Geometric GLM applied to oceanic near-inertial waves

Jacques Vanneste

School of Mathematics and Maxwell Institute
University of Edinburgh, UK
www.maths.ed.ac.uk/~vanneste

with Jin-Han Xie

Geometric view of GLM and glm

Separation between mean flow and 'waves':

- simple mean dynamics,
- simple closure for the waves,
- interpretation of the mean flow, e.g. track particle motion.

Larangian averaging: Andrews & McIntyre's GLM,

$$x = X(a) + \xi(X(a)), \quad \bar{u}^{\mathrm{L}}(X) = \overline{u(X + \xi(X))},$$

GLM is coordinate dependent:

- cannot add points, cannot add vectors at different points,
- $x \in M$ but $X \notin M$; $\nabla \cdot u = 0$ but $\nabla \cdot \bar{u}^{\mathrm{L}} \neq 0$.

Take a geometric approach:

- avoid temptation of coordinate dependence;
- results valid on arbitrary manifolds.

Geometric view of GLM and glm

Separation between mean flow and 'waves':

- simple mean dynamics,
- simple closure for the waves,
- interpretation of the mean flow, e.g. track particle motion.

Larangian averaging: Andrews & McIntyre's GLM,

$$x = X(a) + \xi(X(a)), \quad \bar{u}^{\mathrm{L}}(X) = \overline{u(X + \xi(X))},$$

GLM is coordinate dependent:

- cannot add points, cannot add vectors at different points,
- $x \in M$ but $X \notin M$; $\nabla \cdot u = 0$ but $\nabla \cdot \bar{u}^{\mathrm{L}} \neq 0$.

Take a geometric approach:

- avoid temptation of coordinate dependence;
- results valid on arbitrary manifolds.

Geometric view of GLM and glm

Kinematics: ensemble of flow maps $\phi = \phi^\omega : M \to M$.
Decompose flow maps into mean and perturbation

$$\phi = \xi \circ \bar{\phi}.$$

Taking the time derivative

$$\dot{\xi} \circ \xi^{-1} + \xi_* \bar{u}^{\mathrm{L}} = u, \quad \text{where} \ \ \bar{u}^{\mathrm{L}} = \dot{\bar{\phi}} \circ \bar{\phi}^{-1}$$

ξ_* is the push-forward: $(\xi_* u)^i(x) = (\partial_j \xi^i u^j)(\xi^{-1} x)$.

Need a constraint on ξ to define $\bar{\phi}$:

- GLM (Andrews & McIntyre): $\bar{\xi} = 0$, not geometric,
- glm (Soward & Roberts): $\xi = \mathrm{e}^\eta$ for a vector field η with $\bar{\eta} = 0$ and $\nabla \cdot \eta$,
- alternative: $\overline{\xi^* \dot{\xi}} = 0$, where ξ^* is the pull-back.

Geometric view of GLM and glm

Kinematics: ensemble of flow maps $\phi = \phi^\omega : M \to M$.
Decompose flow maps into mean and perturbation

$$\phi = \xi \circ \bar{\phi}\,.$$

Taking the time derivative

$$\dot{\xi} \circ \xi^{-1} + \xi_* \bar{u}^{\mathrm{L}} = u, \quad \text{where } \bar{u}^{\mathrm{L}} = \dot{\bar{\phi}} \circ \bar{\phi}^{-1}$$

ξ_* is the push-forward: $(\xi_* u)^i(x) = (\partial_j \xi^i u^j)(\xi^{-1} x)$.

Need a constraint on ξ to define $\bar{\phi}$:

- GLM (Andrews & McIntyre): $\bar{\xi} = 0$, not geometric,
- glm (Soward & Roberts): $\xi = \mathrm{e}^\eta$ for a vector field η with $\bar{\eta} = 0$ and $\nabla \cdot \eta$,
- alternative: $\overline{\xi^* \dot{\xi}} = 0$, where ξ^* is the pull-back.

Geometric view of GLM and glm

Kinematics: ensemble of flow maps $\phi = \phi^\omega : M \to M$.
Decompose flow maps into mean and perturbation

$$\phi = \xi \circ \bar{\phi}\,.$$

Taking the time derivative

$$\dot{\xi} \circ \xi^{-1} + \xi_* \bar{u}^{\mathrm{L}} = u, \quad \text{where } \bar{u}^{\mathrm{L}} = \dot{\bar{\phi}} \circ \bar{\phi}^{-1}$$

ξ_* is the push-forward: $(\xi_* u)^i(x) = (\partial_j \xi^i u^j)(\xi^{-1} x)$.

Need a constraint on ξ to define $\bar{\phi}$:

- GLM (Andrews & McIntyre): $\bar{\xi} = 0$, not geometric,
- glm (Soward & Roberts): $\xi = \mathrm{e}^\eta$ for a vector field η with $\bar{\eta} = 0$ and $\nabla \cdot \eta$,
- alternative: $\overline{\xi^* \dot{\xi}} = 0$, where ξ^* is the pull-back.

Geometric view of GLM and glm

Kinematics: ensemble of flow maps $\phi = \phi^\omega : M \to M$.
Decompose flow maps into mean and perturbation

$$\phi = \xi \circ \bar{\phi}\,.$$

Taking the time derivative

$$\dot{\xi} \circ \xi^{-1} + \xi_* \bar{u}^{\mathrm{L}} = u, \quad \text{where } \bar{u}^{\mathrm{L}} = \dot{\bar{\phi}} \circ \bar{\phi}^{-1}$$

ξ_* is the push-forward: $(\xi_* u)^i(x) = (\partial_j \xi^i u^j)(\xi^{-1} x)$.

Need a constraint on ξ to define $\bar{\phi}$:

- GLM (Andrews & McIntyre): $\bar{\xi} = 0$, not geometric,
- glm (Soward & Roberts): $\xi = \mathrm{e}^\eta$ for a vector field η with $\bar{\eta} = 0$ and $\nabla \cdot \eta$,
- alternative: $\overline{\xi^* \dot{\xi}} = 0$, where ξ^* is the pull-back.

Geometric view of GLM and glm

Dynamics: 3D Euler, in terms of the velocity one-form $\upsilon = u_\flat$, dual to u (wrt a metric),

$$\partial_t \upsilon + \mathcal{L}_u \upsilon = -\mathrm{d}\pi, \quad \text{i.e.,} \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\phi^* \upsilon\right) = -\mathrm{d}\left(\phi^* \pi\right).$$

Kelvin's circulation theorem follows:

$$\oint_{\phi \mathcal{C}_0} \upsilon = \oint_{\mathcal{C}_0} \phi^* \upsilon = \text{const.}$$

Averaging leads to a mean-circulation theorem

$$\overline{\oint_{\xi(\bar{\phi}\mathcal{C}_0)} \upsilon} = \oint_{\bar{\phi}\mathcal{C}_0} \overline{\xi^* \upsilon} = \oint_{\bar{\phi}\mathcal{C}_0} \bar{\upsilon}^{\mathrm{L}} = \text{const.}$$

The circulation of the Lagrangian-mean one-form $\bar{\upsilon}^{\mathrm{L}} = \overline{\xi^* \upsilon}$ along contours moving with velocity $\bar{u}^{\mathrm{L}}$ is conserved:

$$\partial_t \bar{\upsilon}^{\mathrm{L}} + \mathcal{L}_{\bar{u}^{\mathrm{L}}} \bar{\upsilon}^{\mathrm{L}} = -\mathrm{d}(\cdots).$$

Geometric view of GLM and glm

Wave-mean flow interaction = relation between $\bar{u}^{\mathrm{L}}$ and $\bar{v}^{\mathrm{L}}$.

Pseudomomentum: $p = \bar{v}^{\mathrm{L}} - \left(\bar{u}^{\mathrm{L}}\right)_\flat$.

Simple relation if $\bar{u}^{\mathrm{L}} = \overline{\xi^* u}$ so that $p = \overline{\xi^* u_\flat} - \left(\overline{\xi^* u}\right)_\flat$:

- GLM: $\bar{u}^{\mathrm{L}}(x) = \overline{u(x + \xi(x))}$ is a coordinate dependent version,
- glm: $\bar{u}^{\mathrm{L}} \neq \overline{\xi^* u}$,
- alternative: $\bar{u}^{\mathrm{L}} = \overline{\xi^* u}$, but mean drifts from ensemble (for $u = O(\epsilon)$, ξ grows secularly).

Soward & Robert's glm appears to be a good compromise.

In practice, need to use coordinates and work pertubatively: $u = \bar{u} + \epsilon u'$ and use Lie-series (cf classical averaging).

Geometric view of GLM and glm

Wave-mean flow interaction = relation between $\bar{u}^{\mathrm{L}}$ and $\bar{v}^{\mathrm{L}}$.

Pseudomomentum: $p = \bar{v}^{\mathrm{L}} - \left(\bar{u}^{\mathrm{L}}\right)_\flat$.

Simple relation if $\bar{u}^{\mathrm{L}} = \overline{\xi^* u}$ so that $p = \overline{\xi^* u_\flat} - \left(\overline{\xi^* u}\right)_\flat$:

- GLM: $\bar{u}^{\mathrm{L}}(x) = \overline{u(x + \xi(x))}$ is a coordinate dependent version,
- glm: $\bar{u}^{\mathrm{L}} \neq \overline{\xi^* u}$,
- alternative: $\bar{u}^{\mathrm{L}} = \overline{\xi^* u}$, but mean drifts from ensemble (for $u = O(\epsilon)$, ξ grows secularly).

Soward & Robert's glm appears to be a good compromise.

In practice, need to use coordinates and work pertubatively: $u = \bar{u} + \epsilon u'$ and use Lie-series (cf classical averaging).

Near-inertial waves

Inertia-gravity waves: fast waves with dispersion relation

$$\omega = \pm(f^2 + N^2k^2/m^2)^{1/2} \quad \text{or} \quad \omega = \pm f(1 + r_\mathrm{d}^2k^2)^{1/2}$$

with r_d radius of deformation ($= NH/(nf\pi)$).

Oceanic inertia-gravity waves important for:

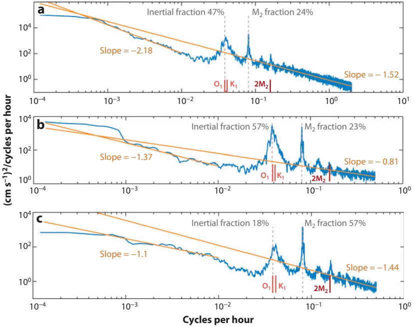
- vertical motion $\Rightarrow$ biology,
- vertical shear, instability, turbulence $\Rightarrow$ diapycnal mixing,
- mixing $\Rightarrow$ pollutant dispersion,
- large-scale ocean circulation, through diapycnal mixing (Munk & Wunsch 2009) and dissipation (Gertz & Straub 2009).

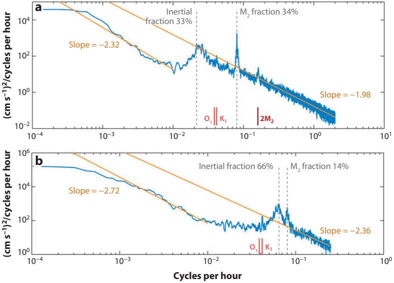
Sources: tides, topography, winds…

Near-inertial waves

Inertia-gravity-wave spectrum is dominated by lowest frequencies: near-inertial waves, NIWs:

$$\omega \approx f, \quad k/m \ll N/f, \quad kr_d \ll 1.$$





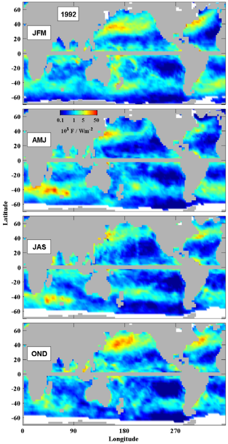
Kinetic energy from current meters at 27° N (150, 1500, 4000 m), 15° N (1000 m) and 50° S (1000 m; Fu et al 1983; Phillips & Rintoul 2000; Ferrari & Wunsch 2009).

Near-inertial waves

About 50% of wave energy in NIWs:

- generated by winds (low frequency) affecting the mixed layer ($k/m \ll 1$),
- f lowest frequency available for resonant interactions,
- subharmonic instability of M_2 tide.

Alford 2003



'Despite their ubiquity, energy, and many years of study, much about the behavior of inertial waves remains obscure.' (Ferrari & Wunsch 2009)

Near-inertial waves

Main issues:

- NIW propagation into ocean interior (weak dispersion),
- role of mean flow in this propagation,
- generation of small vertical scales,
- impact of NIWs on mean flow.

Main theoretical tools: linear wave dynamics,

- WKB approximation (Kunze 1985): takes $kL_{\text{flow}} \gg 1$, but $kL_{\text{flow}} \lesssim 1$,
- Young-Ben Jelloul model (1997): assumes $\omega \approx f$, $kL_{\text{flow}} = O(1)$ to describe slow modulation of NIWs.

Derivation of a wave-mean flow model,
coupling the Young-Ben Jelloul and quasigeostrophic models.

Near-inertial waves

Main issues:

- NIW propagation into ocean interior (weak dispersion),
- role of mean flow in this propagation,
- generation of small vertical scales,
- impact of NIWs on mean flow.

Main theoretical tools: linear wave dynamics,

- WKB approximation (Kunze 1985): takes $kL_{\text{flow}} \gg 1$, but $kL_{\text{flow}} \lesssim 1$,
- Young-Ben Jelloul model (1997): assumes $\omega \approx f$, $kL_{\text{flow}} = O(1)$ to describe slow modulation of NIWs.

Derivation of a wave-mean flow model,
coupling the Young-Ben Jelloul and quasigeostrophic models.

Near-inertial waves

Main issues:

- NIW propagation into ocean interior (weak dispersion),
- role of mean flow in this propagation,
- generation of small vertical scales,
- impact of NIWs on mean flow.

Main theoretical tools: linear wave dynamics,

- WKB approximation (Kunze 1985): takes $kL_{\text{flow}} \gg 1$, but $kL_{\text{flow}} \lesssim 1$,
- Young-Ben Jelloul model (1997): assumes $\omega \approx f$, $kL_{\text{flow}} = O(1)$ to describe slow modulation of NIWs.

Derivation of a wave-mean flow model, coupling the Young-Ben Jelloul and quasigeostrophic models.

Coupled model

Impact of NIWs on mean flow:

- non-dissipative framework,
- time-scale separation $U/(fL) \ll 1$ provides a natural averaging,
- slow modulation of NIW amplitude and mean flow on the same time scale,
- no spatial scale separation,
- averaged model that respects dynamical constraints (momentum, energy conservation, circulation. . .).

Recipe: combine glm (Soward & Roberts 2010), Salmon's variational GLM (2013), and Whitham averaging.

Coupled model

Impact of NIWs on mean flow:

- non-dissipative framework,
- time-scale separation $U/(fL) \ll 1$ provides a natural averaging,
- slow modulation of NIW amplitude and mean flow on the same time scale,
- no spatial scale separation,
- averaged model that respects dynamical constraints (momentum, energy conservation, circulation…).

Recipe: combine glm (Soward & Roberts 2010), Salmon's variational GLM (2013), and Whitham averaging.

Coupled model

Start with hydrostatic Boussineq Lagrangian

$$\mathcal{L}[\boldsymbol{x}, p] = \int \left(\frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \left(fy + \frac{\beta y^2}{2}\right)\dot{x} + bz + p\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}} - 1\right)\right) \mathrm{d}\boldsymbol{a}$$

and introduce $\boldsymbol{x}(\boldsymbol{a}, t) = \boldsymbol{X}(\boldsymbol{a}, t) + \boldsymbol{\xi}(\boldsymbol{X}(\boldsymbol{a}, t), t)$.

To leading order, ξ describes NIWs:
$\partial_t \xi^{(1)} - f\eta^{(1)} = 0, \quad \partial_t \eta^{(1)} + f\xi^{(1)} = 0, \quad \xi_x^{(1)} + \eta_y^{(1)} + \zeta_z^{(1)} = 0.$

Solve in terms of the NIW amplitude: $M(x, y, z, t)$, with

$$\xi^{(1)} + \mathrm{i}\eta^{(1)} = M_z \mathrm{e}^{-\mathrm{i}ft}, \quad \zeta^{(1)} = -\tfrac{1}{2}(\partial_x - \mathrm{i}\partial_y)M\mathrm{e}^{-\mathrm{i}ft} + \text{c.c.}.$$

Whitham average, using $\overline{\boldsymbol{\xi}^{(2)}} = \frac{1}{2}\overline{\boldsymbol{\xi}^{(1)} \cdot \nabla \boldsymbol{\xi}^{(1)}}$ (glm) to obtain $\bar{\mathcal{L}}[\boldsymbol{X}, M, P]$.

Coupled model

Start with hydrostatic Boussineq Lagrangian

$$\mathcal{L}[\boldsymbol{x}, p] = \int \left(\frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \left(fy + \frac{\beta y^2}{2}\right)\dot{x} + bz + p\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}} - 1\right)\right) \mathrm{d}\boldsymbol{a}$$

and introduce $\boldsymbol{x}(\boldsymbol{a}, t) = \boldsymbol{X}(\boldsymbol{a}, t) + \boldsymbol{\xi}(\boldsymbol{X}(\boldsymbol{a}, t), t)$.

To leading order, $\boldsymbol{\xi}$ describes NIWs:
$\partial_t \xi^{(1)} - f\eta^{(1)} = 0, \quad \partial_t \eta^{(1)} + f\xi^{(1)} = 0, \quad \xi_x^{(1)} + \eta_y^{(1)} + \zeta_z^{(1)} = 0.$

Solve in terms of the NIW amplitude: $M(x, y, z, t)$, with

$$\xi^{(1)} + \mathrm{i}\eta^{(1)} = M_z \mathrm{e}^{-\mathrm{i}ft}, \quad \zeta^{(1)} = -\tfrac{1}{2}(\partial_x - \mathrm{i}\partial_y)M\mathrm{e}^{-\mathrm{i}ft} + \mathrm{c.c.}.$$

Whitham average, using $\overline{\boldsymbol{\xi}^{(2)}} = \frac{1}{2}\overline{\boldsymbol{\xi}^{(1)} \cdot \nabla \boldsymbol{\xi}^{(1)}}$ (glm) to obtain $\bar{\mathcal{L}}[\boldsymbol{X}, M, P]$.

Coupled model

Variations δM give the YBJ equation (for $\beta = 0$),

$$(D_t M_z)_z + \frac{\mathrm{i}}{2f}\left(\nabla^2 P M_{zz} + P_{zz}\nabla^2 M - 2\nabla P_z \cdot \nabla M_z\right) = 0.$$

Variations $\delta \boldsymbol{X}^{-1}$ give Lagrangian-averaged primitive equations, with $\nabla_3 \cdot \bar{\boldsymbol{u}}^{\mathrm{L}} = 0$.

Assuming quasigeostrophic mean flow,

$$\bar{\boldsymbol{u}}^{\mathrm{L}} = (\nabla^{\perp}\psi, 0) = f^{-1}(\nabla^{\perp}P, 0),$$

we obtain the coupled YBJ/QG model

$$(D_t M_z)_z + \frac{\mathrm{i}}{2}\left(\nabla^2\psi M_{zz} + (\tfrac{N^2}{f} + \psi_{zz})\nabla^2 M - 2\nabla\psi_z \cdot \nabla M_z\right) = 0,$$

$$\partial_t q + \partial(\psi, q) = 0, \quad \text{with} \quad \left(\nabla^2 + \partial_z\left(\frac{f^2}{N^2}\partial_z\right)\right)\psi = q + F(M^*, M),$$

$$F(M^*, M) = \frac{\mathrm{i}f}{2}\partial(M_z^*, M_z) + \frac{f}{4}\left(2|\nabla M_z|^2 - M_{zz}\nabla^2 M^* - M_{zz}^*\nabla^2 M\right).$$

Coupled model

Variations δM give the YBJ equation (for $\beta = 0$),

$$(D_t M_z)_z + \frac{\mathrm{i}}{2f}\left(\nabla^2 P M_{zz} + P_{zz}\nabla^2 M - 2\nabla P_z \cdot \nabla M_z\right) = 0.$$

Variations $\delta \boldsymbol{X}^{-1}$ give Lagrangian-averaged primitive equations, with $\nabla_3 \cdot \bar{\boldsymbol{u}}^{\mathrm{L}} = 0$.

Assuming quasigeostrophic mean flow,

$$\bar{\boldsymbol{u}}^{\mathrm{L}} = (\nabla^\perp \psi, 0) = f^{-1}(\nabla^\perp P, 0),$$

we obtain the coupled YBJ/QG model

$$(D_t M_z)_z + \tfrac{\mathrm{i}}{2}\left(\nabla^2 \psi M_{zz} + (\tfrac{N^2}{f} + \psi_{zz})\nabla^2 M - 2\nabla\psi_z \cdot \nabla M_z\right) = 0,$$

$$\partial_t q + \partial(\psi, q) = 0, \quad \text{with} \quad \left(\nabla^2 + \partial_z\left(\tfrac{f^2}{N^2}\partial_z\right)\right)\psi = q + F(M^*, M),$$

$$F(M^*, M) = \tfrac{\mathrm{i}f}{2}\partial(M_z^*, M_z) + \tfrac{f}{4}\left(2|\nabla M_z|^2 - M_{zz}\nabla^2 M^* - M_{zz}^*\nabla^2 M\right).$$

Coupled model

The model is Hamiltonian, conserves action and energy:

$$\mathcal{A} = \int |M_z|^2 \, \mathrm{d}\boldsymbol{x} = \text{NIW kinetic energy},$$

$$\mathcal{H} = \frac{1}{2} \int \left(|\nabla\psi|^2 + \frac{f^2}{N^2}(\partial_z\psi)^2 + \frac{N^2}{2}|\nabla M|^2 \right) \mathrm{d}\boldsymbol{x}$$

$$= \text{QG energy + NIW potential energy} \quad .$$

- evolution governed by PV q and NIW amplitude M,
- advecting velocity $\nabla^\perp\psi$ depends on both q and M,
- energy $\mathcal{H}$ is simple in terms of ψ, complicated in terms of q.

Physical implications:

- $\mathcal{A} =$ const: no spontaneous NIW generation,
- $\mathcal{H} =$ const: mean-flow energy decays as $|\nabla M|$ increases.

Coupled model

The model is Hamiltonian, conserves action and energy:

$$\mathcal{A} = \int |M_z|^2 \, \mathrm{d}\boldsymbol{x} \quad = \text{NIW kinetic energy},$$

$$\mathcal{H} = \frac{1}{2} \int \left(|\nabla\psi|^2 + \frac{f^2}{N^2} (\partial_z \psi)^2 + \frac{N^2}{2} |\nabla M|^2 \right) \mathrm{d}\boldsymbol{x}$$

$$= \text{QG energy} + \text{NIW potential energy} \qquad .$$

- evolution governed by PV q and NIW amplitude M,
- advecting velocity $\nabla^\perp\psi$ depends on both q and M,
- energy $\mathcal{H}$ is simple in terms of ψ, complicated in terms of q.

Physical implications:

- $\mathcal{A} = \text{const}$: no spontaneous NIW generation,
- $\mathcal{H} = \text{const}$: mean-flow energy decays as $|\nabla M|$ increases.

Conclusion

Lagrangian mean theories

- think geometrically, avoid coordinate-dependent objects,
- compact notation, unpack only when needed,
- advantages of glm for incompressible fluids.

Near-inertial waves

- use glm in Lagrangian to derive a coupled YBJ-QG model,
- a Hamiltonian subgrid scale model (cf. Gjaja & Holm 1996),
- formulation well suited for numerical integration,
- energy transfer mean flow $\rightarrow$ NIWs: significant in the ocean?
- shallow-water version.