A Proof for the Equivalence of Two Upper Bounds for the Growth of Disturbances from Barotropic Instability

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1 Introduction

If a flow is unstable, disturbances grow, but their growth does not continue indefinitely. Rather, there exists some upper bound to the growth. Shepherd (1988) proposed a method to calculate a fully nonlinear rigorous upper bound for the growth of disturbances from barotropic instability using the conservation of the pseudo-momentum density.

The upper bound, however, was not the tightest bound under the constraints of the conservation of all considered invariants. A tighter bound was obtained by Ishioka and Yoden (1996) by revising the method of Shepherd (1988). They also proposed a new method to calculate the tightest bound under the constraints of the conservation of all considered invariants. They applied these two methods to several basic flow profiles and showed that the values of the two upper bounds were approximately equivalent, with a relative error of ∼1%. This implies that the revised version of the method of Shepherd (1988) can yield the tightest bound under the considered constraints. No proof for the equivalence, however, has yet been reported.

In the present study, a proof for the equivalence is presented, and a more efficient method is proposed to calculate the upper bound.

2 Two upper bounds given by Ishioka and Yoden (1996)

The system under consideration is an incompressible, inviscid, two-dimensional fluid flowing over a rotating sphere. The system is governed by the material conservation of the absolute vorticity q.

Let us consider the time-evolution from the following initial condition:

$$q = q_{\text{initial}}(\mu) + (\text{an infinitesimal disturbance}),$$

where, μ is the sine latitude, and $q_{\text{initial}}(\mu)$ is a non-monotonic function of μ, so that barotropic instability can occur. The problem we consider here is to obtain the upper bound the wave enstrophy can attain in the time evolution.

2.1 The direct bound

Because of the incompressibilty and the material conservation of the absolute vorticity q, any possible distribution of q in the time evolution must be a rearrangement from the initial q distribution. We divide the sphere into M latitudinal belts of equal area. Numbering the belts starting from the south, we define the j-th belt to occupy the interval $-1+(j-1)\Delta\mu \le \mu \le -1+j\Delta\mu$ in the μ-coordinate ($j = 1, 2, \ldots, M$). Here, $\Delta\mu = 2/M$. Normalized by the total area of the sphere, each latitudinal belt has an area of $1/M$. For each latitudinal belt, we define μ_j as $\mu_j = -1 + (j - 1/2)\Delta\mu \quad (j = 1, 2, \ldots, M)$, which we regard as the representative μ value of the j-th belt.

If we introduce r_{ij} ($i, j = 1, 2, \ldots, M$) to indicate the area of the air parcel that is initially in the i-th belt and then moves to the j-th belt, we can describe any rearrangement of air parcels using the matrix (r_{ij}). Considering that an area cannot be negative and each belt has an area of $1/M$, the constraints that must be satisfied by (r_{ij}) are written as follows:

$$r_{ij} \ge 0 \quad (i, j = 1, 2, \ldots, M), \tag{1}$$

$$\sum_{i=1}^{M} r_{ij} = 1/M \quad (j = 1, 2, \ldots, M), \tag{2}$$

$$\sum_{j=1}^{M} r_{ij} = 1/M \quad (i = 1, 2, \ldots, M). \tag{3}$$

We hereinafter refer to any rearrangement of air parcels described using (r_{ij}) that satisfies constraints (1) through (3) as air parcel exchange. Assuming the absolute vorticity of the air parcel that is initially in the i-th belt as $q_i = q_{\text{initial}}(\mu_i)$ ($i = 1, 2, \ldots, M$), the average of the absolute vorticity in the j-th belt can be defined as

$$\overline{q}_j = M \sum_{i=1}^{M} q_i r_{ij} \quad (j = 1, 2, \ldots, M). \tag{4}$$

Using $(\overline{q}_j)$, the total angular momentum D, the zonal enstrophy F_z, and the wave enstrophy F_w can be defined as follows:

$$D = \frac{1}{M}\sum_{j=1}^{M} \mu_j \overline{q}_j, \tag{5}$$

$$F_z = \frac{1}{M}\sum_{j=1}^{M} \frac{1}{2}(\overline{q}_j)^2, \tag{6}$$

$$F_w = \sum_{j=1}^{M}\sum_{i=1}^{M} \frac{1}{2}(q_i - \overline{q}_j)^2 r_{ij}. \tag{7}$$

Now, we consider a maximization problem for F_w under constraints (1) through (3) for (r_{ij}) and the conservation of D:

$$D = D_{\text{initial}} = \frac{1}{M}\sum_{j=1}^{M} \mu_j q_j, \tag{8}$$

Here, D_{initial} is the initial value of D. Finally, we have formulated the procedure to seek the upper bound of F_w under constraints (1) through (3) for (r_{ij}) and constraint (8), which is also considered to be a constraint for (r_{ij}) through (4) and (5). This problem is a convex quadratic programming problem. The upper bound for F_w, which can be computed by solving the convex quadratic programming problem, is referred to as the direct bound.

2.2 The revised Shepherd's bound

After defining $q_{\min}$ and $q_{\max}$ as the minimum and the maximum values of q_i ($i = 1, 2, \ldots, M$), we introduce $Y(\eta)$ as a non-decreasing and piecewise differentiable function of η, for which the domain of definition is $q_{\min} \le \eta \le q_{\max}$. Next, we define $Q(\mu)$ as the inverse of $Y(\eta)$. Using $Y(\eta)$ and $Q(\mu)$, we define the following function:

$$A_Q(\mu, q) = -\int_{Q(\mu)}^{q} \{Y(\eta) - \mu\}\mathrm{d}\eta.$$

This function corresponds to the pseudo-momentum density if we consider $Q(\mu)$ to be the basic state and $q - Q$ to be the perturbation. The quantity $\sum_{j=1}^{M}\sum_{i=1}^{M} A_Q(\mu_j, q_i) r_{ij}$ becomes an invariant for any air parcel exchange that conserves D. Using the mean-value theorem, we obtain the following inequality:

$$F_w \le -\frac{1}{Y'_{\min}}\frac{1}{M}\sum_{j=1}^{M} A_Q(\mu_j, q_j). \tag{9}$$

Here, $Y'_{\min}$ is the minimum of $\mathrm{d}Y/\mathrm{d}\eta$. If we can solve a variational problem of the function $Y(\eta)$ to minimize the right-hand side of (9), we can obtain an upper bound of F_w. This bound is refered to as the revised Shepherd's bound.

3 Outline of the proof

First, we sort the initial profile (q_i) through air parcel exchange so that it becomes non-decreasing with respect to the suffix i. For the realized profile $(\overline{q}_j^{(0)})$, the following inequality holds:

$$\overline{q}_1^{(0)} \le \overline{q}_2^{(0)} \le \ldots \le \overline{q}_M^{(0)}.$$

The total angular momentum corresponding to the sorted profile, $D^{(0)}$, is larger than D_{initial}. Starting from the sorted profile, air parcel exchange is conducted where the gradient of the profile,

$$(\overline{q}_{j+1}^{(0)} - \overline{q}_j^{(0)})/\Delta\mu \quad (j = 1, 2, \ldots, M-1),$$

has the largest value to reduce the gradient. Then, a new profile $(\overline{q}_j^{(1)})$ is obtained and the total angular momentum corresponding to this new profile, $D^{(1)}$, becomes smaller than $D^{(0)}$. Repeating the above procedure, we can finally obtain a profile $(\overline{q}_j^{*})$ the total angular momentum corresponding to which, D^*, equals to D_{initial}.

We determine the $Q(\mu)$ profile that yields the revised Shepherd's bound as a polyline that connects the points $(\mu_j, \overline{q}_j^{*})$ ($j = 1, 2, \ldots, M$). For the above defined $Q(\mu)$ and (r_{ij}) that corresponds to the profile $(\overline{q}_j^{*})$, the equality holds in (9) because $(q_i - \overline{q}_j)r_{ij} \ne 0$ only where $dY/d\eta = Y'_{\min}$.

Remembering that (9) gives an upper bound for F_w, the fact that we have obtained a case in which the equality holds means that $(\overline{q}_j^{*})$ and the corresponding (r_{ij}) gives the maximum of F_w attainable by air parcel exchange that conserves the total angular momentum, and also that the above defined $Q(\mu)$ gives the minimum of the right-hand side of (9). Thus, we have completed the proof for the equivalence of the revised Shepherd's bound and the direct bound.

4 Summary and discussion

We have demonstrated that the two upper bounds are equivalent. The procedure used in the proof is a simple iteration algorithm that does not require an optimization method. Furthermore, the number of required iterations is less than the number of discretizations, M, and very little memory is required because the procedure does not have to deal with (r_{ij}).

Before closing, let us consider why the proposed procedure yields the minimum of F_z (or the maximum of F_w). In the proposed procedure, air parcel exchange occurs at latitudes where $(\overline{q}_{j+1} - \overline{q}_j)/\Delta\mu$ has the maximum value. The reason for this is explained as follows. Let us assume that in two belts μ_a and μ_b ($\mu_a < \mu_b$), zonal mean values of absolute vorticity are $\overline{q}_a$ and $\overline{q}_b$, respectively ($\overline{q}_a < \overline{q}_b$), and that air parcel exchange between these two belts generates zonal mean values of absolute vorticity $\overline{q}_a + \delta q$ and $\overline{q}_b - \delta q$ (at μ_a and μ_b, respectively). The ratio between the increment of F_z and D by the air parcel exchange is written as

$$\frac{\delta F_z}{\delta D} = \frac{\overline{q}_b - \overline{q}_a}{\mu_b - \mu_a}. \tag{10}$$

Since the right-hand side of (10) is the gradient of the line connecting two points $(\mu_a, \overline{q}_a)$ and $(\mu_b, \overline{q}_b)$, (10) implies that air parcel exchange should occur in intervals where the gradient of zonal mean absolute vorticity is the steepest to maximize the decrease of F_z per the decay of D. Therefore, the procedure in the proof can be interpreted as follows. First, the value of D is maximized by sorting the initial distribution of absolute vorticity. Second, air parcel exchange occurs starting from intervals where the gradient of zonal mean absolute vorticity is the steepest. Finally, the profile can be reached that gives the minimum of F_z under the constraint of having the same value of D as the initial profile.

The above interpretation of the procedure seems analogous to annealing, which, in metallurgy, refers to the process of changing the crystal structure of metal to a lower free energy state by increasing and then gradually decreasing the temperature of the metal. If we replace D and F_z with the temperature and the free energy, respectively, the analogy will be clear.

References

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