Inverse cascade and mean flows in wave turbulence

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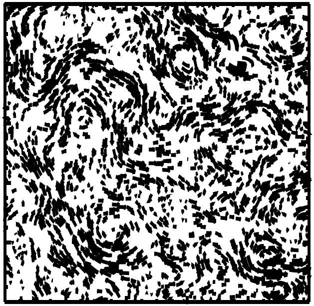
Many thanks to

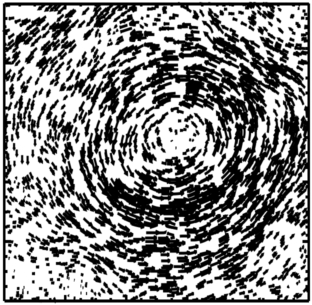
Evgenii A. Kuznetsov
Gregory Falkovich
Pavel Lushnikov
Alexander Korotkevich

for explaining the foundations of the field

References cited in this talk are incomplete and subjective.

Mean flows — coherent structures — condensate

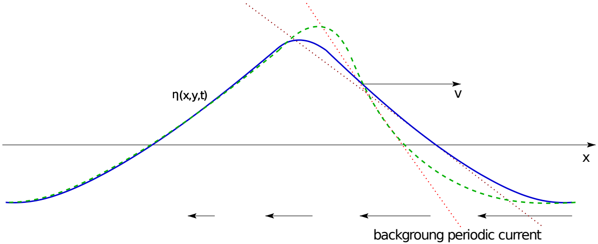




System-size vortex created by inverse cascade as a result of small-scale excitation in experiments by M. Shats at al.

Shats, Xia, Punzmann, Falkovich (2007)

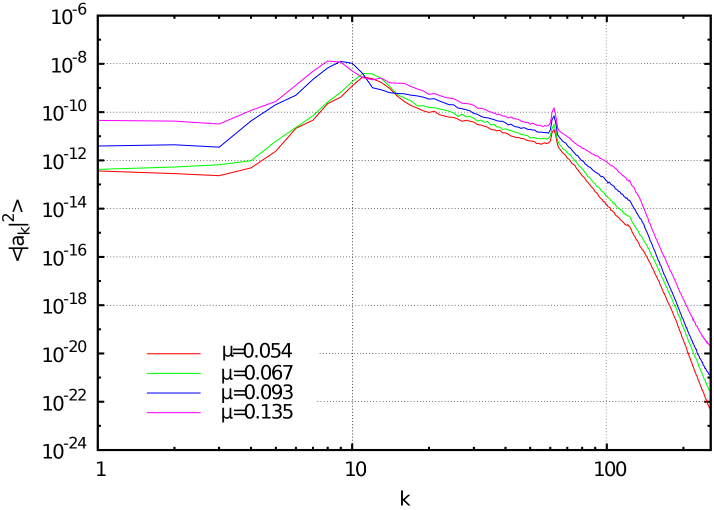
Gravity waves on water surface (A. Korotkevich)



- Formulation in terms of surface elevation $\eta(\mathbf{r}, t)$ and velocity potential on the surface, $\Phi = \phi(\mathbf{r}, \eta, t)$, where $\mathbf{v} = \nabla\phi$.
- Hamiltonian is expanded in powers of steepness, $\mu = \sqrt{|\nabla\eta|^2}$.
- Complex canonical (normal) variables a_k are introduced instead of real $\Phi(\mathbf{r}, t)$ and $\eta(\mathbf{r}, t)$.
- a_k is an elementary excitation (plane wave). Inverse cascase of $|a_k|^2$ is studied.

A. Korotkevich <alexkor@math.unm.edu>, private communications (2014)

Energy spectra of gravity waves (A. Korotkevich)



Mid-range forcing and small-scale damping result in establishing of direct and inverse cascades and accumulation of wave action at small **k**.

Nonlinear Schrödinger (Gross-Pitaevski) equation

$$i\psi_t + \nabla^2\psi \pm |\psi|^2\psi = 0$$

describes the evolution of a temporal envelope of a spectrally narrow wave packet, independent of the origin of the waves and the nature of the nonlinearity

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Benney & Newell (1967) — general settings
Zakharov (1968) — deep water waves
Hasegawa & Tappert (1973) — optical fibers

Why universal?

Linear wave:

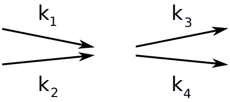
$$\frac{\partial a}{\partial t} + v\frac{\partial a}{\partial x} = 0$$

$$\frac{\partial a_k}{\partial t} + i\omega a_k = 0$$

$$\frac{\partial a_k}{\partial t} = -i\frac{\partial H_2}{\partial a_k^*}$$

$$H_2 = \int \omega_k |a_k|^2 d\mathbf{k}$$

Nonlinearity:



$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$$

$$\mathbf{k} = \mathbf{k}_0 + \mathbf{q}_k, \quad q_k \ll k_0$$

$$H_4 = \dots$$

$$H = H_2 + H_4 = H_2 + \int T_{1234}\, a_1 a_2 a_3^* a_4^*\, \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4$$

Rewrite $\frac{\partial a_k}{\partial t} + i\omega a_k = -i\frac{\partial H_4}{\partial a_k^*}$ for the envelope, $a_k(t) = e^{-i\omega_0 t}\psi(q,t)$,

$$\frac{\partial \psi_q}{\partial t} - i\omega_0\psi_q + i\omega(q)\psi_q = -iT\int \psi_1^*\psi_2\psi_3\, \delta(\mathbf{q} + \mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}_3$$

Why universal?

$$i\frac{\partial \psi_q}{\partial t} + \omega_0\psi_q - \omega(q)\psi_q = T\int \psi_1^*\psi_2\psi_3\,\delta(\mathbf{q}+\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}_3$$

Assume $\omega = \omega(k)$ and expand for small $\mathbf{q}$

$$\omega(q) = \omega_0 + q_i\left(\frac{\partial\omega}{\partial k_i}\right)_0 + \frac{1}{2}q_i q_j\left(\frac{\partial^2\omega}{\partial k_i\partial k_j}\right)_0 = \omega_0 + vq_\parallel + \frac{1}{2}\left(\omega'' q_\parallel^2 + \frac{v}{k_0}q_\perp^2\right)$$

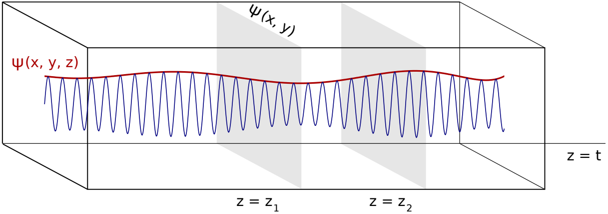
Back to r-space ($\mathbf{k}_0 \parallel \hat{\mathbf{z}}$):

$$\underbrace{i\left(\frac{\partial\psi}{\partial t} + v\frac{\partial\psi}{\partial z}\right)}_{\frac{\partial\psi}{\partial t}\text{ in moving frame}} + \underbrace{\frac{\omega''}{2}\frac{\partial^2\psi}{\partial z^2}}_{\text{dispersion}} + \underbrace{\frac{v}{2k_0}\nabla_\perp^2\psi}_{\text{diffraction}} = \underbrace{T|\psi|^2\psi}_{\text{nonlinearity}}$$

Rescale ψ and spatial coordinates:

$$\boxed{i\psi_t + \nabla^2\psi \pm |\psi|^2\psi = 0}$$

Connection to nonlinear optics



$$\boxed{\frac{1}{c^2}\left(\epsilon E\right)_{tt} - \nabla^2 E = 0}$$

Stationary envelope: $E = \frac{1}{2}\psi(x,y,z)e^{ikz-i\omega t}$, with $\omega = \frac{kc}{\sqrt{\epsilon_0}}$.

Kerr nonlinearity: $\epsilon = \epsilon_0 + \epsilon_2|E|^2 = \epsilon_0 + \epsilon_2|\psi|^2$.

$$\frac{1}{c^2}(i\omega)^2(\epsilon_0 + \epsilon_2|\psi|^2)\psi - \left[\nabla^2\psi + 2ik\psi_z - k^2\psi\right] = 0$$

Neglecting $\frac{\partial^2\psi}{\partial z^2}$ and using $kx \to x$, $\frac{1}{2}kz \to z$, and $\psi|\frac{\epsilon_2}{k\epsilon_0}|^{\frac{1}{2}} \to \psi$,

$$\boxed{i\psi_z + \nabla_\perp^2\psi - T|\psi|^2\psi = 0,} \qquad \text{with} \quad T = \pm 1$$

Connection to hydrodynamics

$$i\psi_t + \nabla^2\psi - T|\psi|^2\psi = 0$$

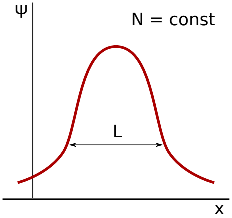
Change of variables: $\psi = Ae^{i\phi}, \quad \rho = A^2, \quad \mathbf{v} = 2\nabla\phi.$

$$\begin{aligned} \mathbf{v}_t + \nabla\frac{|\mathbf{v}|^2}{2} &= -\frac{1}{\rho}\nabla p \\ \rho_t + \nabla(\rho\mathbf{v}) &= 0 \end{aligned}$$

"Equation of state":

$$\frac{1}{\rho}\nabla p = \nabla\left[2T\rho - \frac{1}{\sqrt{\rho}}\nabla^2\sqrt{\rho}\right]$$

Collapses in focusing NSE



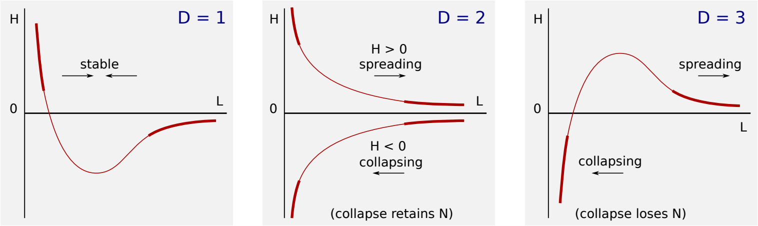
$$i\psi_t + \nabla^2\psi + |\psi|^2\psi = 0$$

Integrals of motion

$$N = \int |\psi|^2 \, d^D r$$
$$\mathcal{H} = \int \left(|\nabla\psi|^2 - \tfrac{1}{2}|\psi|^4 \right) \, d^D r$$

Within the packet

$$|\psi|^2 \sim N/L^D$$
$$\mathcal{H} \sim NL^{-2} - N^2 L^{-D}$$

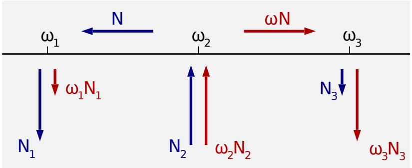


Zakharov & Kuznetsov (1986)

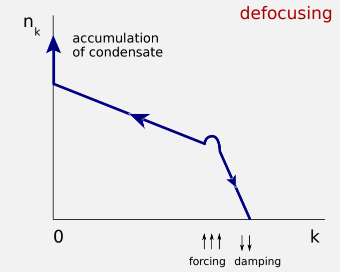
Cascades of turbulence

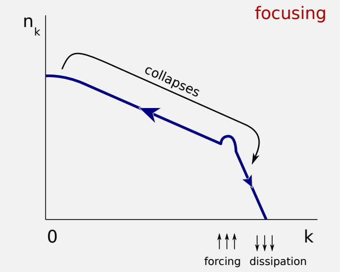
$$\mathcal{H} = \int \omega_k |a_k|^2 d\mathbf{k}$$
$$N = \int |a_k|^2 dk$$

$$N_1 + N_3 = N_2$$
$$\omega_1 N_1 + \omega_3 N_3 = \omega_2 N_2$$



$$N_1 = N_2 \frac{\omega_3 - \omega_2}{\omega_3 - \omega_1} \approx N_2 \qquad \omega_1 N_1 \ll \omega_2 N_2$$
$$N_3 = N_2 \frac{\omega_2 - \omega_1}{\omega_3 - \omega_1} \ll N_2 \qquad \omega_3 N_3 \approx \omega_2 N_2$$





Dyachenko, Newell, Pushkarev, & Zakharov (1992)

Modulational instability

$$\boxed{i\psi_t = -\tfrac{1}{2}\omega''\nabla^2\psi + T|\psi|^2\psi}$$

Exact solution (condensate):

$$\Psi = \sqrt{N_0}e^{-iTN_0 t}$$

For small perturbation $\psi := \Psi + \psi$,

$$i\psi_t = -\tfrac{1}{2}\omega''\nabla^2\psi + 2TN_0\psi + T\Psi^2\psi^* + O(|\psi|^2).$$

In k-space, using $(\psi^*)_k = \psi^*_{-k}$,

$$\begin{aligned} i\tfrac{d}{dt}\psi_k &= \left(\tfrac{1}{2}\omega'' k^2 + 2TN_0\right)\psi_k + T\Psi^2\psi^*_{-k}, \\ -i\tfrac{d}{dt}\psi^*_{-k} &= \left(\tfrac{1}{2}\omega'' k^2 + 2TN_0\right)\psi^*_{-k} + T\Psi^2\psi_k. \end{aligned}$$

Modulational instability

Looking for the solution in the form

$$\psi_k = \alpha e^{-i(TN_0+\Omega_k)t} \qquad \text{and} \qquad \psi^*_{-k} = \beta e^{i(TN_0-\Omega_k)t},$$

rewrite the system as

$$\begin{pmatrix} \frac{1}{2}\omega'' k^2 + TN_0 - \Omega_k & T\Psi^2 \\ T\Psi^{*2} & \frac{1}{2}\omega'' k^2 + TN_0 + \Omega_k \end{pmatrix} \begin{pmatrix} \alpha\, e^{-iTN_0 t} \\ \beta\, e^{iTN_0 t} \end{pmatrix} = 0$$

Bogoliubov dispersion relation:

$$\boxed{\Omega_k^2 = \omega'' TN_0 k^2 + \tfrac{1}{4}\omega''^{\,2} k^4}$$

Instability: $\omega'' T < 0$ (focusing nonlinearity).

Bogoliubov (1947)

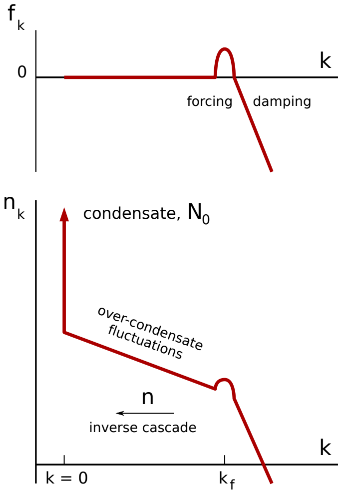
Why turbulence?

- Wide energy spectra; cascades
- Statistical description
- High probability of extreme events (intermittency)
- Coherent structures — condensate or collapses
- Steady (with damping/forcing) or decaying

The rest of this talk is devoted to wave-condensate interactions in steady 2D turbulence described by defocusing NSE.

Defocusing nonlinear Schrödinger equation

$$\boxed{i\psi_t + \nabla^2\psi - |\psi|^2\psi = i\hat{f}\psi}$$



Condensate

$$\Psi = \sqrt{N_0}\exp(-iN_0 t)$$

Notation:

$$N = \overline{|\psi|^2}$$
$$N_0 = |\overline{\psi}|^2$$
$$n = N - N_0 = \int |\psi_k|^2 d^2k$$

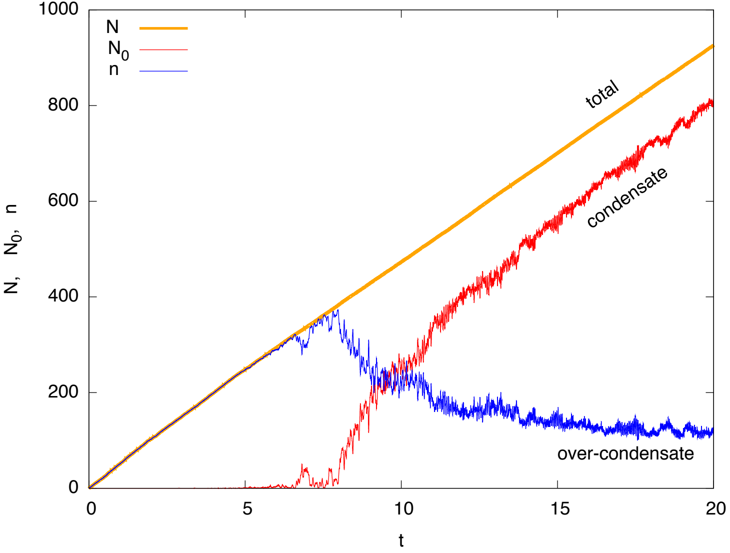
We consider large condensate

$$N_0 \gg n$$

Statistically quasi-steady

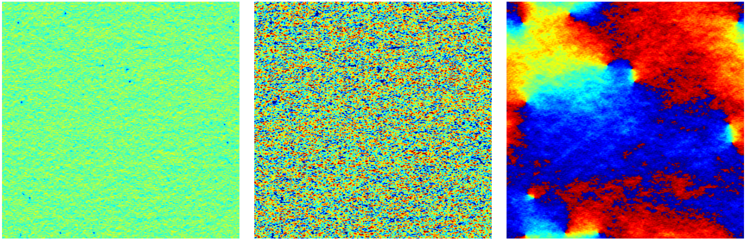
$$t \sim 10^4 \gg \frac{1}{\omega} \sim 10^{-3}$$

Onset of condensate

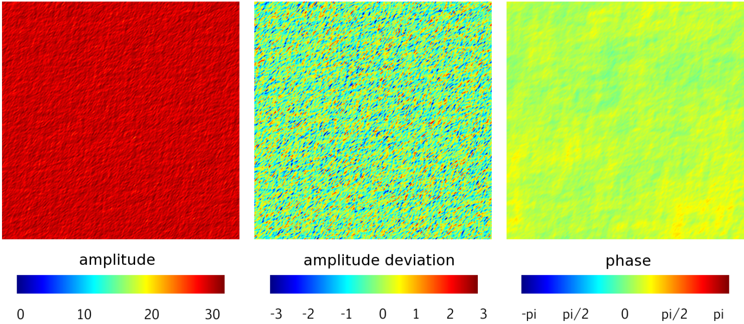


Onset of condensate

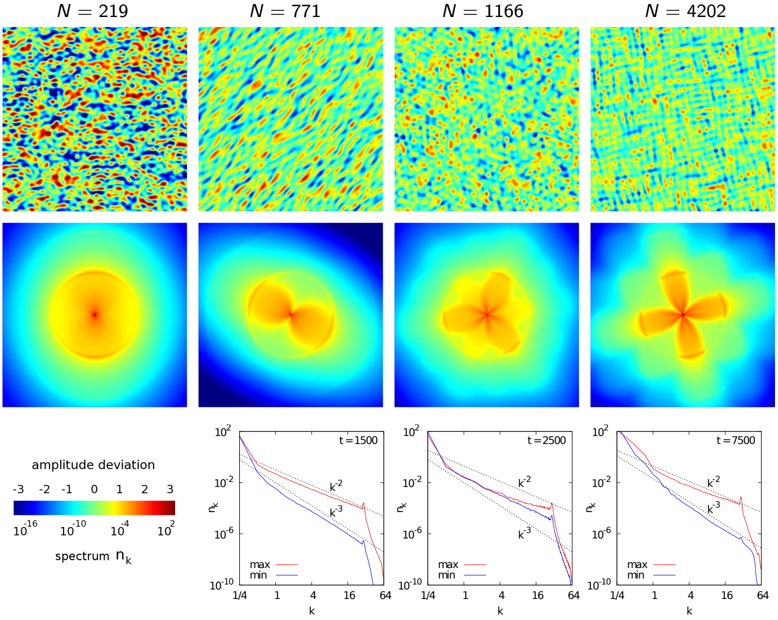
$t = 100: \quad N_0 = 58, \quad n = 160$



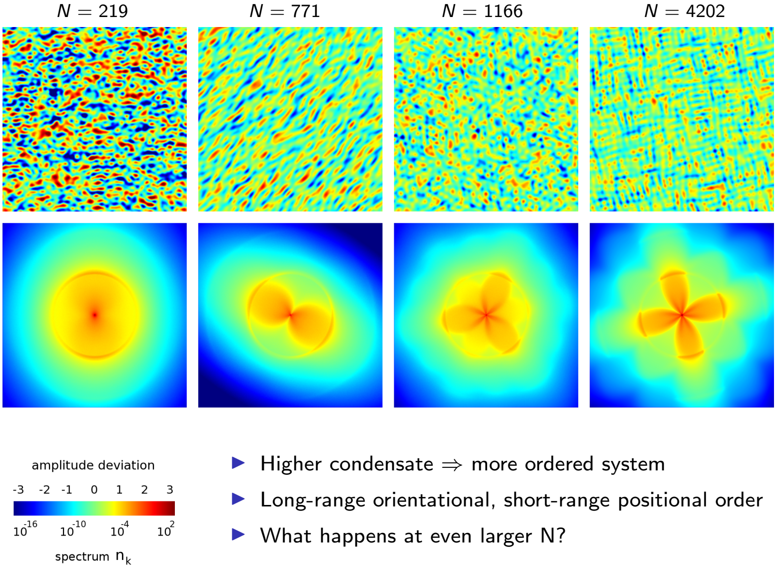
$t = 1500: \quad N_0 = 751, \quad n = 20$



Phase transitions: breakdown of symmeries



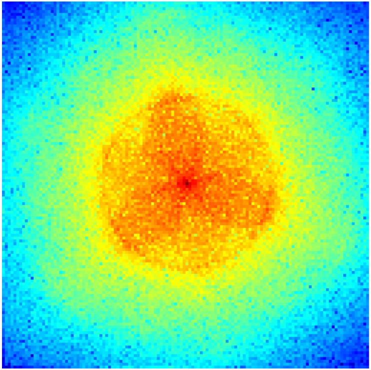
Phase transitions: breakdown of symmeries

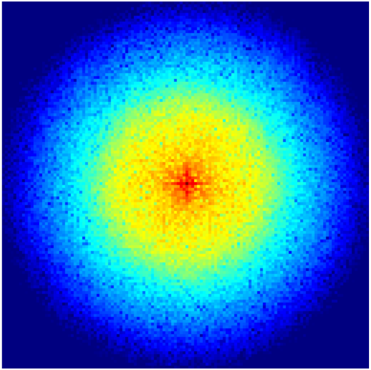


- Higher condensate ⇒ more ordered system
- Long-range orientational, short-range positional order
- What happens at even larger N?

Vladimirova, Derevyanko, & Falkovich (2012)

Effect of forcing





Instability-driven force

$$i\psi_t + \nabla^2\psi - |\psi|^2\psi = i\hat{f}\psi$$

Random force

$$i\psi_t + \nabla^2\psi - |\psi|^2\psi = i\hat{F}$$

Small perturbations

Compare quadratic and cubic terms in Hamiltonian

$$
\begin{aligned}
\langle \mathcal{H}_2 \rangle &= \Omega_k n = N_0^{1/2} k n \\
\langle \mathcal{H}_3 \rangle &= \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} V_{123} \langle \psi_{k_1} \psi_{k_2} \psi^*_{k_3} \rangle \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3) \\
&\simeq \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} |V_{123}|^2 n_1 n_2 \, \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3) \delta(\Omega_1 + \Omega_2 - \Omega_3) \\
&\simeq \frac{|V|^2 n^2 c}{k^3} \frac{k}{c} \simeq \frac{n^2 k}{N_0^{1/2}}
\end{aligned}
$$

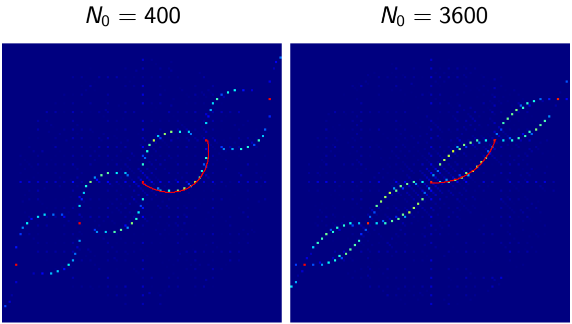
Effective nonlinearity parameter is small,

$$
\frac{\mathcal{H}_3}{\mathcal{H}_2} \simeq \frac{n}{N_0}.
$$

But: weak turbulence assumes random phases.

Angle of interaction: $k/c \sim k/\sqrt{N_0}$, where $c = \sqrt{2N_0}$.

Angle of interaction



Arch grows in k-space from the condensate to a preset mode, $\mathbf{k}_0$.
Arch equation:

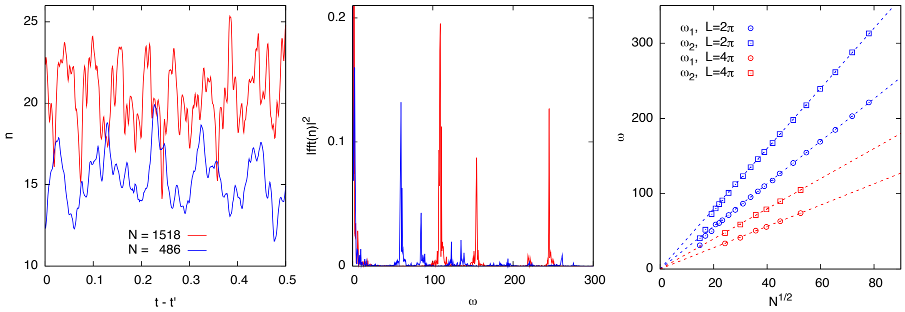
$$\begin{aligned} \omega(k_0) &= \omega(k) + \omega(|\mathbf{k}_0 - \mathbf{k}|) \\ \omega^2(k) &= 2N_0 k^2 + k^4 \end{aligned}$$

Angle of interaction:

$$\phi_{max} \approx \frac{k}{\sqrt{3N_0/2}} \sim \frac{k}{c}$$

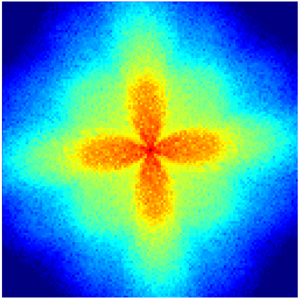
Condensate-turbulence oscillations

- The system periodicly oscillates around a steady state.
- Turbulence and condensate exchange a small fraction of waves.
- Predator-prey model?

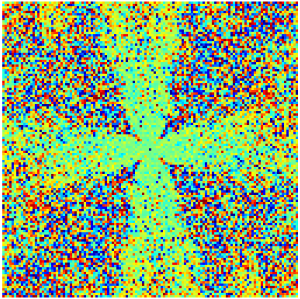


Phase coherence

n_k



$\theta_k = 2\phi_0 - \phi_k - \phi_{-k}$



$$\boxed{2\phi_0 - \phi_k - \phi_{-k} = \pi}$$

Three-wave model

Consider condensate interacting with two waves

$$\psi_{\pm k} = \sqrt{n}\exp(\pm ikx + iN_0 t + i\phi_{\pm k})$$

with $\theta = 2\phi_0 - \phi_k - \phi_{-k}$.

Hamiltonian:

$$H = 2k^2 n + \frac{1}{2}N^2 + 2n(N-2n)(1+cos\theta) + n^2$$

Equations of motion:

$$\begin{aligned} \dot{n} &= 2n(N-2n)\sin\theta \\ \dot{\theta} &= 2k^2 + 2(N-3n) + 2(N-4n)\cos\theta \end{aligned}$$

Stability points:

$\theta = \pi,$	$n = -\frac{1}{2}k^2$	$\Rightarrow$	unphysical
$\theta = 0,$	$n = (4N + k^2)/14$	$\Rightarrow$	too high n

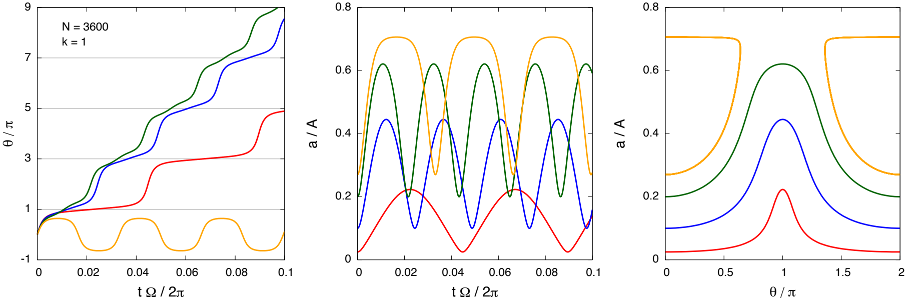
Falkovich (2011), Miller, Vladimirova & Falkovich (2013)

Predictions of three-wave model

$$\begin{aligned} \dot{n} &= 2n(N-2n)\sin\theta \\ \dot{\theta} &= 2k^2 + 2(N-3n) + 2(N-4n)\cos\theta \end{aligned}$$

For $n \ll N$:

- the system spends most of its time around $\theta = \pi$ state
- the frequency of oscillations $2\Omega \approx 2\sqrt{2Nk^2 + k^4}$
- the amplitude $a \equiv \sqrt{n(t)}$ exhibits complicated cusped shape



Individual modes in turbulence

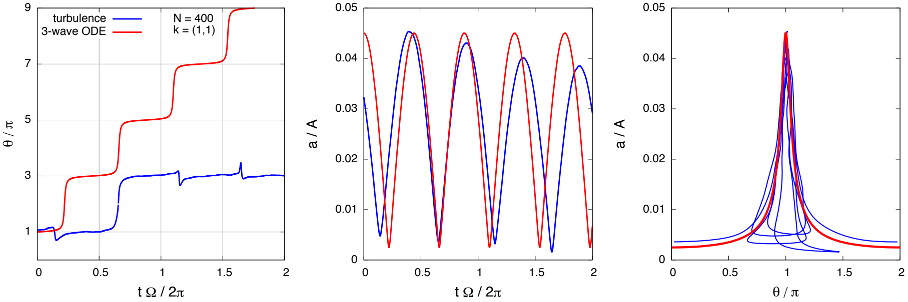
In turbulence, $n \ll N$ condition is well satisfied.

As predicted:

- the system spends most of its time around $\theta = \pi$ state
- the frequency of oscillations approaches $2\Omega = 2\sqrt{2Nk^2 + k^4}$
- the amplitude $a \equiv \sqrt{n(t)}$ exhibits complicated cusped shape

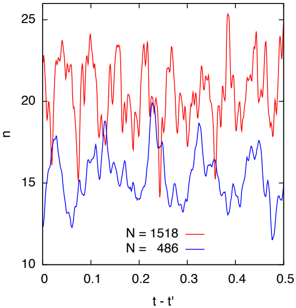
However:

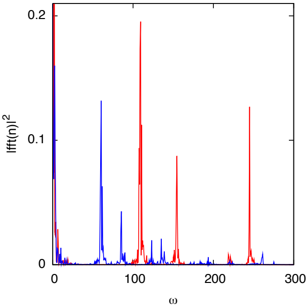
The 3-wave model cannot grasp closed trajectories with $\theta \approx \pi$.

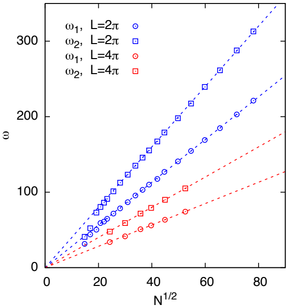


Collective oscillations

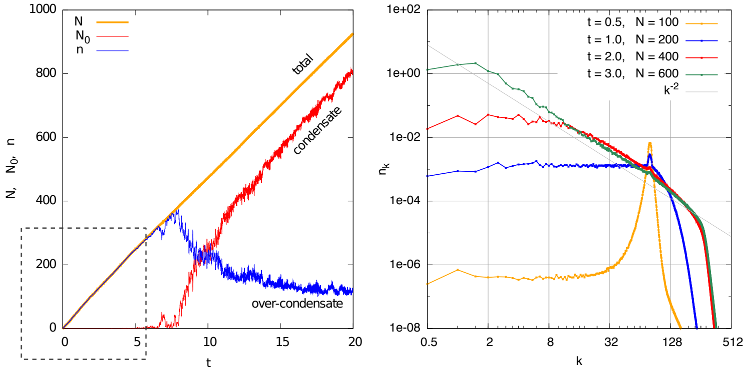
- The system periodically oscillates around a steady state.
- Turbulence and condensate exchange a small fraction of waves.
- The condensate imposes the phase coherence between the pairs of counter-propagating waves (anomalous correlation).
- Collective oscillations are not of a predator-prey type; they are due to phase coherence and anomalous correlations.





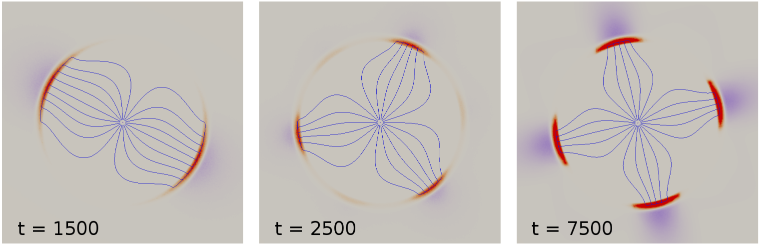


Next: inverse cascade



What is the dynamics of spectra before onset of condensate?

Next: flux in k-space



Can we directly and dynamically measure flux in k-space?

Conclusions

- We consider turbulence in the Gross-Pitaevsky model and study the creation of a coherent condensate via an inverse cascade.
- The growth of the condensate leads to a spontaneous breakdown of symmetries of small-scale over-condensate fluctuations: from the 2-fold to 3-fold to 4-fold (phase transitions).
- At the highest condensate level reached, we observe a short-range positional and long-range orientational order ("hexatic phase").
- The phase transitions happen when the driving term corresponds to an instability and does not occur when pumped by a random force.
- The condensate imposes the phase coherence between the pairs of counter-propagating waves (anomalous correlation).
- Collective oscillations are not of a predator-prey type; they are due to phase coherence and anomalous correlations.