Connection between Zonostrophic Instability and Modulational Instability

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Summary

- Zonostrophic instability [Srinivasan & Young 2012] and modulational instability [Gill 1974, Connaughton et al. 2010] are closely connected. We show that the dispersion relations agree exactly for the case of a single background primary wave.
- Zonal flow as pattern formation:
  - Using CE2, we extend the calculation of zonostrophic instability into the regime of self-consistent nonlinear interactions between zonal flows and fluctuations
  - Results: find nonunique solutions to CE2 with varying jet wavelengths, and merging jets governed by stability boundaries
- For more details, a preprint is available (jbparker@princeton.edu)

Background

Modulational Instability (MI)

- For the unforced, undamped Charney-Hasegawa-Mima equation in an infinite domain, a single, monochromatic wave (the *primary* wave) is an exact solution to the nonlinear equation.

$$\psi(\mathbf{x}) = \psi_0\left(e^{i\mathbf{p}\cdot\mathbf{x}-i\omega t} + e^{-i\mathbf{p}\cdot\mathbf{x}+i\omega t}\right)$$

- Perturb around this wave and linearize the equations. In general, an infinite coupled system must be kept (Gill 1974)
- Consider the *4-Mode-Truncation*: the 4 retained modes are the primary wave $\mathbf{p}$, the secondary wave $\mathbf{q}$, and the sidebands $\mathbf{p}\pm\mathbf{q}$.

Exact dispersion relation [Connaughton et al. 2010]:

$$(q^2+L_d^{-2})\lambda - i\beta q_x = \psi_0^2|\mathbf{p}\times\mathbf{q}|^2(p^2-q^2)\left[\frac{p_+^2-p^2}{(p_+^2+L_d^{-2})(\lambda-i\omega)-i\beta(p_x+q_x)} + \frac{p_-^2-p^2}{(p_-^2+L_d^{-2})(\lambda+i\omega)+i\beta(p_x-q_x)}\right]$$

$$\omega = -\frac{\beta p_x}{p^2+L_d^{-2}}$$

$$\mathbf{p}_\pm = \mathbf{p}\pm\mathbf{q} \qquad \lambda = \text{eigenvalue (growth rate)}$$

Zonostrophic Instability (ZI)

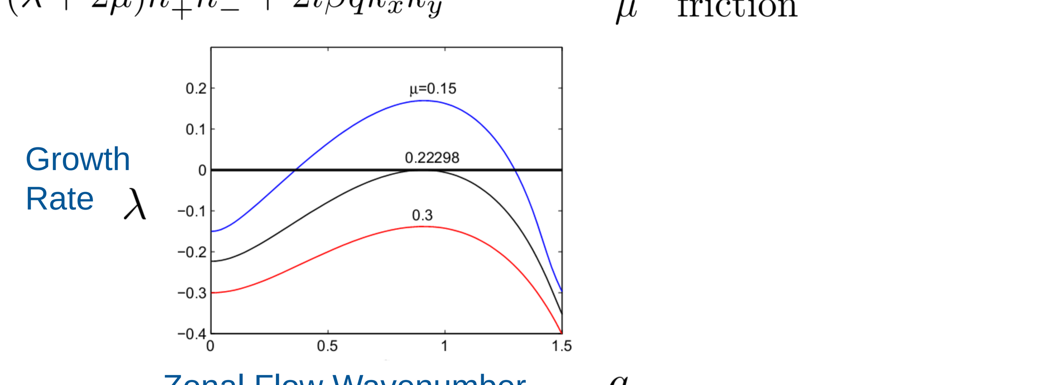
Homogeneous turbulence, described by a correlation function or spectrum, is unstable to a coherent zonal flow perturbation

Dispersion Relation from CE2 of Charney-Hasegawa-Mima equation (allowing for finite deformation radius) [Srinivasan & Young 2012]

$$\frac{\overline{q}^2}{q^2}(\lambda+\mu) = q\Lambda_- - q\Lambda_+,$$

$$\Lambda_\pm = \int \frac{dk_x dk_y}{(2\pi)^2}\frac{k_x^2 k_y(1-\overline{q}^2/\overline{h}_\pm^2)W_H(k_x, k_y \pm \frac{1}{2}q)}{(\lambda+2\mu)\overline{h}_+^2\overline{h}_-^2 + 2i\beta q k_x k_y}$$

λ eigenvalue
q zonal flow wavenumber
W_H background spectrum
μ friction



Connection between ZI and MI

As the background spectrum W_H for ZI, take a single primary wave:

$$W_H(\mathbf{k}) = (2\pi)^2 A\left[\delta(\mathbf{k}-\mathbf{p}) + \delta(\mathbf{k}+\mathbf{p})\right]$$

To correspond to a streamfunction amplitude ψ_0, take $A = \psi_0^2(p^2+L_d^{-2})^2$. Also take $\mu \to 0$.

The 4-mode-truncation MI dispersion relation with $q_x = 0$, and the ZI dispersion relation agree exactly.

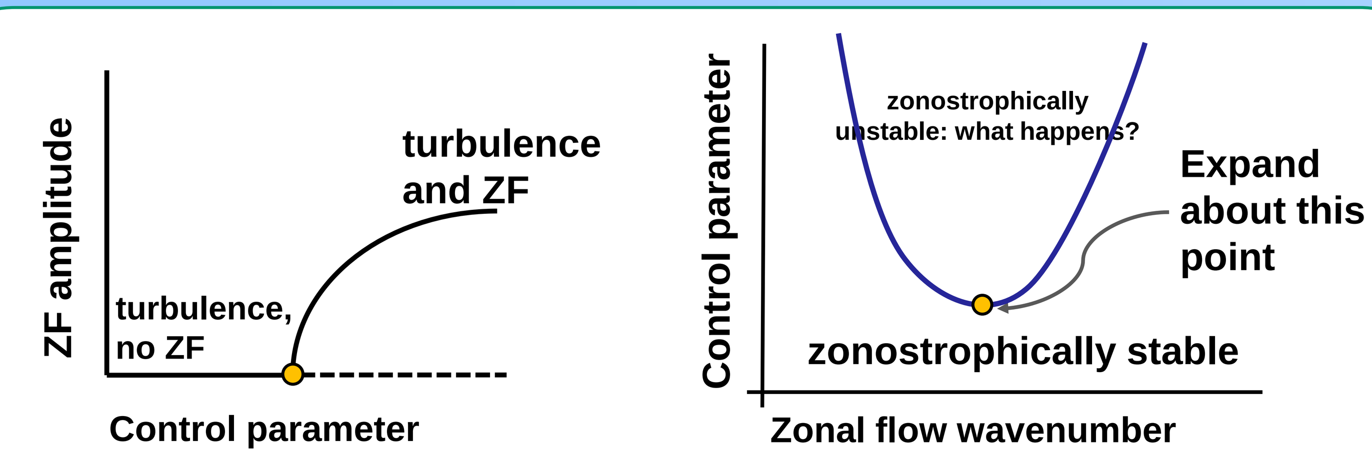
Extension to perturbations that are not zonally symmetric

Don't require $q_x = 0$ now. We use the CE2 formulation of Bakas & Ioannou (2013) which allows for arbitrary coherent structures and not just zonally symmetric structures.

Again, the 4MT MI dispersion relation and the generalized ZI dispersion relation agree exactly.

Note on wave kinetics: CE2 is an exact description of quasilinear behavior, while the wave-kinetic formalism is only valid for long-wavelength (small q) mean fields. The ZI dispersion relation here encompasses the wave-kinetic calculation of modulational instability [Manin & Nazarenko 1994, Smolyakov et al. 2000], but is also valid for arbitrary q.

Bifurcation to Inhomogeneous Turbulence



Multiscale perturbation expansion of CE2 gives an equation for the spatially varying, complex amplitude of the bifurcating mode as a solvability condition at third order

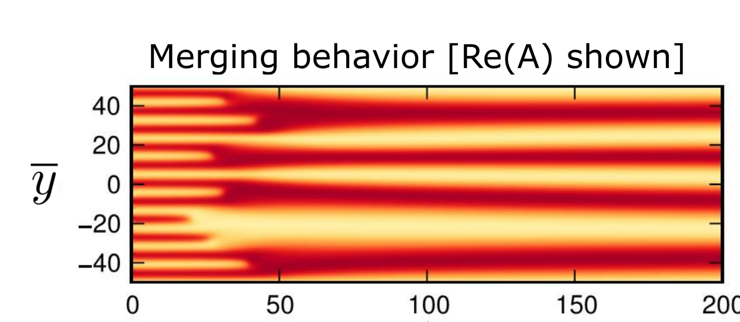
$$\mathbf{u} = \epsilon^{1/2}\mathbf{u}_1 + \epsilon\mathbf{u}_2 + \epsilon^{3/2}\mathbf{u}_3 + \cdots$$

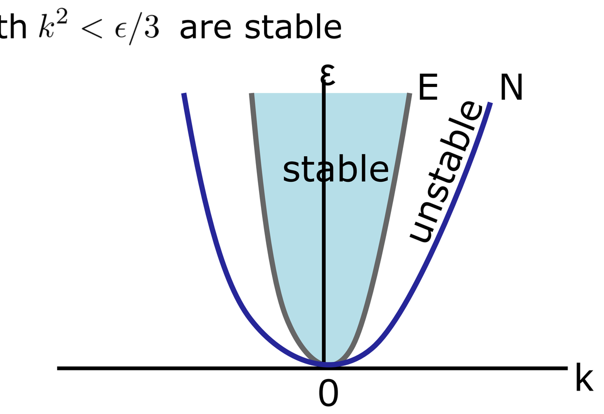
$$\mathbf{u}_1 = A(\overline{y},t)e^{iq_c\overline{y}}\mathbf{v} + \text{ cc.}$$

$$c_0\partial_t A(\overline{y},t) = c_1\epsilon A + c_2\partial_{\overline{y}}^2 A - c_3|A|^2 A$$

"Real Ginzburg-Landau Equation", is constrained by symmetries to have universal behavior. (Can rescale all coefficients to unity)

With all $c_i = 1$,
- For any $k^2 < \epsilon$, $A = \sqrt{\epsilon - k^2}e^{ik\overline{y}}$ is a solution
- Only solutions with $k^2 < \epsilon/3$ are stable





Calculation of Equilibrium & Stability

CE2 Equilibrium

- Assume a given zonal flow wavenumber q
- Expand solution as a Fourier—Galerkin series with coefficients U_p, W_{mnp} to be determined.

$$U(\overline{y}) = \sum_{p=-P}^{P} U_p e^{ipq\overline{y}}$$

$$W(x,y\mid\overline{y}) = \sum_{m=-M}^{M}\sum_{n=-N}^{N}\sum_{p=-P}^{P} W_{mnp}e^{imax}e^{inby}e^{ipq\overline{y}}$$

- Project onto the basis functions to generate a set of nonlinear algebraic equations
- Then use Newton's method to solve for the equilibrium
- Repeat for multiple values of q (multiple solutions)

Stability

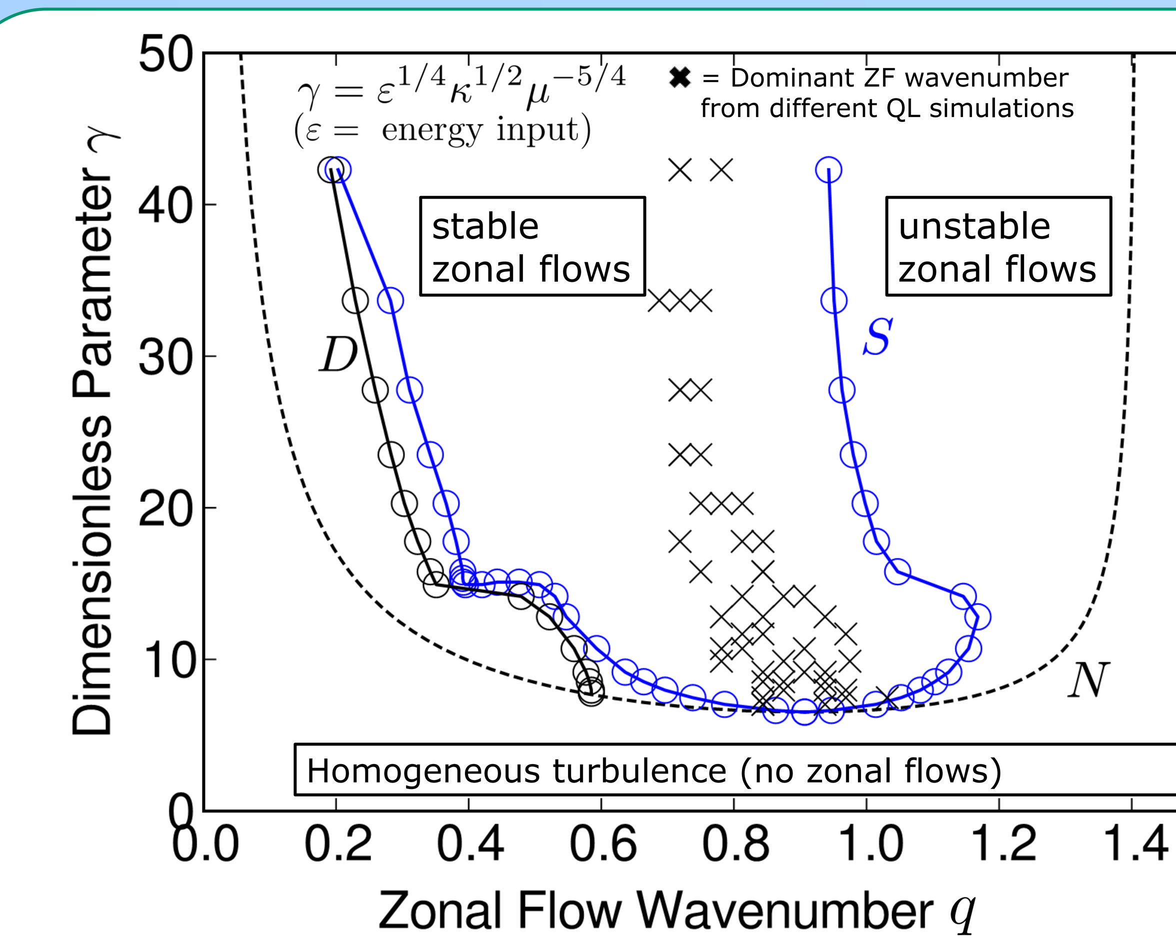
- Consider perturbations $\delta W(x,y\mid\overline{y},t)$, $\delta U(\overline{y},t)$ about an equilibrium
- Equilibrium is periodic in $\overline{y}$: expand perturbation as a Bloch state

$$\delta U(\overline{y},t) = e^{\sigma t}e^{iQ\overline{y}}\sum_p \delta U_p e^{ipq\overline{y}}$$

$$\delta W(x,y\mid\overline{y},t) = e^{\sigma t}e^{iQ\overline{y}}\sum_{mnp}\delta W_{mnp}e^{imax}e^{inby}e^{ipq\overline{y}}$$

- Equilibrium is unstable if there exists an eigenvalue σ with positive real part (for any Q)

Stability Diagram for CE2



Stability diagram for Modified Hasegawa-Mima

Idealized, infinite system: many zonal flow wavelengths correspond to a stable solution

Zonal flow wavenumbers in the unstable region must evolve to get inside the stable region – manifests as merging jets

Acknowledgments

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