Direct Statistical Simulation of Astrophysical Flows

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ABSTRACT

Astrophysical and geophysical flows are amenable to direct statistical simulation (DSS), the calculation of low order statistics that does not rely upon accumulation by direct numerical simulation (DNS) [1]. Anisotropic and inhomogeneous flows, such as those found in the atmospheres of planets, in rotating stars, and in disks, provide the starting point for an expansion in fluctuations about the mean flow, leading to a hierarchy of equations of motion for the equal-time cumulants. The method is described for a general set of evolution equations, and then illustrated for a nonlinear model of a stellar tachocline driven by relaxation to an underlying flow with shear due to Cally [2,3] for which a joint instability arises from the combination of differential rotation and magnetic stress. The reliability of DSS is assessed by comparing statistics so obtained against those accumulated from DNS, the traditional approach. The simplest non-trivial closure, CE2, sets the third and higher cumulants to zero yet yields qualitatively accurate low-order statistics. Physically CE2 retains only the eddy-mean flow interaction, and drops the eddy-eddy interaction. Deficiencies in CE2 can be repaired by retaining the third cumulant. We conclude by discussing possible extensions of the method through the use of better closures and computational methods, and in the range of astrophysical problems that are of interest.

[1] S. M. Tobias, K. Dagon, and J. B. Marston, "Astrophysical Fluid Dynamics via Direct Statistical Simulation," *The Astrophysical Journal* **727**, 127 (2011).

[2] P. S. Cally, "Nonlinear Evolution of 2D Tachocline Instabilities," *Solar Physics* **199,** 231–249 (2001).

[3] Paul S. Cally, Mausumi Dikpati, and Peter A. Gilman, "Clamshell and Tipping Instabilities in a Two-Dimensional Magnetohydrodynamic Tachocline," *The Astrophysical Journal* **582,** 1190–1205 (2003).

Direct Statistical Simulation by Expansion in Cumulants

"An alternative procedure which does not suffer this disadvantage consists of deriving a new system of equations whose unknowns are the statistics themselves…." (E. N. Lorenz, 1967)

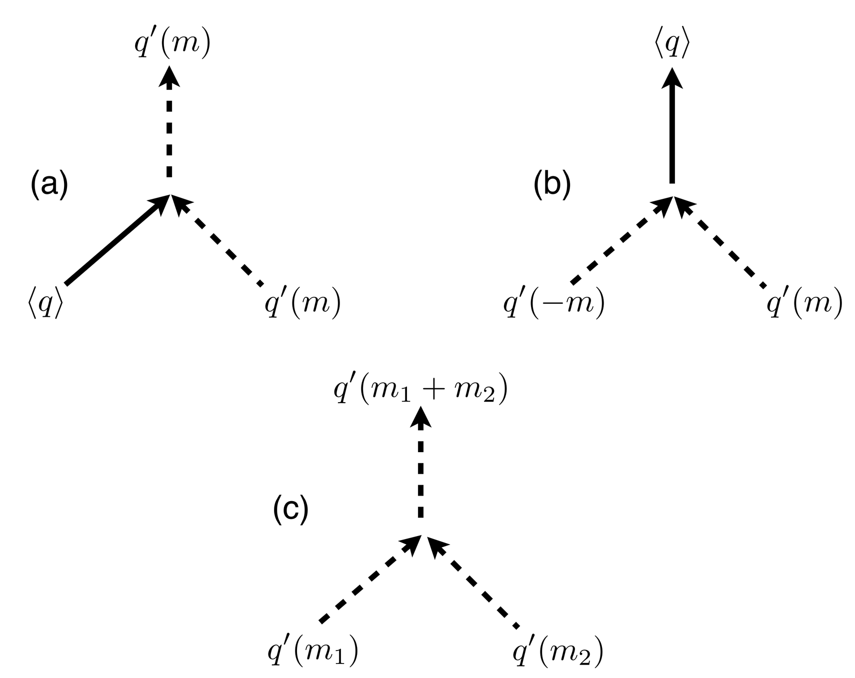
Zonal Average: $$\frac{\partial}{\partial t} q = L[q] + Q[q,\ q]$$

$$q(\theta, \phi) = \overline{q}(\theta) + q'(\theta, \phi)$$

$$\overline{q'} = 0 \qquad \overline{\overline{q}} = \overline{q}$$

$$\frac{\partial}{\partial t}\overline{q} = L[\overline{q}] + \overline{Q[q,\ q]}$$

$$\frac{\partial}{\partial t}\overline{(q\ q)} = \overline{q\ L[q]} + \overline{q\ Q[q,\ q]}$$

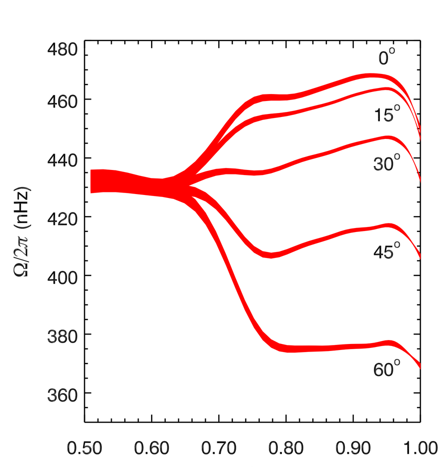


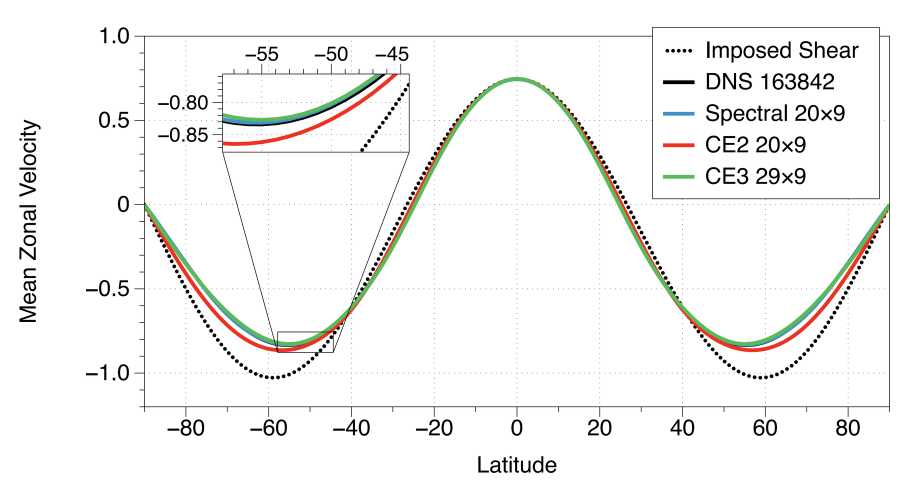
$$\text{CE2}: \overline{q'q'q'} = 0 \ [\text{drop eddy} - \text{eddy interactions of diagram (c)}]$$

$\text{CE3}: \overline{q'q'q'q'} = 0$ See Poster by Wanming Qi for more about CE-N approximations

Hydrodynamic Barotropic Jet Relaxes to Unstable Profile

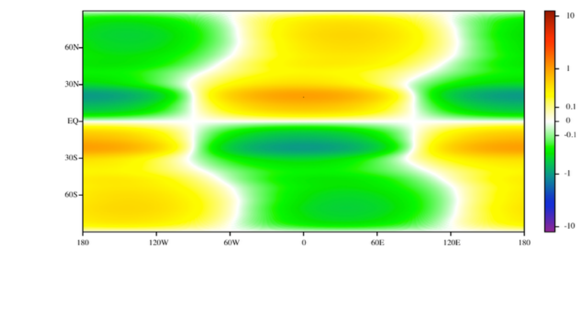
$$\frac{\partial \zeta}{\partial t} + \vec{v}\cdot\vec{\nabla}(\zeta + f) = \frac{\zeta_{jet} - \zeta}{\tau} - \nu_2 \nabla^4 \zeta \qquad \zeta_{jet}(\theta, \phi) = a P_{\ell=3}^{m=0}(\cos\theta)$$



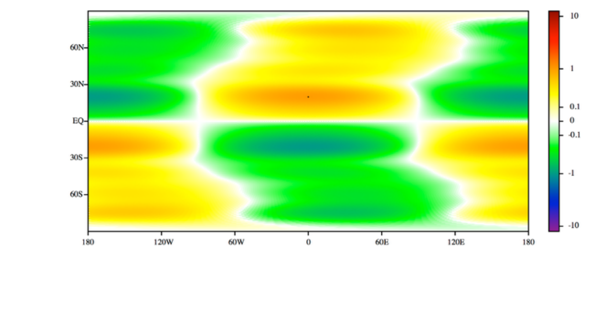


Comparison of Non-Local 2-Point Vorticity Correlations

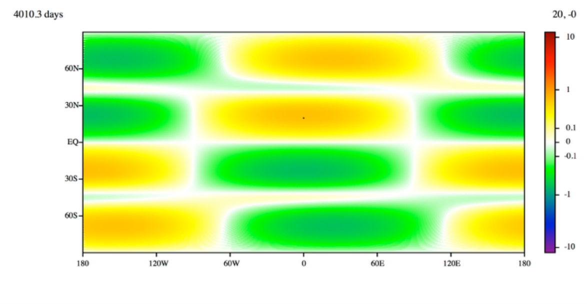
DNS 163,842



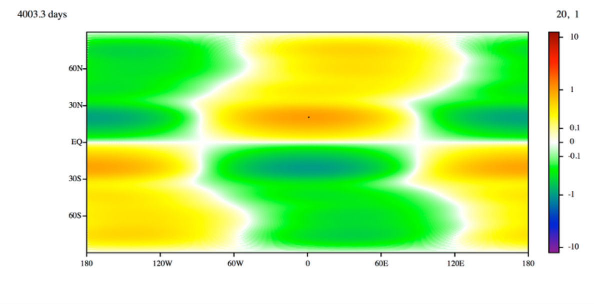
DNS 20 x 9



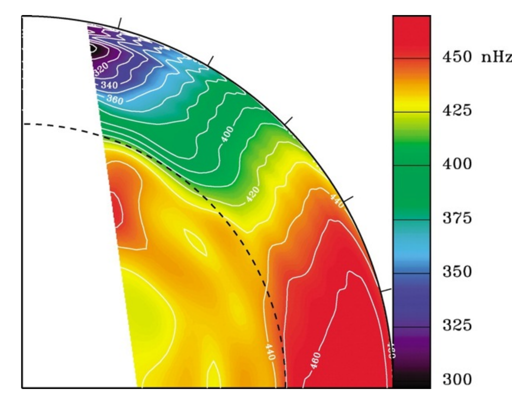
CE2



CE3



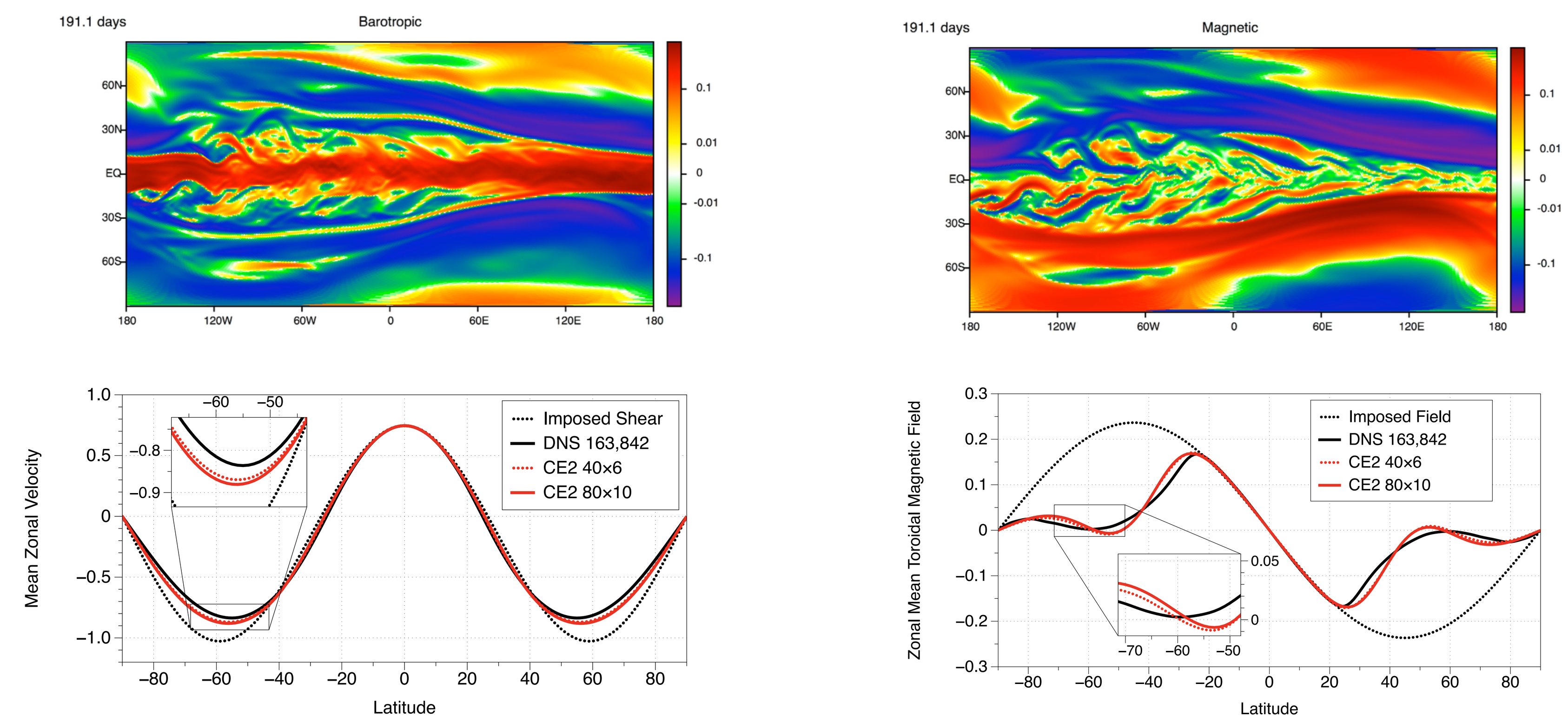
Idealized 2D Model of Solar Tachocline



$$\begin{aligned}
\frac{\partial \vec{v}}{\partial t} &+ (\vec{v}\cdot\vec{\nabla})\vec{v} + 2\vec{\Omega}\times\vec{v} = -\vec{\nabla}p + \vec{j}\times\vec{B} \\
\frac{\partial \vec{B}}{\partial t} &= \vec{\nabla}\times(\vec{v}\times\vec{B}) + \eta\nabla^2\vec{B} \\
\vec{\nabla}\cdot\vec{v} &= 0 \\
\vec{j} &= \vec{\nabla}\times\vec{B}
\end{aligned}$$

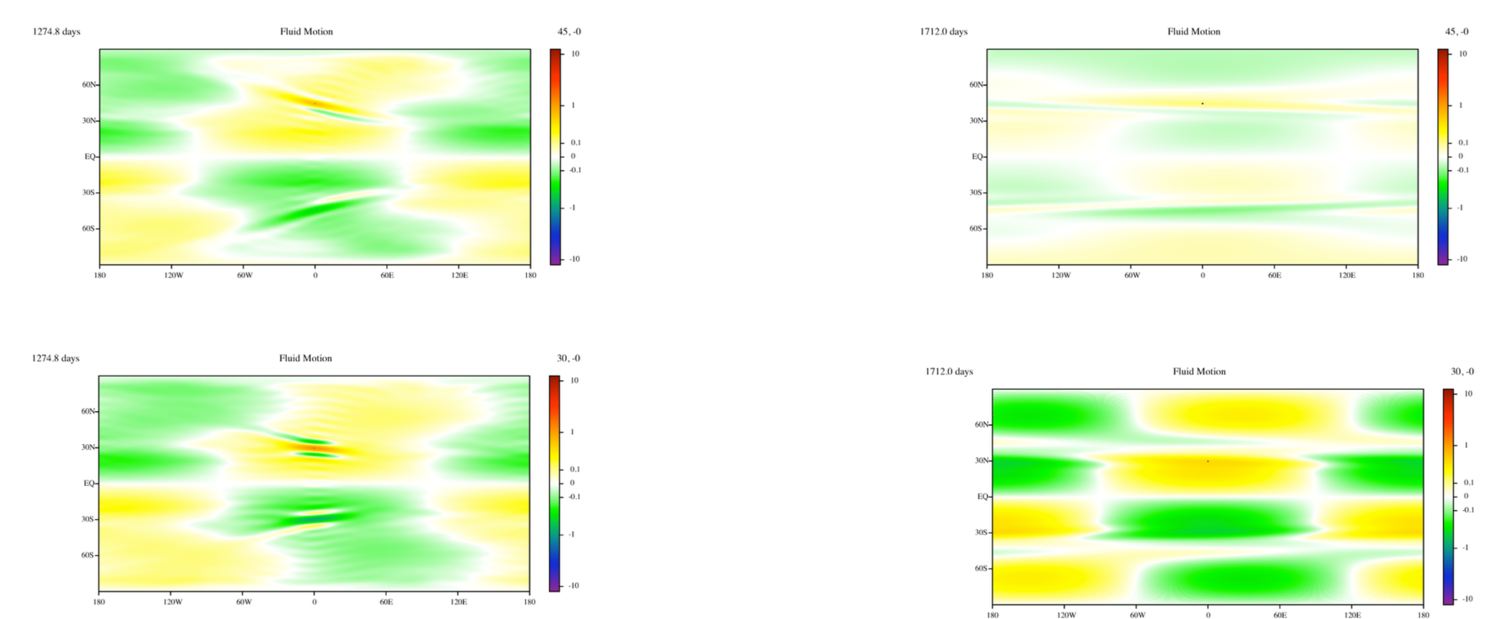
Impose External Magnetic Field

$$\vec{B} = \vec{B}' + \hat{\phi}\ B_0 \sin(\theta)\cos(\theta)$$



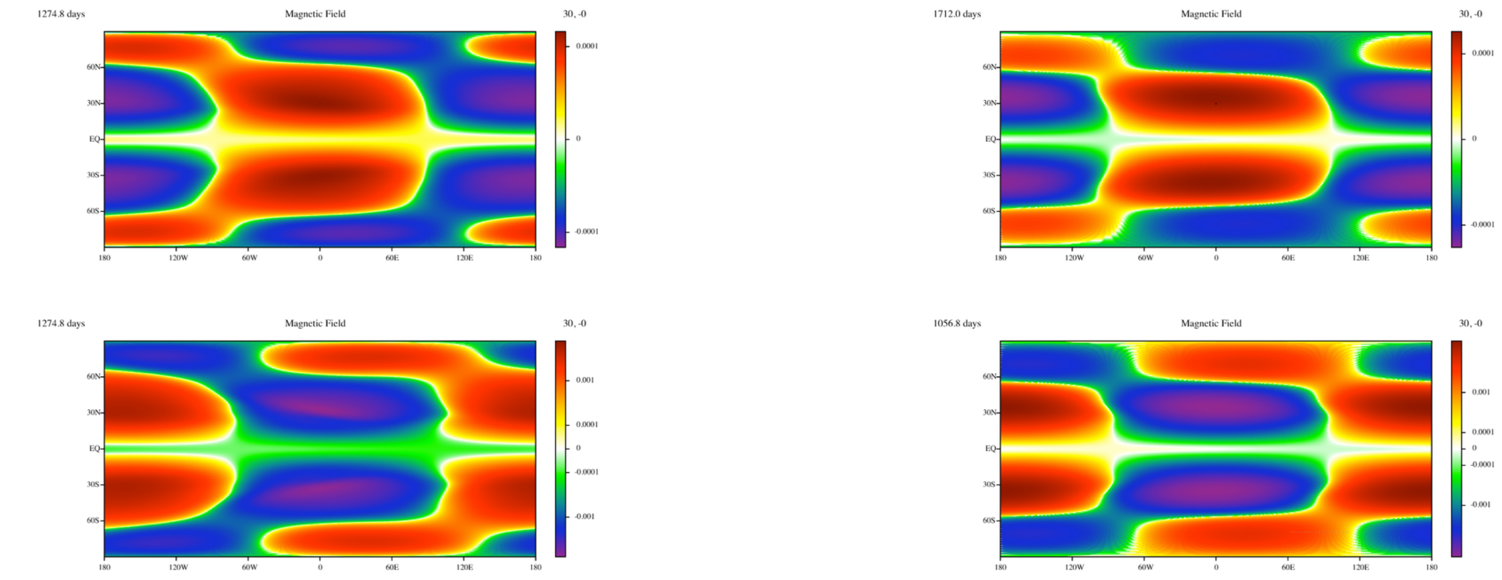
Comparison of Non-Local 2-Point Correlations

DNS CE2



Vorticity Correlations

DNS CE2



Magnetic Potential Correlations

Competition between Reynolds and Maxwell stresses

CONCLUSIONS

• Direct Statistical Simulation, by integrating out fast modes, focuses on the slow modes of most interest. In astrophysics this is a key tool since there is such a wide range of spatial and temporal scales that contribute to the behavior of the flow.
• This approach works for systems with non-trivial mean flows and magnetic fields, such as planetary (including exoplanet) atmospheres, oceans, stars, accretion disks, galaxies, ...
• Accuracy can of the solution can be systematically improved (eg. CE3) and non-perturbative solutions appear to be possible. Complicated large-scale dynamics can also be incorporated using GCE2 (see e.g. talk by JBM)
• DSS can lead to improved understanding, by identifying important interactions driving the large-scale dynamics.
• DSS can be significantly faster than DNS and offers a natural way to incorporate statistical models of subgrid physics.
• Future problems of interest that are amenable to DSS include (H)MRI in disks and experiments and dynamo action in planets and stars.

Supported in part by NSF DMR-0605619