MEAN FIELD AND MEAN FLOWS IN OSCILLATORY DOUBLE-DIFFUSIVE CONVECTION

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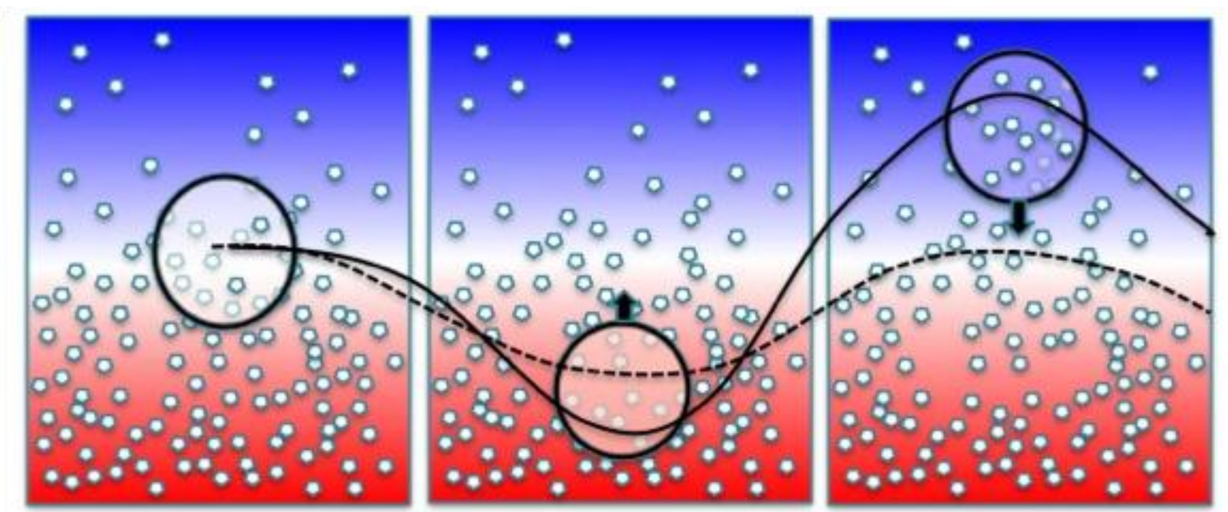
Outline

- What is oscillatory double-diffusive convection?
- Numerical simulations
- Mean-field theory for staircase formation
- Wave/mean-flow interaction for shear layer formation

Oscillatory double-diffusive convection

- A fluid that exhibits a stable compositional gradient alone supports internal gravity waves.
- The addition of a small temperature gradient can destabilize the wave (even if the background density of the system remains stably stratified), as long as temperature diffuses more rapidly than composition

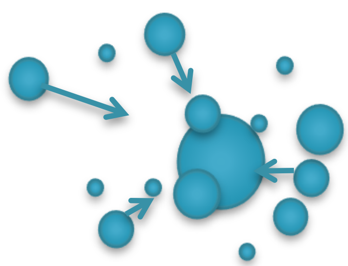


Where is ODDC found?

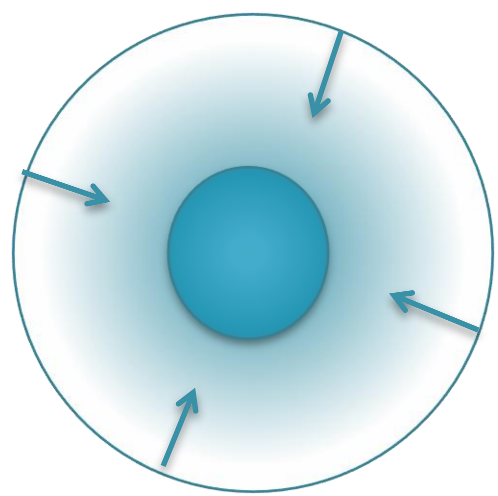
- On Earth:
  - *Polar oceans*: Melting ice releases cold/fresh water on top of warmer, saltier water.
  - *Volcanic lakes*: Geothermal activity warms bottom of lake & releases methane/other dense gases below cold, fresh water.
  - Oscillatory-unstable (but not very much, see later)

Where is ODDC found?

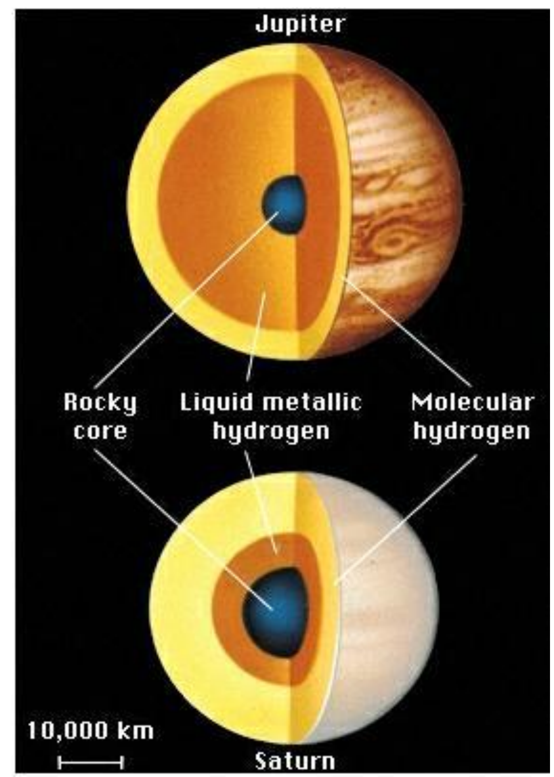
- In Planetary Astrophysics:
  - The core-accretion scenario for giant planet formation leads to an interior structure potentially unstable to oscillatory-convection near core-envelope interface.



Core formation



Gas accretion



Mathematical modeling

Governing equations (Boussinesq approximation, cf. Spiegel & Veronis):

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{\nabla p}{\rho_0} - \frac{\rho}{\rho_0} g\mathbf{e}_z + \nu\nabla^2\mathbf{u}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T - wT_{0z}^{ad} = \kappa_T\nabla^2 T$$

$$\frac{\partial S}{\partial t} + \mathbf{u}\cdot\nabla S = \kappa_S\nabla^2 S$$

$$\nabla\cdot\mathbf{u} = 0$$

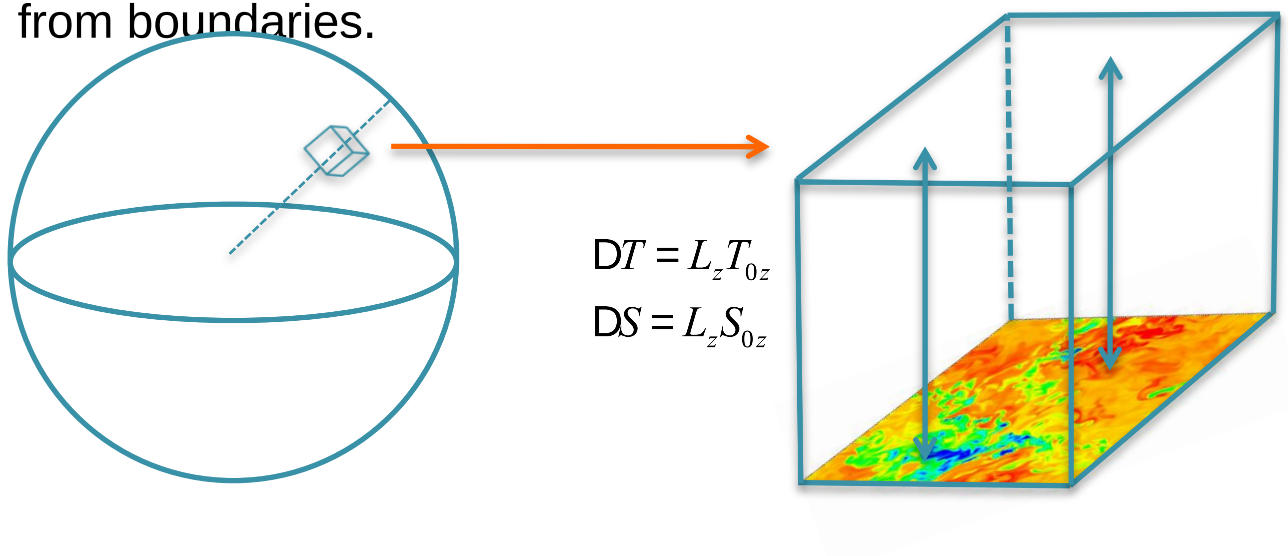
$$\frac{\rho}{\rho_0} = -\alpha T + \beta S$$

- DD-convection scale much, much smaller than system scale, so the Boussinesq (ie. nearly-incompressible fluid) approximation is usually OK.

Mathematical modeling

Model considered:

- Assume **background** temperature and concentration profiles are linear (constant gradients $T_{0z}, T_{0z}^{ad}, S_{0z}$)
- Let $T(x,y,z,t) = zT_{0z} + \tilde{T}(x,y,z,t)$ and $S(x,y,z,t) = zS_{0z} + \tilde{S}(x,y,z,t)$
- Assume that all **perturbations** are triply-periodic in domain (L_x,L_y,L_z):
- This enables us to study the phenomenon with little influence from boundaries.



Mathematical modeling

Governing non-dimensional equations:

$$\frac{1}{\Pr}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + (T - S)\mathbf{e}_\mathbf{z} + \nabla^2\mathbf{u}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T - w = \nabla^2 T$$

$$\frac{\partial S}{\partial t} + \mathbf{u}\cdot\nabla S - R_0^{-1} w = \tau\nabla^2 S$$

$$\nabla\cdot\mathbf{u} = 0$$

Governing parameters:

$$\Pr = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_S}{\kappa_T}$$

$$R_0^{-1} = \frac{\beta S_{0z}}{\alpha(T_{0z} - T_{0z}^{ad})}$$

= Stabilizing S stratification / Destabilizing T stratification

$$[l] = d = \left(\frac{\kappa_T \nu}{\alpha g \left|T_{0z} - T_{0z}^{ad}\right|}\right)^{1/4},$$

$$[t] = \frac{d^2}{\kappa_T}, \quad [T] = d\left|T_{0z} - T_{0z}^{ad}\right|, \quad [S] = \frac{\alpha}{\beta} d\left|T_{0z} - T_{0z}^{ad}\right|$$

Linear theory (basic instability)

Linear stability analysis:

- Assume all perturbations are of the form

$$q(x,y,z,t) = \hat{q} e^{i\mathbf{k}\cdot\mathbf{x} + \lambda t}$$

- Resulting equation for growth rate is a **cubic**

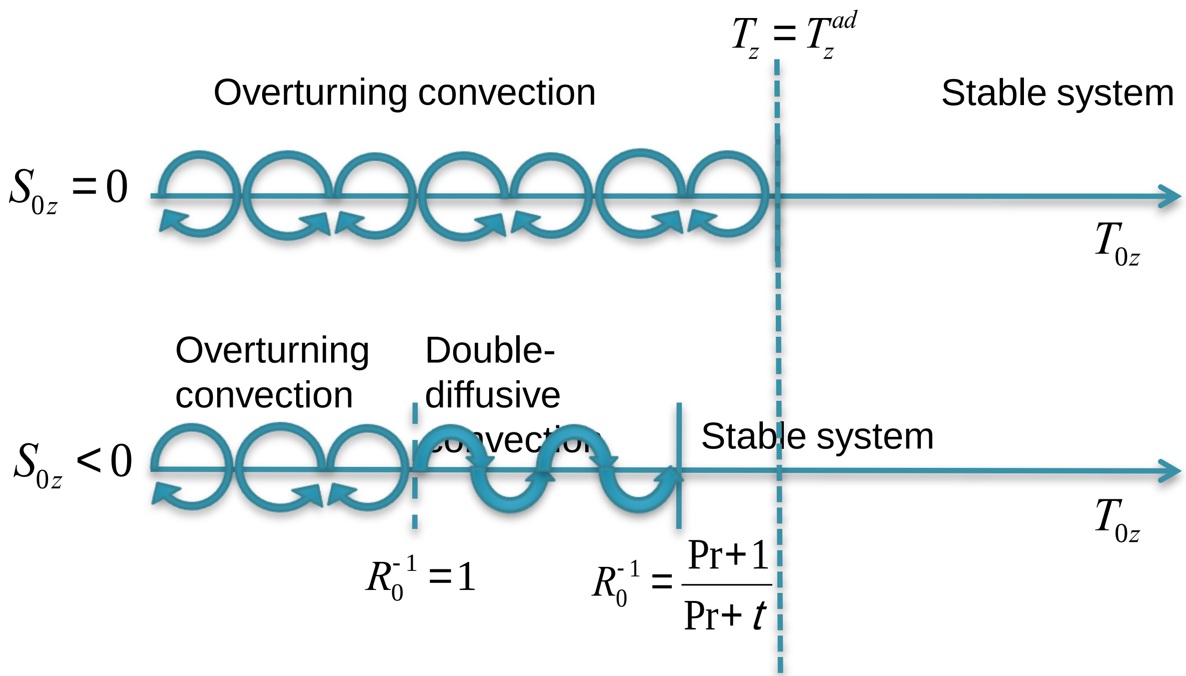
$$\lambda^3 + a\lambda^2 + b\lambda + c = 0$$

where coefficients are functions of $\left(\mathrm{Pr}, \tau, R_0^{-1}, \mathbf{k}\right)$

- Properties of the modes of instability
  - Fastest-growing mode is vertically invariant, horizontal wavelength is of the order of "a few d"
  - Range of instability is $1 < R_0^{-1} < \frac{\mathrm{Pr}+1}{\mathrm{Pr}+\tau}$
  - **Mode is oscillatory**
    - For weak stratification (near overturning convection), $\lambda_R \gg \lambda_I$
    - For strong stratification (close to marginal stability) $\lambda_R \ll \lambda_I$

1.14 in ocean, $O(10^{3\text{-}6})$ in astro

Linear theory (basic instability)



$$R_0^{-1} = \frac{b S_{0z}}{a(T_{0z} - T_{0z}^{ad})}$$

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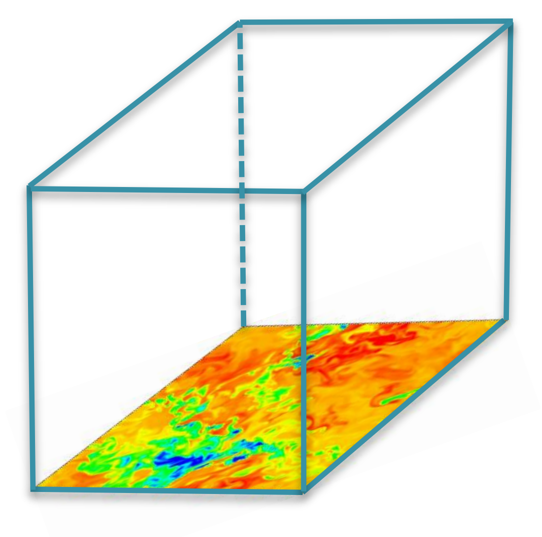
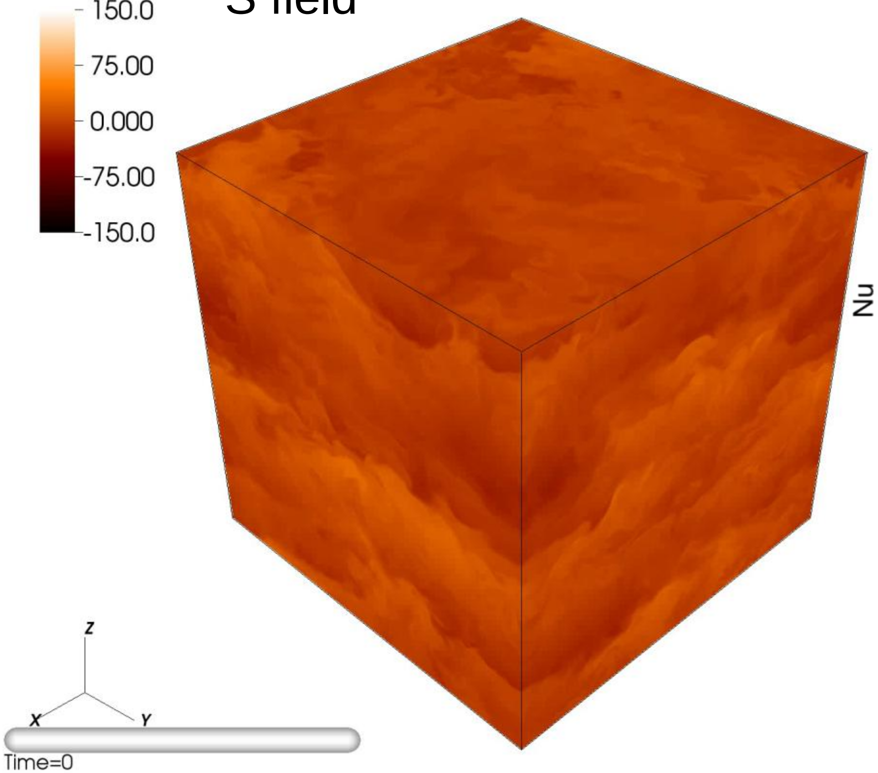


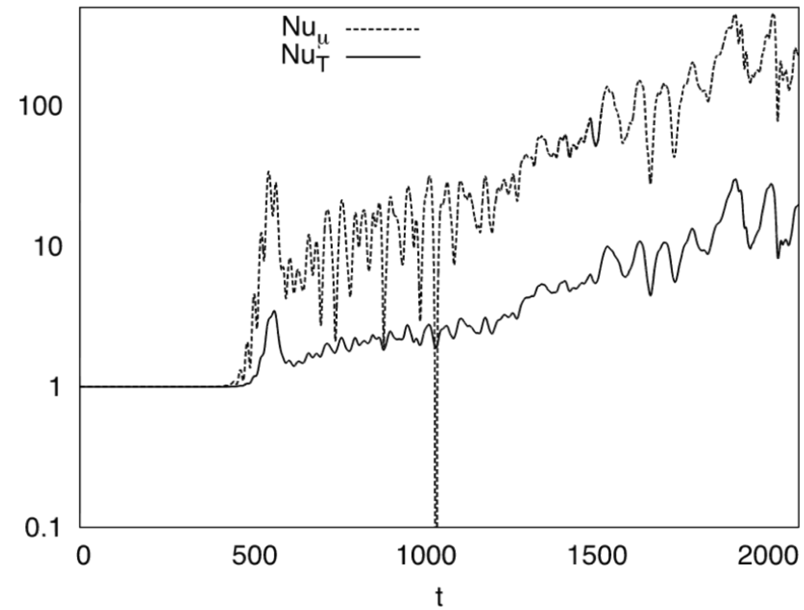
100d x 100d x 100d
Ra = 10^8
Pr = τ= 0.01 or 0.03

Numerical simulations

Example of oscillatory convection close to onset of overturning convection (more unstable case)

S field





$$Nu = \frac{\text{Total flux}}{\text{Diffusive flux}}$$

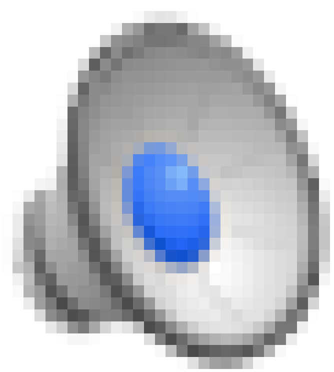
Mirouh, Garaud, Stellmach, Traxler & Wood 2013

Numerical simulations

Example of oscillatory convection close to marginal stability (more stable case)

S field

u field



Moll et al, in prep. 2014

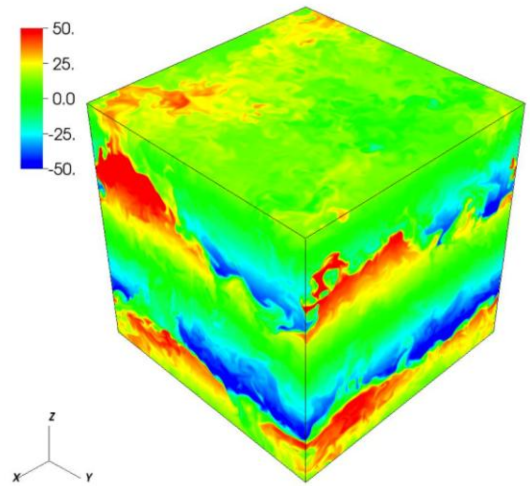
Numerical simulations

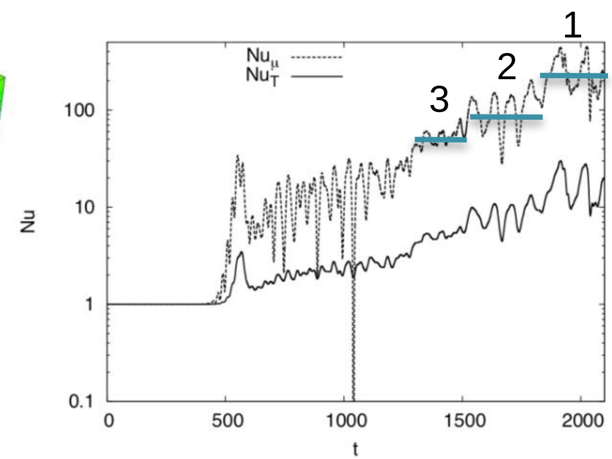
Two outcomes: layers & large-scale gravity wave, with very different transport properties.

$\mathrm{Pr} = 0.03$

$t = 0.03$

$R_0^{-1} = 1.5$

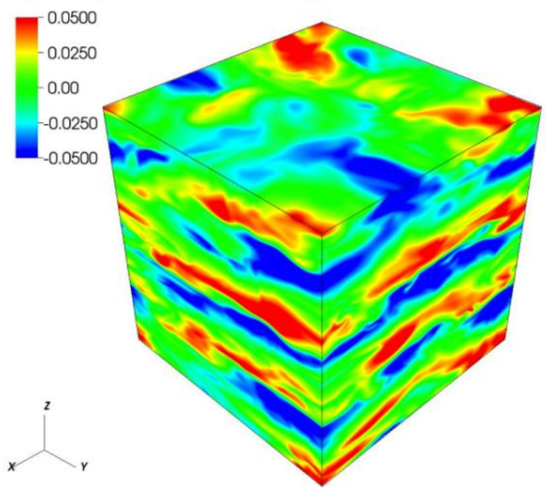


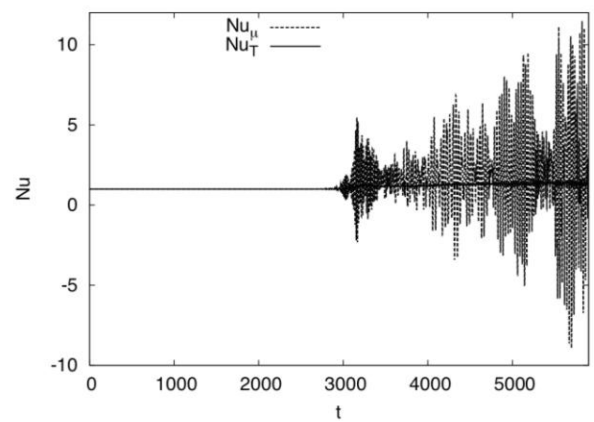


$\mathrm{Pr} = 0.03$

$t = 0.03$

$R_0^{-1} = 5$





Mirouh et al. 2012

Numerical simulations

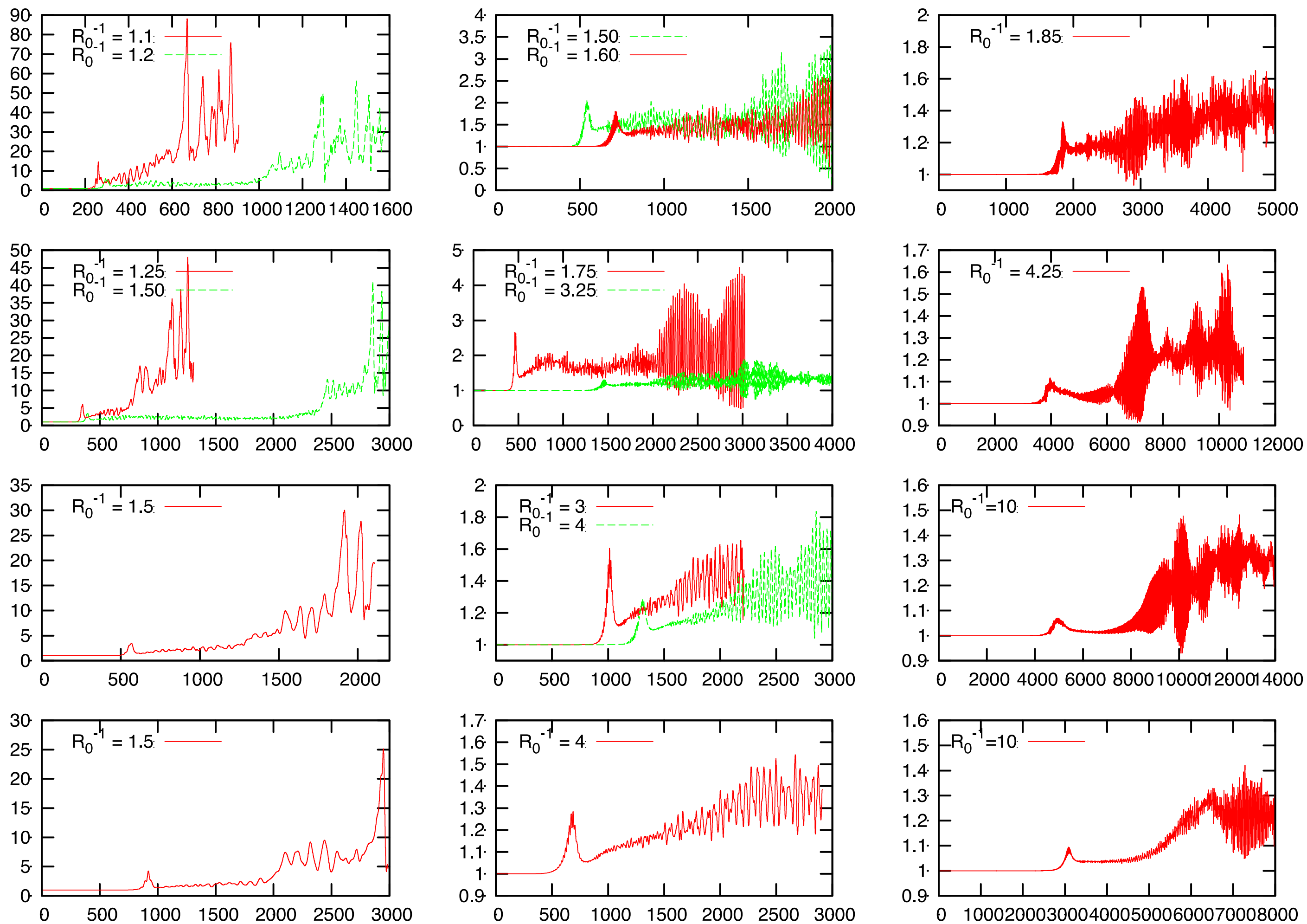
In fact, we always see these two types of solutions.

Pr = t = 0.3

Pr = t = 0.1

Pr = t = 0.03

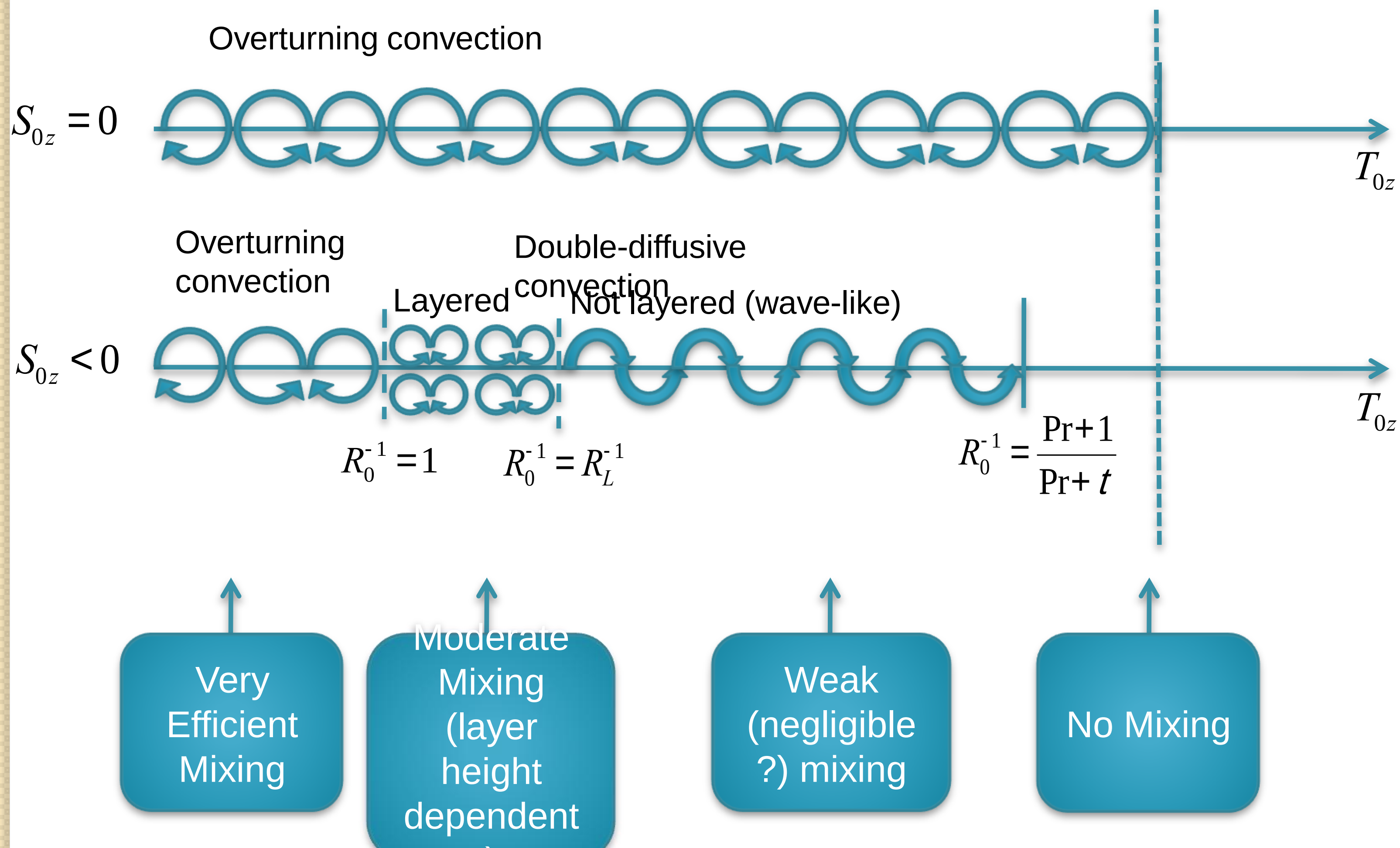
Pr = t = 0.01



Mirouh et al. 2012

Numerical simulations

Schematically:



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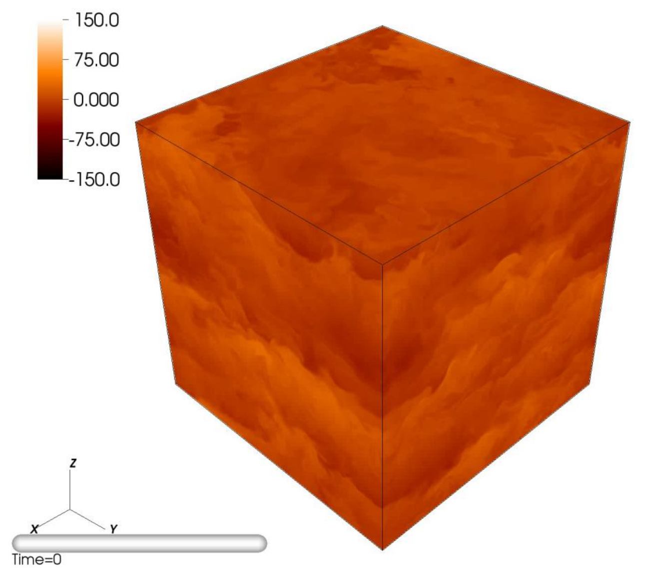
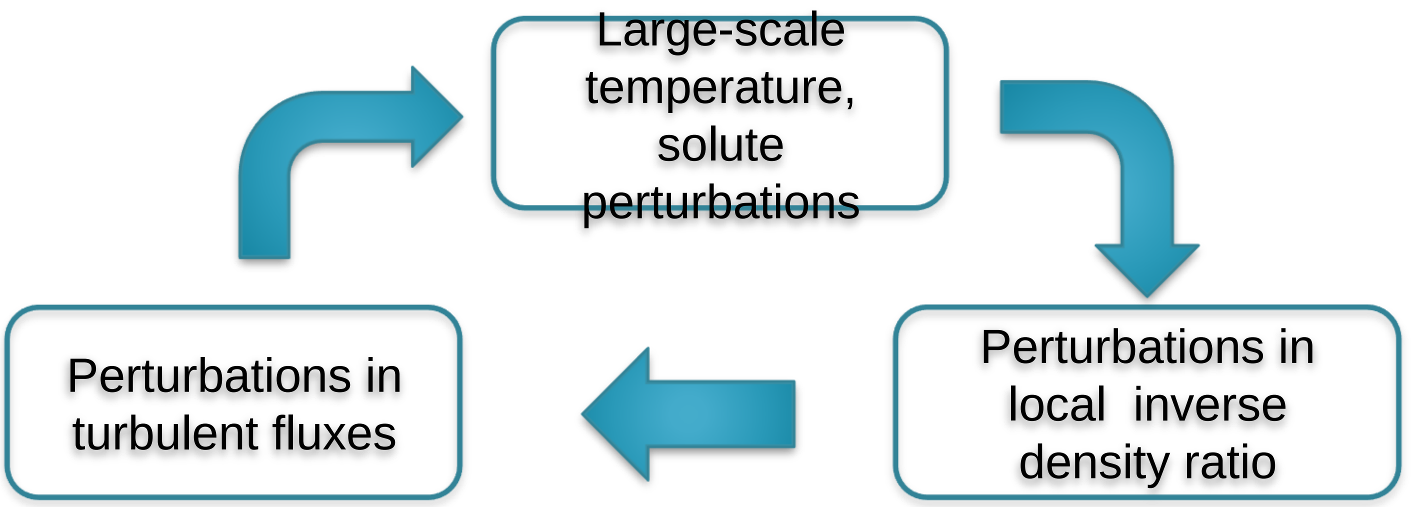


Mean-field theory for layer formation

- The emergence of staircases in DD convection can be understood using “mean-field” theory (Radko 2003)
- **General idea:** large-scale structures form through positive feedback between large-scale temperature/composition perturbation and induced fluxes.



- Different feedback loops can lead to different “mean-field” instabilities, e.g. layering instability, large-scale gravity wave excitation, intrusive instability (Traxler et al. 2011)

Mean-field theory for layer formation

- Horizontally-averaged, filtered equations (ignoring mean flow):

$$\frac{\partial \bar{T}}{\partial t} + \frac{\partial F_T}{\partial z} = 0$$

$$\frac{\partial \bar{S}}{\partial t} + \frac{\partial F_S}{\partial z} = 0$$

Standard closure problem: if the fluxes are known, the problem can be solved for evolution of large-scale fields.

- Assume that

$$F_T = Nu_T \left(1 - \frac{d\bar{T}}{dz}\right)$$

$$F_S = \gamma^{-1} F_T$$

where F_T and F_S are only functions of other non-dimensional quantities:

$$Nu_T = Nu_T(R^{-1}; \mathrm{Pr}, \tau)$$

$$\gamma^{-1} = \gamma^{-1}(R^{-1}; \mathrm{Pr}, \tau)$$

Assumed to be known!

and R^{-1} is the *local* inverse density ratio (a function of z)

$$R^{-1} = \frac{R_0 - \bar{S}_z}{1 - \bar{T}_z}$$

Mean-field theory for layer formation

- Summary of closed model

$$\begin{cases} \dfrac{\partial \bar{T}}{\partial t} + \dfrac{\partial F_T}{\partial z} = 0 \\ \dfrac{\partial \bar{S}}{\partial t} + \dfrac{\partial F_S}{\partial z} = 0 \end{cases} \qquad \begin{cases} F_T = Nu_T \left(1 - \dfrac{d\bar{T}}{dz}\right) \\ F_S = \gamma^{-1} F_T \end{cases} \qquad \begin{cases} Nu_T = Nu_T(R^{-1}; \mathrm{Pr}, \tau) \\ \gamma^{-1} = \gamma^{-1}(R^{-1}; \mathrm{Pr}, \tau) \\ R^{-1} = \dfrac{R_0^{-1} - \bar{S}_z}{1 - \bar{T}_z} \end{cases}$$

- This set of nonlinear equations has a trivial solution

$$\begin{cases} \bar{T} = \bar{S} = 0, \quad F_T, F_S \text{ are constant,} \\ R^{-1} = R_0^{-1} \text{ is constant,} \\ Nu_T = Nu_0, \quad \gamma^{-1} = \gamma_0^{-1} \end{cases}$$

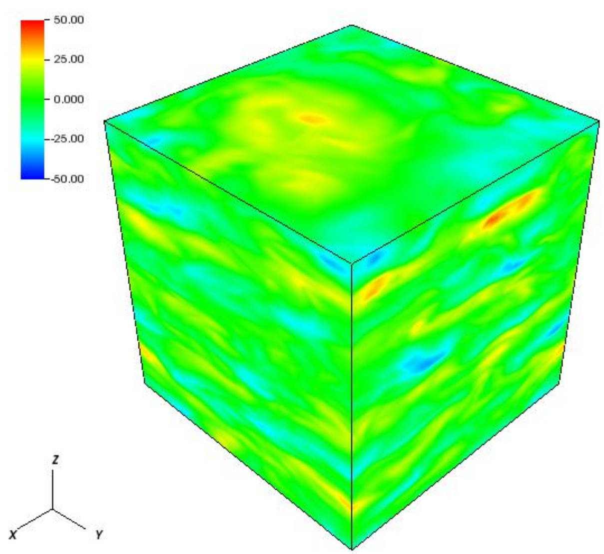
This is the homogeneous ODDC solution

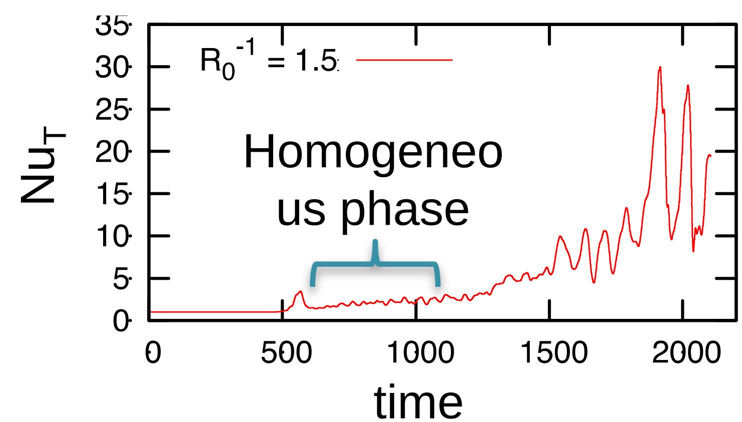
Mean-field theory for layer formation

- Summary of closed model

$$\begin{cases} \dfrac{\partial \overline{T}}{\partial t} + \dfrac{\partial F_T}{\partial z} = 0 \\ \dfrac{\partial \overline{S}}{\partial t} + \dfrac{\partial F_S}{\partial z} = 0 \end{cases} \qquad \begin{cases} F_T = Nu_T \left(1 - \dfrac{d\overline{T}}{dz}\right) \\ F_S = \gamma^{-1} F_T \end{cases} \qquad \begin{cases} Nu_T = Nu_T(R^{-1}; \mathrm{Pr}, t) \\ \gamma^{-1} = \gamma^{-1}(R^{-1}; \mathrm{Pr}, t) \\ R^{-1} = \dfrac{R_0^{-1} - \overline{S}_z}{1 - \overline{T}_z} \end{cases}$$

- This set of nonlinear equations has a trivial solution





Mean-field theory for layer formation

- Consider large-scale, small-amplitude perturbations to that state:

$$R^{-1} = \frac{R_0^{-1} - \bar{S}_z}{1 - \bar{T}_z} \approx (R_0^{-1} - \bar{S}_z)(1 + \bar{T}_z) \approx R_0^{-1} - \bar{S}_z + R_0^{-1}\bar{T}_z$$

- The evolution of the compositional field is given by

$$\frac{\partial S}{\partial t} + \frac{\partial F_S}{\partial z} = 0 \rightarrow \frac{\partial \bar{S}}{\partial t} + \frac{\partial}{\partial z}(\gamma^{-1} F_T) = 0$$

$$\rightarrow \frac{\partial \bar{S}}{\partial t} + \gamma^{-1}\frac{\partial F_T}{\partial z} = -F_T \frac{\partial \gamma^{-1}}{\partial z}$$

- To lowest order,

$$F_T = Nu_T(R^{-1})(1 - \bar{T}_z) \approx Nu_0$$

$$\frac{\partial \gamma^{-1}}{\partial z} = \frac{\partial \gamma^{-1}}{\partial R^{-1}}\frac{\partial R^{-1}}{\partial z} = \frac{\partial \gamma^{-1}}{\partial R^{-1}}(-\bar{S}_{zz} + R_0^{-1}\bar{T}_{zz})$$

$$\rightarrow \frac{\partial \bar{S}}{\partial t} + \ldots = Nu_0 \frac{\partial \gamma^{-1}}{\partial R^{-1}}(\bar{S}_{zz} - R_0^{-1}\bar{T}_{zz}) = 0$$

A "diffusion equation" with "diffusion coefficient"

$$Nu_0 \left(\frac{d\gamma^{-1}}{dR^{-1}}\right)$$

Mean-field theory for layer formation

Radko’sγ-instability criterion: A necessary condition for the γinstability is that the flux ratio should be a decreasing function of density ratio

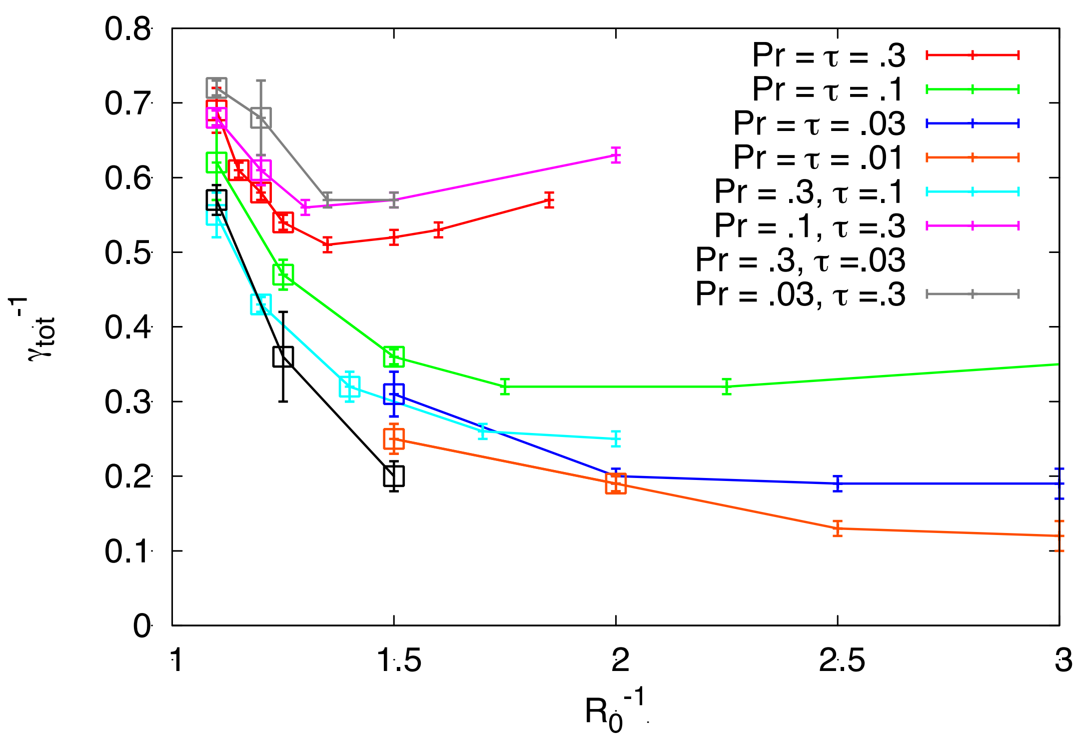
$$\frac{d\gamma^{-1}}{dR^{-1}} < 0$$

The layering instability triggers staircase formation

- Modes of instability are horizontally invariant, vertically sinusoidal perturbations in temperature/composition/density.
- The mode overturns into a staircase when amplitude is

Comparison with simulations

- To test this theory:
  - Measure flux ratio in homogeneous phase of ODDC
  - Check the sign of $\frac{dg^{-1}}{dR^{-1}}$



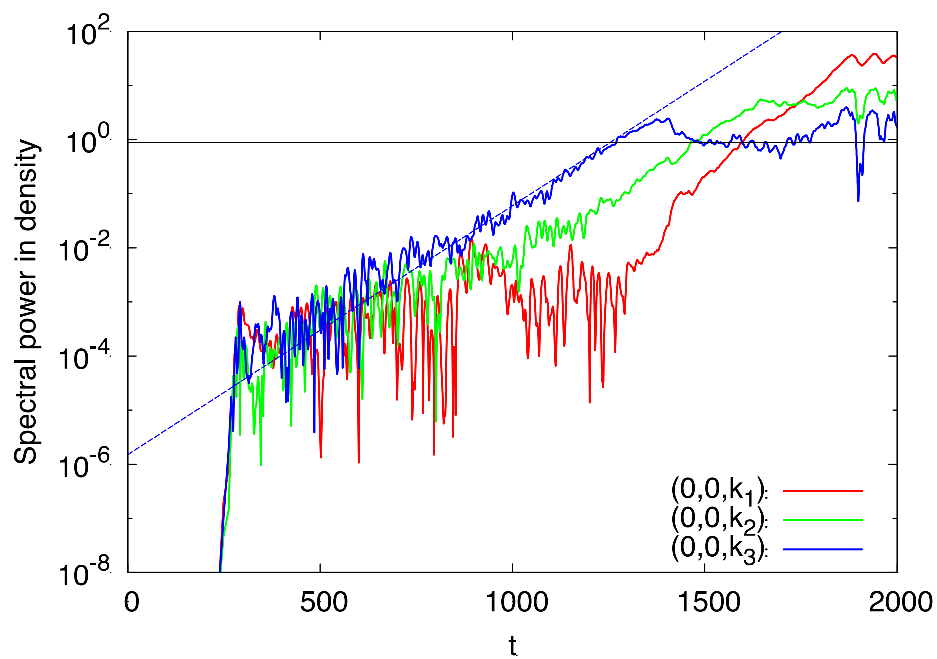
Large symbols : simulations that exhibit layer-formation.

Small symbols: simulations that do not.

Mirouh et al. 2012

Comparison with simulations

- To test this theory:
  - Measure flux ratio in homogeneous phase of ODDC
  - Check the sign of $\dfrac{dg^{-1}}{dR^{-1}}$
  - Pick a simulation, calculate predicted growth rate, compare with actual growth rate.



Mirouh et al. 2012

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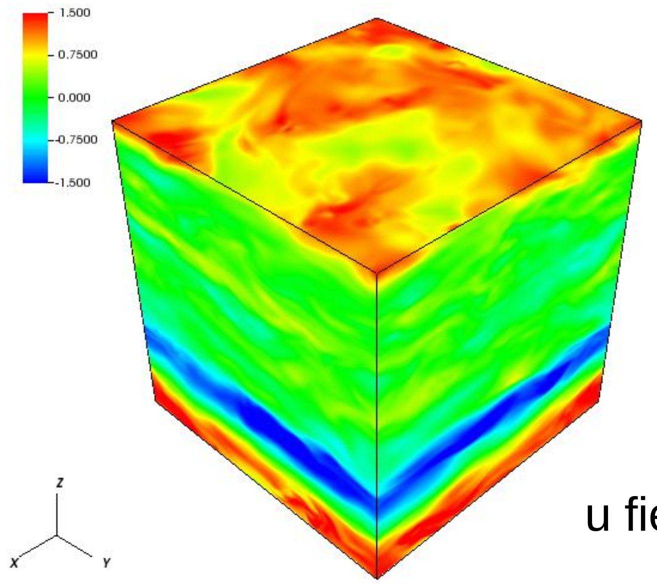


u field

Wave/mean-flow interactions: which one?

- Given that saturated ODDC consists in strongly dissipative, nonlinearly interacting gravity waves, the emergence of persistent shear flows is not surprising.. (cf. talk by Oliver)

- The question lies in origin of the mean flow. Is it:

1. Large-scale ODDC mode + large-scale ODDC mode ➔ mean flow ? (e.g. can be described by reduced system of fully nonlinear equations)
2. Small-scale ODDC modes ➔ mean flow (e.g. can be described by mean-field instability)
3. Large-scale *sheared* ODDC mode ➔ Reynolds stresses ➔ mean flow (e.g. can be described by quasilinear theory)

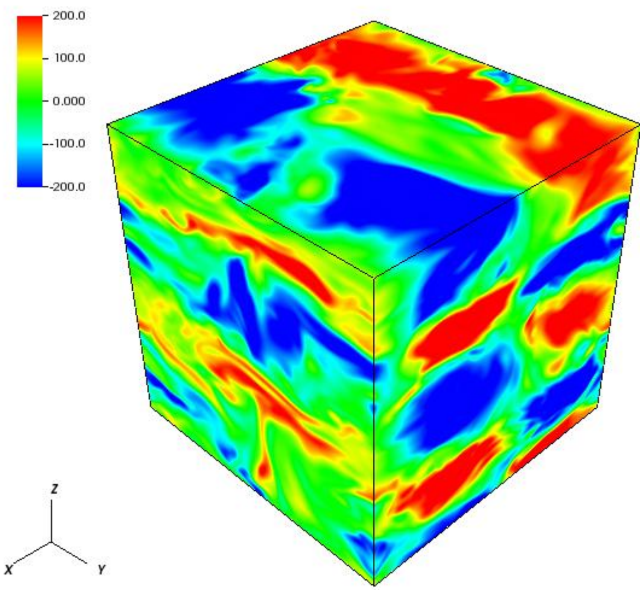
Definition

- In all that follows, we define:

$$\mathbf{k} = (l, m, k)$$

$$l_n = \frac{2p}{L_x} n, \ m_n = \frac{2p}{L_y} n, \ k_n = \frac{2p}{L_z} n$$

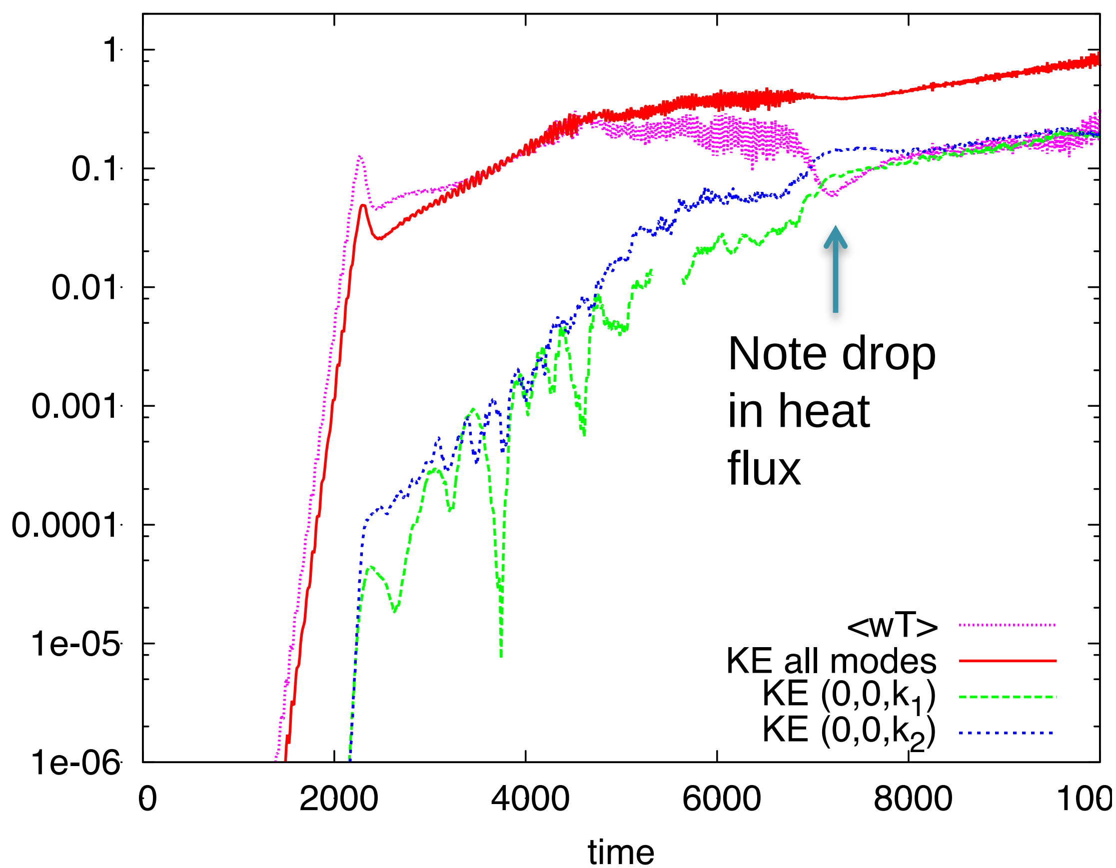
- Example of a flow $\mathbf{k} = (l_1, 0, k_2)$



Shearing modes have a structure of the kind $\mathbf{k} = (0, 0, k_n)$

Energies

The kinetic energy in the shearing modes grows exponentially with nearly constant growth rate for a long time…



Mean-field instability?

By analogy with the γ-instability:

- Horizontally-averaged, filtered equations (ignoring composition):

$$\frac{\partial \bar{T}}{\partial t} + \frac{\partial F_T}{\partial z} = 0$$

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial F_u}{\partial z} = 0$$

- Assume that

$$F_T = Nu_T \left(1 - \frac{d\bar{T}}{dz}\right)$$

$$F_u = \sigma F_T$$

Where Nu_T and the new function σ now depend both on the inverse density ratio R^{-1} and local non-dimensional shearing rate:

$$V = \frac{\bar{u}_z}{\sqrt{\Pr\left(R_0^{-1} - 1\right)}}$$

$$\frac{\partial \sigma}{\partial V} > 0$$

End up "proving" that shear can grow provided

Mean field instability ?

Testing the theory:

- *Slight problem*: by contrast with layering case, we do not have a code that can easily maintain a constant background $s(V)$ shear to measure in idealized homogeneous *sheared* ODDC.
- *Alternative solution*: start from a simulation already in homogeneous ODDC, and gradually increase (periodic) shear using external forcing.

Mean field instability ?

Testing the theory:

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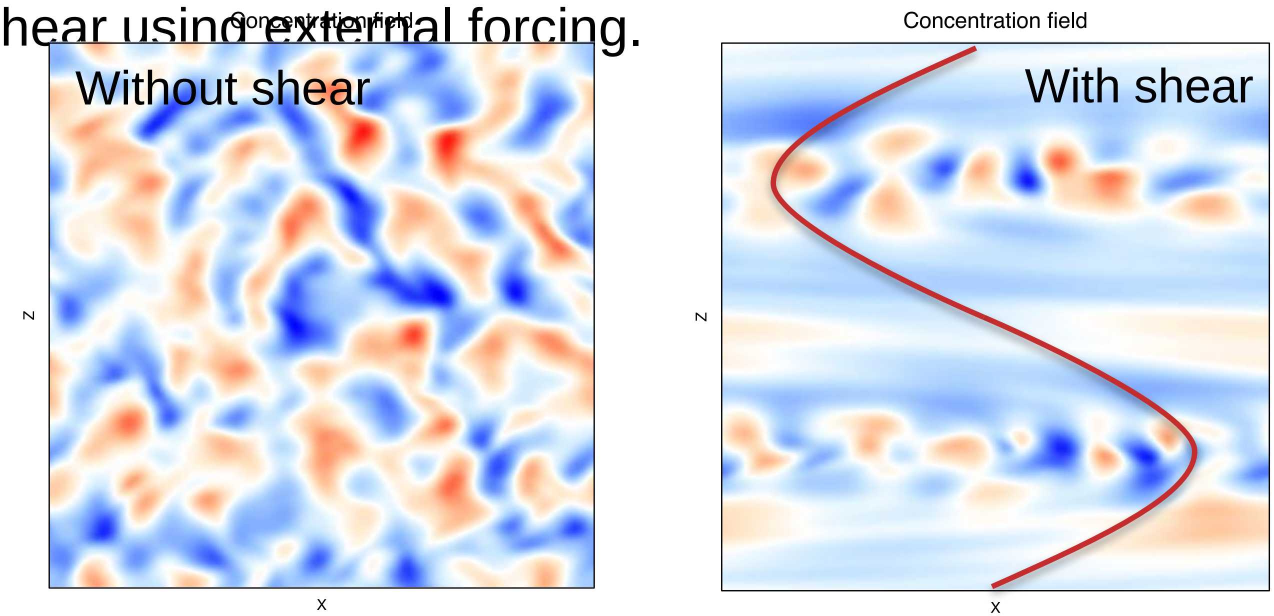
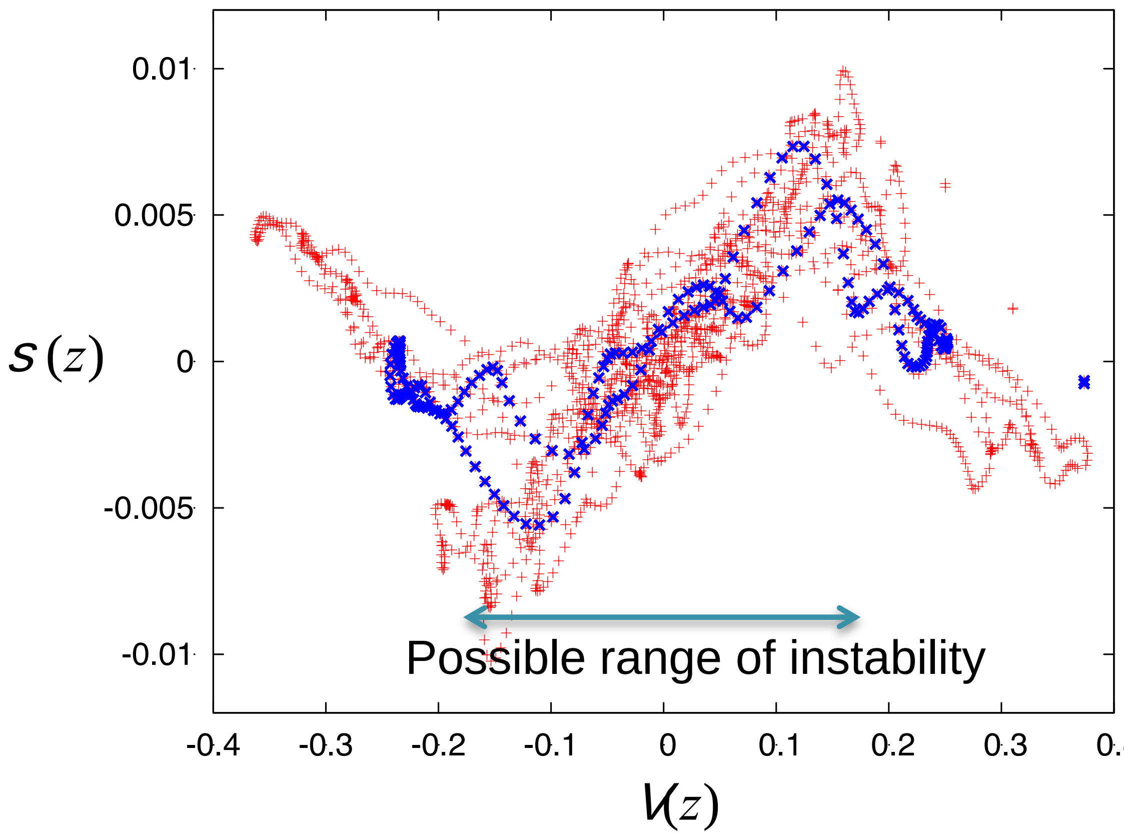


Mean field instability ?

Testing the theory:

- At each position in space, each snapshot in time, measure $s(z)$ and $V(z)$



Mean field instability ?

Testing the theory:

- ✓ For small enough shearing rate, it looks like the "σ-"instability can be excited.
- ✓ Since $\partial \sigma / \partial V$ is more or less constant in that range, this explains why the growth rate of shearing mode is constant.
- ✓ Instability has self-regulating properties

- ✗ The growth rate of the $(0,0,k_2)$ mode is observed to be the same as $(0,0,k_1)$ which contradicts the "anti-diffusive" nature of instability.
- ✗ Theory needs to be tested further by (1) measuring $\sigma(V)$ more systematically and (2) calculating the actual mode growth rates & comparing them with simulations.

Quasilinear theory ?

Idea:

Shear flow modifies basic ODDC instability

$$\frac{1}{\Pr}\left(\frac{\partial \tilde{\mathbf{u}}}{\partial t} + \bar{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}} + \tilde{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}}\right) = -\nabla\tilde{p} + \left(\tilde{T} - \tilde{S}\right)\mathbf{e_z} + \nabla^2\tilde{\mathbf{u}}$$

$$\frac{\partial \tilde{T}}{\partial t} + \bar{\mathbf{u}}\cdot\nabla\tilde{T} - \tilde{w} = \nabla^2\tilde{T}$$

$$\frac{\partial \tilde{S}}{\partial t} + \bar{\mathbf{u}}\cdot\nabla\tilde{S} - R_0^{-1}\tilde{w} = \tau\nabla^2\tilde{S}$$

$$\nabla\cdot\tilde{\mathbf{u}} = 0$$

⇕

Reynolds stresses induced by perturbations accelerate shear flow.

$$\frac{\partial \bar{u}}{\partial t} = \Pr\nabla^2\bar{u} - \frac{\partial}{\partial z}\int \tilde{u}\tilde{w}\,dx$$

Quasilinear theory ?

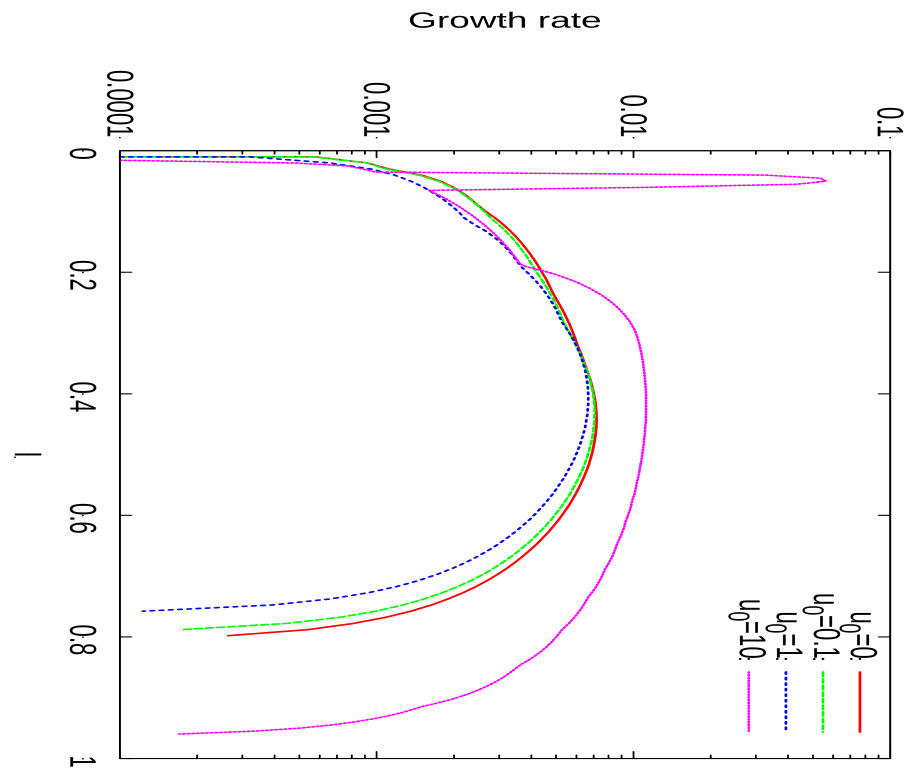
- For sinusoidal "background" shear of the kind

$$\bar{\mathbf{u}} = u_0 \sin(k_1 z)\mathbf{e}_x$$

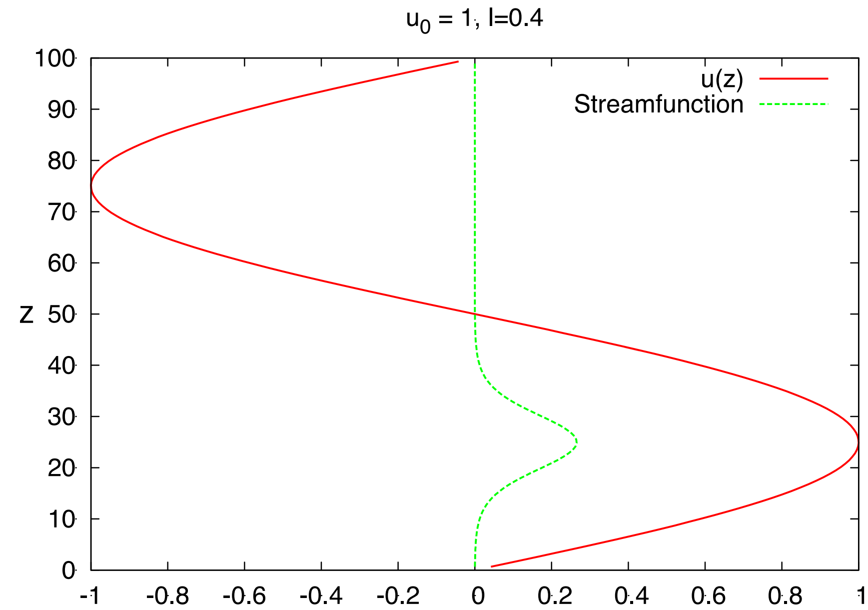
we can use Floquet theory to calculate properties of the perturbations.

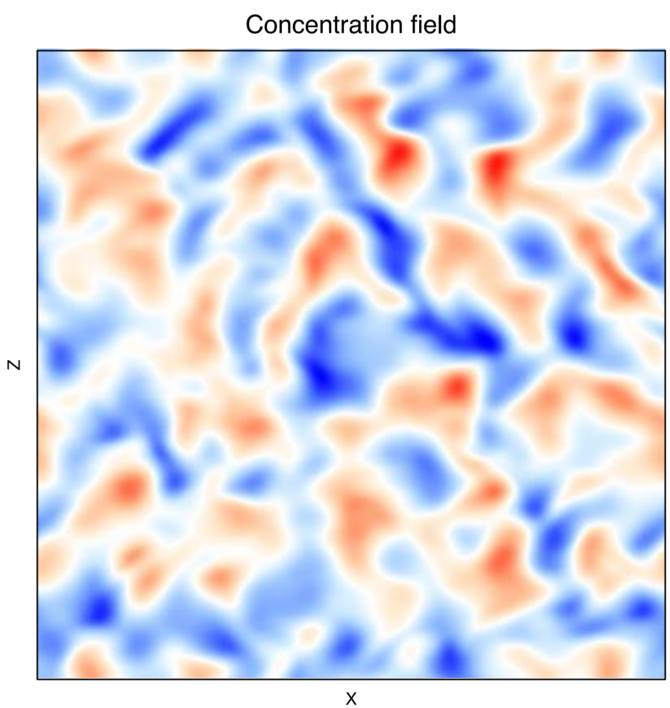
We find that for observed shearing rate in full 3D simulation, "sheared" mode growth rates similar to basic mode growth rate.

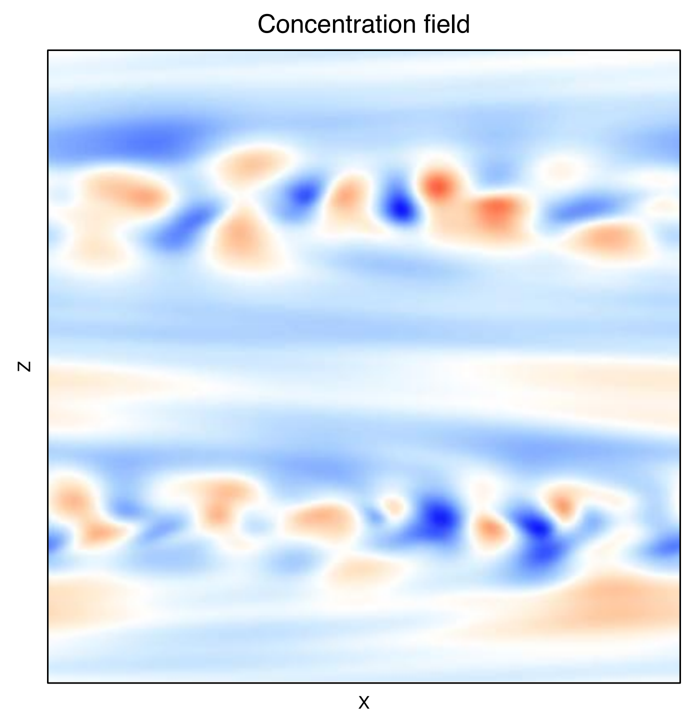
Note that growth rates are invariant for $l \leftrightarrow -l$



For “large” shearing rates, the perturbation are localized in region of little shear.







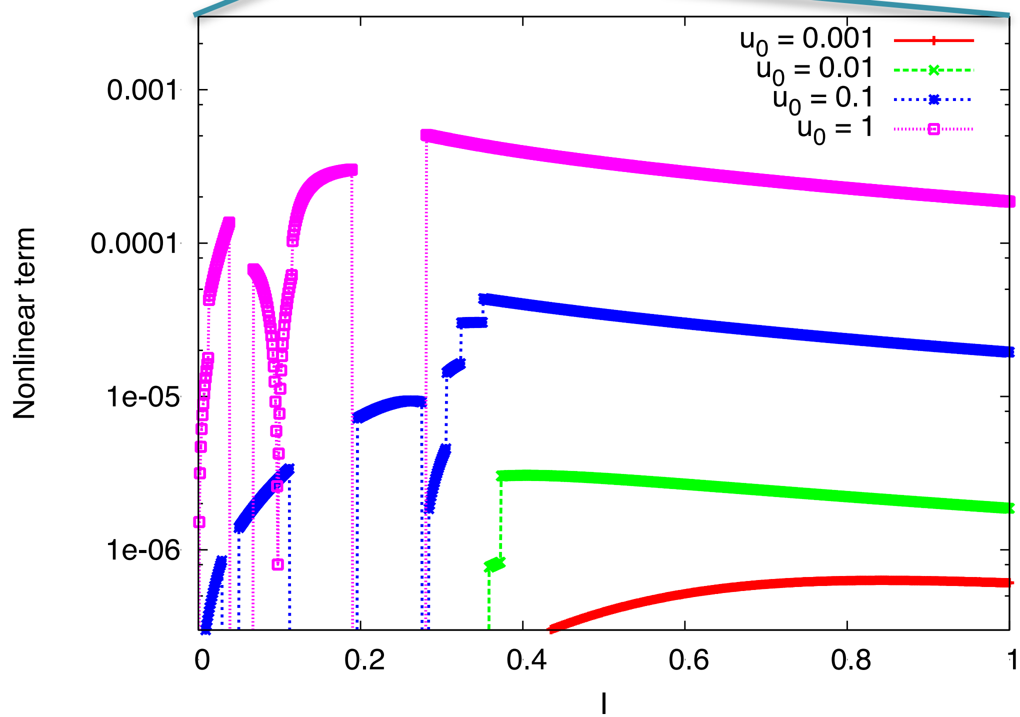
Quasilinear theory ?

- To see if mode can amplify background shear:

$$\int \overline{u}\frac{\partial \overline{u}}{\partial t}dz = \int \overline{u}\,\mathrm{Pr}\frac{\partial^2 \overline{u}}{\partial z^2}dz - \int \overline{u}\frac{\partial}{\partial z}\left(\int \tilde{u}\tilde{w}\,dx\right)dz$$

$$= -\mathrm{Pr}\int\left(\frac{\partial \overline{u}}{\partial z}\right)^2 dz + \int \frac{\partial \overline{u}}{\partial z}\left(\int \tilde{u}\tilde{w}\,dx\right)dz$$

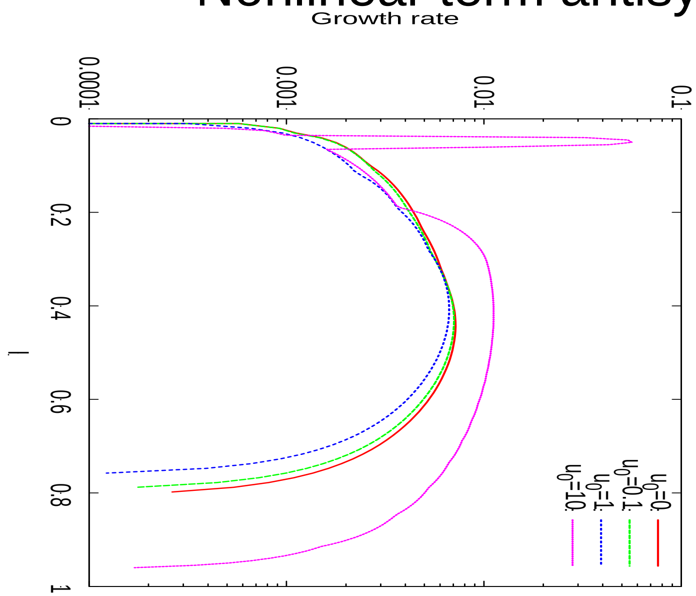
Nonlinear term strictly positive for a wide range of l!

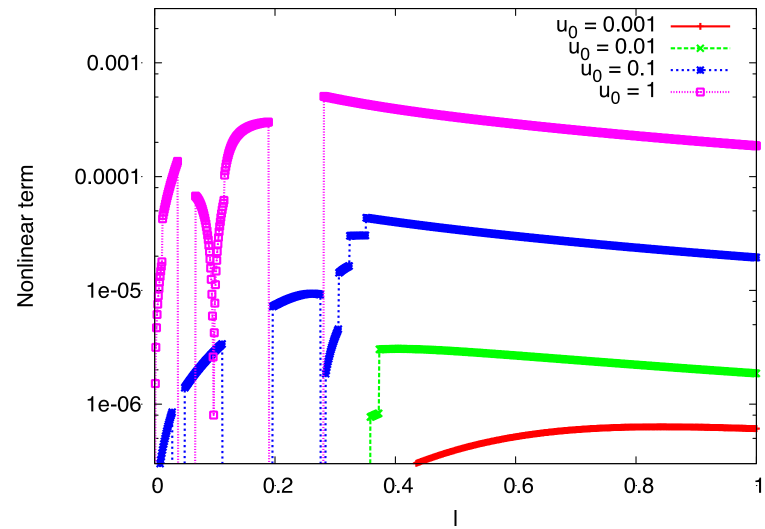


Quasilinear theory ?

Slight problem with the theory:

- Within the scope of the quasilinear theory (so far)
  - Growth rate symmetric in l
  - Nonlinear term antisymmetric in l

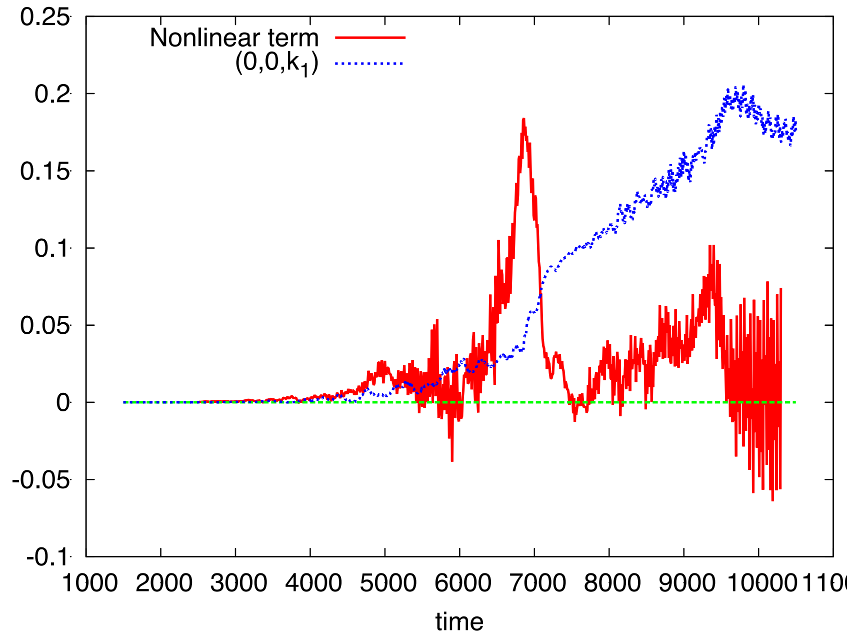




  - We expect as many modes with positive l and negative l, and the total effect of Reynolds stresses should cancel out.

Quasilinear theory ?

- However: the nonlinear terms in actual simulation are clearly mostly positive the whole time so positive l modes are preferred.



- **Question: What causes the asymmetry?**

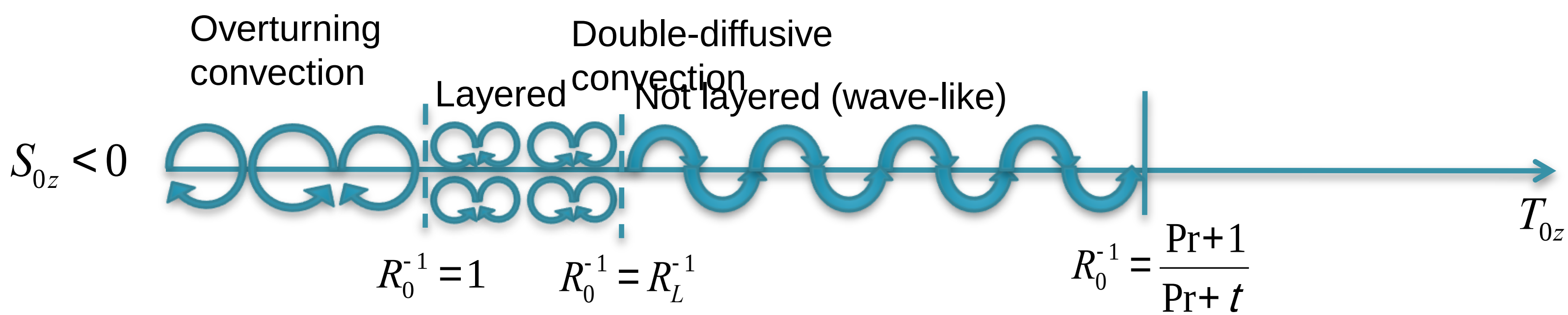
Note: this question is directly re

Why is $\frac{\partial s}{\partial V} > 0$ **for small** v **?**

This remains an open problem …

Summary

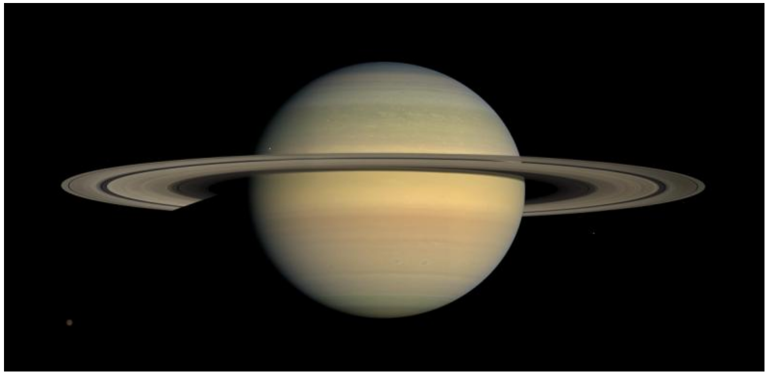
- ODDC exhibits examples of strong interactions between small-scales and large-scales.



- Mean-field theory (theγ–instability) satisfactorily explains layer formation for weakly stratified ODDC systems
- Shear layers are observed to form in low Pr, “strongly” stratified ODDC systems
- Candidates for shear layer formation have been identified:
  - “ σ–instability” (requires scale separation between large and small scales)
  - Quasilinear theory (does not require scale separation)

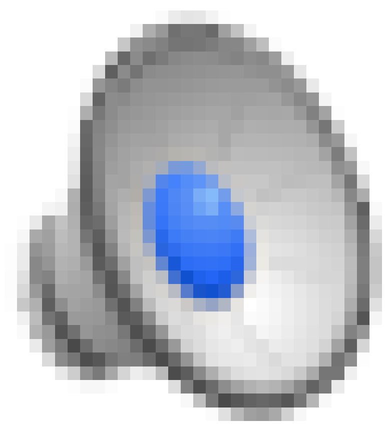
Summary

Both kinds of mean-field/mean flow instabilities have important implications for giant planet structure, evolution, dynamics.



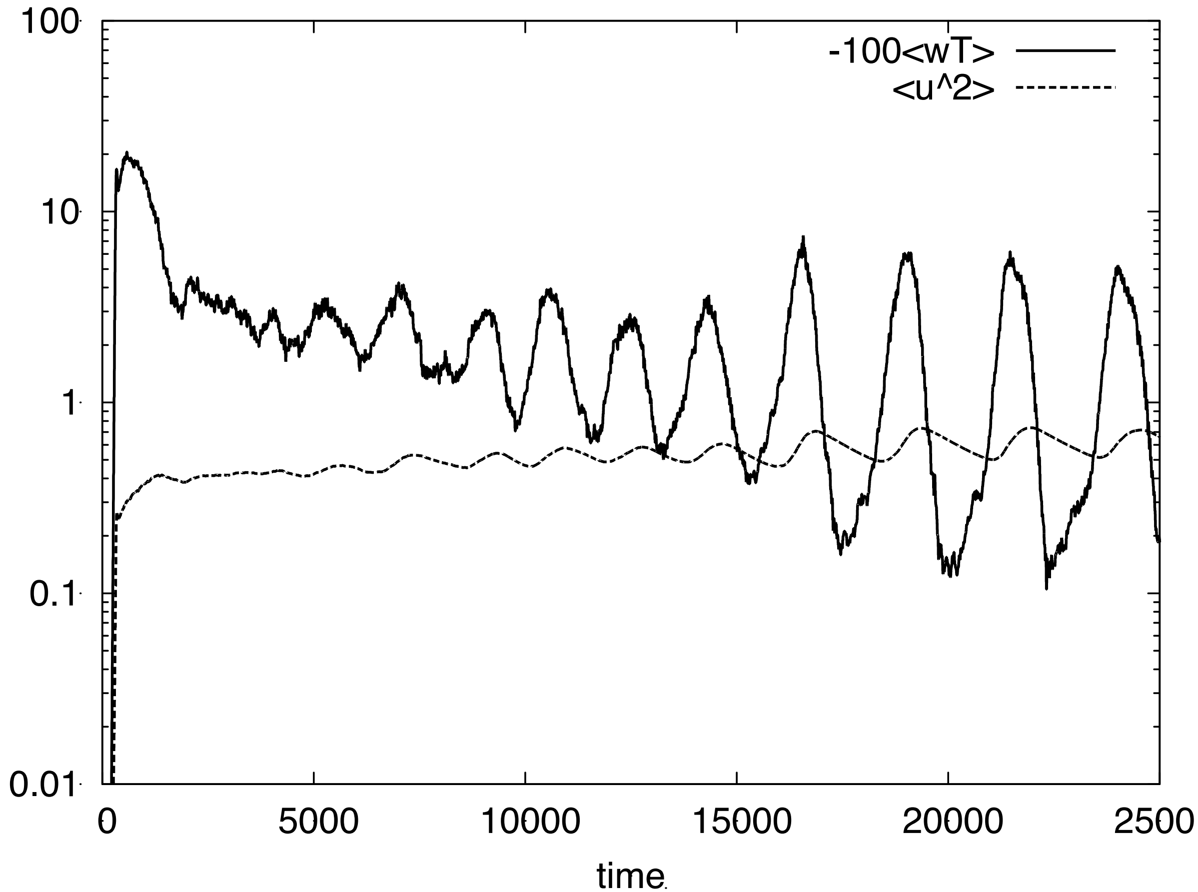
Final side-note

Note: similar dynamics are seen in fingering convection, but only in 2D (not in 3D)… why that is the case is another interesting question!

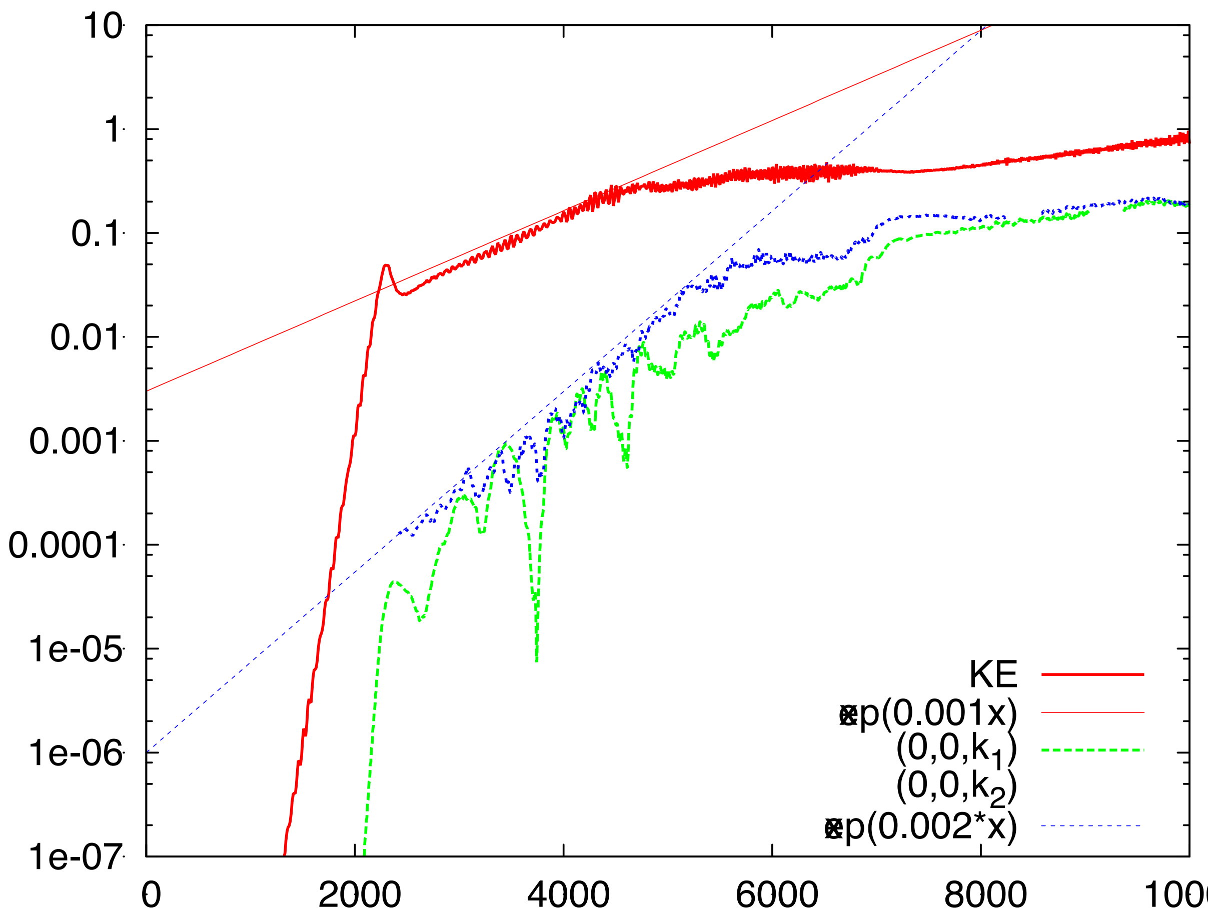


Garaud & Brummell, in prep.

Backup slides



Weakly nonlinear theory ?



Weakly nonlinear theory ?

Requires matching of spatial mode structure:

$$\mathbf{k}^{(1)}+\mathbf{k}^{(2)}=\mathbf{k}^{(S)}\rightarrow$$

$$l^{(1)}+l^{(2)}=0\rightarrow l^{(2)}=-l^{(1)}$$

$$m^{(1)}+m^{(2)}=0\rightarrow m^{(2)}=-m^{(1)}$$

$$k^{(1)}+k^{(2)}=k^{(S)}$$

Examples of “triplets”:

$$\left(l_{1}^{(1)},0,k_{-2}^{(1)}\right)+\left(l_{-1}^{(2)},0,k_{3}^{(2)}\right)=\left(0,0,k_{1}^{(S)}\right)$$

$$\left(l_{1}^{(1)},0,k_{-1}^{(1)}\right)+\left(l_{-1}^{(2)},0,k_{2}^{(2)}\right)=\left(0,0,k_{1}^{(S)}\right)$$

Weakly nonlinear theory ?

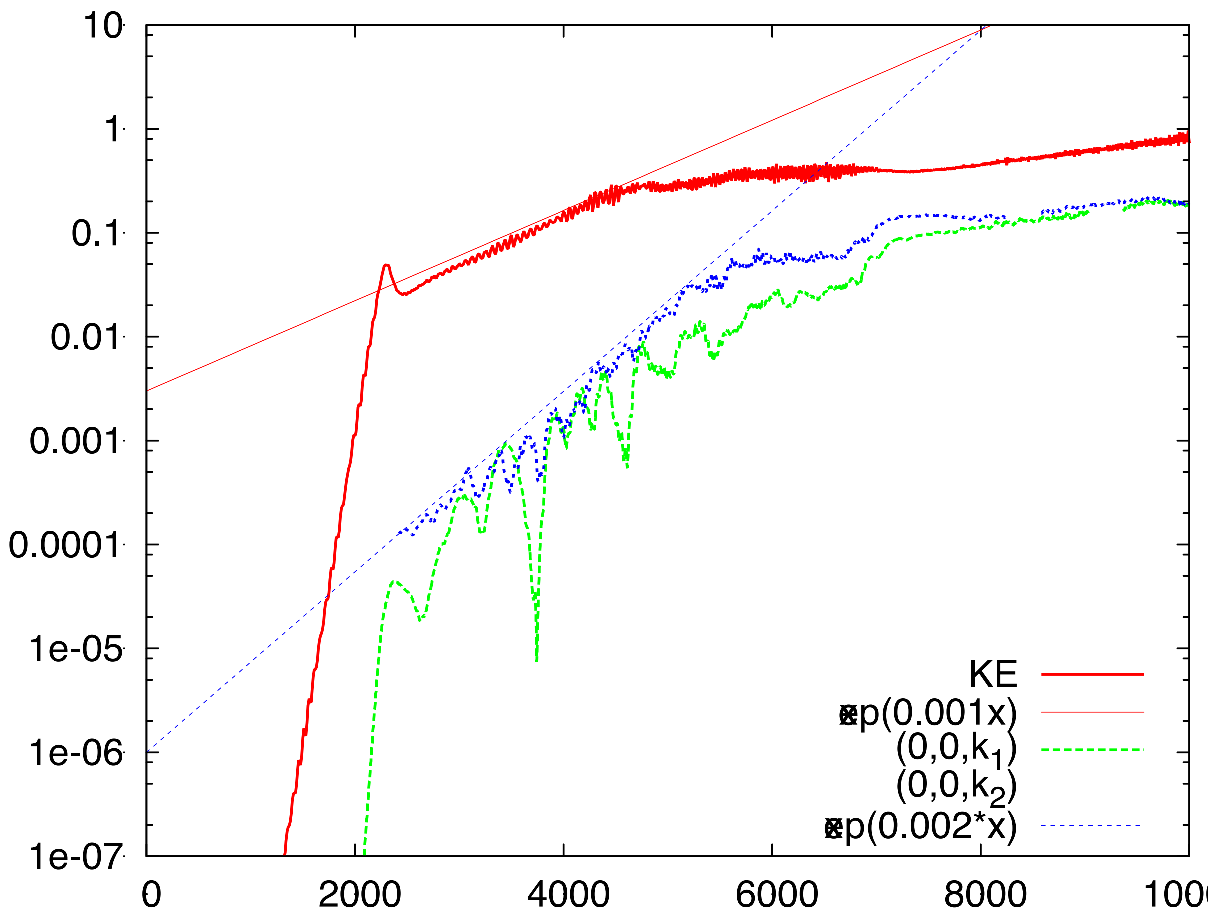
Oscillation frequencies do not match, but this is not a problem given that this is not a steady-state shear flow. In fact, we expect that from

$$\left(\frac{\partial \overline{u}}{\partial t} + \mathbf{u}^{(1)} \cdot \nabla \mathbf{u}^{(2)} \right) = \mathrm{Pr}\, \nabla^2 \overline{u}$$

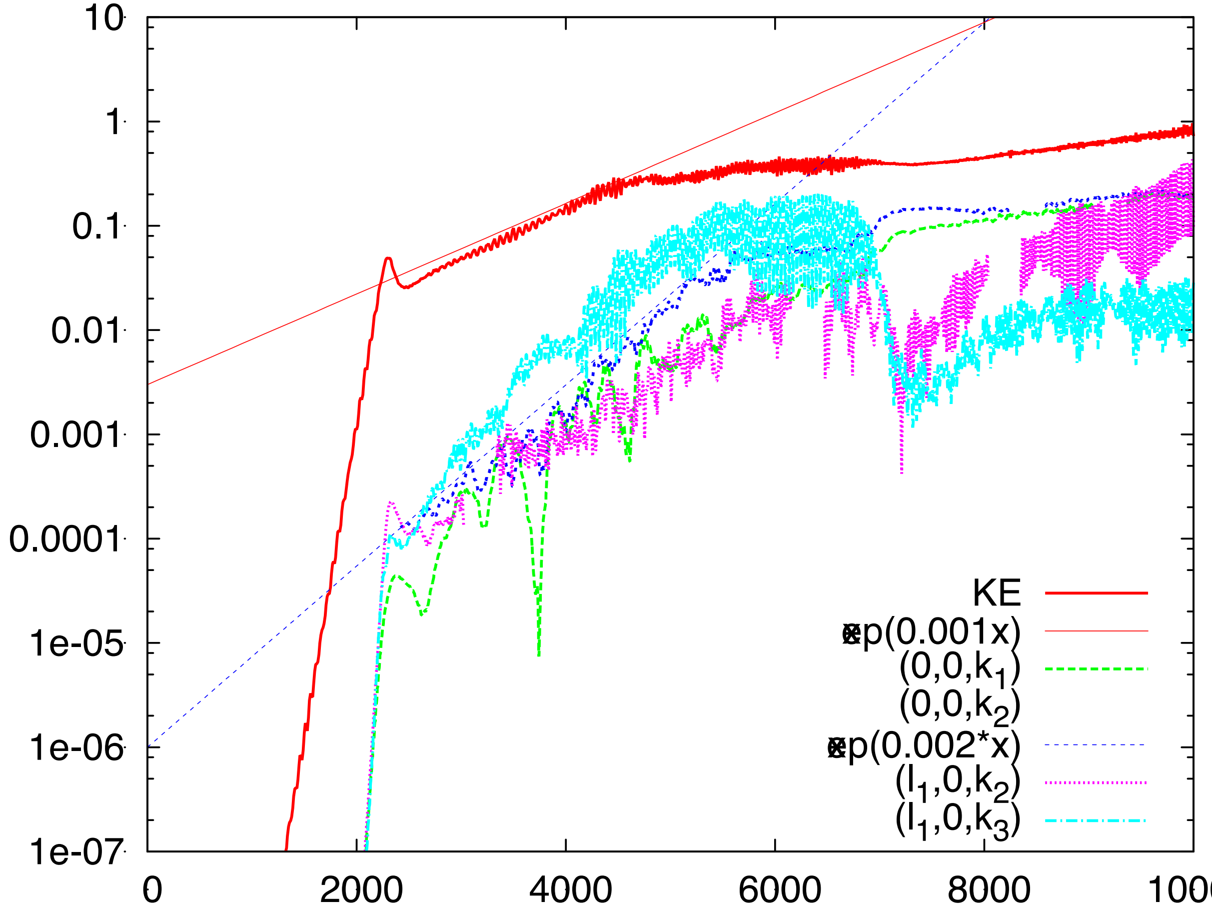
Ubar should grow (more or less) with sum of growth rates of modes (1) and (2):

$$\lambda^{(S)} \approx \lambda^{(1)} + \lambda^{(2)}$$

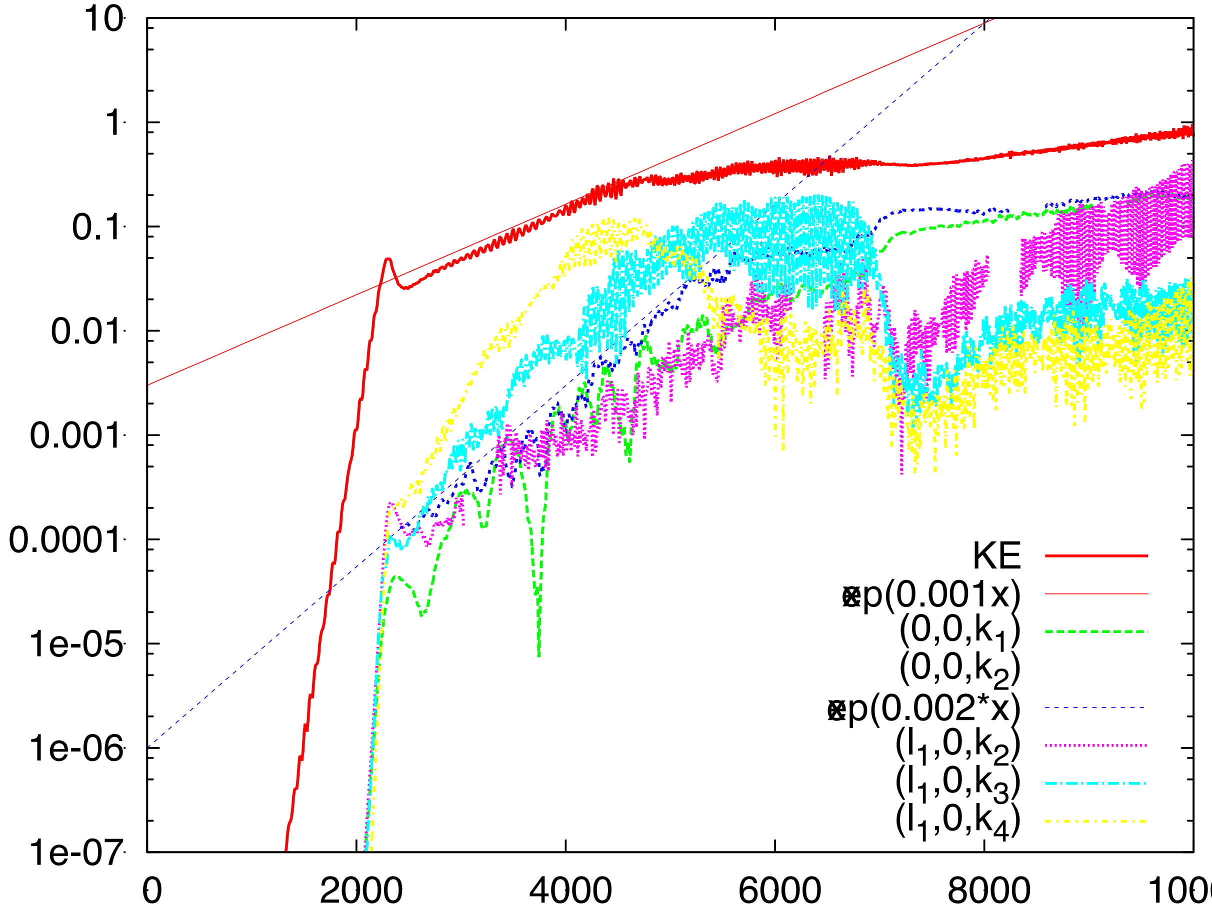
Weakly nonlinear theory ?



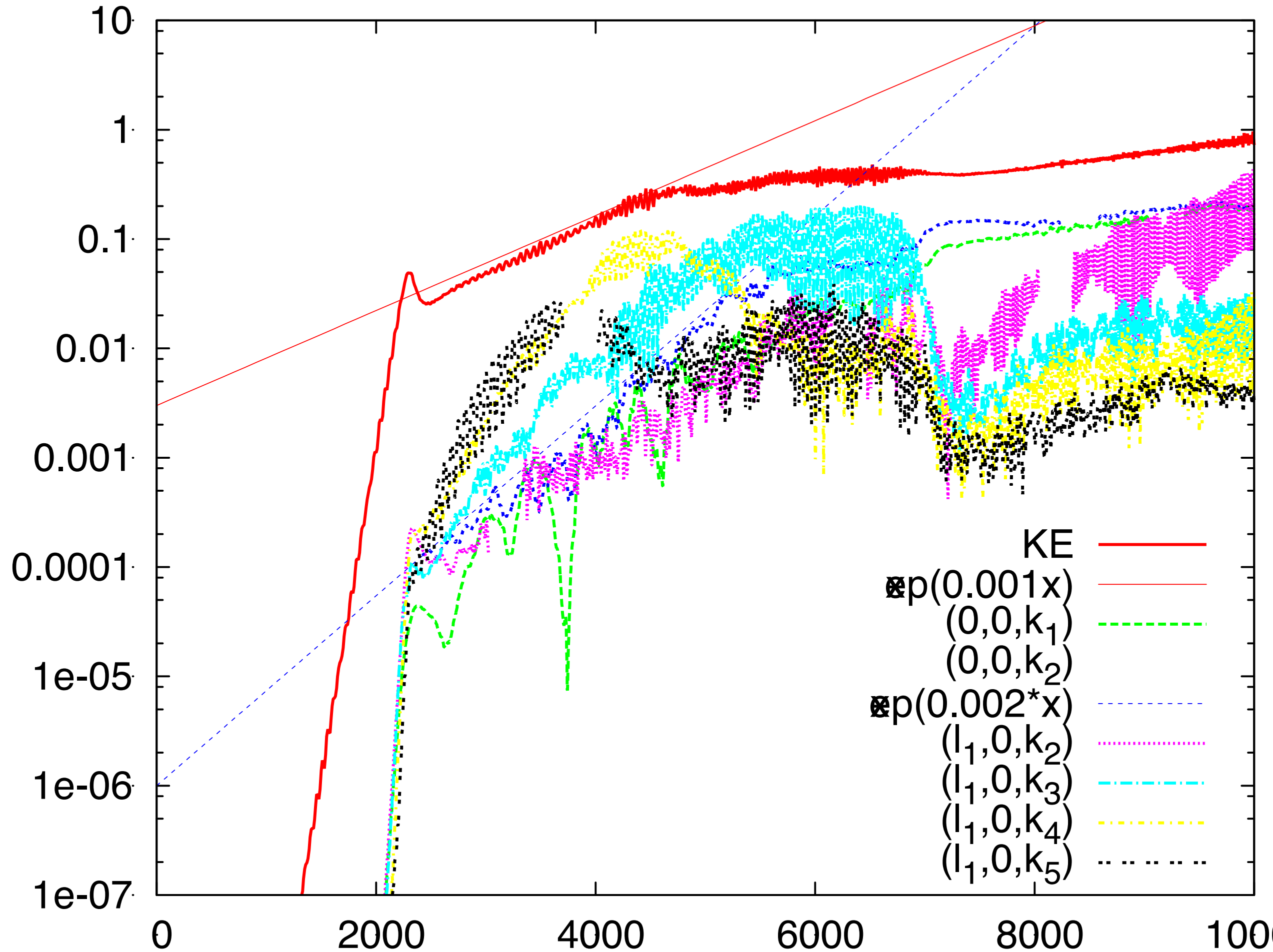
Weakly nonlinear theory ?



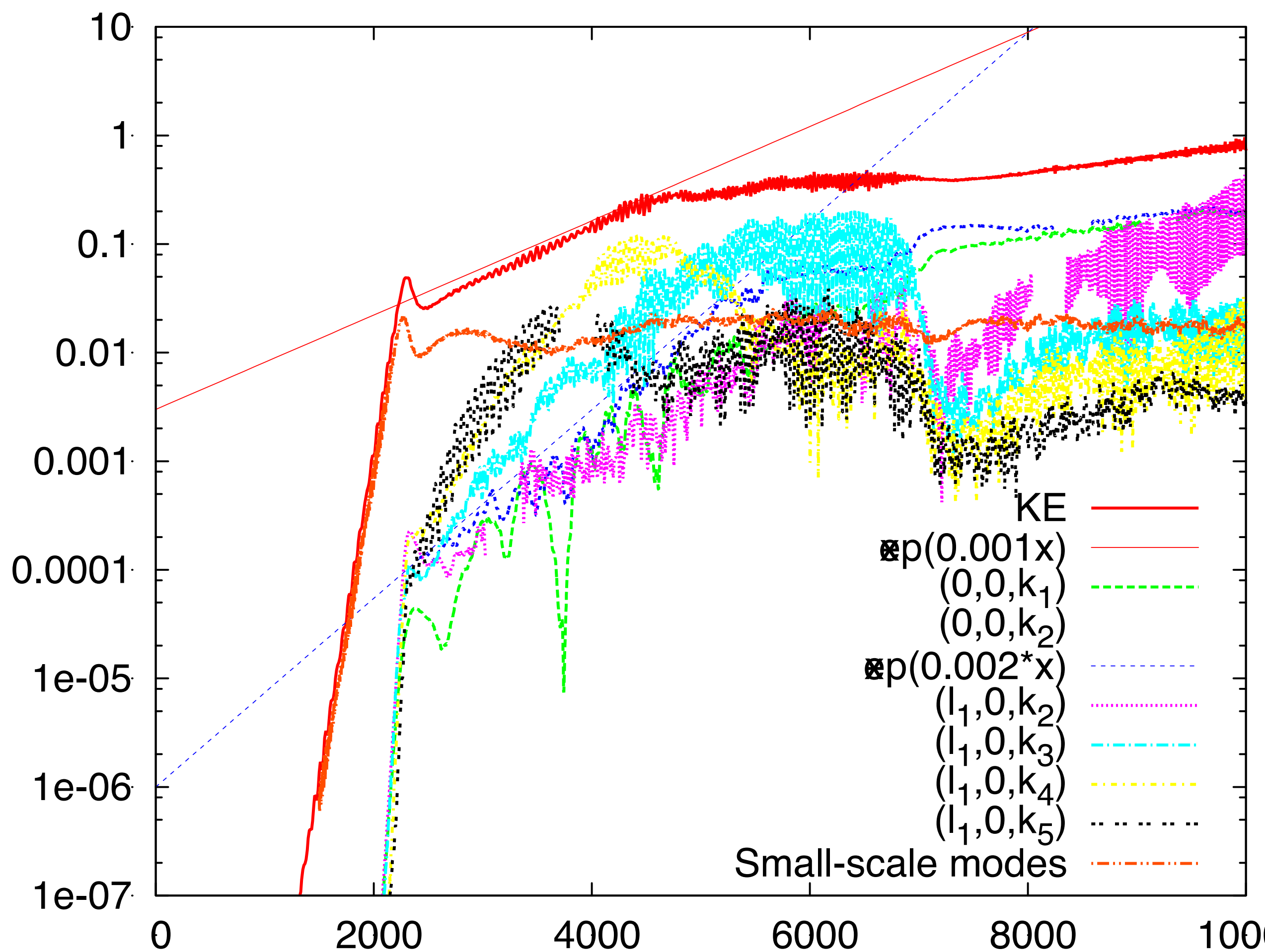
Weakly nonlinear theory ?



Weakly nonlinear theory ?



Weakly nonlinear theory ?



Weakly nonlinear theory? Probably not.

Oscillation frequencies do not match, but this is not a problem given that this is not a steady-state shear flow. In fact, we expect that from

$$\left(\frac{\partial \overline{u}}{\partial t} + \mathbf{u}^{(1)} \cdot \nabla \mathbf{u}^{(2)}\right) = \mathrm{Pr}\,\nabla^2 \overline{u}$$

Ubar should grow (more or less) with sum of growth rates of modes (1) and (2):

$$/^{(S)} \approx /^{(1)} + /^{(2)}$$

Problem : none of the observed triplets seem to work out…