Mean responses to symmetry-breaking perturbations in disordered systems

Michael Proctor
DAMTP, Cambridge

David Hughes
Alastair Rucklidge
Applied Mathematics, Leeds

Motivation: mean field models of MHD turbulence

Can we extract mean coefficients governing long-wavelength instabilities?

Method: Start with homogeneous stationary MHD turbulent state. Perturb basic state with small imposed symmetry breaking terms

Calculate linearised response to perturbation and form mean coefficients governing slow evolution of the mean quantities.

But is this a sensible way to proceed?

Formal calculation of mean field coefficients

- Dynamic mean field theory for long-wavelength instabilities of full MHD states.
- Important to note that when considering MHD states, instability cannot be determined from the induction equation alone Full coupled induction and momentum equations must be used

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U}\cdot\nabla\mathbf{U} = -\nabla P + \mathbf{B}\cdot\nabla\mathbf{B} + Re^{-1}\nabla^2\mathbf{U} + \mathbf{F} \qquad (7)$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{U}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{U} + Rm^{-1}\nabla^2\mathbf{B} \qquad (8)$$

Rm/Re is the magnetic Prandtl number ν/η. The (coupled) equations describing small disturbances (denoted by $\mathbf{b}, \mathbf{u}, p$) are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{U}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{U} = -\nabla p + \mathbf{B}\cdot\nabla\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{B} + Re^{-1}\nabla^2\mathbf{u} \qquad (9)$$

$$\frac{\partial \mathbf{b}}{\partial t} + \mathbf{U}\cdot\nabla\mathbf{b} + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{U} + Rm^{-1}\nabla^2\mathbf{b} \qquad (10)$$

Formal calculation of mean field coefficients

- As for the kinematic problem we separate the disturbance field and flow into mean and fluctuating parts so that $\mathbf{b} = \langle \mathbf{b} \rangle + \mathbf{b}'$, etc.. The equations for the mean parts are

$$
\begin{aligned}
\frac{\partial \langle \mathbf{u} \rangle}{\partial t} + \frac{\partial}{\partial x_j}(\langle U_j \mathbf{u}' + u_j' \mathbf{U} \rangle) &= -\boldsymbol{\nabla} \langle p \rangle \\
&\quad + \frac{\partial}{\partial x_j}(\langle B_j \mathbf{b}' + b_j' \mathbf{B} \rangle) + Re^{-1} \nabla^2 \langle \mathbf{u} \rangle, \qquad (11) \\
\frac{\partial \langle \mathbf{b} \rangle}{\partial t} &= \boldsymbol{\nabla} \times (\langle \mathbf{U} \times \mathbf{b}' + \mathbf{u}' \times \mathbf{B} \rangle) \qquad (12) \\
&\quad + Rm^{-1} \nabla^2 \langle \mathbf{b} \rangle,
\end{aligned}
$$

- For the fluctuating parts we have (assuming $\langle \mathbf{b} \rangle$, $\langle \mathbf{u} \rangle$ uniform correct to leading order)

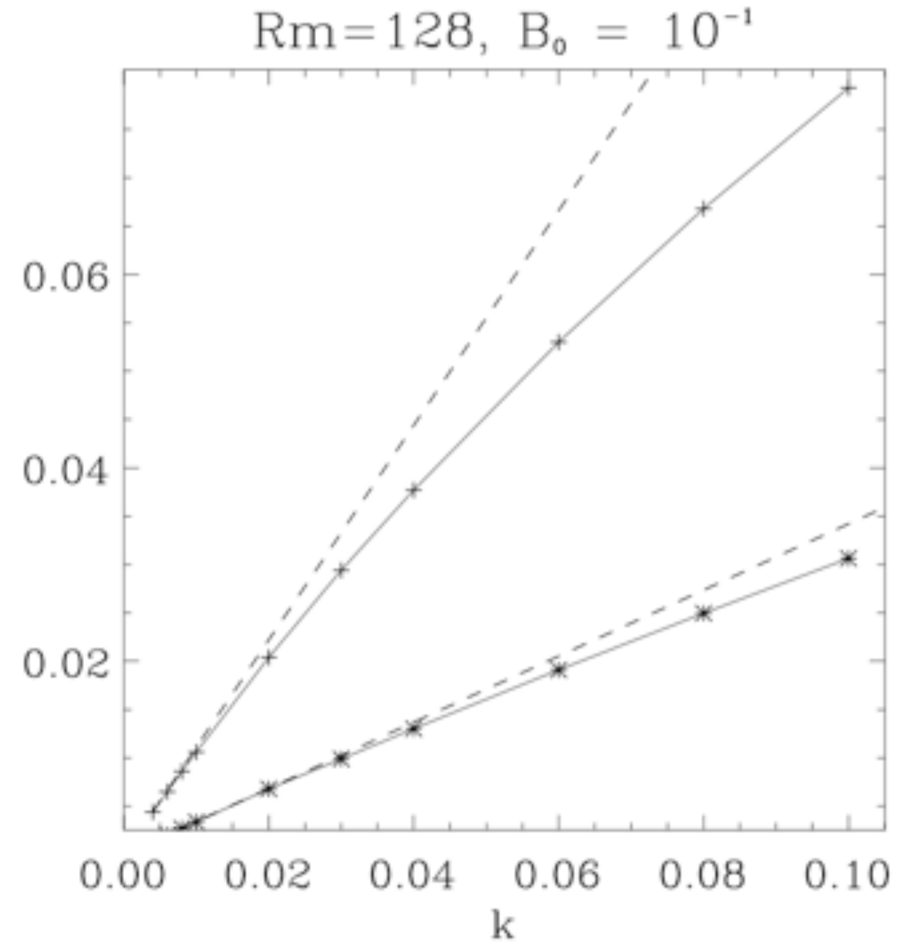
$$
\begin{aligned}
\frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{U} \cdot \nabla \mathbf{u}' + \mathbf{u}' \cdot \nabla \mathbf{U})' + \langle \mathbf{u} \rangle \cdot \nabla \mathbf{U} &= -\nabla p' \qquad (13) \\
&\quad + (\mathbf{B} \cdot \nabla \mathbf{b}' + \mathbf{b}' \cdot \nabla \mathbf{B})' + \langle \mathbf{b} \rangle \cdot \nabla \mathbf{B} + Re^{-1} \nabla^2 \mathbf{u}' \\
\frac{\partial \mathbf{b}'}{\partial t} + (\mathbf{U} \cdot \nabla \mathbf{b}' + \mathbf{u}' \cdot \nabla \mathbf{B})' + \langle \mathbf{u} \rangle \cdot \nabla \mathbf{B} &= \qquad (14) \\
&\quad (\mathbf{B} \cdot \nabla \mathbf{u}' + \mathbf{b}' \cdot \nabla \mathbf{U})' + \langle \mathbf{b} \rangle \cdot \nabla \mathbf{U} + Rm^{-1} \nabla^2 \mathbf{b}'.
\end{aligned}
$$

Formal calculation of mean field coefficients

- Using the above the mean field equations can be written

$$\frac{\partial \langle u_i \rangle}{\partial t} + \frac{\partial}{\partial x_j}(\Gamma^U_{ijl}\langle u_l \rangle + \Gamma^B_{ijl}\langle b_l \rangle) = -\frac{\partial \langle p \rangle}{\partial x_i} + Rm^{-1}\nabla^2 \langle u_i \rangle, \quad (15)$$

$$\frac{\partial \langle b_i \rangle}{\partial t} = \epsilon_{ijk}\frac{\partial}{\partial x_j}(\alpha^B_{kl}\langle b_l \rangle + \alpha^U_{kl}\langle u_l \rangle) + Re^{-1}\nabla^2 \langle b_i \rangle, \quad (16)$$

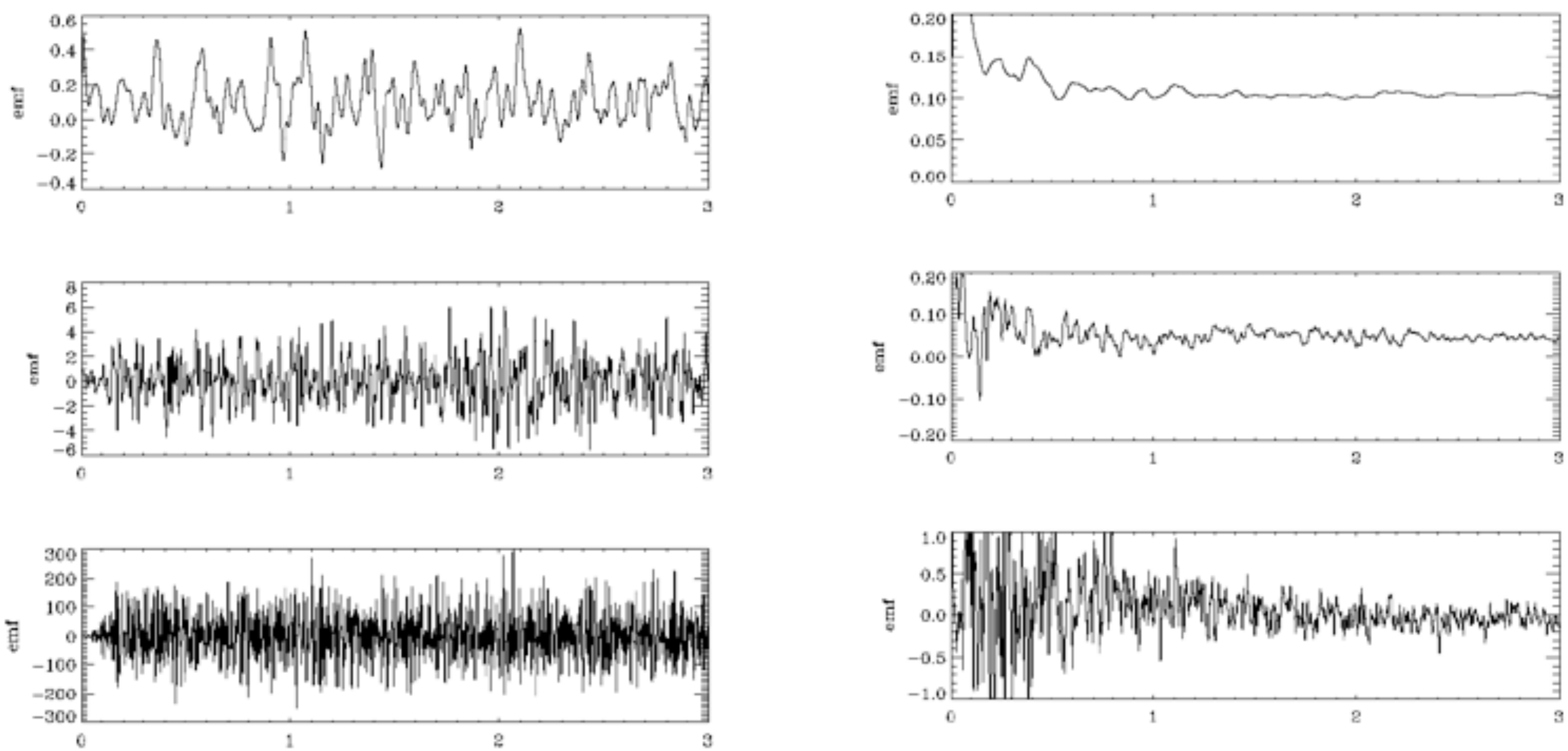


- There are circumstances where this procedure works and yields accurate results
- Example; instability of MHD state with laminar flows depending only on *(x,y)* to long-wavelength three-dimensional disturbances (Courvoisier *et al.* 2011)

Two major problems:

1. Hard to calculate, even numerically as signal noise ratio very small.

2. If equations have positive Lyapunov exponents then linearization will not work as trajectories diverge.

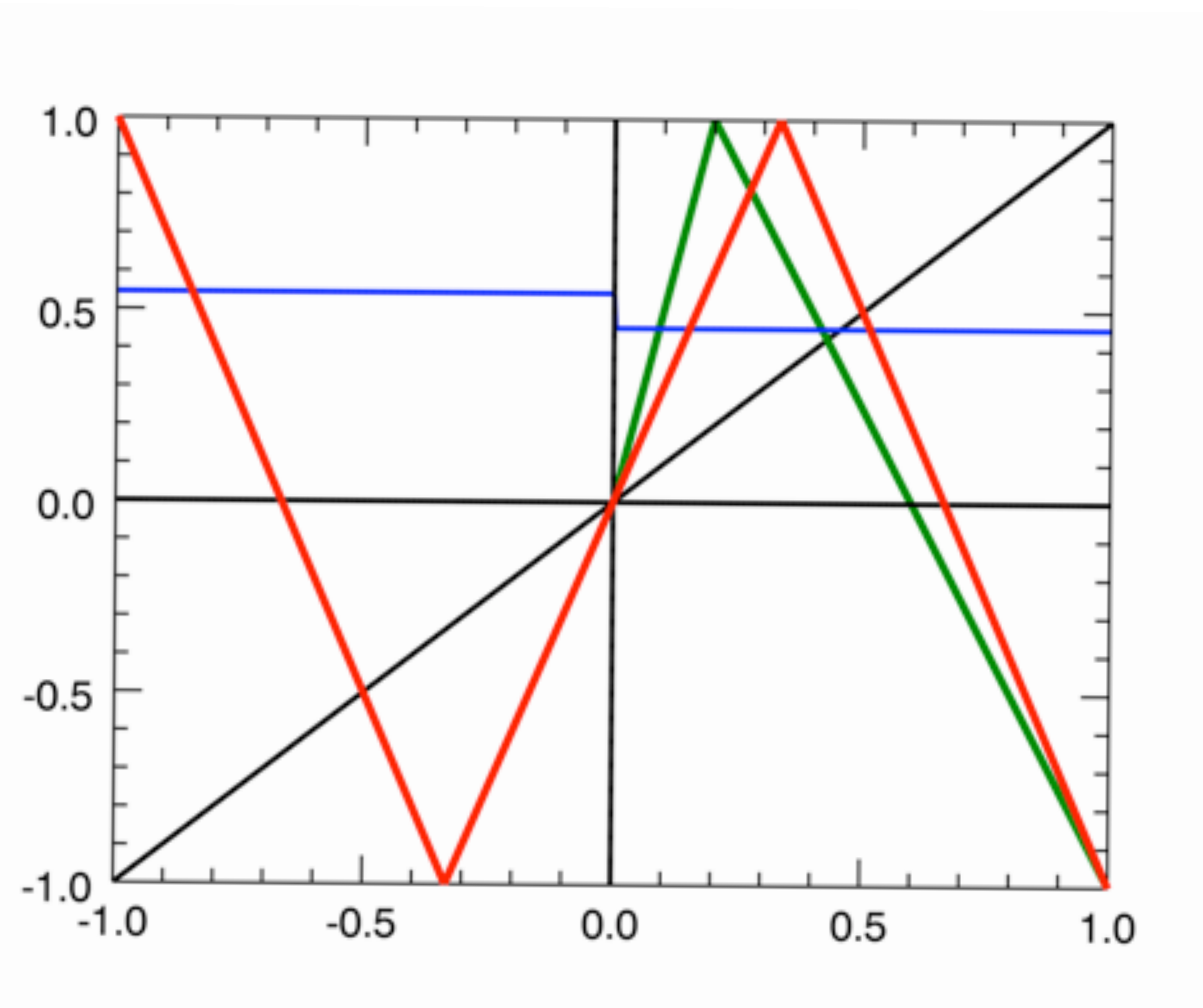


But one might expect that a small symmetry-breaking term would lead to a small change in mean quantities even with large excursions

We consider a number of simpler problems to see if that expectation is satisfied

- Cubic Tent Map
- Cubic logistic map
- Lorenz Map
- Reduced dynamo-type ODE model with stochastic variation

Cubic Tent Map



$f(x)$ defined in $-1 \leq x \leq 1$

$$f(x) = 3x \qquad \frac{1}{3} \leq x \leq 0$$

$$f(x) = -2 - 3x \qquad 1 \leq x \leq -\frac{1}{3}$$

$$f(x) = ax \qquad 0 \leq x \leq \frac{1}{a}$$

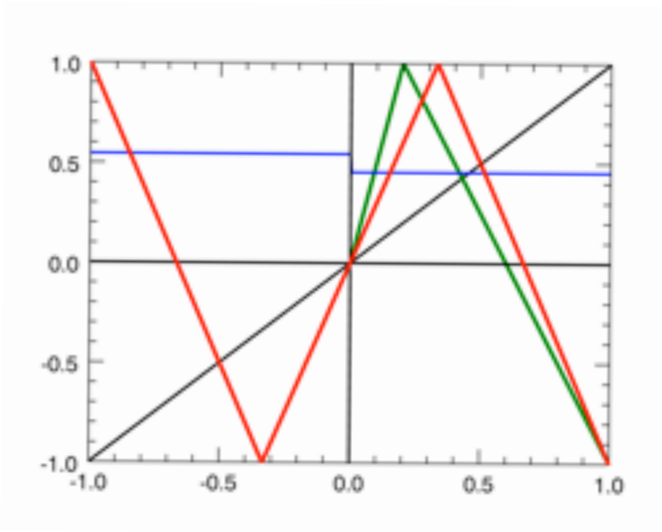
$$f(x) = \frac{1 + a - 2ax}{a - 1} \qquad \frac{1}{a} \leq x \leq 1$$

$a = 3$ gives the symmetric map (red).
Plot shows $a = 5$ (green)
Invariant measure for $a = 5$ shown in blue.

Dynamics of sequence $\{x_0, x_1, \ldots\}$ given by $x_{n+1} = f(x_n)$

Cubic Tent Map

Invariant measure:



$$\mu(x) = \sum_i \frac{\mu(\tilde{x}_i)}{|f'(\tilde{x}_i)|} \; ; \quad \int_{-1}^{1} \mu(x)\, dx = 1$$

$\mu(x)$ *piecewise constant*, with $\mu = \mu_+$, $x > 0$; $\mu = \mu_-$, $x < 0$.

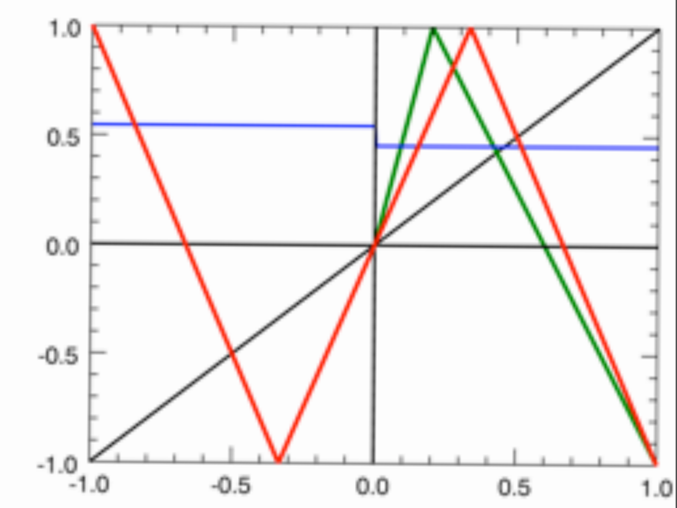
$x > 0 \Rightarrow x$ has three pre-images $\tilde{x}_i$, where $\tilde{x}_1 < 0$ and $0 < \tilde{x}_2 < a^{-1} < \tilde{x}_3 < 1$.

$x < 0$ then $\tilde{x}_{1,2} < 0$ and $\tilde{x}_3 > a^{-1}$.

Moduli of gradients in $x < 0$, $0 < x < a_{-1}$, $a_{-1} < x < 1$ are $3, a, 2a/(a-1)$ respectively. Then get two equations

$$\mu_+ = \frac{1}{3}\mu_- + \left(\frac{1}{a} + \frac{a-1}{2a}\right)\mu_+, \quad \mu_- = \frac{2}{3}\mu_- + \frac{a-1}{2a}\mu_+. \tag{1}$$

Cubic Tent Map



$$\mu_+ = \frac{1}{3}\mu_- + \left(\frac{1}{a} + \frac{a-1}{2a}\right)\mu_+, \quad \mu_- = \frac{2}{3}\mu_- + \frac{a-1}{2a}\mu_+. \tag{7}$$

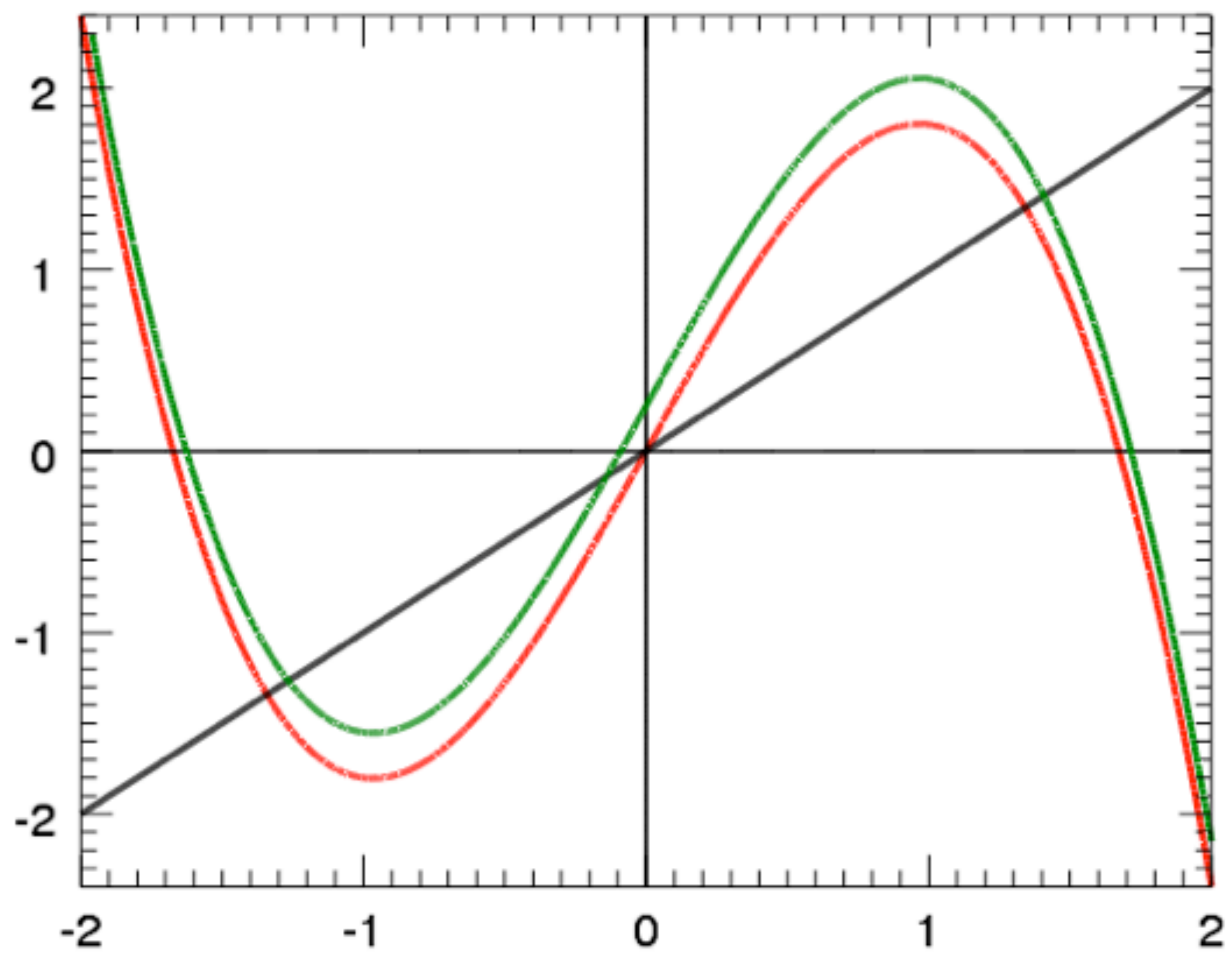
These two equations are both equivalent to $3(a-1)\mu_+ = 2a\mu_-$, and applying the normalisation condition $\mu_+ + \mu_- = 1$ we finally obtain

$$\mu_+ = \frac{2a}{5a-3}, \quad \mu_- = \frac{3(a-1)}{5a-3}. \tag{8}$$

The average of x can now be calculated exactly as

$$\langle x \rangle = \mu_- \int_{-1}^{0} x\, dx + \mu_+ \int_{0}^{1} x\, dx = \frac{1}{2}(\mu_+ - \mu_-) = \frac{3-a}{2(5a-3)}. \tag{9}$$

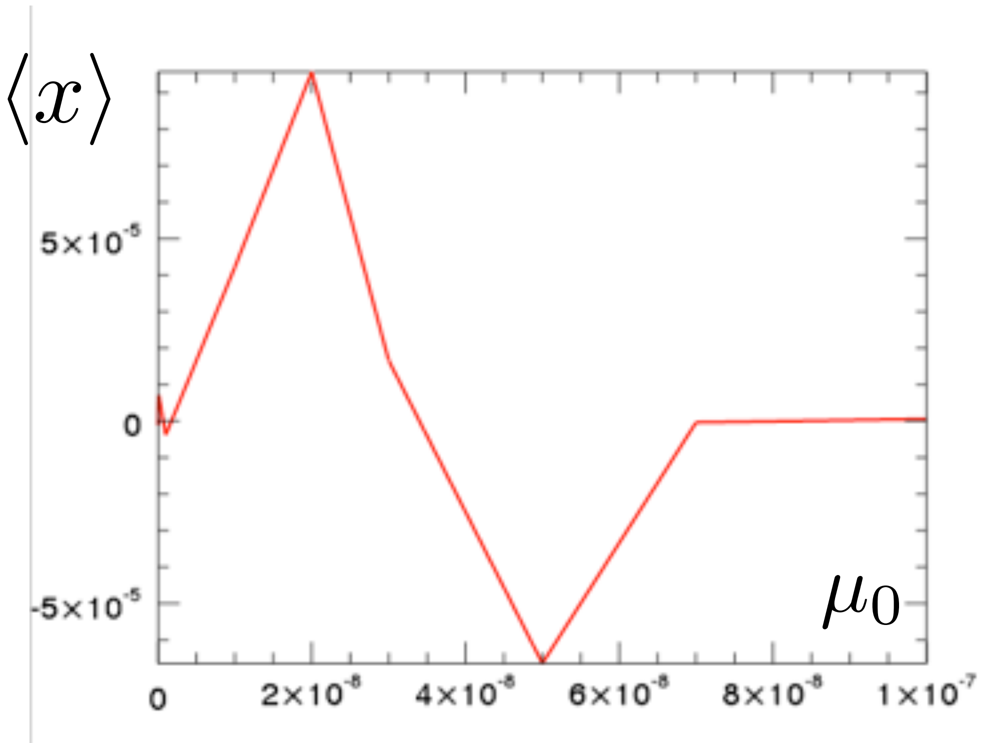
Cubic Logistic Map

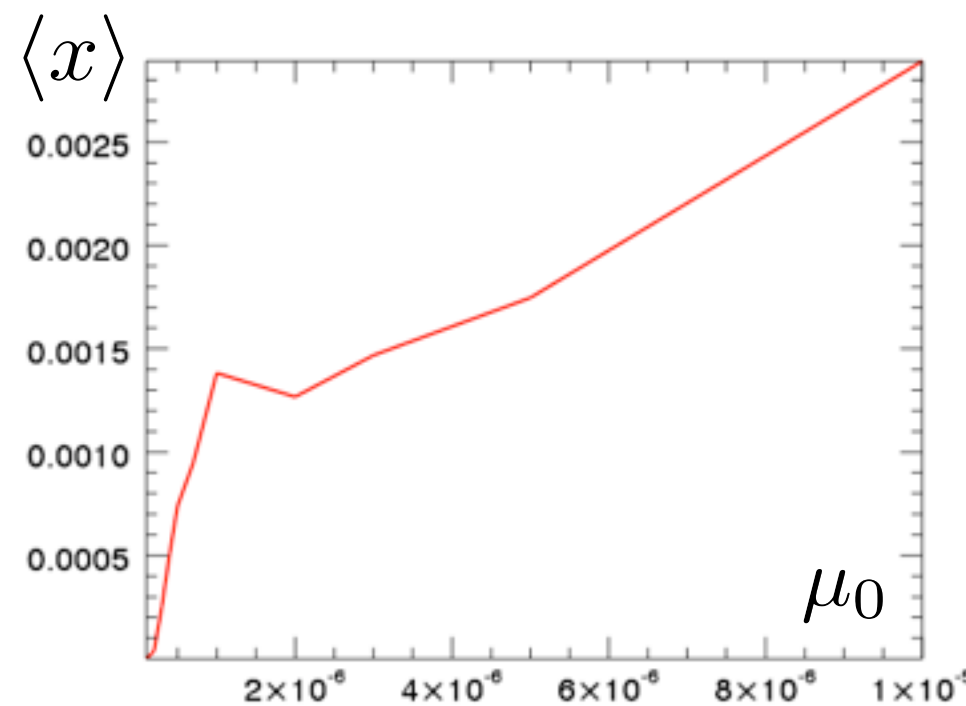


This map into $-2 \leq x \leq 2$ has the form (for $|\mu_0|$ sufficiently small)

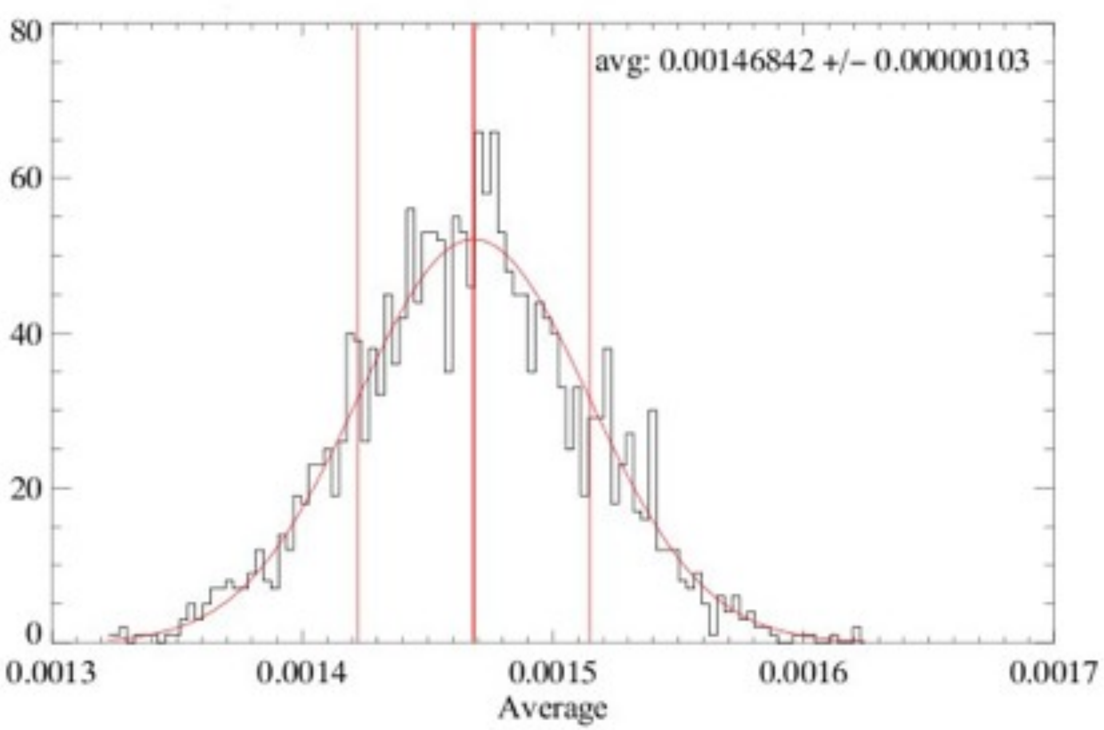
$$f(x) = \mu_0 + 2.8x - x^3$$

Cubic Logistic Map



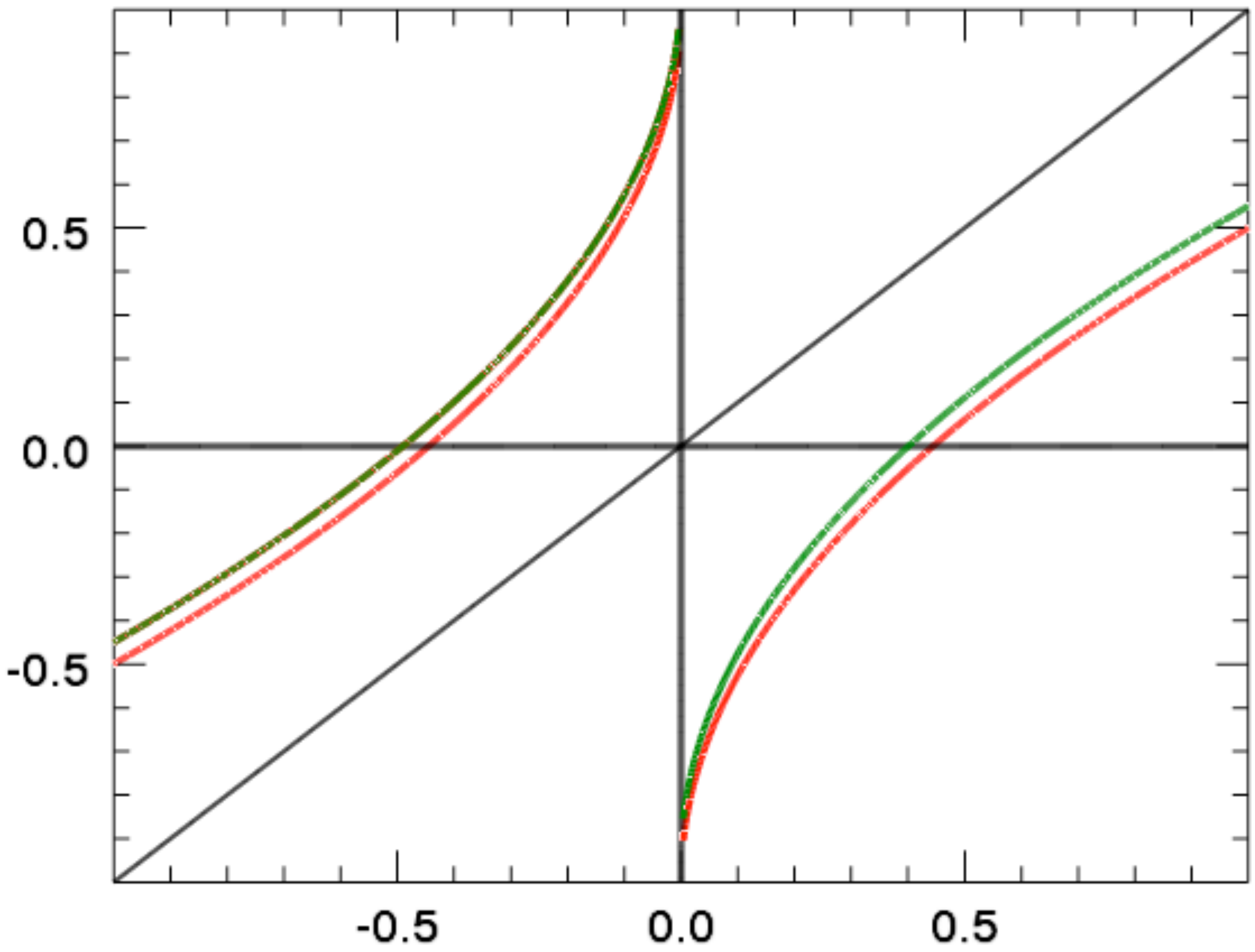


$$\mu_0 = 3 * 10^{-6}$$



- multiple precision used (256 sig.figs)
- for each point, 2000 i.c.s, 10^9 iterations.
- Error ~ 10^{-6} for each point
- No clear relation between μ_0 and <x>
- Attribute failure to existence of dense set of periodic windows - invariant measure highly complex

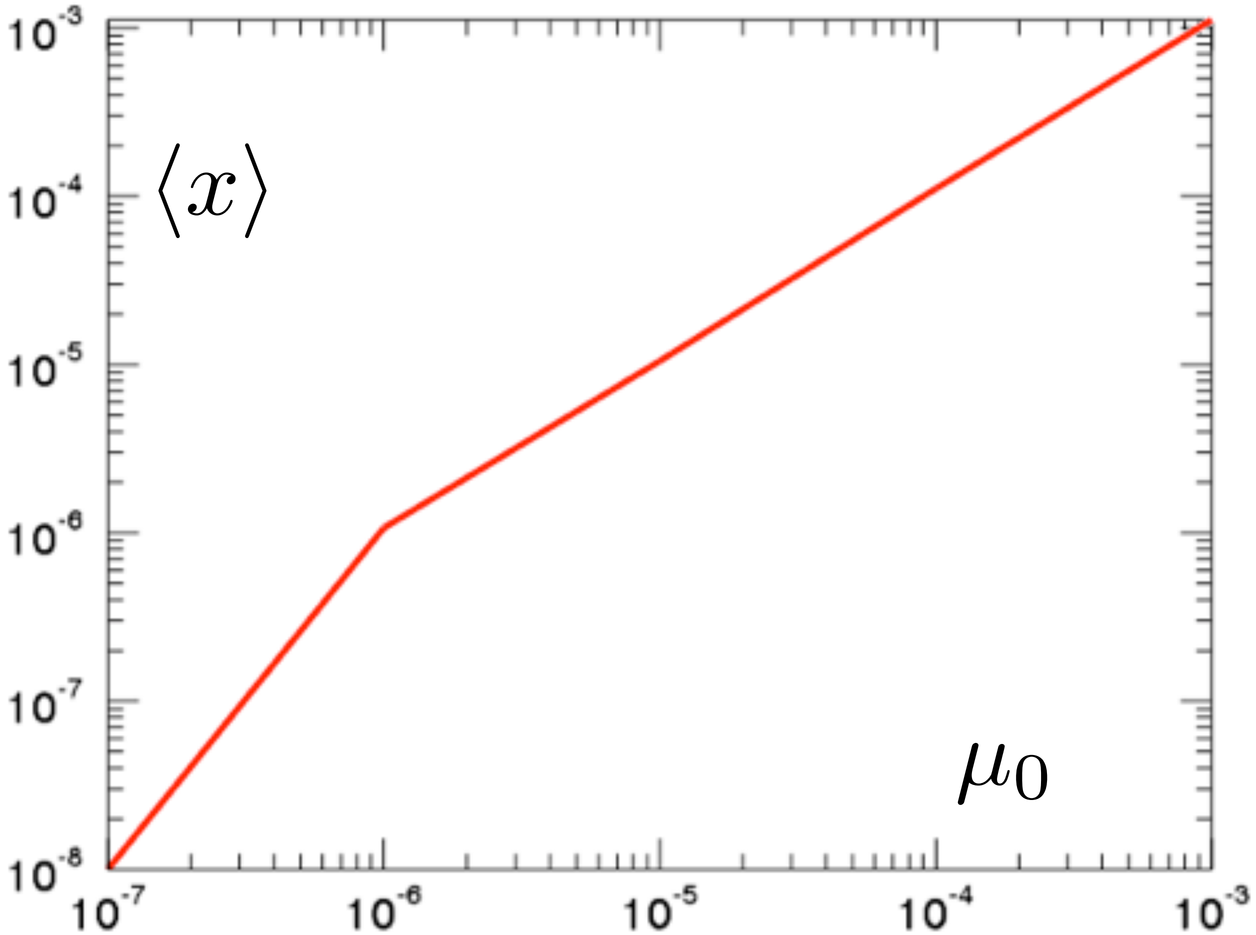
Lorenz Map



This map has no stable periodic orbits so may yield more sensible results

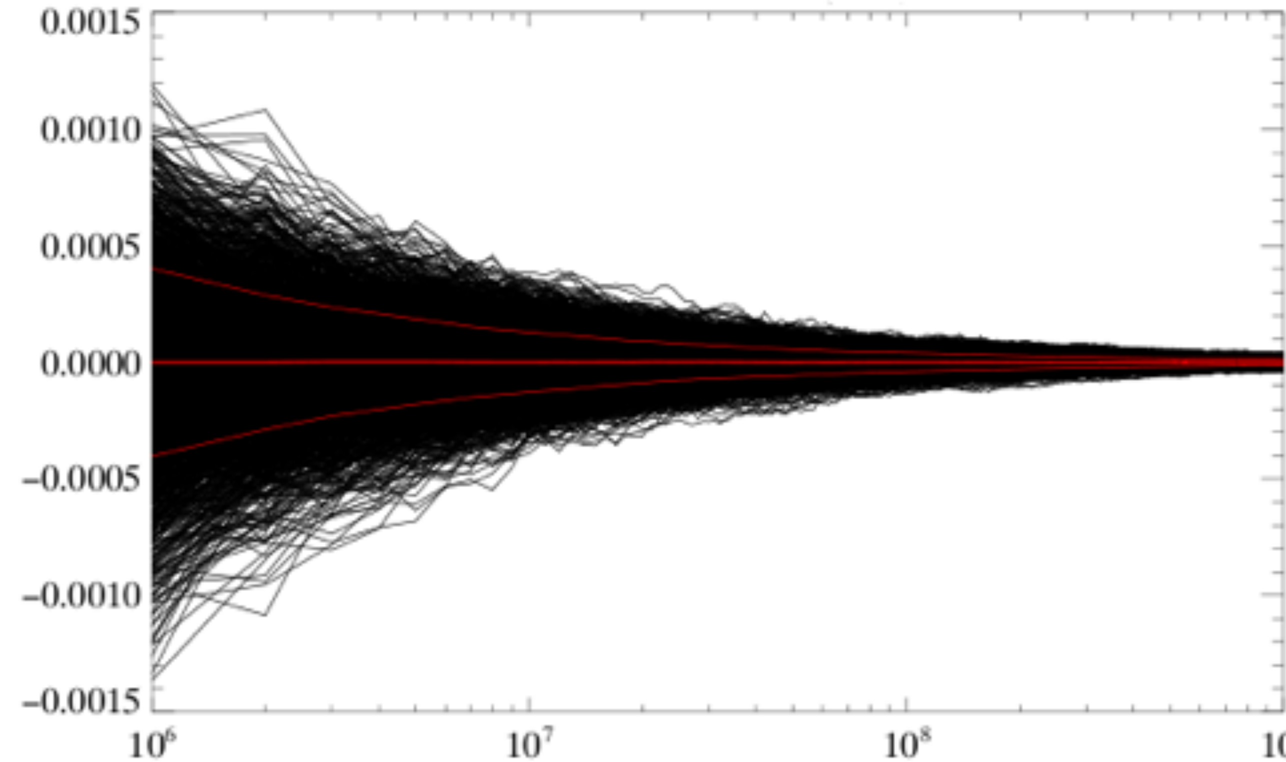
$$f(x) = \mu_0 \,\mathrm{sgn(x)}(-1 + 1.5\sqrt{|x|})$$

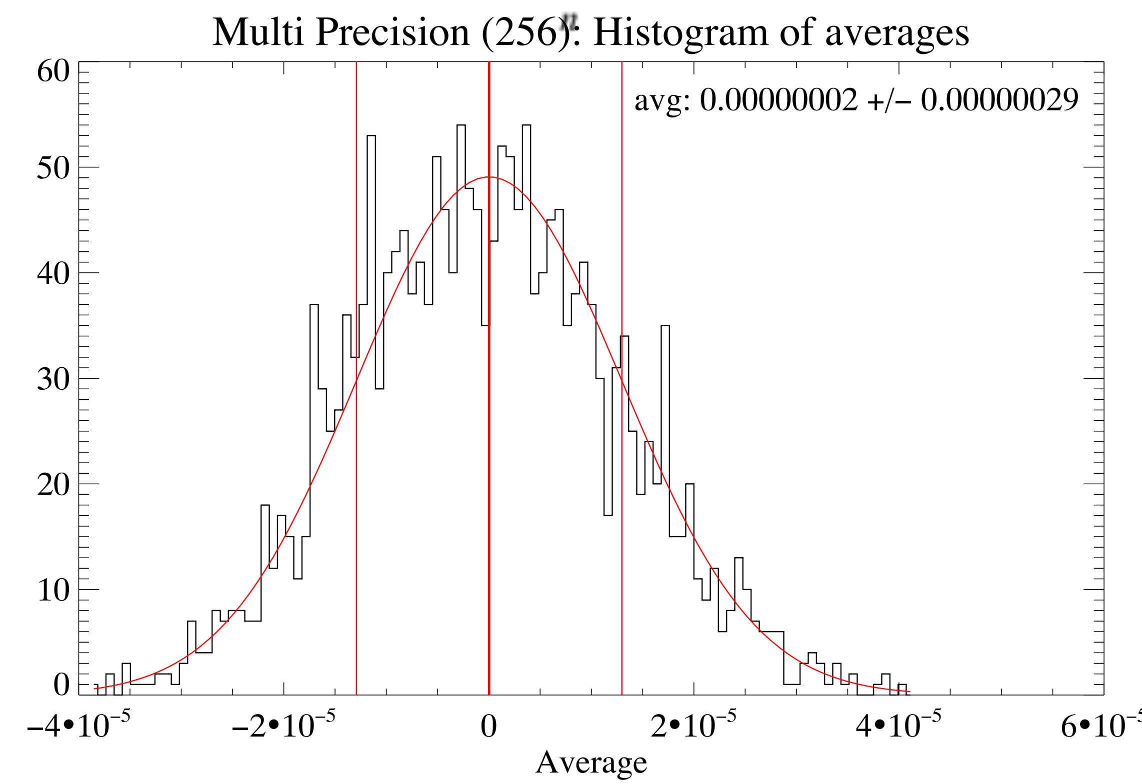
Lorenz Map



for $\mu_0 \gtrsim 10^{-6}$, $\langle x \rangle \sim 1.1\mu_0$

Time series, $\mu_0 = 10^{-8}$





ODE model

Simple 3D ODE model with stochastic forcing, modelled on cut-down dynamo equations of Kennett (1975)

$$\dot{x_0} = -\nu x_0 + F(t); \qquad F(t) = \text{white noise in } [-1, 1]$$

$$\dot{x_1} = \sigma(-x_1 + r x_0 - x_2 x_3)$$

$$\dot{x_2} = -\eta_1 x_2 + x_1 x_3$$

$$\dot{x_3} = -\eta_2 x_3 + x_1 x_2 + \mu_0 x_1$$

When $\mu_0 = 0$ there is a symmetry

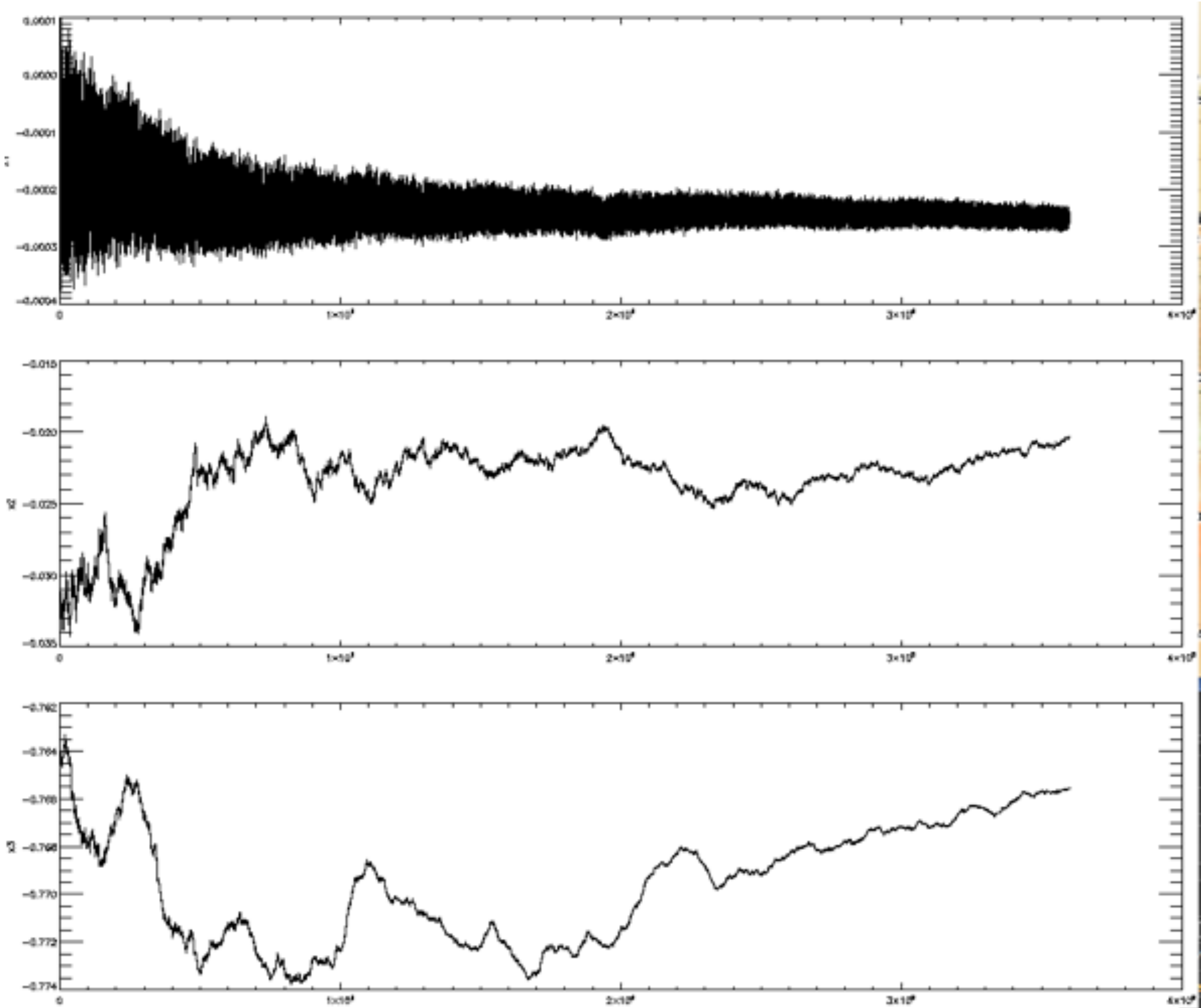
$x_1 \to x_1$, $x_2 \to -x_2$, $x_3 \to -x_3$ [“$B \to -B$”]

Symmetry is broken when $\mu_0 \neq 0$

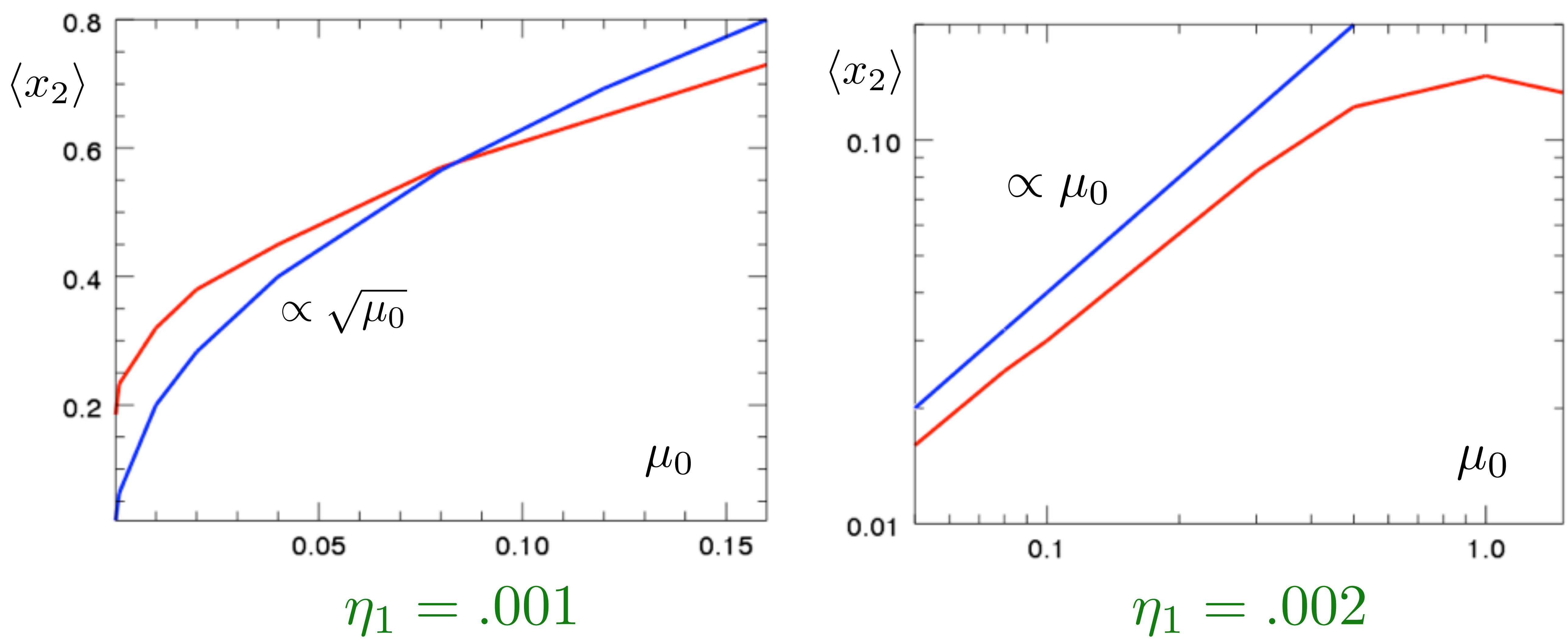
Choose $\nu = 1$,$\sigma = 1$, $r = 2$, $\eta_2 = .001$ and $\eta_1 = .001$ or $.002$

ODE model

- running averages of $>10^6$ iterations for x_1,x_2,x_3
- solutions take a long time to converge so error bars significant



ODE model



In the first case (which has singular behaviour when μ_o=0) there is no linear behaviour, while in the second case there is a clear linear range of response.

Conclusions

- In looking at small symmetry-breaking perturbations to chaotic flows, the nature of the response depends on the structure of the underlying attractor.
- Can speculate that with smooth invariant measures on the attractor there may be some hope of finidng a linear response.
- However there is still the problem of the signal/noise ratio.
- The idea of using linearised equations to find the mean induced response must be abandoned for non-laminar basic states.
- Next steps: Extensive calculation of forced MHD turbulence and effects of small imposed fields.