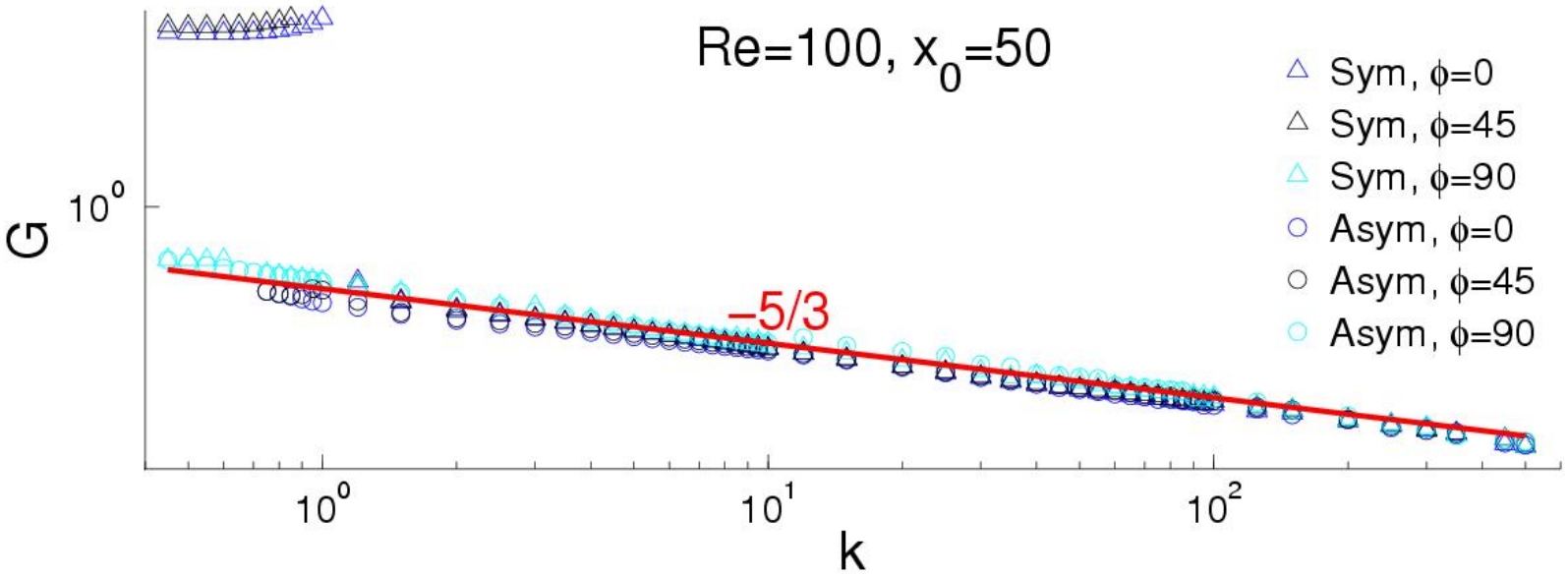

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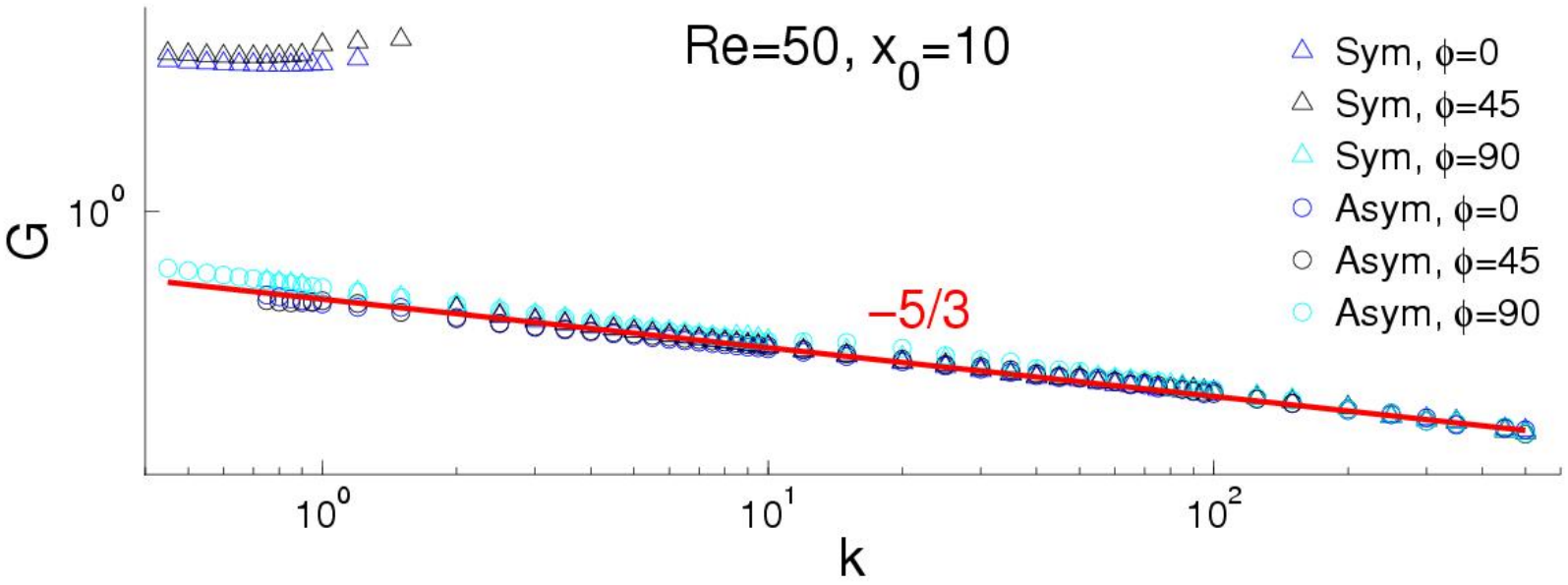
Middle Reynolds number and intermediate wake configuration



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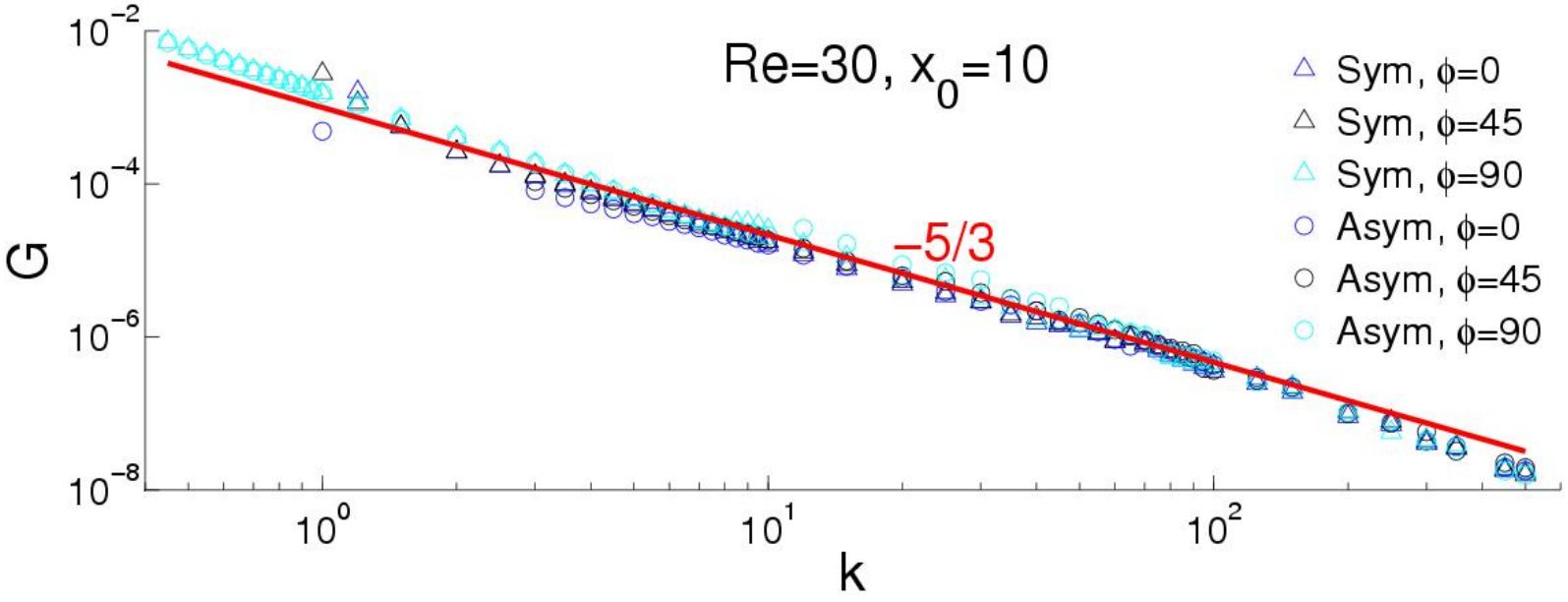

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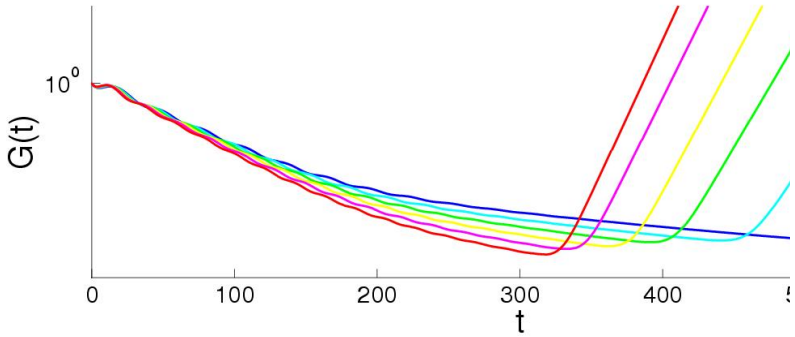

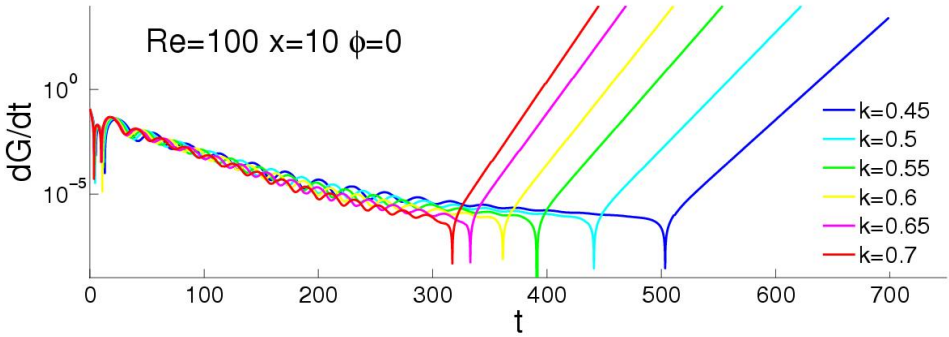

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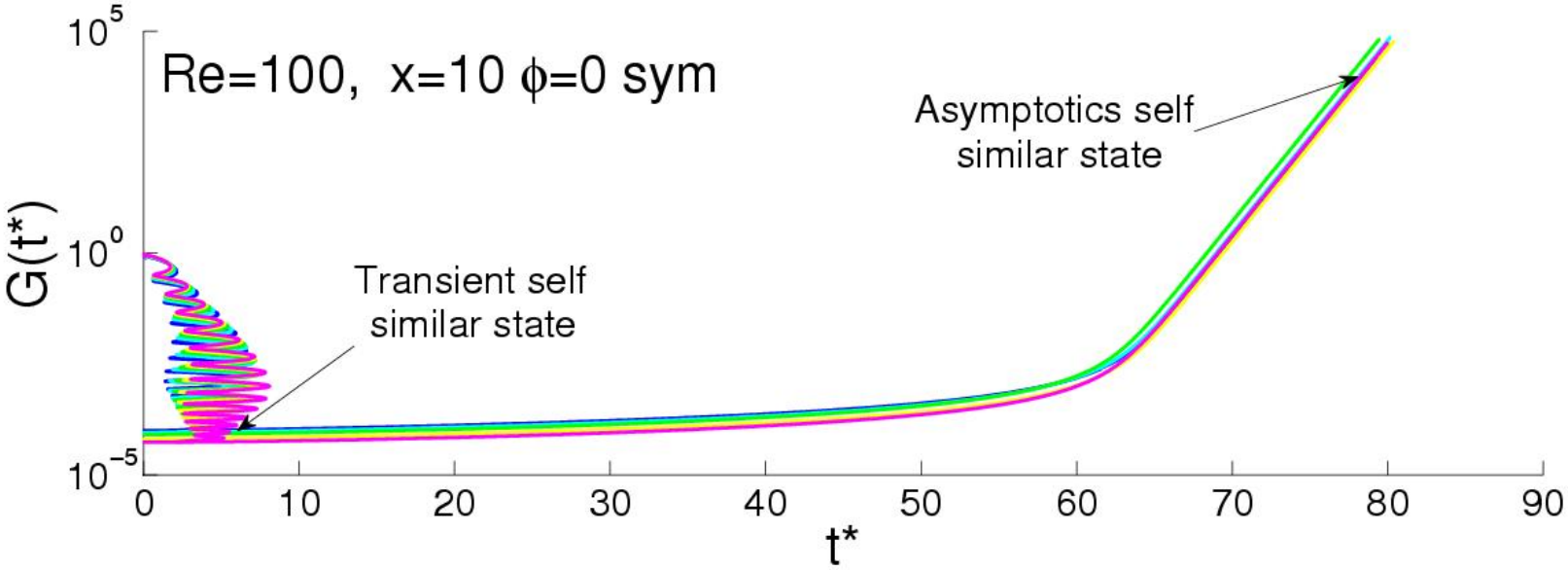

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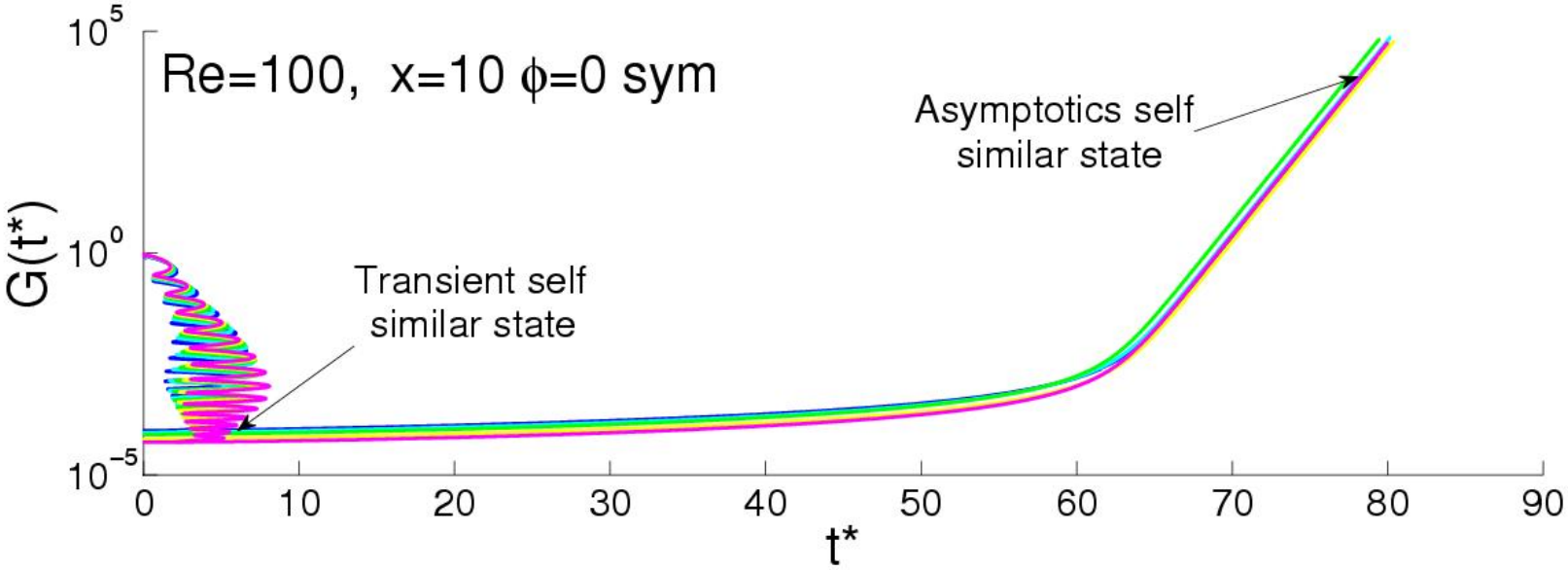

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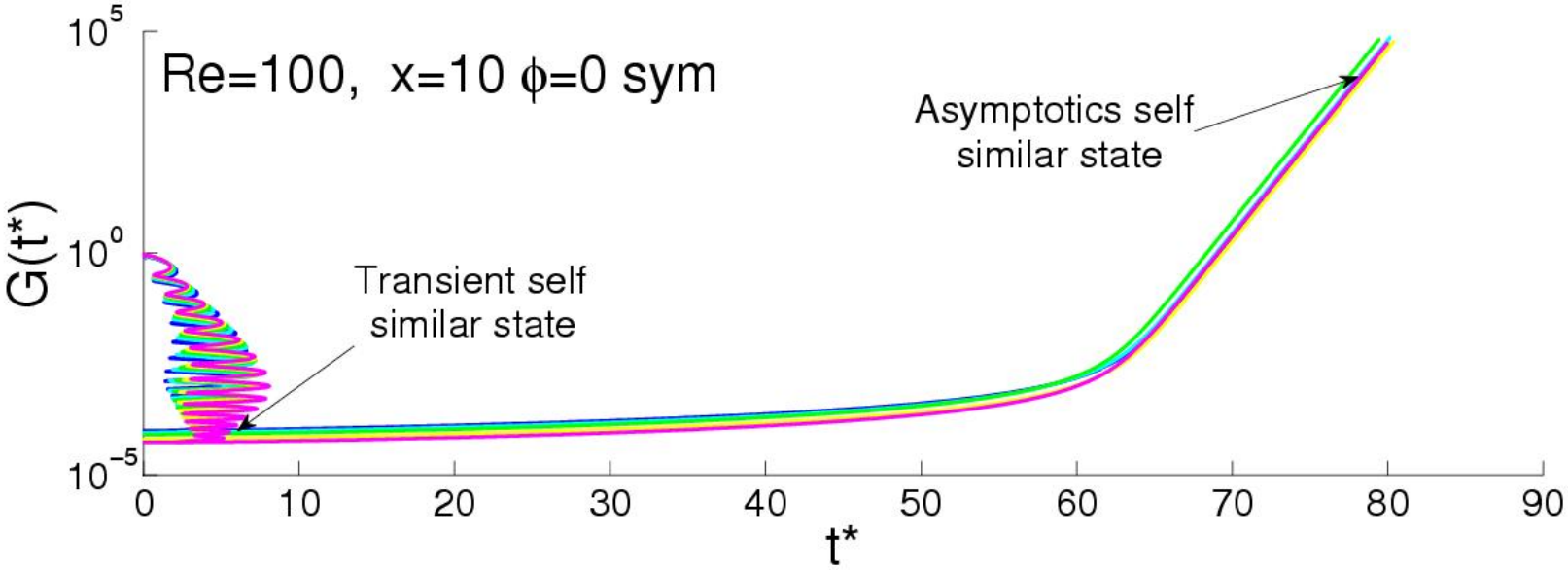

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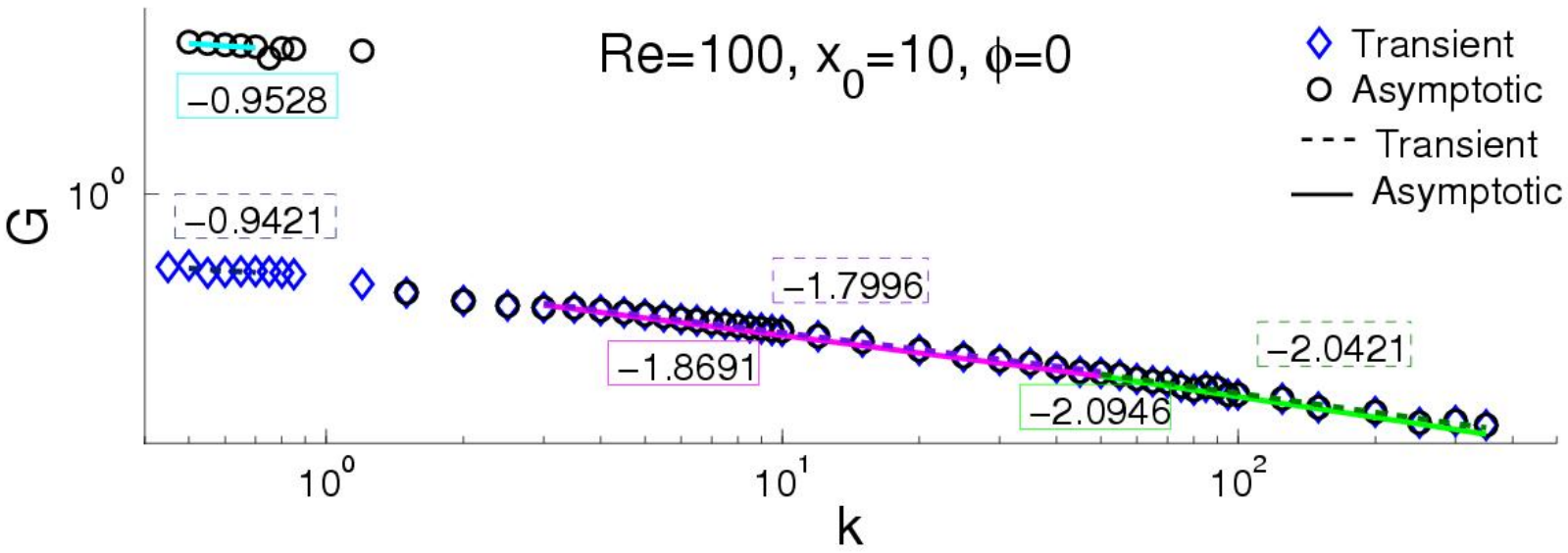

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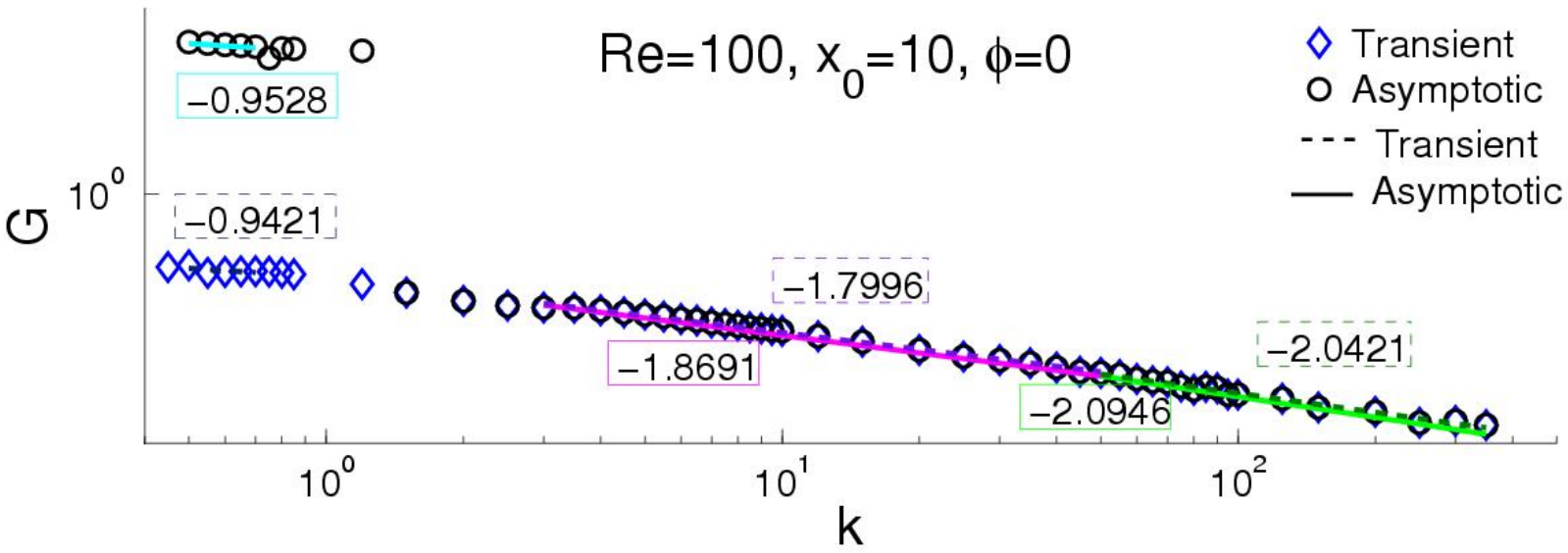

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Coming next $\Rightarrow$ Temporal observation window of a large number of small 3D perturbations injected in a statistical way into the system.